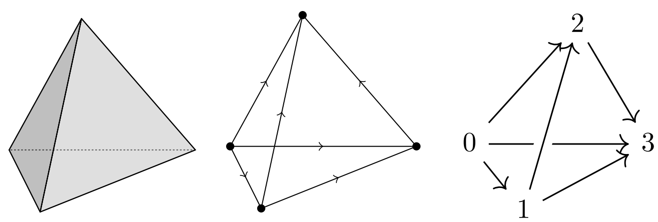
Notes on Higher Categories

Some bastards

UNDER HEAVY CONSTRUCTION



Contents

1

Introduction

These notes grew out of a course given by Prof. Tobias Dyckerhoff at the University of Hamburg. However, much of the presentation and many of the proofs are heavily adapted. In the process I have undoubtedly introduced many mistakes, all of which are entirely my own.

These notes are under very heavy construction at the moment, and the results in them should not be taken as fact. They are a part of my learning process; I write them for this reason.

I write them for another reason as well: I think that there's some pedagogical value in the writings of beginners. The proofs included here are not here because they're the most elegant or the cleanest; instead, they're here because they're the proofs that finally made me, a novice struggling with category theory for the first time, understand the material.

1 Axiomatic framework

These notes are phrased in terms of ZFCU, i.e. the Zermelo-Fraenkel axioms together with the following axioms.

- C: the axiom of choice.
- U: Grothendieck's universe axiom.

An introduction to category theory in the language of Grothendieck universes can be found in the corresponding notes on ordinary category theory.[1]

[1] https://github.com/angusrush/notes-public/blob/master/category_theory/notes.pdf

2 Notational inconsistencies

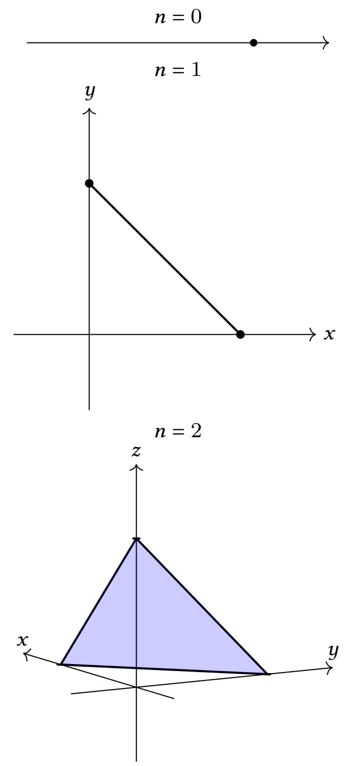
- The unit and counit are sometimes called ϵ and η respectively, and sometimes the other way around. I'm going through and fixing this.

- For a functor $F\colon \mathcal{C} \to \mathcal{D}$ between ordinary categories, I denote the comma category by $(F \downarrow \mathcal{D})$. In the ∞-categorical context, I denote it by $\mathcal{D}_{F/}$. I like the first notaion better in the context of ordinary categories, but prefer the second for simplicial sets.

2

Simplicial sets

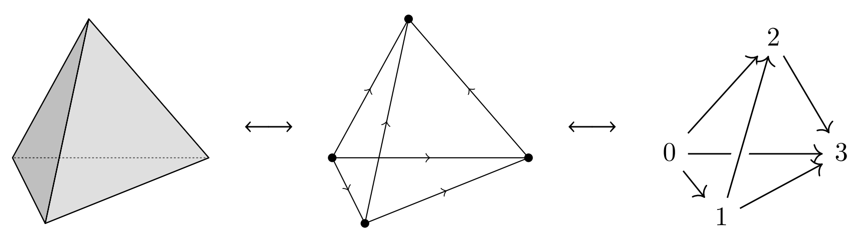
We can easily draw the first few standard n-simplices as subsets of $\mathbb{R}^{n+1}$:



The standard 3-simplex can be identified with a tetrahedron, as in the main text. However, for obvious reasons, we cannot draw it as a subset of $\mathbb{R}^4$.

1 *Simplices as topological spaces and categories*

One of the main themes in the study of higher category theory is the interplay between ordinary topology, simplicial topology, and category theory.



The fundamental topological object we will study is the *geometrical n-simplex.*

Definition 2.1 (geometrical n-simplex)**.** For any $n \geq 0$, the geometrical n-simplex, denoted $|\Delta^n|$, is the set (together with the subspace topology)

$$|\Delta^n| = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

In simplicial topology, one studies spaces glued out of geometrical simplices by replacing the topological simplices by combinatorial models of simplices. These live in a category.

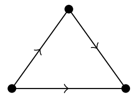
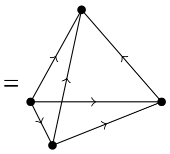
Definition 2.2 (simplex category)**.** Denote by Δ the category whose objects are linearly ordered sets

$$[n] = \{0, \dots, n\}$$

and whose morphisms $[n] \to [m]$ are weakly monotonic[1] maps.

[1] i.e. nondecreasing in the sense that $i \geq j \implies f(i) \geq f(j)$

The objects $[n]$ of Δ are to be interpreted as simplices with ordered vertices. For example,

[0] = •, [1] = , [2] = , [3] = 

We can connect our simplex category to geometric simplices using a functor.

Definition 2.3 (realization functor)**.** The realization functor

$$\rho\colon \Delta \to \mathrm{Top}$$

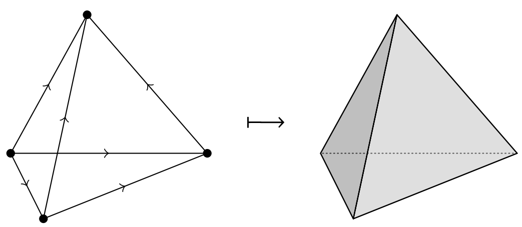
is defined on objects by $[n] \mapsto |\Delta^n|$ and on morphisms $[f]\colon [m] \to [n]$ by

$$f_*\colon |\Delta^m| \to |\Delta^n|; \qquad (t_0, \ldots, t_m) \mapsto (s_1, \ldots, s_m),$$

where

$$s_i = \begin{cases} 0, & f^{-1}(i) = \emptyset \\ \sum_{j \in J^{-1}(i)} t_j & \text{otherwise.} \end{cases}$$

That is, ρ assigns to each combinatorial n-simplex its topological counterpart.

ρ : 

One can also think of these combinatorial simplices categorically. Again, this connection is given by a functor.

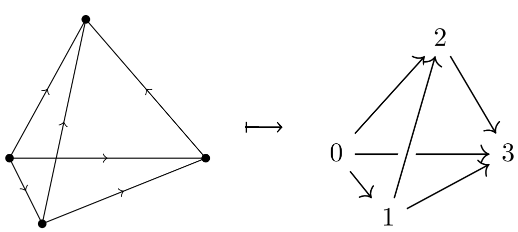
Definition 2.4 (categorification functor)**.** The categorification functor[2] is the functor

$$\chi\colon \Delta \to \mathrm{Cat}$$

which sends the object $[n]$ to the poset category on $[n]$. Note that functors $[m] \to [n]$ are the same thing as weakly monotonic maps $[m] \to [n]$.

[2] This terminology is highly-nonstandard. What we call the "decategorification functor" here is a restriction to a full subcategory of a functor usually referred to in the literature as the "fundamental category" functor.

That is, χ assigns to each combinatorial n-simplex its categorical counterpart.

χ: 

These categories and functors may seem too simple to be of any real use, but one can build astonishingly complicated structures out of them.

2 *Simplicial sets*

A simplicial set is thought of as a collection of standard simplices of various degrees, glued together along their faces. That is, a simplicial set consists of, for each $n \in \mathbb{N}$, a collection of simplices of degree n, together with data specifying which faces should be identified with other faces, and how. This data can be specified efficiently in the following way.

Definition 2.5 (simplicial set)**.** A simplicial set is a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. The category of simplicial sets is the category of such functors, i.e.

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}).$$

We denote this category by Set_Δ.

As discussed briefly in Section 2.3, one thinks of the image under the Yoneda embedding of the object $[n]$ as an n-simplex, called the *standard n-simplex.* One imagines more general simplicial sets as being built by gluing standard simplices together along their faces. Specifically, for any simplicial set K we should interpret $K([n])$ as the set of n-simplices of K. The gluing conditions come from functoriality. We will investigate this in Section 3.

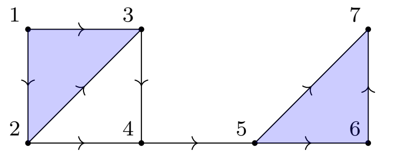
Definition 2.6 (standard simplex)**.** Let $n \geq 0$. The simplicial set

$$\Delta(-, [n]) = \text{よ}([n])$$

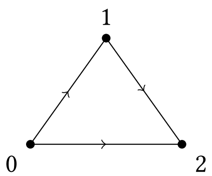
is denoted Δ^n, and called the *standard n-simplex.*

The claim is that the standard n-simplex can be interpreted geometrically, as precisely the sort of picture we have been drawing.[3] To see this, it will be helpful to explicitly analyze some cases in which n is small. Take $n = 2$. To justify the name, the standard 2-simplex had better look something like the 2-simplex we know and love.

[3] More generally, simplicial sets, when interpreted geometrically, can be viewed as one of the many ways to "build topological spaces out of triangles". The simplicial complexes common in introductory courses in (algebraic) topology are another such. However, as we will see, simplicial sets are often more flexible and easier to work with. Moreover, every an *ordered* simplicial complex C, where every 1-simplex has been equipped with a choice of orientation, as in the figure



can be canonically viewed as a simplicial set. Here we view the blue triangles as being filled by a 2-simplex, and the white ones as empty "holes".



Specifically, we should interpret $\Delta^2([n])$ as the set of n-simplices contained in Δ^2. Naïvely, we would expect three 0-simplices, three 1-simplices, and one 2-simplex.

However, the only way to find out what we actually have is to begin calculating. First, we calculate $\Delta^2([0]) = \mathrm{Set}_\Delta([0], [2])$:

$$\Delta^2([0]) = \{0 \mapsto 0, 0 \mapsto 1, 0 \mapsto 2\}.$$

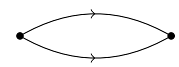
To avoid formulae whose lengths quickly get out of hand, we will drop the functional notation, denoting the above by $\{0, 1, 2\}$.

This is good. We said that $\Delta^2([0])$ should be interpreted as the set of 0-simplices, and indeed $\Delta^2([0])$ has three elements, which represent in an obvious way the vertices of the familiar 2-simplex. Now calculating $\Delta^2([1])$, we find

$$\Delta^2([1]) = \{\{0,1\} \mapsto \{0,1\}, \{0,2\}, \{1,2\}, \{0,0\}, \{1,1\}, \{2,2\}\}.$$

This is not so good; we can interpret the three edges $\{0,1\}$, $\{0,2\}$, and $\{1,2\}$ as the edges between 0 and 1, 0 and 2, and 1 and 2 that we were expecting, but we also have three more. We interpret these as 'degenerate edges,' which are all bunched up at the vertices. This is one difference between simplicial complexes and simplicial sets: simplicial sets can have degenerate edges.[4]

[4] Another difference is that an n-simplex need not be specified by its set of vertices. For example, the following picture is not a simplicial complex, but it is a simplicial set.



the reason for this is that, in a simplicial complex, there is at most one n-simplex with a given set of vertices. In the drawing above, however, both 1-simplices have the same set of vertices.

Continuing, we find

$$\Delta^2([2]) = \{\{0,0,0\}, \{0,0,1\}, \{0,0,2\}, \ldots\}.$$

We recognize these as degenerate faces. The only nondegenerate face will be $\{0,1,2\}$.

Note that contrary to what you might expect, $\Delta^n([k])$ is not empty for $k > n$; it is simply filled with an enormous number of degenerate simplices.[5]

[5] To be precise, $\binom{n+k+1}{k}$ simplices.

Example 2.7. Let X be a set. Denote the power set of X by $\mathcal{P}(X)$. Let $\mathcal{J} \subseteq \mathcal{P}(\{0,1,\ldots,n\})$. There is a simplicial set $\Delta^{\mathcal{J}}$ whose n-simplices are defined to be

$$\Delta^{\mathcal{J}}([n]) \subseteq \Delta([m],[n])$$

consisting of those f such that $\mathrm{im}(f) \subset J$ for some $J \in \mathcal{J}$.

Here are some specific examples of $\mathcal{J}$ which will be useful to us.

- If $\mathcal{J} = \mathcal{P}(\{0,\ldots,n\})$, then
  $$\Delta^{\mathcal{J}}([n]) = \Delta^n,$$
  the standard n-simplex.
- If
  $$\mathcal{J} = \mathcal{P}(\{0,1,\ldots,n\}) \setminus \{\{0,1,\ldots,n\}\},$$
  then the simplicial set we obtain is called the *boundary* of Δ^n, and denoted $\partial\Delta^n$.

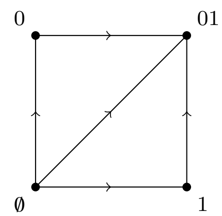
  The sets $\Delta^n([i])$ and $\partial\Delta^n([i])$ are the same for $i < n$.
- Fix some i such that $0 \leq i \leq n$. Then with
  $$\mathcal{J} = \mathcal{P}(\{0,1,\ldots,n\}) \setminus \{\{0,1,\ldots,n\}, \{0,1,\ldots,i-1,i+1,\ldots,n\}\},$$
  we obtain the *ith horn* of Δ^n, which we denote Λ^n_i.

  For example, the 2-simplices in the horn Λ^2_1 consists of the subset of monotone functions $[2] \to [2]$ whose range lies entirely within one of the sets $\{0,1,2,01,12,22\}$. That is
  $$\{000, 001, 011, 111, 112, 122, 222\}.$$

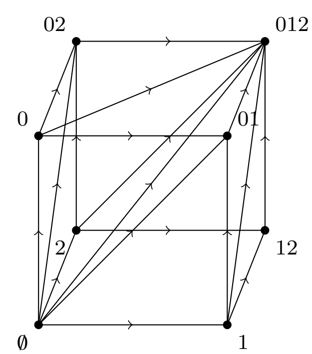
We can construct another interesting simplicial set from the power set $\mathcal{P}([n])$. The power set itself has the structure of a poset — its objects are ordered by inclusion of subsets. We can define a simplicial set $C^{[n]}$ by setting

$$C^{[n]}([k]) = \{f : [k] \to [n] \mid f \text{ order-preserving}\}$$

Drawing only non-degenerate simplices, C^1 will look like



Drawing only non-degenerate 1-simplices, C^2 looks like



Where the 2- and 3-simplices then fill the cube.

We can draw Λ^2_i, for $i = 0, 1, 2$, as follows.

$$\Lambda^2_0 = \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}, \quad \Lambda^2_1 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}, \quad \Lambda^2_2 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

Definition 2.8 (inner, outer horn)**.** If $i = 0$ or $i = n$, then we say that Λ^n_i is an outer horn. Otherwise, if $0 < i < n$, we say that Λ^n_i is an inner horn.

Definition 2.9 (simplicial complex)**.** A set $\mathcal{K} \subset \mathcal{P}(\{0, 1, \dots, n\})$ is called an simplicial complex if for every $\mathcal{J} \in \mathcal{K}$ and $I \subseteq \mathcal{J}$ with $I \neq \emptyset$, $I \in \mathcal{K}$.

Example 2.10. Let $\mathcal{K}$ be any simplicial complex. We can interpret $\mathcal{K}$ as a simplicial set by associating

$$\mathcal{K} \mapsto \Delta^{\mathcal{K}}.$$

What would have been called the simplices of $\mathcal{K}$ correspond to the nondegenerate elements of the simplicial set $\Delta^{\mathcal{K}}$.

Note that there are many simplicial sets which are not obtained from simplicial complexes in this way. That is, the theory of simplicial sets is a strict generalization of that of simplicial complexes.

Simplicial sets are functors, but the functorial notation is often unnecessarily burdensome. Therefore, it is standard to denote the n-simplices of a simplicial set K by K_n rather than $K([n])$. This is the notation that we will use from now on.

3 *Which data does a simplicial set give us?*

3.1 *Face and degeneracy maps*

Let K be a simplicial set. By definition evaluating K on the object $[n]$ gives a set K_n, which we interpret as the set of n-simplices making up K. We have claimed that the functoriality of K provides instructions for how these simplices should be glued together. In this section we will shine some light on this claim.

First, we note that we can understand how K acts on morphisms $\phi\colon [n] \to [m]$ by understanding how it behaves on simpler functions, known as the *face* and *degeneracy functions.*

Definition 2.11 (face function)**.** Let $n \geq 0$. For any k with $0 \leq k < n$, we the ith face function,

$$\partial_i\colon [n-1] \to [n]; \qquad j \to \begin{cases} j, & j < i \\ j+1, & j \geq i \end{cases}.$$

$$\sigma_k \quad = \quad \begin{tikzcd}[row sep=small]
\vdots & \vdots \\
k-2 \arrow[r, mapsto] & k-2 \\
k-1 \arrow[r, mapsto] & k-1 \\
k \arrow[r, mapsto] & k \\
k+1 \arrow[dr, mapsto] & k+1 \\
k+2 \arrow[dr, mapsto] & k+2 \\
\vdots & \vdots
\end{tikzcd}$$

Applying a simplicial set K to the ith face function gives a map from K_n to K_{n-1}.

Definition 2.12 (face map)**.** Let $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. Corresponding to the face function $\partial_i\colon [n] \to [n-1]$ (see Definition 2.11), there is a map

$$d_i\colon X_n \to X_{n-1}.$$

This map is also known as the ith face map.

If we were being pedantic, we would denote the face functions by ∂_i^n instead of ∂_i, but keeping track of these indices turns out to be more confusing than elucidating.

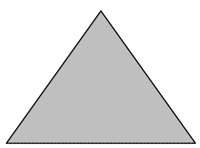
Acting on an n-simplex σ with a face map gives one of the $(n-1)$-simplices making up the boundary of σ. Recall that our combinatorial models for simplices have ordered vertices—the vertices of an n-simplex run from 0 to n. The ith face map takes an n-simplex to its ith face; that is, the face opposite the vertex labelled by i.

Example 2.13. Consider the simplicial set $K = \Delta^2$.

Take $\sigma \in \Delta^2_2$ given by the identity

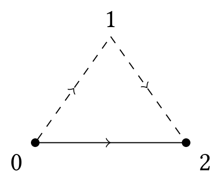
$$\sigma\colon \quad \begin{tikzcd}[row sep=small]
0 \arrow[r, mapsto] & 0 \\
1 \arrow[r, mapsto] & 1 \\
2 \arrow[r, mapsto] & 2
\end{tikzcd} .$$

This corresponds to the single nondegenerate 2-simplex in Δ^2, which is what we intuitively think of as the 'whole' 2-simplex.



Applying d_1 corresponds to pulling back by ∂_1 as follows.

$$d_1\sigma = \sigma \circ \partial_1 = \quad \begin{tikzcd}[row sep=small]
0 \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0 \\
1 \arrow[dr, mapsto] & 1 \arrow[r, mapsto] & 1 \\
 & 2 \arrow[r, mapsto] & 2
\end{tikzcd} \quad = \quad \begin{tikzcd}[row sep=small]
0 \arrow[r, mapsto] & 0 \\
1 \arrow[dr, mapsto] & 1 \\
 & 2
\end{tikzcd}$$

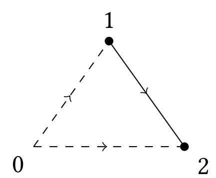


Similarly, d_0 takes σ to the edge between 1 and 2, and d_2 takes σ to the edge between 0 and 1.

Let $f \in \Delta^2_1 = \Delta([1], [2])$ be one of the six 1-simplices. For example, take

$$f\colon \quad \begin{tikzcd}[row sep=small] 0 \arrow[dr, mapsto] & 0 \\ 1 \arrow[dr, mapsto] & 1 \\ & 2 \end{tikzcd}$$

corresponding to the solid edge from 1 to 2.



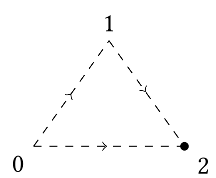
Again we compute. The face map d_0 comes from the face function ∂_0, which we can draw like this.

$$\partial_0\colon \quad \begin{tikzcd}[row sep=small] 0 \arrow[dr, mapsto] & 0 \\ & 1 \end{tikzcd}$$

The map d_0 acts by precomposing with ∂_0.

$$d_0 f = f \circ \partial_0 = \quad \begin{tikzcd}[row sep=small] 0 \arrow[dr, mapsto] & 0 \arrow[dr, mapsto] & 0 \\ & 1 \arrow[dr, mapsto] & 1 \\ & & 2 \end{tikzcd} \quad = \quad \begin{tikzcd}[row sep=small] 0 \arrow[ddr, mapsto] & 0 \\ & 1 \\ & 2 \end{tikzcd}$$

Therefore, d_0 takes the edge between 1 and 2 to the vertex 2.



Put more concisely, and as should now be clear from the form of ∂_0, we can see that d_0 takes any 1-simplex from a to b to the zero-simplex b. That is, taking the vertices of a 1-simplex as numbered in order, d_0 *deletes the zeroth vertex.vertex*

This is a general feature: the ith face d_i map acts on an n-simplex by deleting the ith vertex, returning the $(n-1)$-simplex opposite.

So we understand how the face maps work, at least on n-simplices.

Definition 2.14 (degeneracy function)**.** Let $n > 0$. For every i with $0 \leq i \leq n$, there is a function

$$\sigma_k \colon [n] \to [n-1]; \qquad j \mapsto \begin{cases} j, & j \leq k \\ j-1, & j > k \end{cases}$$

called the ith degeneracy function.

$$\sigma_k \quad = \quad \begin{tikzcd} \vdots & \vdots \\ k-2 \arrow[r, mapsto] & k-2 \\ k-1 \arrow[r, mapsto] & k-1 \\ k \arrow[r, mapsto] & k \\ k+1 \arrow[ur, mapsto] & k+1 \\ k+2 \arrow[ur, mapsto] & k+2 \\ \vdots & \vdots \end{tikzcd}$$

Definition 2.15 (degeneracy map)**.** Let $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. Corresponding to the degeneracy function $\sigma_k \colon [n] \to [n-1]$ there is a corresponding map

$$s_k \colon X_{n-1} \to X_n.$$

This map is also known as the kth degeneracy map.

Up until now, most of the simplices we have been talking about have been *nondegenerate*. However, simplicial sets also contain degenerate simplices. Recall, for example, that the standard 2-simplex actually contains six 1-simplices, three of which correspond to the edges of the triangle, and three of which correspond to edges which are 'bunched up' at the vertices. The nondegenerate edges correspond to the injective maps $[1] \to [2]$, and the degenerate edges correspond to the constant maps.

The ith degeneracy map takes an $(n-1)$-simplex to a degenerate n-simplex

The face and degeneracy functions are interesting because they generate all morphisms; that is, any morphism $\phi \colon [n] \to [m]$ can be written as a composition of face and degeneracy functions.

Proposition 2.16. Let $\phi \colon [n] \to [m]$ be a weakly monotonic function. Then ϕ can be written as a composition of face and degeneracy maps.

Sketch of proof. Since ϕ is by definition nondecreasing, we can specify it by, starting from 0, a sequence □

Theorem 2.17 (simplicial identites)**.** The face maps and degeneracy maps satisfy the following conditions, known as the *simplicial identities.*

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i, & i &< j \\ d_i \circ s_j &= s_{j-1} \circ d_i, & i &< j \\ d_j \circ s_j &= 1 = d_{j+1} \circ s_j \\ d_i \circ s_j &= s_j \circ d_{i-1}, & i &> j+1 \\ s_i \circ s_j &= s_{j+1} \circ s_i, & i &\leq j. \end{aligned}$$

Fact 2.18. A simplicial set X is equivalently the following data.

- The sets X_n, $n \geq 0$.
- The face and degeneracy maps satisfying the simplicial identities.

3.2 Morphisms of simplicial sets; simplices and the Yoneda lemma

So far, we have talked a lot about simplicial sets, i.e. functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$. From category theory, one feels that the natural notion of a morphism of simplicial sets should be that of a natural transformation. However, simplicial sets are geometric objects, and we have some idea, on a geometrical level, of what we would like morphisms between them to look like. Fortunately, these notions coincide.

Consulting the definition of a simplicial set, one sees that a morphism between simplicial sets $f : X \to Y$ should consist of, for each $n \geq 0$, a map $f_n : X_n \to Y_n$, compatible with the morphisms in Δ. In fact, because of Proposition 2.16, it suffices to demand that the f_i should be compatible with face and degeneracy maps.

What am I trying to say?

1. Morphims of simplicial sets. These are what you'd expect; n-simplices are sent to n-simplices in such a way that
2. Fully faithfulness: restricting to maps between standard simplices gives a copy of Δ in Set_Δ.
3. The n-simplices of a simplicial set X are in natural bijection to maps of the standard n-simplex into X.

4 Basic properties of the category of simplicial sets

Any functor category inherits a lot of structure from it codomain category. In the case of $\mathrm{Set}_\Delta = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$, we are especially lucky, as Set has an enormous amount of structure. In this section we note some of the structure Set_Δ inherits from Set.

4.1 Existence of limits and colimits

Proposition 2.19. The category Set_Δ has small limits and colimits. These are computed level-wise, i.e. n-simplicices of a (co)limit of a diagram of simplicial sets are given by the the limit of the diagram's restriction to n-simplices.

Proof. The category Set has small limits and colimits, which implies by Theorem A.3 that any functor category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ does.

Limits and colimits are computed pointwise thanks to Corollary A.5. □

Example 2.20 (products). By Corollary A.4, we can compute this limit pointwise; that is

$$(X \times Y)_n = X_n \times Y_n.$$

Let us understand this this by studying $\Delta^1 \times \Delta^1$ in some detail.

The 0-simplices of $\Delta^1 \times \Delta^1$ are ordered pairs $\Delta^1_0 \times \Delta^1_0 = \{0,1\} \times \{0,1\}$. We can visualize this like so.

$$\begin{tikzcd}
(0,1) \bullet & \bullet (1,1) \\
(0,0) \bullet & \bullet (1,0)
\end{tikzcd}$$

Similarly, the 1-simplices $(a,b) \to (c,d)$ are pairs of a 1-simplex $a \to b$ and a 1-simplex $c \to d$. There are 3 1-simplices in Δ^1_1: $\{a = \{0,0\}, b = \{0,1\}, c = \{1,1\}\}$. Thus, we have the following 1-simplices in $\Delta^1 \times \Delta^1$ (where we do not draw degenerate simplices).

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \\
(0,0) \arrow[u] \arrow[r] \arrow[ur] & (1,0) \arrow[u]
\end{tikzcd}$$

Continuing in the same way, one finds that there are two non-degenerate 2-simplices, filling the above triangles. Thus, as one might expect, the product of two intervals is a square.

Example 2.21 (pushouts). Corollary A.4 also immediately implies that Set_Δ has pushouts, which we can also compute poinwise. Recall that we think of pushouts as given by gluing, at least when

The fact that limits and colimits are computed pointwise implies immediately that morphisms of simplicial sets are computed pointwise.

Lemma 2.22. We have the following.

1. A morphism of simplicial sets is a monomorphism if and only if for each n, the component f_n is a monomorphism (i.e. injection) of sets.
2. A morphism of simplicial sets is a epimorphism if and only if for each n, the component f_n is a epimorphism (i.e. surjection) of sets.

Proof. Consequence of Corollary A.5 □

4.2 Cartesian closure

Note 2.23. At the moment, the order of a lot of Cartesian products in this section is a bit out of whack, since I'm straddling two different conventions. I'll fix this if I have time.

Thanks to Proposition 2.19, the category Set_Δ has products. It turns out that it also has exponential objects (i.e. an internal hom with respect to the Cartesian product). This makes it into a *Cartesian closed category.*

Before we prove that such an internal hom exists, let us assume that Set_Δ admits an internal hom $\mathrm{Maps}(A, B)$, and see where we end up.

Lemma 2.24. Let B, C be simplicial sets. Should it exist, the internal hom $\mathrm{Maps}(B, C)$ must be given, up to natural isomorphism, by the functor

$$\mathrm{Set}_\Delta(B \times \Delta^\bullet, C),$$

where $\Delta^\bullet$ denotes the Yoneda embedding $\Delta \hookrightarrow \mathrm{Set}_\Delta$.

Proof. By definition, $\mathrm{Maps}(B, C)$ needs to be part of a hom-set adjunction

$$\mathrm{Set}_\Delta(A \times B, C) \cong \mathrm{Set}_\Delta(A, \mathrm{Maps}(B, C)).$$

In particular, when $A = \Delta^n$, we have that

$$\mathrm{Set}_\Delta(\Delta^n \times B, C) \cong \mathrm{Set}_\Delta(\Delta^n, \mathrm{Maps}(B, C)).$$

However, by the Yoneda lemma, there is an isomorphism

$$\mathrm{Set}_\Delta(\Delta^n, \mathrm{Maps}(B, C)) \cong \mathrm{Maps}(B, C)_n,$$

natural in n. Thus the internal hom, should it exist, must be given level-wise by

$$\mathrm{Maps}(B, C)_n \cong \mathrm{Set}_\Delta(B \times \Delta^n, C).$$

The naturality of the Yoneda embedding now implies that this level-wise isomorphism must really be a natural isomorphism, i.e. that

$$\mathrm{Maps}(B, C) \cong \mathrm{Set}_\Delta(B \times \Delta^\bullet, C).$$

□

Theorem 2.25. For simplicial sets B and C, the simplicial set $\mathrm{Maps}(B, C)$ defined by the formula
$$\mathrm{Maps}(B, C) = \mathrm{Set}_\Delta(B \times \Delta^\bullet, C),$$
is a model for the internal hom $\mathrm{Maps}(B, C)$. That is, for each simplicial set B, there is a bijection
$$\mathrm{Set}_\Delta(A \times B, C) \cong \mathrm{Set}_\Delta(A, \mathrm{Maps}(B, C)),$$
natural in A and C.

Proof. We define two mutually inverse maps. First, let $f\colon A \times B \to C$. We define a map $\tilde{f}\colon A \to \mathrm{Maps}(B, C)$ level-wise as follows: for any $\sigma \in A_n$, define $\tilde{f}_n(\sigma)\colon \Delta^n \times B \to C$ to be the composition
$$\Delta^n \times B \xrightarrow{\sigma \times B} A \times B \xrightarrow{f} C.$$

Now we define a map in the other direction. First, we define a family of morphisms which turn out to be the components of the counit. Consider a map
$$\mathrm{ev}\colon \mathrm{Maps}(B, C) \times B \to C$$
defined level-wise by
$$\mathrm{ev}_n\colon \mathrm{Set}_\Delta(\Delta^n \times B, C) \times B_n \to C_n \qquad (f, b) \mapsto f_n(\mathrm{id}_{\Delta^n}, b).$$

The, given $\tilde{f}\colon A \to \mathrm{Maps}(B, C)$, we define $f\colon A \times B \to C$ to be the composition
$$A \times B \xrightarrow{\tilde{f} \times B} \mathrm{Maps}(B, C) \times B \xrightarrow{\mathrm{ev}} C. \tag{2.1}$$

It is not *too* hard to show that these are really mutually inverse and check naturality. □

Note 2.26. We have
$$\mathrm{Maps}(K, S)_0 = \mathrm{Set}_\Delta(K \times \Delta^0, S) = \mathrm{Set}_\Delta(K, S),$$
i.e. maps $K \to S$ correspond to vertices of $\mathrm{Maps}(K, S)$. The 1-simplices of $\mathrm{Maps}(K, S)$ corrspond to maps $\Delta^1 \times K \to S$, i.e. homotopies between maps, and so on.

5 *Constructing simplicial sets from cosimplicial objects*

So far, all the concrete examples of simplicial sets we have seen have been of the form $\Delta^{\mathcal{K}}$ for some $\mathcal{K} \subset \mathcal{P}(\{0, \dots, n\})$. These examples are useful, but not very interesting in their own right.

In this section, we define two ways of creating a simplicial set out of existing mathematical data; in one case, a category, and in the other, a topological space. We will do this using *cosimplicial objects*, i.e. functors
$$\alpha\colon \Delta \to \mathcal{C}$$

where $\mathcal{C}$ is some locally small category.

The idea is as follows. Given a cosimplicial object $\alpha\colon \Delta \to \mathcal{C}$ and any object $C \in \mathcal{C}$, we can define a corresponding simplicial set $\mathcal{N}_\alpha(C)$ by first Yoneda embedding, and then composing the resulting hom functor with α:

$$C \mapsto \mathcal{N}_\alpha(C) = \mathcal{C}(\alpha(-), C).$$

That $\mathcal{N}_\alpha(C)$ is a simplicial set follows immediately from the contravariance of the first slot of the hom functor.

The n-simplices of $\mathcal{N}_\alpha(C)$ are then given by maps of the $\mathcal{C}$-model of the n-simplex into C:

$$\mathcal{N}_\alpha(C)_n = \mathcal{C}(\alpha([n]), C).$$

In fact, by the functoriality of hom this construction gives us a functor $\mathcal{C} \to \mathrm{Set}_\Delta$ taking $C \mapsto \mathcal{N}_\alpha(C)$.

The power of defining simplicial sets via cosimplicial objects comes from Lemma A.10, which provides us immediately with a left adjoint to the functor $\mathcal{N}_\alpha$. This left adjoint provides a weak inverse, i.e. a way of constructing from any simplicial set K an object of the category $\mathcal{C}$. More specifically, Lemma A.10 guarantees us an adjunction

$$\mathcal{Y}_!\alpha : \mathrm{Set}_\Delta \longleftrightarrow \mathcal{C} : \mathcal{N}_\alpha,$$

where $\mathcal{Y}_!\alpha$ is the Yoneda extension of α.

The alert reader will remember that we have already defined two cosimplicial objects: the *realization functor* $\rho\colon \Delta \to \mathrm{Top}$ (Definition 2.3) and the *categorification functor* $\chi\colon \Delta \to \mathrm{Cat}$. (Definition 2.4). We said that they would provide a connection between the theory of simplicial sets and the theory of topological spaces and categories respectively. Indeed, these are the cosimplicial objects we will use.

5.1 Nerves

Definition 2.27 (nerve of a category)**.** Let $\mathcal{C}$ be a small category. By composing the Yoneda embedding with the categorification functor (Definition 2.4)

$$\chi\colon \Delta \to \mathrm{Cat},$$

we obtain a simplicial set called the nerve of $\mathcal{C}$ and denoted $N(\mathcal{C})$. That is,

$$N(\mathcal{C}) = \mathrm{Fun}(\chi(-), \mathcal{C}).$$

Let us examine this construction in some detail. We have

$$N(\mathcal{C})_0 = \mathrm{Fun}([0], \mathcal{C}).$$

Since $[0]$ is nothing else but the category with one object and one morphism, $\mathrm{Fun}([0], \mathcal{C})$ simply picks out the objects of $\mathcal{C}$. That is, the zero-simplices of $N(\mathcal{C})$ simply consist of the objects of $\mathcal{C}$.

Similarly, the category $[1]$ consists of two object and a morphism between them, so the functors in $\mathrm{Fun}([1], \mathcal{C})$ pick out diagrams of the form

$$X_0 \longrightarrow X_1$$

which is tantamount to picking out morphisms in $\mathcal{C}$. Functors in $\mathrm{Fun}([2], \mathcal{C})$ pick out commuting triangles.

$$\begin{tikzcd} & X_1 \arrow[dr] & \\ X_0 \arrow[ur] \arrow[rr] & & X_2 \end{tikzcd}$$

In general, $N(\mathcal{C})([n])$ picks out chains of morphisms,

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n \ ,$$

together with all possible compositions. One should picture these as forming an n-dimensional simplex.[6]

[6] We have already encountered a nerve in a previous sidebar. One can view the power set $\mathcal{P}([n])$ as a poset, and thus as a category. The cubes C^n, which we defined above as simplicial sets, are nothing more than the nerves of the categories $\mathcal{P}([n])$.

To summarize, one pictures the construction of the nerve of a category $\mathcal{C}$ in the following way.

1. One imagines for each object of the category $\mathcal{C}$ a zero-simplex, i.e. a little dot.
2. One imagines for each morphism between objects x and y an edge between the corresponding zero-simplices. So far, we have constructed a multidigraph, i.e. a directed graph with multiple edges between vertices.
3. Our multidigraph has lots of little triangles formed by morphisms. We fill in each of the commuting triangles with a 2-simplex.
4. Now we proceed inductively. Given the set of $(n-1)$-simplices of $N(\mathcal{C})$, we add an n-simplex whenever we have all the $(n-1)$-simplices forming its boundary.

Note that the n-simplices for $n > 2$ don't tell us very much; we add them whenever we have their boundaries. This is known as *2-coskeletality,* and is a reflection of the fact that categories only have objects and morphisms, and no higher data. We will meet the notion of coskeletality in Section 7.

Proposition 2.28. The nerve is left is right adjoint to the functor $\text{よ}_!\chi$, where $\chi\colon \Delta \to \mathrm{Cat}$ is the categorification functor (Definition 2.4), which assigns $[n]$ to itself considered as a poset category, and $\text{よ}_!$ denotes the Yoneda extension.

$$\text{よ}_!\chi : \mathrm{Set}_\Delta \leftrightarrow \mathrm{Cat} : N$$

Proof. The nerve is of the form $\mathcal{C} \mapsto \mathrm{Set}_\Delta(\chi(-), \mathcal{C})$. The result follows from Lemma A.10. ◻

The functor $\text{よ}_!\chi$ provides a way of constructing, from any simplicial set, a category. We will not say too much about it now. Rather, we will come back to it (in a slightly weaker form) in Section 2, when we have the tools to understand it properly.

5.2 Singular sets

Definition 2.29 (singular set of a topological space)**.** Let X be a topological space. The functor

$$\mathrm{Top}(\rho(-), X) : \Delta^{\mathrm{op}} \to \mathrm{Set}$$

is called the singular set of X, and denoted $\mathrm{Sing}(X)$.

Unlike the nerve, which creates a reasonably faithful simplicial likeness of any category, the singular set creates a caricature. Consider, for example, the geometric 2-simplex $|\Delta^2|$. We would hope that the singular set of $|\Delta^2|$ would be something like Δ^2, but alas: rather than three 0-simplices, there are uncountably many, one for each continuous map of the point $|\Delta^0|$ into $|\Delta^2|$. Similarly, there is a 1-simplex for each continuous map of the interval into $|\Delta^2|$.

However, things are not as hopeless as they may appear. We will later see that these extra simplices are in some sense superficial, and that the simplicial set $\mathrm{Sing}(|\Delta^2|)$ is, in an appropriate sense, equivalent to Δ^2.

By Lemma A.10, the singular set functor is right adjoint to the Yoneda extension of the realization functor ρ, which we call the *geometric realization.*

Definition 2.30 (geometric realization)**.** Let K be an simplicial set. The geometric realization of K, denoted $|K|$, is the image

$$|K| = (\text{よ}_! \rho)(K).$$

We now have the tools to study the geometric realization in some detail. This will be the content of Section 6.

5.3 A colimit formula for simplicial sets

There is one more cosimplicial object we have met, which may seem almost too simple to count: the Yoneda embedding $\text{よ} : \Delta \to \mathrm{Set}_\Delta$ itself! Applying the same procedure as above, we find the functor

$$\mathrm{Set}_\Delta \to \mathrm{Set}_\Delta; \qquad K \mapsto \mathrm{Set}_\Delta(\text{よ}(-), K).$$

This is (by the Yoneda lemma) simply the identity functor.

However, Lemma A.10 now implies a very useful formula.

Proposition 2.31. Let K be any simplicial set. Then K is the colimit of the functor

$$\Delta_{/K} \to \Delta \xrightarrow{\text{よ}} \mathrm{Set}_\Delta .$$

Proof. Lemma A.9. □

The objects of the category $(\Delta \downarrow K)$ consist of all morphisms $\Delta^n \to K$ for any n. This is simply the set of all simplices of K. The morphisms between simplices are all ways of mapping simplices into each other which

This vindicates the assertion that we have been making all along: that a simplicial set should be viewed as being built by gluing together its simplices.

6 *Geometric realization*

In Subsection 5.2, we saw that we could turn any topological space X into a simplicial set using the singular set functor (Definition 2.29)

$$\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{Set}_\Delta; \qquad \mathrm{Sing}(X) = \mathrm{Top}(\rho(-), X).$$

We then saw that Sing has a left adjoint, the geometric realization functor, which provides a weak inverse to Sing. The geometric realization has the following formula.

$$|K| = (よ_! \rho)(K).$$

As we will see in this section, the geometric realiztion takes a simplicial set K to a topological space which is created by assigning to each n-simplex in K to an actual n-simplex in Top, and gluing these together in the appropriate way.

The goal of this section is to derive and understand the following formula for the geometric realization.

Theorem 2.32. Let $K\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. The geometric realization of K can be computed by the formula

$$|K| = \left(\coprod_{n \geq 0} K_n \times |\Delta^n| \right) / \sim,$$

where $\sim$ is the equivalence relation generated by the relation

$$(f^*(x), y) \sim (x, f_*(y)), \qquad \text{for all } f\colon [m] \to [n].$$

Proof. By the colimit formula for left Kan extensions (Fact A.6), we can write

$$|K| = \operatorname{colim}\left[(\Delta \downarrow K) \to \Delta \xrightarrow{\rho} \mathrm{Top} \right]. \tag{2.2}$$

Let f be a morphism $[m] \to [n]$. Then by the contravariance of any simplicial set K, f induces a morphism $f^*\colon K_n \to K_m$. Similarly, by the covariance of the geometric realization functor $\rho\colon \Delta \to \mathrm{Top}$, f induces a map $f_*\colon |\Delta^m| \to |\Delta^n|$.

By the coequalizer formula for colimits (Fact A.1), we can express the colimit in Equation 2.2 as the following coequalizer.

$$\coprod_{\substack{f \in \mathrm{Morph}(\Delta/K) \\ f\colon [m] \to [n]}} |\Delta^m| \overset{A}{\underset{B}{\rightrightarrows}} \coprod_{\substack{([n], \alpha) \in \mathrm{ob}(\Delta/K) \\ \alpha : \Delta^n \to K}} |\Delta^n| \longrightarrow \mathrm{coeq}$$

Here, the morphisms A and B are defined via the univeral property for coproducts by their components

$$A_f = \iota_{|\Delta^m|}, \qquad B_f = \iota_{|\Delta^n|} \circ f_*.$$

Let us understand the first coproduct, taken over the set of morphisms in $(\Delta \downarrow K)$. Recall that a morphism $([m], f) \to ([n], f')$ in $(\Delta \downarrow K)$ is a morphism $g\colon [m] \to [n]$ in Δ making the following diagram commute.

$$g\colon [m] \to [n]; \qquad \begin{tikzcd} \Delta^m \arrow[rr, "\Delta^g"] \arrow[dr, "f"'] & & \Delta^n \arrow[dl, "f'"] \\ & K & \end{tikzcd}$$

This means that morphisms in $(\Delta \downarrow K)$ can be uniquely parametrized by the following pieces of data.

- A morphism $g\colon [m] \to [n]$.
- A morphism $f'\colon \Delta^n \to K$.

The other morphism f must be the composite $f' \circ \Delta^g$.

This means that we are really taking the coproduct over two things: morphisms $[m] \to [n]$, and morphisms $\Delta^n \to K$. But by the Yoneda lemma

$$\mathrm{Set}_\Delta(\Delta^n, K) \cong K_n,$$

so we write the first coproduct as

$$\coprod_{f\colon [m]\to[n]} \left(\coprod_{x \in K_n} |\Delta^m| \right)$$

which (by slight abuse of notation, taking K_n with the discrete topology) is simply

$$\coprod_{f\colon [m]\to[n]} K_n \times |\Delta^m|.$$

□

Let $K\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. The <u>geometric realization</u> of K is the topological space

$$|K| = \left(\coprod_{n \geq 0} K_n \times |\Delta^n| \right) / \sim,$$

where $\sim$ is the equivalence relation generated by

$$(f^*(x), y) \sim (x, f_*(y)), \qquad \text{for all } f\colon [m] \to [n].$$

Recall the face functions σ_i and degeneracy functions ∂_i. Applying a simplicial set to these gave us the face and degeneracy maps s_i and d_i respectively. Applying the Yoneda embedding gives maps $S_i\colon \Delta^n \to \Delta^{n-1}$ and $D_i\colon \Delta^n \to \Delta^{n+1}$. The factorization of any morphism in Δ into a composition of face and degeneracy maps immediately gives us the following.

Corollary 2.33. The geometric realization of a simplicial set X can be computed using the formula

$$|K| = \left(\coprod_{n \geq 0} K_n \times |\Delta^n| \right) /\sim,$$

where $\sim$ is the equivalence relation generated by

$$(d_i(x), y) \sim (x, D_i(y)), \qquad (s_i(x), y) \sim (x, S_i(y)),$$

for all i.

The first identification tells us that we should identify any two standard n-simplices which share a face along that face. The second tells us that we should identify degenerate simplices to their corresponding nondegenerate simplices; that is, that degenerate simplices don't add anything new.

Theorem 2.34. The geometric realization $|X|$ of any simplicial set X is a CW complex. The cells of $|X|$ correspond to the nondegenerate simplices of X.

The adjunction

$$|\cdot| : \mathrm{Set}_\Delta \leftrightarrow \mathrm{Top} : \mathrm{Sing}$$

gives us a way of translating between the language of simplicial sets and the language of simplices in topological spaces.

7 *Skeleta and coskeleta*

For any $n \geq 0$, denote by $\iota_{\leq n} \colon \Delta_{\leq n} \hookrightarrow \Delta$ the full subcategory inclusion on objects $[0], [1], \ldots, [n]$. We can restrict any simplicial set along this functor, giving its so-called *n-truncation.* This extends to a so-called *n-truncation functor.*

Definition 2.35 (truncation)**.** Let $n \geq 0$. The n-truncation functor is the pullback

$$\iota_{\leq n}^* = \mathrm{tr}_{\leq n} \colon \mathrm{Set}_\Delta \to \mathrm{Set}_{\Delta_{\leq n}}.$$

One visualizes the n-truncation of a simplicial set as forgetting simplices of degree greater than n.

As usual, this is a forgetful functor, so left and right adjoints (should they exist) provide algoritmic ways of attempting to solve the impossible problem of recovering the lost simplices.

Definition 2.36 (skeleton, coskeleton)**.** Let $n \geq 0$.

- The left adjoint to the n-truncation functor is called the n-skeleton functor, and denoted

$$\mathrm{sk}_{\leq n} \colon \mathrm{Set}_{\Delta_{\leq n}} \to \mathrm{Set}_\Delta .$$

- The right adjoint to the n-truncation functor is called the n-coskeleton functor, and denoted

$$\mathrm{sk}_{\leq n} \colon \mathrm{Set}_{\Delta_{\leq n}} \to \mathrm{Set}_{\Delta}\,.$$

Note that the skeleton and coskeleton functors are simply the left and right Kan extension functors, i.e.

$$\mathrm{sk}_{\leq n} = (\iota_{\leq n})_!, \qquad \mathrm{cosk}_{\leq n} = (\iota_{\leq n})_*.$$

For any simplicial set K, we can define the n-skeleton of K to be the simplicial set built by first forgetting simplices of degree greater than n, then adding them back in using the n-skeleton functor. Since the skeleton functor is a Kan extension, the n-skeleton of a simplicial set is the left Kan extension along the restriction. That is,

$$\mathrm{sk}_{\leq n}(K) = (\mathrm{sk}_{\leq n} \circ \mathrm{tr}_{\leq n})(K).$$

Similarly, we define its n-coskeleton to be

$$\mathrm{cosk}_{\leq n}(K) = (\mathrm{cosk}_{\leq n} \circ \mathrm{tr}_{\leq n})(K).$$

Example 2.37. Let K be any simplicial set. Let us compute the n-skeleton of K. By definition of the left Kan extension

$$\begin{tikzcd}
\Delta^{\mathrm{op}}_{\leq n} \arrow[rr, "K \circ \iota^{\mathrm{op}}_{\leq n}"] \arrow[dr, "\iota^{\mathrm{op}}_{\leq n}"'] & & \mathrm{Set} \\
& \Delta^{\mathrm{op}} \arrow[ur, "(\iota^{\mathrm{op}}_{\leq n})_!(K \circ \iota^{\mathrm{op}}_{\leq n})"'] &
\end{tikzcd}$$

we have, by the colimit formula, the following expression for the m-simplices of $\mathrm{sk}_{\leq n} K$.

$$(\mathrm{sk}_{\leq n} K)_m = \mathrm{colim}\left[(\iota^{\mathrm{op}}_{\leq n} \downarrow [m]) \to \Delta^{\mathrm{op}}_{\leq n} \to \mathrm{Set}\right]$$

When $m \leq n$, then the category $(\iota^{\mathrm{op}}_{\leq n} \downarrow [m])$ has the terminal object $([m], \mathrm{id}_{[m]})$, so the colimit is

$$(K \circ \mathrm{tr}_{\leq n})([m]) = K_m.$$

Thus, the n-skeleton of K agrees with K on simplices of degree less than or equal to n.

Now let $m \geq n$. In this case, the category $(\mathrm{tr}_{\leq n} \downarrow [m])$ does not have a terminal object.

Do at some point: building simplicial set via pushouts as in saturated hull of boundary inclusion is monos.

8 Kan complexes

Definition 2.38 (Kan complex)**.** A Kan complex is a simplicial set such that for every $n > 0$ and every $0 \leq i \leq n$, for every map

$$\Lambda^n_i \xrightarrow{f} K$$

there exists a map $\bar{f}$ (possibly not unique!) making the following diagram commute.

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

The above condition is known as the *horn-filling condition,* and $\bar{f}$ is called the *horn filler.* Roughly, the horn-filling condition says that any horn in K can be completed to a simplex.

A priori, it is not clear that this condition is interesting. However, we will now show that every singular set (i.e. simplicial set of the form $\mathrm{Sing}(X)$ for some topological space X) is a Kan complex. If we will in the end be interested in translating between topological spaces and simplicial sets, it will be useful to have a necessary condition for a simplicial complex to be a singular set.

Theorem 2.39. For any topological space X, the set $\mathrm{Sing}(X)$ is a Kan complex.

Proof. The injection

$$j\colon |\Lambda^n_i| \to |\Delta^n|$$

admits a continuous right inverse

$$p\colon |\Delta^n| \to |\Lambda^n_i|.$$

That is,

$$p \circ j = \mathrm{id}_{|\Lambda^n_i|}.$$

By applying the functor $|\cdot|$, we can take any diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathrm{Sing}(X) \\ \Delta^n & \end{tikzcd}$$

to the diagram

$$\begin{tikzcd} |\Delta^n_i| \arrow[r, "|f|=f'"] \arrow[d, "j"'] & |\mathrm{Sing}(X)| \\ |\Delta^n| & \end{tikzcd},$$

now in Top. The counit gives us a morphism $\eta_X\colon |\mathrm{Sing}(X)| \to X$

$$\begin{tikzcd} |\Lambda^n_i| \arrow[r, "\eta_X \circ f'"] \arrow[d, "j"'] & X \\ |\Delta^n| & \end{tikzcd}.$$

But the map j has a left inverse, which we can compose with f' as follow.

$$\begin{tikzcd} {|\Lambda^n_i|} \arrow[r, "f'"] \arrow[d, shift left, "j"] & X \\ {|\Delta^n|} \arrow[u, bend left, "p"] \arrow[ur, "f\circ p"'] & \end{tikzcd}$$

Applying the Sing functor again, we find

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\epsilon_{\Lambda^n_i}"] \arrow[d, hook] & \mathrm{Sing}(|\Lambda^n_i|) \arrow[r] \arrow[d] & \mathrm{Sing}(X) \\ \Delta^n \arrow[r, "\epsilon_{\Delta^n}"] & \mathrm{Sing}(|\Delta^n|) \arrow[ur] & \end{tikzcd} .$$

The square commutes by the naturality of ϵ. The triangle commutes because it is the image of a commuting triangle. Thus, taking the composition gives us a morphism $\Delta^n \to \mathrm{Sing}(X)$. $\square$

Proposition 2.40. Let $\mathcal{C}$ be a small category.

1. All inner horns of $N(\mathcal{C})$ have a filler; that is, any morphism

$$\Lambda^n_i \xrightarrow{f} N(\mathcal{C})$$

   where $0 < i < n$ can be extended to a commuting diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, "\bar{f}"'] & \end{tikzcd} .$$

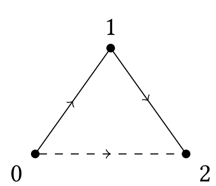
2. The simplicial set $N(\mathcal{C})$ has *all* horn fillers, i.e. is a Kan complex, if and only if $\mathcal{C}$ is a groupoid. Furthermore, this horn filler is unique for all $n \geq 2$.

Proof. There are no inner horns in degrees 0 or 1, so we start with a $(2,1)$-horn.

Consider a horn

$$\Lambda^2_1 \longrightarrow N(\mathcal{C}) .$$

To build from this a map $\Delta^2 \to N(\mathcal{C})$, we have to fill in images of everything in Δ^2 missing from Λ^2_1, i.e. the dashed 1-simplex and the full 2-simplex.



When we map the horn Λ^2_1 into $N(\mathcal{C})$, we find two morphisms as follows.

$$\begin{tikzcd} & a \arrow[dr, "g"] & \\ b \arrow[ur, "f"] & & c \end{tikzcd}$$

A map of a 2-simplex into $N(\mathcal{C})$ consists of a commuting triangle in $\mathcal{C}$. Such a map making the diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

□

9 Kan fibrations

Definition 2.41 (Kan fibration)**.** Let $p\colon K \to S$ be a morphism of simplicial sets. We say that p is a Kan fibration if for every i with $0 \le i \le n$ and every diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \arrow[d, "p"] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

there exists a (not necessarily unique!) morphism $\Delta^n \to K$ making the diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & S \end{tikzcd}$$

commute.

This is a sort of a relative version of horn filling. Note that if we take $S = *$, the constant simplicial set (which is the terminal object in the category of simplicial sets), we recover something resembling notion of a horn filler. More precisely:

Corollary 2.42. A Kan fibration $K \to \Delta^0$ is the same as a Kan complex.

Proof. The bottom right triangle

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

contains no information. □

3

Lifting problems

One often encounters problems of the following form.

> Given some commuting square
>
> $$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$
>
> construct a dashed arrow making the triangles formed commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

These problems are known as lifting problems. They turn out to be very important. In this chapter, we develop some powerful machines which enable us to easily solve lifting problems.

In Section 1, we define the notion of a *saturated set,* which is a set of morphisms closed under certain operations. We see that these operations are closed under intersection, and thus one can define the *saturated hull* of any set of morphisms to be the smallest saturated hull containing them.

We then define the notion of a *lifting property;* roughly, a morphism f has the left lifting property with respect to a set $\mathcal{S}$ of morphisms if for all $s \in \mathcal{S}$, and all commuting with f on the left and some $s \in \mathcal{S}$ on the right, one can find a dashed arrow as follows.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[d, "s"] \\ Y \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Similarly, a morphism g has the *right lifting property* with respect to $\mathcal{S}$ if one can always find a dashed arrow as follows.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "s"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

For any set $\mathcal{S}$ of morphisms, we denote the set of all morphisms which have the left lifting property with respect to it by ${}_\perp\mathcal{S}$, and the set of all morphisms with the right lifting property with respect to it by $\mathcal{S}_\perp$. We then show that for any set $\mathcal{M}$, the set ${}_\perp\mathcal{M}$ is saturated, immediately implying that the saturated hull of any set $\mathcal{S}$ is contained in ${}_\perp(\mathcal{S}_\perp)$.

In Section 2, we define the notion of a compact object. Roughly speaking, a compact object is an object which satisfies a sort of finiteness condition.

In Section 3, we prove a technical lemma called the *small object argument.* This lemma provides a factorization for morphisms whose domain is compact. This factorization immediately implies that for sets of morphisms whsoe domain is compact, we have the equality $\overline{M} = {}_\perp(\mathcal{M}_\perp)$.

In Section 4, we define the set of anodyne morphisms to be the set of all morphisms with the left lifting property with respect to Kan fibrations, and show that the set of

1 Saturated sets

Denote by $\mathcal{C}$ a category with small colimits.

Definition 3.1 (saturated)**.** A subset $\mathcal{S} \subset \mathrm{Morph}(\mathcal{C})$ is said to be saturated if the following conditions hold.

1. $\mathcal{S}$ contains all isomorphisms.
2. $\mathcal{S}$ is closed under pushouts, i.e. for any pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r] & B' \end{tikzcd}$$

   with $i \in \mathcal{S}$, then $i' \in \mathcal{S}$.

3. $\mathcal{S}$ is closed under retracts, [1] i.e. for any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}"] & A' \arrow[r] \arrow[d, "i"] & A \arrow[d, "i'"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B' \arrow[r] & B \end{tikzcd}$$

   with $i \in \mathcal{S}$, we also have $i' \in \mathcal{S}$.

4. $\mathcal{S}$ is closed under countable composition, i.e. for any chain of morphisms indexed by $\mathbb{N}$

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} \cdots$$

   the map

$$A_0 \to \operatorname*{colim}_{n\in\mathbb{N}} A_n = A_\infty$$

   is in $\mathcal{S}$.

[1] An object X_0 in some category is a retract of an object X if the identity on X_0 factors as $X_0 \hookrightarrow X \hookrightarrow X_0$. For example, any topological space is a retract of the point. This axiom says that $\mathcal{S}$ is closed under retracts in the category of morphisms $\mathrm{Fun}([1], \mathcal{C})$.

5. $\mathcal{S}$ is closed under small coproducts: if the family of morphisms

$$\{i_j : A_j \to B_j\}_{j \in J} \subset \mathcal{S}$$

is in $\mathcal{S}$, where J is a small set, then the morphism $\coprod_j i_j$ defined using the universal property of coproducts

$$\textstyle\coprod_{j \in J} A_j \xrightarrow{\coprod i_j} \coprod_{j \in J} B_j$$

is in J.

Saturatedness is a useful property. By showing that a certain set of morphisms is saturated, which is often possible based on general principles, the closure properties allow you to manipulate the morphisms in powerful ways.

Example 3.2. The set

$$\{f : X \to Y \mid f \text{ is mono}\} \subset \mathrm{Morph}(\mathrm{Set}_\Delta)$$

is saturated.

1. Clearly, every isomorphism is a monomorphism.

2. Since colimits are computed pointwise (Corollary A.4) it suffices to check that every pushout of monos is mono in Set.

3. Consider simplicial sets and maps as follows, with i a monomorphism.

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d, "j"'] \arrow[rr, bend left, "\mathrm{id}"] & A' \arrow[r, "g"'] \arrow[d, "i"] & A \arrow[d, "j"] \\ B \arrow[r, "k"] \arrow[rr, bend right, "\mathrm{id}"'] & B' \arrow[r, "h"] & B \end{tikzcd}$$

Since $g \circ f = \mathrm{id}$, we must have that f is a monomorphism, so $i \circ f$ is a monomorphism. Thus, $k \circ j$ is a monomorphism, so j must be a monomorphism.

4. Again, it suffices to check that countable composition respects monomorphisms in Set. Again, this is immediate from the universal property.

5. We can check this in Set. Again it is immediate from the universal property.

Definition 3.3 (saturated hull)**.** Let $\mathcal{M}$ be any set of morphisms in $\mathcal{C}$. We define the saturated hull $\overline{M}$ of $\mathcal{M}$ to be the smallest saturated set containing $\mathcal{M}$, i.e. the intersection of all saturated sets containing $\mathcal{M}$.

Note 3.4. This always exists because the set of all morphisms of $\mathcal{C}$ is a saturated set, and because the intersection of saturated sets is again saturated (because properties of the form 'closure under x' are respected by intersection).

Example 3.5. Consider the set $\mathcal{M}$ of boundary fillings

$$\mathcal{M} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}.$$

We claim that the saturated hull $\overline{\mathcal{M}}$ is the set of monomorphisms. Note that by Example 3.2, the set of all monomorphisms is saturated.

To see this, consider some monomorphism $f\colon X \to Y$. We think of f as including X as a subobject of Y. Our goal is to construct f from boundary inclusions using pushouts, retracts, countable composition, and coproducts.

Take the coproduct

$$X^0 \equiv X \amalg \left(\coprod_{\Delta^0 \in Y_0 \smallsetminus f_0(X_0)} \Delta^0 \right) = X \amalg \left\{ \begin{matrix} \text{0-skeleton of } Y \text{ which} \\ \text{is not part of } X \end{matrix} \right\}.$$

Since $\partial\Delta^0 = \emptyset$, we can build a map $g^0\colon X \hookrightarrow X^0$ as a coproduct of boundary fillings.

$$X = X \amalg \left(\coprod_{\Delta^0 \in Y_0 \smallsetminus f_0(X_0)} \partial\Delta^0 \right) \xhookrightarrow{g^0} X \amalg \left(\coprod_{\Delta^0 \in Y_0 \smallsetminus f_0(X_0)} \Delta^0 \right).$$

Since saturated sets are closed under coproducts, $g^0 \in \overline{\mathcal{M}}$.

We get a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g^0"'] & & Y \\ & X^0 \arrow[ur, "f^0"'] & \end{tikzcd}$$

where g^0 is simply the inclusion, and f^0 takes X to $f(X)$ and the simplices of Y we added to X to themselves.

To summarize: we have a factorization $f^0\colon X^0 \to Y$ which is an isomorphism on 0-simplices. It is trivially an isomorphism on all 0-cells (i.e. boundaries of 0-simplices), i.e. inclusions $\partial\Delta^0 = \emptyset \hookrightarrow X \to Y$. In fact, since $\partial\Delta^1 = \Delta^0 \amalg \Delta^0$, it is an isomorphism on all 1-cells (i.e. boundaries of 1-simplices). Thus, we have the following solid maps. We call the pushout X^1.

$$\begin{tikzcd} \displaystyle\coprod_{\substack{\Delta^1 \in Y_1 \smallsetminus f_1(X_1) \\ \text{nondegenerate}}} \partial\Delta^1 \arrow[r] \arrow[d, hook] & X^0 \arrow[d, dashed, "g^1"] \\ \displaystyle\coprod_{\substack{\Delta^1 \in Y_1 \smallsetminus f_1(X_1) \\ \text{nondegenerate}}} \Delta^1 \arrow[r, dashed] & X^1 \end{tikzcd}$$

Note that since g^1 is a coproduct of boundary inclusions, it is in $\overline{\mathcal{M}}$.

Schematically, we have

$$X^1 = X^0 \coprod_{\substack{\text{0-skeleton} \\ \text{of } Y}} \left\{ \begin{matrix} \text{1-skeleton of } Y \text{ which} \\ \text{is not in } X \end{matrix} \right\}.$$

Thus, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[dr, "g^0"'] & & & Y \\
& X^0 \arrow[urr, "f^0"] \arrow[dr, "g^1"'] & & \\
& & X^1 \arrow[uur, "f^1"'] &
\end{tikzcd}$$

where $g^1 \in \overline{\mathcal{M}}$ and f^1 is a monomorphism and an isomorphism on 1-simplices.

We continue this process, constructing sets X^2, X^3, etc., at each step adding another level of the skeleton via boundary inclusions of nondegenerate simplices and pushouts. At the kth step, we have a simplicial set X^k and a factorization

$$\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[dr, dotted, no head] & & & Y \\
& X^{k-1} \arrow[urr, "f^{k-1}"] \arrow[dr, "g^k"'] & & \\
& & X^k \arrow[uur, "f^k"'] &
\end{tikzcd}$$

where $g^k \in \overline{\mathcal{M}}$ and f^k is a monomorphism and an isomorphism on k-simplices.

By construction, we have

$$\operatorname*{colim}_{n\in\mathbb{N}} X^n \equiv X^\infty = Y,$$

since the map f^∞ is an isomorphism on n-simplices for all n. The countable composition of the gs is equal to f, and we get a factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g^\infty"'] & & Y \\
& X^\infty \arrow[ur, equal] &
\end{tikzcd}$$

where g^∞ is the countable composition of elements of $\overline{\mathcal{M}}$, and hence is in $\overline{\mathcal{M}}$.

Definition 3.6 (lifting property)**.** Let $\mathcal{C}$ be a category, and let $i\colon A \to B$ and $p\colon K \to S$ be morphisms in $\mathcal{C}$. Suppose that for every commuting square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "p"] \\
B \arrow[r] & S
\end{tikzcd}$$

there exists a morphism $B \to K$ making the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

commute. Then we say either of the following, which are equivalent.

- The morphism i has the left lifting property with respect to the morphism p.
- The morphism p hs the right lifting property with respect to the morphism i.

For any set of morphisms $\mathcal{M}$ we define two sets, ${}_{\perp}\mathcal{M}$ and $M_{\perp}$ as follows.

$$\begin{aligned} {}_{\perp}\mathcal{M} &= \{f \mid f \text{ has the left lifting property with respect to all morphisms in } \mathcal{M}\} \\ \mathcal{M}_{\perp} &= \{g \mid g \text{ has the right lifting property with respect to all morphisms in } \mathcal{M}\} \end{aligned}$$

Example 3.7. Let $\mathcal{C} = \mathrm{Set}_{\Delta}$. Let

$$\mathcal{M} = \{f : \Lambda_i^n \to \Delta^n \mid 0 \leq i \leq n \text{ and } n > 0\}.$$

That is, $\mathcal{M}$ is the set of all horn inclusions of the n-simplex. Then what is $\mathcal{M}_{\perp}$?

Any morphism $g \in \mathcal{M}_{\perp}$ needs to satisfy the following property: for any commuting square as follows

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, "f"'] & K \arrow[d, "g"] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

there exists some morphism $\Delta^n \to K$ making the following diagram commute.

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, "f"'] & K \arrow[d, "g"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

Checking Definition 2.41, we see that this says precisely that g is a Kan fibration. That is, $\mathcal{M}_{\perp}$ is the set of all Kan fibrations.

Checking that a set of morphisms is saturated by checking each condition individually is difficult and time-consuming. The next theorem provides a very efficient way of producing saturated sets of morphisms.

Theorem 3.8. Let $\mathcal{M}$ be any set of morphisms in $\mathcal{C}$. Then ${}_{\perp}\mathcal{M}$ is saturated.

Proof. We need to check conditions 1–5 in Definition 3.1.

1. Let $f : K \to S$ be an isomorphism. We need to check that f has the left lifting property. Suppose we are given a commuting square as follows.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & K \arrow[d] \\ B \arrow[r] & S \end{tikzcd}$$

We can define a morphism $h : B \to K$ by taking $h = \alpha \circ f^{-1}$.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f", shift left] & K \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "h"] \arrow[u, bend left, "f^{-1}"] & S \end{tikzcd}$$

2. Suppose $f \in {}_{\perp}\mathcal{M}$, and let $\tilde{f}$ be any pushout.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "\tilde{f}"] \\ B \arrow[r] & B' \end{tikzcd}$$

We need to show that $\tilde{f} \in {}_{\perp}\mathcal{M}$. To this end, consider a lifting problem as follows, with $m \in \mathcal{M}$.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\tilde{f}"'] & X \arrow[d, "m"] \\ B' \arrow[r] & Y \end{tikzcd}$$

Stacking this square next to the pushout square above, we find the following.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "\tilde{f}"'] & X \arrow[d, "m"] \\ B \arrow[r] & B' \arrow[r] & Y \end{tikzcd}$$

We can solve the outer lifting problem.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "\tilde{f}"'] & X \arrow[d, "m"] \\ B \arrow[r] \arrow[urr, dashed] & B' \arrow[r] & Y \end{tikzcd}$$

This gives us a cocone under the pushout, which gives us a unique dotted morphism, which is the lift we are after.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "\tilde{f}"] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, dashed, bend right] & B' \arrow[dr, dotted] & \\ & & X \end{tikzcd}$$

3. Suppose $i \in {}_{\perp}\mathcal{M}$, and consider a commuting square as follows.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B \arrow[r] & B' \end{tikzcd} \tag{3.1}$$

We need to show that i' has the left lifting property. To this end, consider a commuting square as follows.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] & K \arrow[d] \\ B' \arrow[r] & S \end{tikzcd}$$

Sticking this onto the right of Equation 3.1, we obtain a new diagram.

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d, "i'"] & K \arrow[d] \\
B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B \arrow[r] & B' \arrow[r] & S
\end{tikzcd}
\tag{3.2}
$$

The commuting square

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & K \arrow[d] \\
B \arrow[r, "\alpha"] & S
\end{tikzcd}
$$

is a lifting problem we can solve, giving us a morphism $B \to K$ making the diagram commute. Adding this onto our original diagram and pre-composing it with α gives us the morphism required.

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d, "i'"] & K \arrow[d] \\
B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[urrr, dashed] & B \arrow[r] & B' \arrow[r] & S
\end{tikzcd}
\tag{3.3}
$$

4. Consider a chain of morphisms

$$
A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} \cdots
$$

with $i_k \in {}_{\perp}\mathcal{M}$ for all k. We need to show that the map $A_0 \to A_\infty$ induced by the colimit has the left lifting property.

Consider a commuting square as follows.

$$
\begin{tikzcd}
A_0 \arrow[r] \arrow[d, "i_0"] \arrow[dddd, bend right] & K \arrow[dddd] \\
A_1 \arrow[d, "i_1"] & \\
A_2 \arrow[d, "i_2"] & \\
A_3 \arrow[d, dotted, no head] & \\
A_\infty \arrow[r] & S
\end{tikzcd}
$$

Suppose we have a map $A_k \to K$ which makes all squares and triangles formed commute. This gives us a commuting square as below.

$$\begin{tikzcd}
A_0 \arrow[r, dashed] \arrow[d, dashed, "i_0"] \arrow[dddd, dashed, bend right] & K \arrow[dddd] \\
A_1 \arrow[d, dotted, no head] & \\
A_k \arrow[uur] \arrow[d, "i_k"] & \\
A_{k+1} \arrow[d, dotted, no head] \arrow[dr] & \\
A_\infty \arrow[r, dashed] & S
\end{tikzcd}$$

Because i_k has the left lifting property, we get a map $A_{k+1} \to K$ making the diagram commute.

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, "i_0"] \arrow[dddd, bend right] & K \arrow[dddd] \\
A_1 \arrow[d, dotted, no head] & \\
A_k \arrow[uur, dashed] \arrow[d, "i_k"] & \\
A_{k+1} \arrow[d, dotted, no head] \arrow[dr, dashed] \arrow[uuur, dashed] & \\
A_\infty \arrow[r] & S
\end{tikzcd}$$

By induction, this gives us a collection of maps $A_k \to K$, one for each $k \in \mathbb{N}$, which make the triangles

$$\begin{tikzcd}
A_k \arrow[rr, "i_k"] \arrow[dr] & & A_{k+1} \arrow[dl] \\
& K &
\end{tikzcd}$$

commute. That is, we get a cocone under K. But this gives us a map $A_\infty \to K$ as required.

5. Similar.

$\square$

We get a similar, but dual and slightly weaker result for sets with the right lifting property.

Proposition 3.9. Let $\mathcal{M}$ be any set of morphisms in $\mathcal{C}$. The set $\mathcal{M}_\perp$ has the following properties.

1. $\mathcal{M}_\perp$ contains all isomorphisms.

2. $\mathcal{M}_\perp$ is closed under *finite* composition.
3. $\mathcal{M}_\perp$ is closed under retracts.
4. $\mathcal{M}_\perp$ is closed under pullbacks.
5. $\mathcal{M}_\perp$ is closed under small products.

Corollary 3.10. Let $\mathcal{M}$ be a set of morphisms in $\mathcal{C}$. Then

$$\overline{\mathcal{M}} \subset {}_\perp(\mathcal{M}_\perp).$$

Proof. The set $\mathcal{M}$ clearly has the left lifting property for the set of all morphisms which have the right lifting property for $\mathcal{M}$, so $\mathcal{M} \subset {}_\perp(\mathcal{M}_\perp)$. Since ${}_\perp(\mathcal{M}_\perp)$ is a saturated set, it contains $\overline{\mathcal{M}}$. □

It turns out that under some conditions, we have an equality

$$\overline{\mathcal{M}} = {}_\perp(\mathcal{M}_\perp).$$

2 *Compact objects*

In many situations, objects which obey some sort of finiteness condition are especially well-behaved. In the situation of lifting problems, the small object argument, due to Quillen, guarantees us that any set of morphisms whose domains are small provides a weak factorization system.

Definition 3.11 (filtered poset)**.** A poset $(I, \leq)$ is called filtered (or $\aleph_0$*-filtered*) if every finite subset of I has an upper bound.

More generally, the poset $(I, \leq)$ is κ-filtered, for some regular cardinal κ, if every subset of I cardinality less than κ has an upper bound.

In this chapter, we will only worry about $\aleph_0$-filtered posets.

Example 3.12. Let X be a set, and consider the poset of finite subsets $\mathcal{P}_{\text{fin}}(X)$. This is a filtered poset since every finite collection $X_i \in \mathcal{P}_{\text{fin}}(X)$ is contained in the set $\bigcup_i X_i$, which is again finite.

Definition 3.13 (compact object)**.** An object $X \in \mathcal{C}$ is said to be compact if for every diagram

$$\phi\colon I \longrightarrow \mathcal{C}; \qquad i \mapsto Y_i$$

with I a filtered poset, the criteria below are satisfied.

Denote by Y_∞ the colimit of ϕ, and write the colimit maps

$$\eta_i\colon A_i \longrightarrow Y_\infty.$$

1. There exists some $i \in I$ and $f_i \colon X \to Y_i$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty \end{tikzcd}$$

   commute.

2. Given maps $f_i \colon X \to Y_i$ and $f_j \colon X \to Y_j$ making the diagrams

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty \end{tikzcd}$$

   commute, there exist k with $k \geq i, j$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[d, "f_j"'] & Y_i \arrow[d] \\ Y_j \arrow[r] & Y_k \end{tikzcd} .$$

Lemma 3.14. Suppose that $\mathcal{C}$ is locally small. Then $X \in \mathcal{C}$ is compact only if the contravariant Yoneda embedding

$$\tilde{\mathcal{Y}} \colon X \mapsto \mathcal{C}(X, -)$$

commutes with small filtered colimits.

Proof. Suppose that $\tilde{\mathcal{Y}}$ commutes with small filtered colimits. Then there is a bijection

$$\mathcal{C}(X, Y_\infty) \cong \mathcal{C}(X, \operatorname*{colim}_{i \in I} Y_i) \cong \operatorname*{colim}_{i \in I} \mathcal{C}(X, Y_i). \tag{3.4}$$

Since each inclusion $Y_i \hookrightarrow Y_j$ gives an inclusion

$$\mathcal{C}(X, Y_i) \hookrightarrow \mathcal{C}(X, Y_j),$$

□

Example 3.15. A set $X \in \mathrm{Set}$ is compact if and only if X is finite.

Suppose X is compact. Consider the poset $\mathcal{P}_{\mathrm{fin}}(X)$ from Example 3.12. The colimit over this poset is X since every element of X is contained in some finite subset, so any map out of X is completely determined by maps out of its finite subsets which agree on intersections. We can view this as a a diagram in Set via the inclusion map.

Now consider the identity map $\mathrm{id}_X \colon X \to \operatorname{colim} X_i$. Since X is compact, there is some set X_i such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "\mathrm{id}_X"'] & X_i \arrow[d] \\ & X \end{tikzcd}$$

By definition, the set X_i is finite. Since f_i must be injective, X must also be finite.

Now suppose that X is finite. Let

$$\phi : I \to \mathrm{Set}$$

with I a filtered poset be a diagram, and consider a map $f : X \to \operatorname{colim} \phi$. In set, we have the formula

$$Y_\infty \equiv \operatorname{colim} \phi = \left(\coprod_{i \in I} Y_i \right) / \sim .$$

Denote by ι_i the map $Y_i \to Y_\infty$.

For every $x \in X$, there is some (certainly not unique!) $i_x \in \mathbb{N}$ such that Y_{i_x} contains an element e such that $\iota_{i_x}(e) = f(x)$. The set

$$\{Y_{i_x}\}_{x \in X}$$

is finite, so it has an upper bound Y_i. Therefore, each of element of x is represented by some element of Y_i, so there is a factorization as follows.

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d] \\ & Y_\infty \end{tikzcd}$$

Now suppose that we have maps $f_i : X \to Y_i$ and $f_j : X \to Y_j$ making the following diagrams commute.

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d] \\ & Y_\infty \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_j"] \arrow[dr, "f"'] & Y_j \arrow[d] \\ & Y_\infty \end{tikzcd}$$

Example 3.16. In Set_Δ, the compact objects are precisely those simplicial sets with finitely many nondegenerate simplices.

3 *The Small Object Argument*

The small object argument is the proof of following statement.

Lemma 3.17. Let $\mathcal{C}$ be a locally small category with small colimits. Let $\mathcal{M}$ be a small set of morphisms in $\mathcal{C}$ such that for each $i : A \to B \in \mathcal{M}$, the domain A of i is compact.[2] Then every morphism $f : X \to Y$ in $\mathcal{C}$ admits a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

with $h \in \overline{\mathcal{M}}$ and $g \in \mathcal{M}_\perp$.

[2] If $\mathcal{C}$ is presentable, then this condition is satisfied vacuously, since every object is small.

Proof. Let $f: X \to Y$ be a morphism in $\mathcal{C}$. Any commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd},$$

with i in $\mathcal{M}$, is certainly completely specified by the data of the morphsim $i: A \to B$, the map $A \to X$, and the map $B \to Y$. Therefore, the collection of all such commuting diagrams is indexed by a small set. Denote this set by J. We can write the set of all commuting squares as follows.

$$\left\{ \begin{tikzcd} A_j \arrow[r] \arrow[d, "i_j"'] & X \arrow[d, "f"] \\ B_j \arrow[r] & Y \end{tikzcd} \right\}_{j \in J}$$

Form the coproducts

$$A = \coprod_{j \in J} A_j \qquad \text{and} \qquad B = \coprod_{j \in J} B_j.$$

By the universal property for coproducts, specifying a map out of a coproduct is the same as specifying a map from each summand. Taking $\iota_{B_j} \circ i_j: A \to B_j$ gives us a map $\amalg_j i_j: A \to B$.

Note that the following diagram commutes.

$$\begin{tikzcd} \coprod_{j \in J} A_j \arrow[r] \arrow[d, "\amalg_j i_j"'] & X \arrow[d, "f"] \\ \coprod_{j \in J} B_j \arrow[r] & Y \end{tikzcd}$$

Now consider the following pushout.

$$\begin{tikzcd} \coprod_{j \in J} A_j \arrow[r] \arrow[d, "\amalg_j i_j"'] & X \arrow[d, "h_1"'] \arrow[ddr, bend left, "f"] & \\ \coprod_{j \in J} B_j \arrow[r] \arrow[rrd, bend right] & X_1 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "g_1"] & \\ & & Y \end{tikzcd}$$

Note that this gives us a factorization $f = g_1 \circ h_1$, but it's not the one we want: while we have that $h_1 \in \overline{\mathcal{M}}$ since it's the pushout of something in $\overline{\mathcal{M}}$, we do not yet have that $g_1 \in \mathcal{M}_\perp$. We need to keep looking.

We do this whole process again, considering the set of diagrams

$$\left\{ \begin{tikzcd} A_j \arrow[r] \arrow[d, "i_j"'] & X_1 \arrow[d, "g_1"] \\ B_j \arrow[r] & Y \end{tikzcd} \right\}$$

Take the pushout as before.

$$\begin{tikzcd}
\coprod_{j\in J} A_j \arrow[r] \arrow[d, "\amalg_j i_j"'] & X_1 \arrow[d, "h_2"'] \arrow[ddr, bend left, "g_1"] & \\
\coprod_{j\in J} B_j \arrow[r] \arrow[drr, bend right] & X_2 \arrow[dr, "g_2"] & \\
& & Y
\end{tikzcd}$$

Repeating this process, we get a chain of morphisms $h_i\colon X_{i-1} \to X_i$, each of which is in $\overline{\mathcal{M}}$.

$$X \xrightarrow{h_1} X_1 \xrightarrow{h_2} X_2 \xrightarrow{h_3} X_3 \cdots\cdots\cdots X_\infty$$

We can take the countable composition without leaving $\overline{\mathcal{M}}$, giving us a map $h\colon X \to X_\infty$.

For each X_i, we also constructed a map $g_i\colon X_i \to Y$. The universal property for colimits allows us to turn this into a map $g\colon X_\infty \to Y$. It turns out that this map is in $\mathcal{M}_\perp$!

In order to see that, consider the following lifting problem, with $i \in \mathcal{M}$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X_\infty \arrow[d, "g"] \\
B \arrow[r] & Y
\end{tikzcd}$$

By assumption, Y is compact, so any map $Y \to X_\infty$ factors through some X_j.

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd, "i"'] & & X_\infty \arrow[dd, "g"] \\
& X_j \arrow[ur] \arrow[dr] & \\
B \arrow[rr] & & Y
\end{tikzcd} \tag{3.5}$$

The lower-left square should look familiar.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X_j \arrow[d, "g_j"] \\
B \arrow[r] & Y
\end{tikzcd}$$

This is one of the diagrams that we took the coproduct of in the jth step above. The pushout gave us the following commuting diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X_j \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & X_{j+1} \arrow[dr] & \\
& & Y
\end{tikzcd}$$

Nestling this diagram into Equation 3.5, we find

$$\begin{tikzcd}
A \arrow[rrr] \arrow[dr] \arrow[ddd, "i"'] & & & X_\infty \arrow[ddd, "g"] \\
& X_j \arrow[urr] \arrow[dr] & & \\
& & X_{j+1} \arrow[uur] \arrow[dr] & \\
B \arrow[urr] \arrow[rrr] & & & Y
\end{tikzcd}$$

This gives us a map $B \to X_{j+1} \to X_\infty$ which makes the diagram commute. We have solved the lifting problem! □

One important use of the small object argument is to prove the following.

Corollary 3.18. Let $\mathcal{C}$ be a locally small category with small colimits, and let $\mathcal{M}$ be a small set of morphisms in $\mathcal{C}$ such that for each $i\colon A \to B \in \mathcal{M}$, the domain A of i is compact as in Lemma 3.17. Then

$$\overline{\mathcal{M}} = {}_\perp(\mathcal{M}_\perp).$$

Proof. We have $\overline{\mathcal{M}} \subseteq {}_\perp(\mathcal{M}_\perp)$ by Corollary 3.10.

Consider some morphism $f \in {}_\perp(\mathcal{M}_\perp)$. Then by Lemma 3.17, f has a factorization

$$f = g \circ h,$$

with $g \in \mathcal{M}_\perp$ and $h \in \overline{\mathcal{M}}$. We can write this as a commuting square.

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "{}_\perp(\mathcal{M}_\perp)\ni f"'] & Z \arrow[d, "g\in\mathcal{M}_\perp"] \\
Y \arrow[r, "\mathrm{id}_Y"'] & Y
\end{tikzcd}$$

Because f has the left lifting property with respect to g, we get a morphism $r\colon Y \to Z$ making the diagram commute.

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\
Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, dashed, "r"] & Y
\end{tikzcd}$$

Unfolding this, we find the following diagram.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"'] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[r, "\mathrm{id}_X"'] \arrow[d, "h"'] & X \arrow[d, "f"'] \\
Y \arrow[r, "r"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Z \arrow[r, "g"] & Y
\end{tikzcd}$$

This says precisely that f is a retract of h, so $f \in \overline{\mathcal{M}}$. □

4 *Anodyne morphisms*

Definition 3.19 (anodyne morphism)**.** A morphism of simplicial sets is called anodyne if it has the left lefting property with respect to all Kan fibrations (Definition 2.41).

Example 3.20. By definition, horn fillings $\Lambda_i^n \hookrightarrow \Delta^n$ are anodyne.

Lemma 3.21. The set of anodyne morphisms is the saturated hull of the set of horn inclusions.

Proof. We have the following string of equalities.

$$\begin{aligned} {}_{\perp}\{\text{Kan fibrations}\} &= {}_{\perp}(\{\text{horn fillings}\}_{\perp}) && \text{(By Example 3.7)} \\ &= \overline{\{\text{horn fillings}\}} && \text{(By Corollary 3.10)} \end{aligned}$$

where the last equality holds because horns have finitely many non-degenerate simplices, and hence are compact by Example 3.16. □

Corollary 3.22. The set of all anodyne morphisms is saturated.

Definition 3.23 (smash product)**.** Let $f\colon A \to B$ and $g\colon A' \to B'$ be morphisms of simplicial sets. Consider the following diagram.

$$\begin{tikzcd} A \times B \arrow[r, "f\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times g"'] & A' \times B \arrow[d, "\mathrm{id}\times g"] \\ A \times B' \arrow[r, "f\times\mathrm{id}"'] & A' \times B' \end{tikzcd}$$

Taking the pushout, we find a morphism

$$f \wedge g\colon A' \times B \coprod_{A\times B} A \times B' \to A' \times B',$$

called the smash product of f and g.

$$\begin{tikzcd} A \times B \arrow[r, "f\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times g"'] & A' \times B \arrow[d] \arrow[ddr, bend left, "\mathrm{id}\times g"] & \\ A \times B' \arrow[r] \arrow[drr, bend right, "f\times\mathrm{id}"'] & A' \times B \coprod_{A\times B} A \times B' \arrow[dr, dashed, "f\wedge g"] & \\ & & A' \times B' \end{tikzcd}$$

Lemma 3.24. Let f, g, and h be morphisms of simplicial sets. We have the following.

1. If f and g are monic, then $f \wedge g$ is monic.
2. There is a natural isomorphism $(f \wedge g) \wedge h \simeq f \wedge (g \wedge h)$

Proof.

1. The pushout square defining the smash product takes place in Set_Δ. By Corollary 2.22, it suffices to check that each component is monic. Therefore, consider the defining diagram now as a map of sets.

$$\begin{tikzcd}
A \times B \arrow[r, "f\times \mathrm{id}"] \arrow[d, "\mathrm{id}\times g"'] & A' \times B \arrow[d, hook] \arrow[ddr, bend left, "\mathrm{id}\times g"] & \\
A \times B' \arrow[r, hook] \arrow[drr, bend right, "f\times \mathrm{id}"'] & A' \times B \coprod_{A\times B} A \times B' \arrow[dr, dashed, "f\wedge g"] & \\
 & & A' \times B'
\end{tikzcd}$$

In the context of sets, we are allowed to think of f and g as including $A \subset A'$ and $B \subset B'$ respectively. The wedge product $f \wedge g$ is thus the inclusion of the union

$$(A' \times B) \cup (A \times B') \subset A' \times B'.$$

2. Draw a cube.

□

We will need this lemma many times.

Lemma 3.25. The following lifting problems are equivalent.

$$\begin{tikzcd}
A' \times B \coprod_{A\times B} A \times B' \arrow[r] \arrow[d] & X \arrow[d] \\
A' \times B' \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r] \arrow[d] & \mathrm{Maps}(A', X) \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & \mathrm{Maps}(A, X) \underset{\mathrm{Maps}(A,Y)}{\times} \mathrm{Maps}(A', Y)
\end{tikzcd}$$

Proof. First, consider the first. By expanding the pushout into its cocone and replacing maps out of it with maps out of the cocone, we find the following diagram (where we have now labelled maps).

$$\begin{tikzcd}
A \times B \arrow[r, "\alpha\times \mathrm{id}_B"] \arrow[d, "\mathrm{id}_A\times \beta"'] & A' \times B \arrow[dd, "\mathrm{id}_{A'}\times \beta"'] \arrow[dr, "a"] & \\
A \times B' \arrow[rr, "c"] \arrow[dr, "\alpha\times \mathrm{id}_B"'] & & X \arrow[d, "b"] \\
 & A' \times B' \arrow[r, "d"'] \arrow[ur, dashed, "\ell"] & Y
\end{tikzcd}$$

Re-arranging to avoid overlaps, we find the following diagram.

$$\begin{tikzcd}
& A \times B \arrow[dl, "\alpha\times\mathrm{id}_B"'] \arrow[dr, "\mathrm{id}_A\times\beta"] & \\
A' \times B \arrow[dr, "\mathrm{id}_{A'}\times\beta"] \arrow[dddr, bend right, "a"'] & & A \times B' \arrow[dl, "\alpha\times\mathrm{id}_{B'}"'] \arrow[dddl, bend left, "c"] \\
& A' \times B' \arrow[d, "d"'] \arrow[dd, dashed, bend left, "\ell"] & \\
& Y & \\
& X \arrow[u, "b"] &
\end{tikzcd}$$

This diagram is made of several commuting squares and one triangle, which tell us the following.

1. The top square commutes trivially.
2. The left square tells us that $b \circ a = d \circ (\mathrm{id}_{A'} \times \beta)$.
3. The right square tells us that $b \circ c = d \circ (\alpha \times \mathrm{id}_{B'})$.
4. The outer square tells us that $a \circ (\alpha \times \mathrm{id}_B) = c \circ (\mathrm{id}_A \times \beta)$.
5. The triangle involving the lift tells us that $d = b \circ \ell$.

We will denote the adjuncts to a, c, d, and ℓ by

$$\tilde{a} \colon B \longrightarrow [A', X], \qquad \tilde{c} \colon B' \longrightarrow [A, X], \qquad \tilde{d} \colon B' \longrightarrow [A', Y], \qquad \tilde{\ell} \colon B' \longrightarrow [A', X].$$

Expanding the second diagram in the same way as the first, we find the following.

$$\begin{tikzcd}
& B' \arrow[ddl, bend right, "\tilde{b}"'] \arrow[ddr, bend left, "\tilde{d}"] \arrow[dd, dashed, bend left, "\tilde{\ell}"] & \\
& B \arrow[u, "\beta"] \arrow[d, "\tilde{a}"'] & \\
& \mathrm{Maps}(A', X) \arrow[dl, "{[\alpha,\mathrm{id}_X]}"'] \arrow[dr, "{[\mathrm{id}_{A'},c]}"] & \\
\mathrm{Maps}(A, X) \arrow[dr, "{[\mathrm{id}_A,c]}"'] & & \mathrm{Maps}(A', Y) \arrow[dl, "{[\alpha,\mathrm{id}_Y]}"] \\
& \mathrm{Maps}(A, Y) &
\end{tikzcd}$$

□

Lemma 3.26. Fix some morphism $i \colon A \longrightarrow A'$ in Set_Δ. The set

$$\mathcal{S} = \{f \mid i \wedge f \text{ is anodyne}\}$$

is saturated.

Proof. Let $f: B \to B'$. By definition, $i \wedge f$ is anodyne if and only if every lifting problem

$$\begin{tikzcd} A' \times B \coprod_{A\times B} A \times B' \arrow[r] \arrow[d, "i\wedge f"'] & X \arrow[d, "g"] \\ A' \times B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where g is a Kan fibration has a solution. However, by Lemma 3.25, we may equivalently say that $i \wedge f$ is anodyne if and only if every lifting problem of the following form has a solution.

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f"'] & \mathrm{Maps}(A', X) \arrow[d] \\ B' \arrow[r] \arrow[ur, dashed] & \mathrm{Maps}(A, X) \underset{\mathrm{Maps}(A,Y)}{\times} \mathrm{Maps}(A', Y) \end{tikzcd}$$

Defining

$$\mathcal{R} = \{\mathrm{Maps}(A', X) \to \mathrm{Maps}(A, X) \underset{\mathrm{Maps}(A,Y)}{\times} \mathrm{Maps}(A', Y) \mid X \to Y \text{ Kan fibration}\},$$

we see that

$$\{f \mid f \wedge i \text{ anodyne}\} = {}_{\perp}\mathcal{R}.$$

Thus by Theorem 3.8 $\{f \mid f \wedge i \text{ anodyne}\}$ is saturated. □

Theorem 3.27. Consider the following sets of morphisms.

$$\mathcal{M}_1 = \left\{\Lambda^n_i \to \Delta^n \mid n > 0,\ 0 \le i \le n\right\}$$

$$\mathcal{M}_2 = \left\{\{e\} \times \Delta^n \coprod_{\{e\}\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \to \Delta^1 \times \Delta^n \,\middle|\, n \ge 0,\ e = 0, 1\right\}$$

$$\mathcal{M}_3 = \left\{\{e\} \times S \coprod_{\{e\}\times S} \Delta^1 \times S \to \Delta^1 \times S' \,\middle|\, S \to S' \text{ monic},\ e = 0, 1\right\}$$

Each of these sets generate the set of all anodyne morphisms in the sense that

$$\overline{\mathcal{M}_1} = \overline{\mathcal{M}_2} = \overline{\mathcal{M}_3} = \{\text{anodyne morphisms}\}.$$

Proof. We will show the following inclusions.

1. $\mathcal{M}_2 \subset \overline{\mathcal{M}_1}$
2. $\mathcal{M}_1 \subset \overline{\mathcal{M}_3}$
3. $\overline{\mathcal{M}_2} = \overline{\mathcal{M}_3}$

This will imply that

$$\overline{\mathcal{M}_2} \subset \overline{\mathcal{M}_1} \subset \overline{\mathcal{M}_3} \subset \overline{\mathcal{M}_2}.$$

1. Fix a morphism i in $\mathcal{M}_2$ with $e = 0$, i.e. an inclusion

$$C^0 = \{0\} \times \Delta^n \coprod_{\{0\}\times\partial\Delta^n} \Delta^1 \times \Delta^n \xhookrightarrow{i} \Delta^1 \times \Delta^n = C.$$

Consider the poset $[1] \times [n]$, which we can draw as follows.

$$\begin{tikzcd} 10 \arrow[r] & 11 \arrow[r] & 12 \arrow[r] & \cdots \arrow[r] & 1(n-1) \arrow[r] & 1n \\ 00 \arrow[r] \arrow[u] & 01 \arrow[r] \arrow[u] & 02 \arrow[r] \arrow[u] & \cdots \arrow[r] & 0(n-1) \arrow[r] \arrow[u] & 0n \arrow[u] \end{tikzcd}$$

The simplicial set C is the nerve of this poset. An $(n+1)$-simplex in C is thus given by a map of posets $[n+1] \to [1] \times [n]$. An $(n+1)$-simplex is nondegenerate if and only if it is a monomorphism. There are exactly $n+1$ nondegenerate simplices in $\Delta^1 \times \Delta^n$, of the form

$$\begin{tikzcd} & & & 1i \arrow[r] & 1(i+1) \arrow[r] & \cdots \arrow[r] & 1n \\ 00 \arrow[r] & 01 \arrow[r] & \cdots \arrow[r] & 0i \arrow[u] & & & \end{tikzcd}$$

for $0 \leq i \leq n$. We denote the above $(n+1)$-simplex by σ_i.

Consider the nondegenerate $(n+1)$-simplex σ_n given by

$$\sigma_n = \begin{tikzcd} & & & & 1n \\ 00 \arrow[r] & 01 \arrow[r] & 02 \arrow[r] & \cdots \arrow[r] & 0n \arrow[u] \end{tikzcd} .$$

The ith face of σ_n is given by skipping over the ith vertex of σ_n. For example, $d_1\sigma_n$ is in blue below.

$$\begin{tikzcd} & & & & 1n \\ 00 \arrow[r] \arrow[rr, bend left] & 01 \arrow[r] & 02 \arrow[r] & \cdots \arrow[r] & 0n \arrow[u] \end{tikzcd}$$

Each of the faces $d_0\sigma_n$, $d_1\sigma_n$, etc., is in C^0, except $d_n\sigma_n$.

$$\begin{tikzcd} & & & & & 1n \\ 00 \arrow[r] & \cdots \arrow[r] & 0(n-2) \arrow[r] & 0(n-1) \arrow[r] \arrow[urr] & 0n \arrow[ur, phantom] & \end{tikzcd}$$

This means that $\sigma_n \cap C^0 \cong \Lambda^{n+1}_n$, giving us a pushout

$$\begin{tikzcd} \Lambda^{n+1}_n \arrow[r, "{(d_0\sigma_n,\ldots,d_{n-2}\sigma_n,-,d_n\sigma_n)}"] \arrow[d] & C^0 \arrow[d, hook, "i_0"] \\ \Delta^n \arrow[r, "\sigma_n"] & C^1 = C^0 \cup \sigma_n \end{tikzcd}$$

exhibiting i_0 as a pushout of a horn inclusion. Hence, i_0 is in $\overline{\mathcal{M}}_1$.

Reasoning along the same lines shows that σ_{n-1} has two faces missing from C^0: $d_n\sigma_{n-1}$ and $d_{n-1}\sigma_{n-1}$. However, $d_{n-2}\sigma_{n-2}$ is in C^1, so we again have a pushout

$$\begin{tikzcd} \Lambda^{n+1}_{n-1} \arrow[r, "{(\ldots,d_{n-2}\sigma_{n-1},-,d_n\sigma_{n-1},\ldots)}"] \arrow[d] & C^1 \arrow[d, hook, "i_1"] \\ \Delta^n \arrow[r, "\sigma_n"] & C^2 = C^1 \cup \sigma_{n-1} \end{tikzcd}$$

Proceeding inductively, we see that we have a string of inclusions

$$C^0 \xhookrightarrow{i_0} C^1 \xhookrightarrow{i_1} C^2 \xhookrightarrow{i_2} \cdots \xhookrightarrow{i_{n-1}} C^n$$

where $C^i \hookrightarrow C^{i+1}$ is given by a pushout

$$\begin{tikzcd}
\Lambda^{n+1}_{n-i} \arrow[r] \arrow[d] & C^i \arrow[d, hook] \\
\Delta^{n+1} \arrow[r] & C^{i+1} = C^i \cup \sigma_{n-i}
\end{tikzcd} .$$

The composition of these is i. Each i_k is in $\overline{\mathcal{M}}_1$, so i is a composition of morphisms in $\overline{\mathcal{M}}_1$, hence is in $\overline{\mathcal{M}}_1$

The case $e = 1$ is dual.

2. Consider the map $i\colon [n] \to [1] \times [n]$ sending i to $(i, 0)$, and the map $r\colon [1] \times [n] \to [n]$ as follows.

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & k \arrow[r] & k+1 \arrow[r] & \cdots \arrow[r] & n \\
0 \arrow[r] \arrow[u] & 1 \arrow[r] \arrow[u] & \cdots \arrow[r] & k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & \cdots \arrow[r] & k \arrow[u]
\end{tikzcd}$$

3. For any morphisms $i\colon A \to A'$ in Set_Δ, the set

$$\{f\colon B \to B' \mid f \wedge i \in \overline{\mathcal{M}}_2\}$$

is saturated because by Lemma 3.25 it is given by

$${}_\perp\{\mathrm{Maps}(A', X) \to \mathrm{Maps}(A, Y) \times_{\mathrm{Maps}(A', Y)} \mathrm{Maps}(A', X) \mid X \to Y \text{ Kan fibration}\}.$$

$$\mathcal{E} = \{\{e\} \to \Delta^1 \mid e = 0, 1\}.$$

Then

It is easy to see that $\overline{\mathcal{M}}_2 \subset \overline{\mathcal{M}}_3$ since the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ is monic.

□

This theorem gives us

Corollary 3.28. Let f be anodyne, and let $g\colon X \to Y$ be a monomorphism. Then $f \wedge g$ is anodyne.

Proof. □

Example 3.29. Any horn filling $\Lambda^n_i \hookrightarrow \Delta^n$ is anodyne, and boundary fillings $\partial\Delta^m \to \Delta^m$ are mono, so pushouts

$$\partial\Delta^m \times \Delta^n \coprod_{\partial\Delta^m \times \Lambda^n_i} \Delta^m \times \Lambda^n_i \to \Delta^m \times \Delta^n$$

are anodyne.

5 *A technical lemma of great importance*

Lemma 3.30. Let $i\colon A \to B$ be a monomorphism, and let $p\colon K \to S$ be a Kan fibration (Definition 2.41). Consider the following pushout.

$$\begin{tikzcd}
\mathrm{Maps}(B,K) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathrm{Maps}(A,K) \underset{\mathrm{Maps}(A,S)}{\times} \mathrm{Maps}(B,S) \arrow[r] \arrow[d] & \mathrm{Maps}(A,K) \arrow[d] \\
& \mathrm{Maps}(B,S) \arrow[r] & \mathrm{Maps}(A,S)
\end{tikzcd}$$

The dashed arrow is a Kan fibration.

Proof. We need to show that the dashed arrow solves right lifting problems of the following form,

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "j"'] & \mathrm{Maps}(B,K) \arrow[d, dashed] \\
B' \arrow[r] \arrow[ur, dotted] & \mathrm{Maps}(A,K) \underset{\mathrm{Maps}(B,K)}{\times} \mathrm{Maps}(B,S)
\end{tikzcd}$$

with $j\colon A' \to B'$ a horn filling. It will actually be easier to solve this problem in the generality where j is any anodyne morphism; this will certainly suffice because by Theorem 3.27, all horn fillings are anodyne.

By Lemma 3.25, this is equivalent to the following lifting problem.

$$\begin{tikzcd}
A' \times B \coprod_{A' \times A} B' \times A \arrow[r] \arrow[d] & K \arrow[d, "p"] \\
B' \times B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

The vertical arrow on the left is the smash product $i \wedge j$. By Corollary 3.28, this is anodyne. By assumption, p is a Kan fibration, so the lift exists. ◻

Corollary 3.31. Let K be a Kan complex, and let B be a simplicial set. Then the simplicial set $\mathrm{Maps}(B,K)$ is a Kan complex.

Proof. Take $A = \emptyset$ and $S = *$. Then we have that the map

$$\mathrm{Maps}(B,K) \to \mathrm{Maps}(\emptyset,K) \underset{\mathrm{Maps}(\emptyset,*)}{\times} \mathrm{Maps}(B,*) = *$$

is a Kan fibration. But by Corollary 2.42, this means that $\mathrm{Maps}(B,K)$ is a Kan complex. ◻

Corollary 3.32. For $i\colon A \to B$ a monomorphism of simplicial sets and K a Kan complex, the map

$$\mathrm{Maps}(i, \mathrm{id}_K)\colon\ \mathrm{Maps}(B, K) \to \mathrm{Maps}(A, K)$$

is a Kan fibration.

Proof. Setting $S = *$, we find that the map

$$\mathrm{Maps}(B, K) \to \mathrm{Maps}(A, K) \underset{\mathrm{Maps}(A,*)}{\times} \mathrm{Maps}(B, *) = \mathrm{Maps}(A, K)$$

is a Kan fibration, which is what we are after. □

4

Simplicial homotopy

1 Homotopy of vertices

Definition 4.1 (homotopy of vertices)**.** Let S be a simplicial set. We say that $x, y \in K_0$ are homotopic if there is an edge $f \colon x \to y$.

Theorem 4.2. If $S = K$ is a Kan complex, the above notion is an equivalence relation.

Proof. We have that x is homotopic to x because of the edge $s_0 x$.

Given $f \colon x \to y$ and $g \colon y \to z$, we can fill the following horn.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed] & & z \end{tikzcd}$$

This gives a homotopy from x to z.

Similarly, suppose we have a homotopy $x \to y$. Then filling the horn

$$\begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr, "s_0 x"'] & & x \end{tikzcd}$$

gives us a homotopy $y \to x$. □

Thus, we can think of partitioning a Kan complex into disconnected components.

2 Homotopy of edges

We will often want to consider edges in a simplicial set up to some notion of homotopy.

Definition 4.3 (homotopy of edges)**.** Let $f \colon x \to y$ and $g \colon x \to y$ be edges in some simplicial set K. We say that f and g are homotopic if there exists a 2-simplex of the following form.

$$\begin{tikzcd} & w \arrow[dr, "s_0 w"] & \\ v \arrow[ur, "f"] \arrow[rr, "g"'] & & w \end{tikzcd}$$

Unfortunately, this is in general not an equivalence relation since it is not symmetric. It becomes one if we restrict our attention to Kan complexes. However, it turns out we will not have to make any use of outer horn fillers. This will be important later on.

Theorem 4.4. For any set with inner horn fillers, that is fillers $\Lambda^n_i \to \Delta^n$ for $0 < i < n$, homotopy of edges is an equivalence relation.

In particular, it is an equivalence for any Kan complex K.

Proof. First, we show reflexivity. For any edge f, the simplicial identities (Theorem 2.17) ensure that the degenerate simplex $\sigma_f = s_1 f$ satisfies

$$d_1\sigma_f = f, \qquad d_2\sigma_f = f, \qquad \text{and} \qquad d_0\sigma_f = s_0 d_0 f.$$

To see that it is symmetric, create from a 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

the 1-skeleton of a 2-simplex whose front and back are as follows.

$$\begin{tikzcd} y \arrow[r, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[ur, "g"] \arrow[r, "f"'] & y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "s_0 y"] \arrow[dr, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[r, "f"'] & y \end{tikzcd}$$

This gives us the boundaries of four 1-simplices. Each of these can be filled except for the bottom simplex of the first diagram. Doing so gives us a 1-horn, which we can fill to find a 2-simplex as follows.

$$\sigma' = \begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

This says precisely that $g \sim f$.

To check transitivity, suppose we are given two simplices as follows.

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "g"] \arrow[rr, "h"'] & & y \end{tikzcd}$$

We can create a horn $\Lambda^3_2 \to X$ as follows.

$$\begin{tikzcd} y \arrow[r, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[r, "h"'] \arrow[ur, "g"] & y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "s_0 y"] \arrow[dr, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[r, "h"'] & y \end{tikzcd}$$

Filling this gives what we want. □

Theorem 4.5. Two edges f and g in a Kan complex K are homotopic in the above sense if and only if there exists a square of the following form.

$$\begin{tikzcd} v \arrow[r, "f"] \arrow[d, "s_0 v"'] \arrow[dr] & w \arrow[d, "s_0 w"] \\ v \arrow[r, "g"'] & w \end{tikzcd}$$

Proof. Suppose we have a 2-simplex exhibiting f and g as homotopic. We can form a square by attaching the 2-simplex $s_0 g$ as to the bottom.

$$\begin{tikzcd} v \arrow[r, "f"] \arrow[d, "s_0 v"'] \arrow[dr, "g"] & w \arrow[d, "s_0 w"] \\ v \arrow[r, "g"'] & w \end{tikzcd}$$

Given a square as above, call the bottom 2-simplex η and the upper simplex μ. Create the horn

$$\begin{tikzcd} v \arrow[r, "s_0 v"] \arrow[d, "g"'] \arrow[dr, "d_1 \eta"] & v \arrow[d, "g"] \\ w \arrow[r, "s_0 w"'] & w \end{tikzcd} \qquad \begin{tikzcd} v \arrow[r, "s_0 v"] \arrow[d, "g"'] & v \arrow[d, "g"] \arrow[dl, "g"'] \\ w \arrow[r, "s_0 w"'] & w \end{tikzcd}$$

where the top left simplex is η, the top right is $s_0 g$, the bottom right is $s_1 g$, and the bottom left is unfilled. Filling it, we find that $g \sim d_1 \eta$. But μ exhibits $f \sim d_1 \eta$, so $f \sim g$. □

3 *Homotopy of maps*

Definition 4.6 (homotopy)**.** Let $f, g\colon B \to K$ be simplicial sets. A homotopy from f to g is a morphism

$$H\colon \Delta^1 \times B \to K$$

such that the restriction of H to $\{0\} \times B$ is f, and the restriction of H to $\{1\} \times B$ is g.

In this case, we will write

$$f \overset{H}{\simeq} g$$

To put it another way, H is a homotopy above if the pullback along the inclusion $B \to$

$\Delta^1 \times B$ sending $\sigma \mapsto (0, \sigma)$ is equal to f, and similarly for g.

$$\begin{tikzcd}
\Delta^0 \times B \arrow[d, hook, "\{0\}\times\mathrm{id}"'] \arrow[dr, "f"] & \\
\Delta^1 \times B \arrow[r, "H"] & K \\
\Delta^0 \times B \arrow[u, hook', "\{1\}\times\mathrm{id}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Note that chasing elements around, this immediately implies that, for all σ,

$$H(0, \sigma) = f(\sigma) \qquad \text{and} \qquad H(1, \sigma) = g(\sigma).$$

Definition 4.7 (relative homotopy)**.** Suppose we are further given a monomorphism $i\colon A \to B$ making the following diagram commute.

$$\begin{tikzcd}
& B \arrow[dr, "f"] & \\
A \arrow[ur, "i"] \arrow[dr, "i"'] & & K \\
& B \arrow[ur, "g"'] &
\end{tikzcd}$$

Denote $f \circ i = g \circ i = u$. Then we say that H is a homotopy relative to A if the diagram

$$\begin{tikzcd}
\Delta^1 \times B \arrow[r, "H"] & K \\
\Delta^1 \times A \arrow[u, "\mathrm{id}\times i"] \arrow[r, "\pi_A"'] & A \arrow[u, "u"']
\end{tikzcd}$$

commutes. We will write

$$f \overset{H}{\simeq} g \quad (\text{rel. } A).$$

The intuitive meaning of this diagram is that H leaves $A \subset B$ fixed. Indeed, again chasing elements around, we find that

$$H(e, \sigma) = f(\sigma) = g(\sigma)$$

for all $\sigma \in A$ and $e = 0, 1$.

We would like to use homotopy to define an equivalence relation on maps of simplicial sets. However, in a general simplicial set this is impossible. However, we saw that even in the case that f and g are inclusions of a vertex, this is impossible without control of outer horns. Consider the maps

$$\iota_0,\ \iota_1 \colon \Delta^0 \to \Delta^1.$$

which send the point to the zeroth and first vertices. The morphisms ι_0 and ι_1 satisfy

$$\iota_0 \simeq \iota_1,$$

since $H = \mathrm{id}_{\Delta^1}$ makes the following diagram commute.

$$\begin{tikzcd}
\Delta^0 \times \Delta^0 = \Delta^0 \arrow[d, "d^1\times\mathrm{id}"'] \arrow[dr, "\iota_0"] & \\
\Delta^1 \times \Delta^0 = \Delta^1 \arrow[r, "H=\mathrm{id}_{\Delta^1}"] & \Delta^1 \\
\Delta^0 \times \Delta^0 = \Delta^0 \arrow[u, "d^0\times\mathrm{id}"] \arrow[ur, "\iota_1"'] &
\end{tikzcd}$$

However, there is no morphism $H\colon \Delta^1 \to \Delta^1$ which sends $0 \mapsto 1$ and $1 \mapsto 0$, so there does not exist any H such that

$$\iota_1 \overset{H}{\simeq} \iota_0.$$

The way out of this is to specialize to Kan complexes.

Proposition 4.8. Let $i\colon A \to B$ be a monomorphism of simplicial sets, let K be a Kan complex, and let $u\colon A \to K$ be a morphism. Then the relation *is homotopic relative to* A defines an equivalence relation on the set

$$\{f : B \to K \mid f \circ i = u\}.$$

Proof. By Note 2.26, we can think of u as a map $\Delta^0 \to \mathrm{Maps}(A, K)$ which picks out u as a zero-simplex.

Consider the pullback square

$$\begin{tikzcd}
\mathrm{Maps}(B,K)^u \arrow[r] \arrow[d] & \mathrm{Maps}(B,K) \arrow[d, "\mathrm{Maps}(i,\mathrm{id}_K)"] \\
\Delta^0 \arrow[r, "u"'] & \mathrm{Maps}(A,K)
\end{tikzcd}$$

By Corollary 3.32, the map $\mathrm{Maps}(i, \mathrm{id}_K)$ is a Kan fibration. Thus, since Kan fibrations are saturated (hence closed under pullback), we have that the induced map $\mathrm{Maps}(B, K)^u \to \{u\}$ is also a Kan fibration. But by Corollary 2.42, this means that $\mathrm{Maps}(B, K)^u$ is a Kan complex.

The elements of $\mathrm{Maps}(B, K)^u_0$ are ordered pairs $(u, f) \in \{u\} \times \mathrm{Set}_\Delta(B, K)$ such that $u = i \circ f$. That is, $\mathrm{Maps}(B, K)^u_0$ consists of those morphisms $B \to K$ which agree with u when restricted to A along i.

Thus, showing that homotopy between maps $B \to K$ relative to A is an equivalence relation is the same as showing homotopy between vertices of the Kan complex $\mathrm{Maps}(B, K)^u$ is an equivalence relation, which is true by Theorem 4.4. □

4 *Homotopy groups*

Definition 4.9 (homotopy group)**.** Let K be a Kan complex. Define the zeroth homotopy group of K, denoted $\pi_0(K)$, to be the set of homotopy classes of vertices of K. This is well-defined by Theorem 4.2.

For $n \geq 1$, define the nth homotopy group of K at v, denoted $\pi_n(K, v)$, to be the set of homotopy classes of n-simplices $\alpha\colon \Delta^n \to K$ relative to the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$, such that the following diagram commutes.

$$\begin{tikzcd}
\Delta^n \arrow[r, "\alpha"] & X \\
\partial\Delta^n \arrow[u, hook] \arrow[r] & \Delta^0 \arrow[u, hook, "v"']
\end{tikzcd}$$

This diagram says precisely that the boundary of Δ^n is collapsed to the point v.

Definition 4.10. Let $\alpha, \beta\colon \Delta^n \to K$ be n-simplices whose boundaries are trivial. Denote their equivalence classes by $[\alpha]$ and $[\beta]$ respective. Define their composition, denoted $[\alpha] \cdot [\beta]$ as follows. First, define an inner $(n+1)$-horn with faces

$$(\overbrace{\{v\}, \cdots, \{v\}}^{n-1}, \alpha, -, \beta).$$

This has a filler σ such that

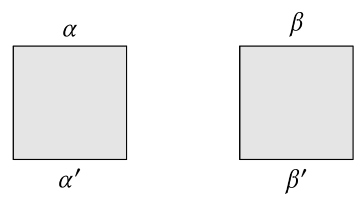
$$\partial\sigma = (\{v\}, \cdots, \{v\}, \alpha, d_n\sigma, \beta)$$

Define $[\alpha] \cdot [\beta] = [d_n\sigma]$.

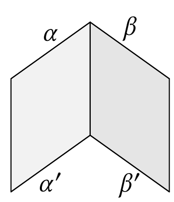
It remains to check that this composition is well-defined.

Proposition 4.11. The composition defined above is well-defined, i.e. if $\alpha' \overset{H}{\simeq} \alpha$ and $\beta' \overset{H'}{\simeq} \beta$, then $[\alpha'] \cdot [\beta'] = [\alpha] \cdot [\beta]$.

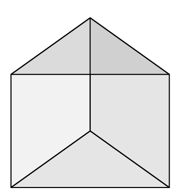
Sketch of proof. We have homotopies between the n-simplices we want to multiply,



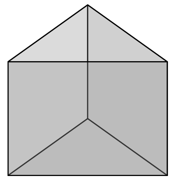
which we extend to a homotopy between the horns defining our multiplication law.



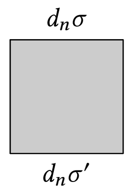
These horns have fillers, giving us the result of our multiplication.



Using the general theory of anodyne morphisms, we can extend our homotopy between the horns to a homotopy between the full filled simplices.



Restricting to the filled faces gives us a homotopy between the result of multiplication.



□

Proof. By definition, we have an $(n+1)$-simplices σ and σ' such that

$$\partial\sigma = (\{v\}, \ldots, \{v\}, \alpha, d_n\sigma, \beta)$$

and

$$\partial\sigma' = (\{v\}, \ldots, \{v\}, \alpha', d_n\sigma', \beta').$$

We need to show that $d_n\sigma \simeq d_n\sigma'$.

First, we construct a homotopy between the horns

$$\eta = (\{v\}, \ldots, \{v\}, \alpha, -, \beta) \qquad \text{and} \qquad \eta' = (\{v\}, \ldots, \{v\}, \alpha', -, \beta').$$

By the distributive law for products and coproducts

$$A \times (B \amalg C) \simeq (A \times B) \amalg (A \times C),$$

a homotopy between horns $\Lambda^{n+1}_i \to K$ consists of a homotopy for each face. Therefore, we can choose our homotopy H'' to have components $(\mathrm{id}_v, \ldots, \mathrm{id}_v, H, -, H')$. This gives us a map $\Delta^1 \times \Lambda^{n+1}_n \to K$. Our fillers give us two maps $\sigma, \sigma' \colon \Delta^{n+1} \to K$, which we can view as a map $\partial\Delta^1 \times \Delta^{n+1} \to K$. This gives us the following commutative diagram

$$\begin{tikzcd}
\partial\Delta^1 \times \Lambda^{n+1}_n \arrow[r, hook] \arrow[d, hook] & \partial\Delta^1 \times \Delta^{n+1} \arrow[d, "{(\sigma,\sigma')}"] \\
\Delta^1 \times \Lambda^{n+1}_n \arrow[r, "H''"] & K
\end{tikzcd}$$

This gives us a map

$$h \colon \Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Lambda^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1} \to K.$$

Note that by Example 3.29, we get an anodyne extension

$$\Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Lambda^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1} \hookrightarrow \Delta^1 \times \Delta^n.$$

This means that (by Corollary 2.42) we can find a lift for the following diagram.

$$\begin{tikzcd}
\Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Lambda^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1}. \arrow[r, "h"] \arrow[d, hook] & K \\
\Delta^1 \times \Delta^{n+1} \arrow[ur, dashed, "\exists I"'] &
\end{tikzcd}$$

Restricting I to the nth face of Δ^{n+1} gives us a homotopy

$$\Delta^1 \times \Delta^n \to K$$

as required. □

Theorem 4.12. Let K be a Kan complex, and let $v \in K_0$ be a vertex of K. Then the multiplication from Definition 4.10 makes $\pi_n(K, v)$ into a group for $n \geq 1$.

Proof. First we check associativity. Let $\alpha, \beta, \gamma \colon \Delta^n \to K$ be n-simplices with trivial boundary, representing elements $[\alpha]$, $[\beta]$, $[\gamma] \in \pi_n(K, v)$. By definition, we get an $(n+1)$-simplex σ_{n-1} such that

$$\partial\sigma_{n-1} = (v, v, \ldots, v, \alpha, d_n\sigma_{n-1}, \beta).$$

giving us the result of the multiplication $[\alpha] \cdot [\beta] = [d_n\sigma_{n-1}]$.

Similarly, we have some σ_{n+2} such that

$$\partial\sigma_{n+2} = (v, v, \ldots, v, \beta, d_n\sigma_{n+2}, \gamma)$$

so $[\beta] \cdot [\gamma] = [d_n\sigma_{n+2}]$.

Form from these a simplex σ_{n+1} with boundary

$$\partial\sigma_{n+1} = (v, v, \ldots, v, d_n\sigma_{n-1}, d_n\sigma_{n+1}, \gamma)$$

giving us the result of the multiplication $([\alpha] \cdot [\beta]) \cdot [\gamma] = [d_n\sigma_{n-1}] \cdot [\gamma] = [d_n\sigma_{n+1}]$.

These simplices fit together to give a horn

$$\Lambda^{n+2}_n \to K$$

with boundary

$$(v, v, \ldots, v, \sigma_{n-1}, -, \sigma_{n+1}, \sigma_{n+2}).$$

Filling this gives an $(n+1)$-simplex σ with boundary

$$\partial\sigma = (v, v, \ldots, v, \sigma_{n-1}, d_n\sigma, \sigma_{n+1}, \sigma_{n+2}).$$

Using the simplicial identities, we find

$$\begin{aligned}\partial d_n\sigma &= (d_0 d_n\sigma, d_1 d_n\sigma, \ldots, d_{n-2}d_n\sigma, d_{n-1}d_n\sigma, d_n d_n\sigma, d_{n+1}d_n\sigma)\\ &= (d_{n-1}d_0\sigma, d_{n-1}d_1\sigma, \ldots, d_{n-1}d_{n-2}\sigma, d_{n-1}d_{n-1}\sigma, d_n d_{n+1}\sigma, d_n d_{n+2}\sigma)\\ &= (v, v, \ldots, v, \alpha, d_n\sigma_{n+1}, d_n\sigma_{n+2}).\end{aligned}$$

This is an n-simplex exhibiting $[\alpha]\cdot[d_n\sigma_{n-2}] = [d_n\sigma_{n+1}]$.

Thus, we have

$$\begin{aligned}([\alpha]\cdot[\beta])\cdot[\gamma] &= [d_n\sigma_{n-1}]\cdot[\gamma]\\ &= [d_n\sigma_{n+1}]\\ &= [\alpha]\cdot[d_n\sigma_{n+2}]\\ &= [\alpha]\cdot([\beta]\cdot[\gamma]).\end{aligned}$$

The degenerate n-simplex v functions as a unit on the left because there exists an $(n+1)$-horn $s_n\omega$ with

$$\begin{aligned}\partial(s_n\omega) &= (d_0 s_n\omega, \ldots, d_{n-1}s_n\omega, d_n s_n\omega, d_{n+1}s_n\omega)\\ &= (s_{n-1}d_0\omega, \ldots, s_{n-1}d_{n-1}\omega, \omega, \omega)\\ &= (v, \ldots, v, v, \omega, \omega),\end{aligned}$$

so $[e]\cdot[\omega] = [\omega]$.

It functions as a unit on the right because of $s_{n-2}\omega$.

Furthermore. □

We would like $\pi_0(K)$ and $\pi_n(K, v)$ to be the simplicial analogs of the topological notions of homotopy groups. Recall that we can turn any simplicial set into a topological space via the geometric realization functor $|\cdot|$. Therefore, one would hope that the simplicial homotopy groups would agree with the homotopy groups of the geometrical realization. At least for π_o this turns out to be the case.

Theorem 4.13. For any Kan complex K, we have $\pi_0(K) \cong \pi_0(|K|)$ as sets.

Proof. Define a map $K_0 \to |K|$ sending $v \to |v|$.

Suppose $v \sim v'$ in K. Then there is an edge f between v and k. Applying $|\cdot|$, we find an edge $|v| \to |v'|$. Thus, the map above respects equivalence classes, and descends to a map $\pi_0(K) \to \pi_0(|K|)$.

Now some homotopy stuff I don't understand. □

Theorem 4.14. Let $p\colon K \to S$ be a Kan fibration. Let v be a vertex of K, and let $w = p(v)$. Let F be the fiber over w given by the following pullback equare.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ \{w\} \arrow[r, hook] & S \end{tikzcd}$$

There is the following long exact sequence.

$$\cdots \longrightarrow \pi_{n+1}(K,v) \longrightarrow \pi_n(F,v) \longrightarrow \pi_n(K,v) \longrightarrow \pi_n(S,w) \longrightarrow \pi_{n-1}(F,v) \longrightarrow \cdots$$

5

Model categories

1 Localization of categories

Many areas of mathematics have a notion of *weak equivalence,* i.e. a class of morphisms which are not isomorphisms but which one wants to think of as equivalences.

- In topology, one often wants to study spaces up to homotopy equivalence.
- In homological algebra, one is often interested in chain complexes only up to quasi-isomorphism.
- In ordinary category theory, one wants to consider two categories to be the same if they are equivalent.

The difficulty in treating these classes of morphisms as equivalences is that we would like equivalences to be invertible, whereas morphisms belonging to the above classes are in general not invertible. Therefore, we would like to solve the following problem: given a category $\mathcal{C}$ and a set of morphisms $\mathcal{W}$ in $\mathcal{C}$, how can we change $\mathcal{C}$ so as to make each morphism in $\mathcal{W}$ an isomorphism? There are two ways that one could go about this.

1. One could freely add formal inverses for all morphisms in $\mathcal{W}$. This is, modulo issues of size, always possible, but difficult to work with. This is analogous to the localization of a ring.
2. One can try to find a replacement for each object of $\mathcal{C}$ such that weak equivalences between these objects turn out to be invertible. This is what one does in homological algebra by taking projective resolutions.

Whatever approach one takes, the result is called a the localization of $\mathcal{C}$ by $\mathcal{W}$.

Definition 5.1 (localization of a category)**.** Let $\mathcal{C}$ be a category, and $\mathcal{W}$ a set of morphisms of $\mathcal{C}$. The localization of $\mathcal{C}$ at $\mathcal{W}$ is a pair $(\mathcal{C}[\mathcal{W}^{-1}], \pi)$ where $\mathcal{C}([\mathcal{W}^{-1}])$ is a category and $\pi\colon \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is a functor, such that the following conditions hold.

- For all $w \in \mathcal{W}$, $\pi(w)$ is an isomorphism.
- For any category $\mathcal{A}$ and functor $F \colon \mathcal{C} \to \mathcal{A}$ such that $F(w)$ is an isomorphism for all $w \in \mathcal{W}$, there exists a functor $\tilde{F} \colon \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{A}$ and a natural isomorphism $\tilde{F} \circ \pi \Rightarrow F$.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\pi"'] & {} & \mathcal{A} \\ & \mathcal{C}[\mathcal{W}^{-1}] \arrow[ur, "\exists \tilde{F}"'] \arrow[u, phantom, "\Uparrow"] & \end{tikzcd}$$

Note that the above diagram is not required to commute on the nose.

- For every category $\mathcal{A}$, the pullback functor

$$\pi^* \colon \mathrm{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{A}) \to \mathrm{Cat}(\mathcal{C}, \mathcal{A})$$

is fully faithful.

Note that by the third axiom, the functor $\tilde{F}$ is guaranteed to be unique up to isomorphism: fully faithfulness tells us precisely that functors $\mathcal{C} \to \mathcal{A}$ and functors $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{A}$ sending $\mathcal{W}$ to isomorphisms are in bijection up to isomorphism

As previously alluded to, there is an explicit construction of the localization of a category at any set of weak equivalences, but it is difficult to work with. Fortunately, model categories provide a way of computing and working with localizations which is practical and avoids issues of size.

2 *Model categories*

Model categories provide a distillation of the structures appearing in various flavors of homotopy theory.

Definition 5.2 (model category)**.** A model category is a category $\mathcal{C}$ together with three sets of morphisms

- $\mathcal{W}$, the set of *weak equivalences*
- $\mathcal{F}\mathrm{ib}$, the set of *fibrations*
- $\mathcal{C}\mathrm{of}$, the set of *cofibrations*

subject to the following axioms.

(M1) The category $\mathcal{C}$ has all small limits and colimits.

(M2) The set $\mathcal{W}$ satisfies the so-called *two-out-of-three law:* if the diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Z \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

commutes and two out of three of f, g, and h are weak equivalences, then the third is as well.

(M3) The sets $\mathcal{W}$, $\mathcal{F}\mathrm{ib}$, and $\mathcal{C}\mathrm{of}$ are all closed under retracts: given the following diagram, if g belongs to any of the above sets, then so does f.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z \arrow[r] & W \arrow[r] & Z \end{tikzcd}$$

(M4) We can solve a lifting problem

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] & P \end{tikzcd}$$

if all of the following conditions hold.

- The morphism i is a cofibration.
- The morphism p is a fibration.
- At least one of i and p is a weak equivalence.

(M5) Any morphism $f \colon X \to Y$ admits the following factorizations.

- We can write $f = p \circ i$, where p is a fibration and i is both a cofibration and a weak equivalence.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathcal{C}\mathrm{of}\cap\mathcal{W}\ni i"'] & & B \\ & X \arrow[ur, "p\in\mathcal{F}\mathrm{ib}"'] & \end{tikzcd}$$

- We can write $f = p \circ i$, where p is both a fibration and a weak equivalence, and i is a cofibration.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathcal{C}\mathrm{of}\ni i"'] & & B \\ & X \arrow[ur, "p\in\mathcal{F}\mathrm{ib}\cap\mathcal{W}"'] & \end{tikzcd}$$

To simplify terminology, we will use the following terms.

- We will call morphisms which are both weak equivalences and fibrations *trivial fibrations*:

$$\mathcal{W} \cap \mathcal{F}\mathrm{ib} = \{\text{trivial fibrations}\}.$$

- We will call morphisms which are both weak equivalences and cofibrations *trivial cofibrations*:

$$\mathcal{W} \cap \mathcal{C}\mathrm{of} = \{\text{trivial cofibrations}\}.$$

- We will call an object $X \in \mathcal{C}$ *cofibrant* if the unique morphism $\emptyset \to X$ is a cofibration.

- We will call an object $Y \in \mathcal{C}$ *fibrant* if the unique morphism $X \to *$ is a fibration.

We will see some examples of model categories after a few lemmas.

Lemma 5.3 (lifting lemma)**.** Let $\mathcal{C}$ be any category, and let $f = p \circ i$ be a morphism in $\mathcal{C}$.

1. If f has the left lifting property with respect to p, then f is a retract of i.
2. If f has the right lifting property with respect to i then f is a retract of p.

Proof. Suppose f has the left-lifting property with respect to p. Then the following diagram admits a lift.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[ur, dashed] \arrow[r, equal] & B \end{tikzcd}$$

Expanding, we find the following diagram.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ B \arrow[r, dashed] \arrow[rr, bend right, "\mathrm{id}_B"'] & X \arrow[r, "p"] & B \end{tikzcd}$$

This exhibits f as a retract of i.

The second case is dual. □

The axioms for a model category are over-specified, in the sense that providing $\mathcal{W}$ and either $\mathcal{F}\mathrm{ib}$ and $\mathcal{C}\mathrm{of}$ is sufficient to derive the other one.

Lemma 5.4. Let $\mathcal{C}$ be a model category. We have the following.

1. $\mathcal{C}\mathrm{of}_\perp = \mathcal{F}\mathrm{ib} \cap \mathcal{W}$
2. $\mathcal{C}\mathrm{of} = {}_\perp(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$
3. $(\mathcal{C}\mathrm{of} \cap \mathcal{W})_\perp = \mathcal{F}\mathrm{ib}$
4. $\mathcal{C}\mathrm{of} \cap \mathcal{W} = {}_\perp\mathcal{F}\mathrm{ib}$

Proof. We prove 4. The proof of 1 is the same, mutatis mutandis.

Let $f \in \mathcal{C}\mathrm{of} \cap \mathcal{W}$. Then by Axiom (M4), $f \in {}_\perp\mathcal{F}\mathrm{ib}$.

Now suppose $f \in {}_\perp\mathcal{F}\mathrm{ib}$. By Axiom (M5), we can always find a factorization $f = p \circ i$, where p is a fibration and i is a trivial cofibration.

By Axiom (M4), f has the left lifting property with respect to p, so by Lemma 5.3, f is a retract of i. But since by Axiom (M3) both $\mathcal{W}$ and $\mathcal{C}\mathrm{of}$ are closed under retracts, f is also a trivial cofibration as needed.

From 4, 3 follows immediately. The proof of 1 is precisely the same as that of 4, and 2 follows from 1 immediately. □

Corollary 5.5. In any model category, the sets $\mathcal{C}\mathrm{of}$ and $\mathcal{C}\mathrm{of} \cap \mathcal{W}$ are saturated (Definition 3.1).

Proof. By Theorem 3.8, any set which is of the form ${}_{\perp}\mathcal{M}$ is saturated. □

Example 5.6. The following are examples of model structures.

1. The category Top carries the so-called *Quillen model structure,* with
   - $\mathcal{W}$ = weak homotopy equivalences,
   - $\mathcal{F}\mathrm{ib}$ = Serre fibrations,
   - $\mathcal{C}\mathrm{of} = {}_{\perp}(\mathcal{F}\mathrm{ib} \cap \mathcal{W})$.
2. The category Set_{Δ} has a model structure called the *Kan model structure* with
   - $\mathcal{W}$ = weak homotopy equivalences, i.e. Kan fibrations $X \to Y$ such that $\pi_0(X) \to \pi_0(Y)$ is a bijection and $\pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $x \in X$.
   - $\mathcal{F}\mathrm{ib}$ = Kan fibrations,
   - $\mathcal{C}\mathrm{of}$ = monomorphisms.

   Note that in this case, $\mathcal{W} \cap \mathcal{C}\mathrm{of} = {}_{\perp}\mathcal{F}\mathrm{ib} = \{\text{anodyne morphisms}\}$.
3. For any small ring R, the category $\mathrm{Ch}_{\geq}(R)$ of bounded-below chain complexes of small left R-modules has a model structure, called the *projective model structure,* with
   - $\mathcal{W}$ = Quasi-isomorphisms
   - $\mathcal{F}\mathrm{ib}$ = morphisms f of complexes such that f_n is an epimorphism for all $n \geq 0$
   - $\mathcal{C}\mathrm{of}$ = morphisms f of complexes such that f_n is a monomorphism for all $n \geq 0$, and $\mathrm{coker}\, f_n$ is projective for all $n \geq 0$.

Example 5.7. Let $\mathcal{C}$ be a category with model structure $(\mathcal{W}, \mathcal{F}\mathrm{ib}, \mathcal{C}\mathrm{of})$. Then $\mathcal{C}^{\mathrm{op}}$ is a model category with model structure $(\mathcal{W}, \mathcal{C}\mathrm{of}, \mathcal{F}\mathrm{ib})$.

2.1 A model structure on the category of categories

Proving that three sets of morphisms form the fibrations, cofibrations, and weak equivalences of a model structure can be an immense undertaking. There are a few model structures, however, which are simple enough that the conditions can be checked relatively simply.

Consider the category I with the following objects and morphisms (excluding identity morphisms).

$$\begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd}$$

Denote the functor including $0 \hookrightarrow I$ by ι.

The following is a model structure on Cat.

- $\mathcal{W}$ = categorical equivalences
- $\mathcal{F}\mathrm{ib}$ = *isofibrations*, i.e. functors with the right lifting property with respect to ι
- $\mathcal{C}\mathrm{of}$ = functors which are injective on objects

We verify the axioms one by one. This section is incomplete and a bit wrong at the moment. I'll come back to it.

2.1.1 Axiom M1: Limits and colimits

We already know that Cat has all limits and colimits.

2.1.2 Axiom M2: Two-out-of-three

We have the following results.

- If a functor is a weak equivalence, then its weak inverse is also a weak equivalence.
- Composition of weak equivalences gives again a weak equivalence (by the horizontal composition formula).

Thus, the same reasoning that shows that the 2/3 law holds for bona-fide isomorphisms also holds for weak equivalences.

2.1.3 Axiom M3: Closure under retracts

First suppose that G is an equivalence of categories. We need to show that F is also an equivalence of categories.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "R"] \arrow[d, "F"'] & \mathcal{C} \arrow[r, "S"] \arrow[d, "G"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{B} \arrow[r, "T"'] & \mathcal{D} \arrow[r, "U"'] & \mathcal{B} \end{tikzcd}$$

Since both $S \circ R$ and $U \circ T$ compose to the identity, T and R are injective on objects and morphisms, and S and U are similarly surjective. Thus G is:

- **Essentially surjective:** We can map any $b \in \mathcal{B}$ to $\mathcal{D}$ with T, then lift it to something in $\mathcal{C}$ using G's weak inverse, which we'll call G^{-1}. Mapping the result of this to $\mathcal{A}$ with S gives us an object
$$(S \circ G^{-1} \circ T)(b) \in A.$$
The claim is that under F this maps to something isomorphic to b. Evaluating on F, we have
$$F((S \circ G^{-1} \circ T)(b)) = (U \circ G \circ G^{-1} \circ T)(b).$$
We have a natural isomorphism $G \circ G^{-1} \Rightarrow \mathrm{id}$, which tells us that
$$(U \circ G \circ G^{-1} \circ T)(b) \cong (U \circ T)(b) = b.$$

- **Fully faithful:** The logic is very similar. We can map any morphism f in $\mathcal{B}$ to a morphism in $\mathcal{D}$ with T, and (because G is fully faithful) find a unique preimage in $\mathcal{C}$ which we'll call $G^{-1}(T(f))$. Applying S gives us a morphism

$$(S \circ G^{-1} \circ T)(f) \quad \text{in } \mathcal{A}.$$

Then

$$\begin{aligned}(F \circ S \circ G^{-1} \circ T)(f) &= (U \circ G \circ G^{-1} \circ T)(f) \\ &= (U \circ T)(f) \\ &= f.\end{aligned}$$

Now suppose that G is in $\mathfrak{C}$. We need to show that $F \in \mathfrak{C}$.

Applying the forgetful functor to Set we have the following diagram.

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "s"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "t"'] & D \arrow[r, "u"'] & B \end{tikzcd}$$

The map r is injective because the identity factors through it, and g is injective by assumption. Suppose we are given $a, a' \in A$ such that $f(a) = f(a')$. Then $t(f(a)) = t(f(a'))$, so $g(r(a)) = g(r(a'))$. But g and r are both injective, so $a = a'$.

Now suppose that G is in $\mathfrak{F}$. We need to show that $F \in \mathfrak{F}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "R"] \arrow[d, "F"'] & \mathcal{C} \arrow[r, "S"] \arrow[d, "G"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{B} \arrow[r, "T"'] & \mathcal{D} \arrow[r, "U"'] & \mathcal{B} \end{tikzcd}$$

To see this, note that we can augment any lifting problem

$$\begin{tikzcd} (*) \arrow[r] \arrow[d, hook] & \mathcal{A} \arrow[d, "F"] \\ (* \leftrightarrow \bullet) \arrow[r] & \mathcal{B} \end{tikzcd}$$

as follows; composing with R gives us a lifting problem we can solve, yielding the dashed arrow.

$$\begin{tikzcd} (*) \arrow[r] \arrow[d, hook] & \mathcal{A} \arrow[r, "R"] \arrow[d] & \mathcal{C} \arrow[r, "S"] \arrow[d] & \mathcal{A} \arrow[d] \\ (* \leftrightarrow \bullet) \arrow[r] \arrow[urr, dashed] & \mathcal{B} \arrow[r] & \mathcal{D} \arrow[r] & \mathcal{B} \end{tikzcd}$$

We can compose this with S to get a lift to A, which is what we want.

2.1.4 Axiom M4: Solving lifting problems

Consider a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "i"'] & \mathcal{B} \arrow[d, "P"] \\ \mathcal{C} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

where i is injective on objects and P is both an isofibration and a weak equivalence.

Pick any $d \in \mathcal{D}$. Since P is an equivalence of categories, there is some object $b \in \mathcal{B}$ such that $P(b) \cong d$. Define a functor $I \to \mathcal{D}$ which sends $* \mapsto b$ and $\bullet \mapsto P(b)$, and a functor $(*) \to \mathcal{B}$ which picks out b. This gives us a commuting diagram

$$\begin{tikzcd} (*) \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ I \arrow[r] & \mathcal{D} \end{tikzcd}$$

Since P is an isofibration, we have a lift $I \to \mathcal{B}$, which maps $\bullet$ to an object $\hat{d} \in \mathcal{B}$ such that $P(\hat{d}) = d$. Thus, P is surjective on objects.

Suppose first that c is in the image of i. Then (because i is injective on objects) we can lift c uniquely to an object of A, then map the result to $\mathcal{B}$ with F. We define this to be $\ell(c)$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "i"'] & \mathcal{B} \arrow[d, "P"] \\ \mathcal{C} \arrow[r, "G"'] \arrow[ur, dashed, "\ell"] & \mathcal{D} \end{tikzcd}$$

This makes both triangles formed commute trivially.

If c is not mapped to by i, then we define $\ell(c)$ to be any lift $\widehat{G(c)}$. This makes the lower triangle commute by definition, and the upper triangle is irrelevant since i never hits these objects.

Since P is an equivalence of categories, the maps

$$P_{b,b'} \colon \mathcal{B}(b, b') \to \mathcal{D}(P(b), P(b')); \qquad \left[b \xrightarrow{f} b'\right] \mapsto \left[P(b) \xrightarrow{P(f)} P(b')\right]$$

are bijections, hence can be inverted. Thus we can define ℓ on morphisms $f \colon c \to c'$ in $\mathcal{C}$ by

$$\ell(f) = P^{-1}_{\ell(c),\ell(c')}(G(f)).$$

The upper triangle

$$\begin{tikzcd} \left[a \xrightarrow{f} a'\right] \arrow[r, mapsto] \arrow[d, mapsto] & \left[F(a) \xrightarrow{F(f)} F(a')\right] \overset{!}{=} \left[\ell(i(a)) \xrightarrow{\ell(i(f))} \ell(i(b))\right] \\ \left[i(a) \xrightarrow{i(f)} i(a')\right] \arrow[ur, mapsto] & \end{tikzcd}$$

commutes since $\ell(i(a)) = F(a)$ and $\ell(i(a')) = F(a')$, so

$$\begin{aligned}\ell(i(f)) &= P^{-1}_{\ell(i(a)),\ell(i(a'))}(G(i(f)))\\ &= P^{-1}_{F(a),F(a')}(P(F(f)))\\ &= F(f).\end{aligned}$$

To see that the lower triangle

$$\begin{tikzcd} & & \left[\ell(c) \xrightarrow{\ell(f)} \ell(c')\right] \arrow[d, mapsto] \\ \left[c \xrightarrow{f} c'\right] \arrow[r, mapsto] \arrow[urr, mapsto] & \left[G(c) \xrightarrow{G(f)} G(c')\right] \stackrel{!}{=} & \left[P(\ell(c)) \xrightarrow{P(\ell(f))} P(\ell(c'))\right] \end{tikzcd}$$

note that we have already seen that $P(\ell(c)) = G(c)$ and $P(\ell(c')) = G(c')$, and for $f\colon c \to c'$ we defined

$$\ell(f) = P^{-1}_{\ell(c),\ell(c')}(G(f)),$$

so

$$P(\ell(f)) = G(f).$$

Now we are in the situation where i is both injective and an equivalence of categories, and P is an isofibration.

Let $c \in \mathcal{C}$. Becase i is essentially surjective we can always pick $c' \in \mathcal{C}$ such that c' is in the image of i, i.e. such that there exists some a_c with $i(a_c) = c' \cong c$. For each such c, pick such a c' and a_c. If c is already in the image of i, pick $c' = c$; that is, demand that $a_{i(a)} = a$.

This allows us to define, for each $c \in \mathcal{C}$ functors as follows.

$$* \longmapsto F(a_c)$$

$$(* \leftrightarrow \bullet) \longmapsto G(c') \leftrightarrow G(c)$$

Then P's isofibricity gives a functor $I \to \mathcal{B}$ making the diagram

commute, which sends $\bullet$ to some object $b_c \in \mathcal{B}$. By the commutativity, $P(b_c) = G(c)$.

Define the lift ℓ by $\ell(c) = b_c$. The upper triangle is

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & F(a) \stackrel{!}{=} \ell(i(a)) \\ i(a) \arrow[ur, mapsto] & \end{tikzcd}$$

which commutes because $\ell(i_a) = F(a_a) = F(a)$.

The lower triangle

$$\begin{tikzcd} & \ell(c) \arrow[d, mapsto] \\ c \arrow[r, mapsto] \arrow[ur, mapsto] & G(c) \stackrel{!}{=} P(\ell(c)) \end{tikzcd}$$

commutes because $P(\ell(c)) = P(b_c) = G(c)$.

By the same logic as the previous problem, any morphism $f: c \to c'$ lifts well-definedly to a map $\hat{f}: a_c \to a_{c'}$. We define

$$\ell(f) = F(\hat{f}).$$

The upper triangle

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & F(f) \overset{!}{=} \ell(i(f)) \\ i(f) \arrow[ur, mapsto] & \end{tikzcd}$$

commutes since $\ell(i(f)) = f$ by definition of $\hat{f}$. The lower triangle

$$\begin{tikzcd} & \ell(f) \arrow[d, mapsto] \\ f \arrow[r, mapsto] \arrow[ur, mapsto] & P(\ell(f)) \overset{!}{=} G(f) \end{tikzcd}$$

commutes because

$$\begin{aligned} P(\ell(f)) &= P(F(\hat{f})) \\ &= G(i(\hat{f})) \\ &= G(f). \end{aligned}$$

2.1.5 Axiom M5: Factorization

For any functor $F: \mathcal{C} \to \mathcal{D}$, define a functor $G: \mathcal{C} \to \mathcal{L}$ on objects by

$$x \mapsto (x, Fx, \mathrm{id}_{Fx}),$$

and on morphisms by $f \mapsto f$. The functor G' sending

$$(c, d, f) \mapsto c.$$

is a weak inverse to G. As a check: these form an equivalence of categories since

$$G' \circ G: x \mapsto (x, Fx, \mathrm{id}_{Fx}) \mapsto x$$

and

$$G \circ G': (c, d, f) \mapsto c \mapsto (c, Fc, \mathrm{id}_{Fc});$$

the map

$$G(\mathrm{id}_c): (c, d, f) \to (c, Fc, \mathrm{id}_{Fc})$$

is a natural isomorphism since the naturality square

$$\begin{tikzcd} (c, d, f) \arrow[r, "G(\mathrm{id}_c)"] \arrow[d, "g"'] & (c, Fc, \mathrm{id}_{Fc}) \arrow[d, "g"] \\ (c', d', f') \arrow[r, "G(\mathrm{id}_{c'})"'] & (c', Fc', \mathrm{id}_{Fc'}) \end{tikzcd}$$

commutes.

It is also clear that G is injective on objects since $(c, d, f) \neq (c', d', f') \implies c \neq c'$, so $G \in \mathfrak{C}$.

Define also a functor $H\colon \mathcal{L} \to \mathcal{D}$ sending $(c, d, f) \mapsto d$ and $g\colon c \to c'$ to $F(g)$. Then

$$H \circ G\colon c \mapsto (c, Fc, \mathrm{id}_{Fc}) \mapsto Fc; \qquad g \mapsto g \mapsto Fg.$$

It remains only to show that $H \in \mathfrak{F}$, we need to show that it is an isofibration. Consider the following diagram.

$$\begin{tikzcd}
(*) \arrow[rrr] \arrow[ddd, hook] & & & \mathcal{L} \arrow[ddd, "H"] \\
& * \arrow[r, mapsto] \arrow[d, mapsto] & (c, d, f) \arrow[d, mapsto] & \\
& * \arrow[r, mapsto] & d & \\
(* \leftrightarrow \bullet) \arrow[rrr] & & & \mathcal{D}
\end{tikzcd}$$

We need to show that given an isomorphism $r\colon d \to d'$ and an object (c, d, f) in $\mathcal{L}$, we can find an object (c', d', f') which is isomorphic to (c, d, f). We can; taking $c' = c$, and $f' = r \circ f$ gives us what we want since the following diagram commutes.

$$\begin{tikzcd}
& (c, d, f) \leftrightarrow (c, d', r \circ f) \arrow[d, mapsto] \\
(* \leftrightarrow \bullet) \arrow[ur, mapsto] \arrow[r, mapsto] & c \leftrightarrow c'
\end{tikzcd}$$

Part 2: We define G to be the canonical injection, and H to be defined on objects as follows.

$$H\colon x \mapsto \begin{cases} Fx, & x \in \mathcal{C} \\ x, & x \in \mathcal{D} \end{cases}$$

The hom-sets in $\mathcal{R}$ are already appropriate hom-sets in $\mathcal{D}$, so we don't need to do anything to define H on morphisms: we simply define it to be the identity on hom-sets.

To see that H is an equivalence of categories, note that it is essentially surjective because it is surjective 'on the nose,' and that it is fully faithful because H is the identity on hom-sets.

It is obvious that G is injective on objects.

Now consider the following diagram.

$$\begin{tikzcd}
(*) \arrow[r, "a"] \arrow[d] & \mathcal{R} \arrow[d] \\
(* \leftrightarrow \bullet) \arrow[r, "b"] & \mathcal{D}
\end{tikzcd}$$

Define a functor $\ell\colon I \to \mathcal{R}$ sending $* \mapsto a(*)$ and $\bullet \mapsto b(\bullet)$. This makes both triangles commute: the upper triangle is

$$\begin{tikzcd} * \arrow[r, mapsto] \arrow[d, mapsto] & a(*) \\ * \arrow[ur, mapsto] & \end{tikzcd}$$

and the lower triangle is

$$a(*) \leftrightarrow b(\bullet)$$

$$* \leftrightarrow \bullet$$

3 *The Homotopy category of a model category*

In Subsection 3.1, we define the notion of a *left homotopy* of maps $A \to B$, denoted $\overset{l}{\sim}$, and show that for any cofibrant object A and any object B, left homotopy yields an equivalence relation on $\mathcal{C}(A, B)$.

$$\overbrace{A}^{\mathcal{C}\text{of}} \xrightarrow{f} B$$

In Subsection 3.2, we define a dual notion, *right homotopy*, denoted $\overset{r}{\sim}$, and show that for any object A and any fibrant object B, right homotopy yields an equivalence relation on $\mathcal{C}(A, B)$.

$$A \xrightarrow{f} \overbrace{B}^{\mathcal{F}\text{ib}}$$

In Subsection 3.3, we show that when looking at maps from a cofibrant object to a fibrant object

$$\overbrace{A}^{\mathcal{C}\text{of}} \xrightarrow{f} \overbrace{B}^{\mathcal{F}\text{ib}}$$

a left homotopy $f \overset{l}{\sim} f'$ gives rise to a right homotopy and vice versa, and thus the two notions agree. Thus, for maps from a cofibrant object to a fibrant object, we do not keep track of left- and right homotopies, calling them simply *homotopies.*

This allows us to define, for two maps between fibrant-cofibrant objects, a notion of homotopy equivalence; we say that a map $f\colon A \to B$, where A and B are fibrant-cofibrant, is a homotopy equivalence if it has a homotopy inverse g, i.e. a map g such that

$$f \circ g \sim \mathrm{id}_B, \qquad \text{and} \qquad g \circ f \sim \mathrm{id}_A.$$

Next, we show that this notion is preserved under composition: if f and g are homotopy equivalences, then so is $g \circ f$. This allows us to prove Whitehead's theorem, which tells us that any weak equivalence between fibrant-cofibrant objects is a homotopy equivalence.

Finally, we define the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category, whose objects are the fibrant-cofibrant objects of $\mathcal{C}$, and whose morphisms $X \to Y$ are the homotopy classes of morphisms $X \to Y$.

3.1 Left homotopy

Definition 5.8 (cylinder object)**.** Let $\mathcal{C}$ be a model category, and let $A \in \mathcal{C}$. A cylinder object for A is an object C together with a factorization

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i=(i_0,i_1)"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

such that i is a cofibration and σ is a weak equivalence.

In any model category, every object A admits a cylinder object. Axiom (M5) guarantees that we can find an object C and maps i and c as above where i is a cofibration and σ is a trivial fibration, hence certainly a fibration. However, many model categories have an object such I such that any object A has a cylinder object built from A and I, for example by taking the product $A \times I$.

Example 5.9. In the category Top with the Quillen model structure of Example 5.6, for each object X there is a cylinder object $X \times I$, where I is the unit interval $[0, 1]$.

Example 5.10. In Set_Δ equipped with the Kan model structure, for each simplicial set S there is a cylinder object, given by $\Delta^1 \times S$.

$$\begin{tikzcd} S \amalg S \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "(s_0\times\mathrm{id},s_1\times\mathrm{id})"'] & S \\ \Delta^1 \times S \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

In this case, i is a cofibration because it is an inclusion (hence a monomorphism as required) and π_2 is a weak equivalence (i.e. a homotopy equivalence) with homotopy inverse $s \mapsto (0, s)$.

Example 5.11. In the model category on categories, defined in Subsection 2.1, any category $\mathcal{C}$ has cylinder object $\mathcal{C} \times I$, taken with the below factorization.

$$\begin{tikzcd} \mathcal{C} \amalg \mathcal{C} \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i=(\mathrm{id}\times\{*\},\mathrm{id}\times\{\bullet\})"'] & \mathcal{C} \\ \mathcal{C} \times I \arrow[ur, "\pi"'] & \end{tikzcd}$$

The map i is obviously injective on objects, and the map π is an equivalence since the map $c \mapsto (c, 0)$ is a weak inverse.

Example 5.12. Let R be a small ring, and consider the category $\mathrm{Ch}_{\geq}(R)$ with the projective model structure of Subsection 2.1. Denote by $I_\bullet$ the chain complex[1]

[1] This is the chain complex corresponding, under the Dold-Kan correspondence, to the unit interval.

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{(\mathrm{id},-\mathrm{id})} R \oplus R \ .$$

Then each chain complex $A_\bullet$ has a cylinder object

$$\begin{tikzcd} A \oplus A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d] & A \\ A \otimes I \arrow[ur] & \end{tikzcd} .$$

Definition 5.13 (left homotopy). Let $\mathcal{C}$ be a model category, and let $f, g\colon A \to B$ be two morphsims in $\mathcal{C}$. A left homotopy between f and g consists of a cylinder object C for A (Definition 5.8) and a map $H\colon C \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

In this case, we will write

$$f \overset{l}{\sim} g.$$

Example 5.14. In the projective model structure on $\mathrm{Ch}_{\geq 0}(R)$, a left homotopy with respect to the cylinder object of Example 5.11 is a chain homotopy.

Example 5.15. In the Quillen model structure on Top with cylinder objects as in Example 5.9, a left homotopy between f and g is a homotopy in the standard sense.

Example 5.16. In the Kan model structure on Set_Δ with cylinder objects as in Example 5.10, left homotopy reproduces Definition 4.6.

Lemma 5.17. Let A be a cofibrant object. If C is a cylinder object for A, then the maps i_1 and i_2 are trivial cofibrations.

Proof. Since A is cofibrant, by definition the map $\emptyset \to A$ is a cofibration. Consider the following pushout square.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \amalg A \end{tikzcd}$$

Since by Corollary 5.5 cofibrations are closed under pushout, the canonical injections $\iota_\alpha\colon A \to A \amalg A$, $\alpha = 1, 2$, are also cofibrations. Since $i_\alpha = i \circ \iota_\alpha$ is a composition of cofibrations, it is again a cofibration by closure under countable composition.

Since the diagram

$$\begin{tikzcd} A \arrow[d, hook, "\iota_\alpha"] \arrow[dr, "\mathrm{id}"] \arrow[dd, bend right, "\iota_\alpha"'] & \\ A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i"] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commutes, we can write $\mathrm{id}_A = \sigma \circ i_\alpha$ (with id_A a weak equivalence because it is an isomorphsim and σ a weak equivalence by assumption), so by the two-out-of-three law i_α is a weak equivalence. □

Proposition 5.18. If A is a cofibrant object, then left homotopy of maps $A \to B$ is an equivalence relation.

Proof. To see that left homotopy is an equivalence relation, let C be any cylinder object over A.

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i=(i_0,i_1)"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Then the following diagram commutes.

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,f)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "f\circ\sigma"'] & \end{tikzcd}$$

This means that $f \circ \sigma$ is a left homotopy from f to itself.

Next, we show that left homotopy is symmetric. First, we note that in any category with coproducts, there is a natural isomorphism $\mathrm{swap}_A : A \amalg A \to A \amalg A$ which switches the components. Because $\mathfrak{Cof}$ is saturated and thus contains all isomorphisms, swap_A is a cofibration. This gives us a new factorization exhibiting C as a cylinder object of A.

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i\circ\mathrm{swap}_A"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Let H be a left homotopy $f \overset{l}{\sim} g$ as follows.

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

The diagram

$$\begin{tikzcd} A \amalg A \arrow[d, "\mathrm{swap}_A"'] \arrow[dr, "(g,f)"] & \\ A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

exhibits a left homotopy $g \overset{l}{\sim} f$.

It remains only to show that left homotopy is transitive. To this end, suppose $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$,

$$\begin{tikzcd} A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "{(i_0,i_1)}"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd} \qquad \begin{tikzcd} A \amalg A \arrow[r, "{(g,h)}"] \arrow[d, "{(i'_0,i'_1)}"'] & B \\ C' \arrow[ur, "H'"'] & \end{tikzcd}$$

where C and C' are cylinder objects for A with the following factorizations.

$$\begin{tikzcd} A \amalg A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{(i_0,i_1)}"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} A \amalg A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{(i'_0,i'_1)}"'] & B \\ C' \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

Consider the following diagram, where the top left square is a pushout square.

$$\begin{tikzcd} A \arrow[r, "i'_0"] \arrow[d, "i_1"'] & C' \arrow[d, "j_1"'] \arrow[ddr, bend left, "H'"] & \\ C \arrow[r, "j'_0"] \arrow[drr, bend right, "H"'] & C \coprod_A C' \arrow[dr, dashed, "\exists! \tilde{H}"] & \\ & & B \end{tikzcd}$$

The following diagram commutes.

$$\begin{tikzcd} A \amalg A \arrow[r, "{(f,h)}"] \arrow[d, "{(j'_0 \circ i_0, j_1 \circ i'_0)}"'] & B \\ C \amalg_A C' \arrow[ur, "\tilde{H}"'] & \end{tikzcd}$$

It remains only to show that $C \amalg_A C'$ is a cylinder object for A. Define a map $\tilde{\sigma}$ as follows.

$$\begin{tikzcd} A \arrow[r, "i'_0"] \arrow[d, "i_1"'] & C' \arrow[d, "j_1"'] \arrow[ddr, bend left, "\sigma'"] & \\ C \arrow[r, "j'_0"] \arrow[drr, bend right, "\sigma"'] & C \coprod_A C' \arrow[dr, dashed, "\exists! \tilde{\sigma}"] & \\ & & A \end{tikzcd}$$

Since $j_1 \in \mathcal{W} \cap \mathcal{C}of$ and $\sigma' \in \mathcal{W}$, we have by Axiom (M2) that $\tilde{\sigma}$ is a weak equivalence.

Thus, to show that $C \amalg_A C'$ is a cylinder object with the factorization

$$\begin{tikzcd} A \amalg A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{(j'_0 \circ i_0, j_1 \circ i'_0)}"'] & A \\ C \amalg_A C' \arrow[ur, "\tilde{\sigma}"'] & \end{tikzcd}$$

we need only show that $(j'_0 \circ i_1, j_1 \circ i'_0)$ is a cofibration.

First note that the following square is a pushout square, where δ is the codiagonal.

$$\begin{tikzcd}
(A \amalg A) \amalg (A \amalg A) \arrow[r, "(i_0,i_1)\amalg(i_0',i_1')"] \arrow[d, "(\mathrm{id},\delta,\mathrm{id})"'] & C \amalg C' \arrow[d, "(j_0',j_1)"] \\
A \amalg A \amalg A \arrow[r] & C \amalg_A C
\end{tikzcd}$$

To see this, note maps

$$(\alpha, \beta) \colon C \amalg C' \to Z \qquad \text{and} \qquad (a, b, c) \colon A \amalg A \amalg A$$

which form a cocone satisfy $\beta \circ i_0' = b$ and $\alpha \circ i_1 = f$.

$$\begin{tikzcd}
(A \amalg A) \amalg (A \amalg A) \arrow[r, "(i_0,i_1)\amalg(i_0',i_1')"] \arrow[d, "(\mathrm{id},\delta,\mathrm{id})"'] & C \amalg C' \arrow[d] \arrow[ddr, bend left, "(\alpha,\beta)"] & \\
A \amalg A \amalg A \arrow[r, "k"] \arrow[drr, bend right, "(a,b,c)"'] & C \amalg_A C \arrow[dr, dashed, "\exists!"] & \\
& & Z
\end{tikzcd}$$

Thus, there is a unique map

$$C \amalg_A C \to Z$$

making the above diagram commute.

Note that because $(i_0, i_1) \amalg (i_0', i_1')$ is a cofibration, k must be also.

Also, the map $A \amalg A \to A \amalg A \amalg A$ is a cofibration because the following diagram is a pushout.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d] \\
A \amalg A \arrow[r] & A \amalg A \amalg A
\end{tikzcd}$$

Thus, the composition

$$A \amalg A \to A \amalg A \amalg A \to C \amalg_A C'$$

is a cofibration. □

3.2 Right homotopy

There is a dual notion to left fibrations, called right fibrations. The theory of right fibrations is dual to that of left fibrations.

Definition 5.19 (path object)**.** Let B be an object of a model category. A path object for B is an object P, together with a factorization

$$\begin{tikzcd} & P \arrow[d, "p"] \\ B \arrow[ur, "s"] \arrow[r, "{(\mathrm{id},\mathrm{id})}"'] & B \times B \end{tikzcd}$$

where p is a fibration and s is a weak equivalence.

Definition 5.20 (right homotopy)**.** A right homotopy between morphisms $f, g\colon A \to B$ consists of a path object P and a factorization as follows.

$$\begin{tikzcd} & P \arrow[d, "p"] \\ A \arrow[ur, "H"] \arrow[r, "{(f,g)}"'] & B \times B \end{tikzcd}$$

Lemma 5.21. Let B be a fibrant object. If P is a cylinder object for B, then the maps p_0 and p_1 are trivial fibrations.

Proof. Apply Lemma 5.17 to the opposite model category (Example 5.7). □

Proposition 5.22. Let B a fibrant object. Right homotopy of maps $f, g\colon A \to B$ is an equivalence relation.

Proof. Apply Proposition 5.18 to the opposite model category (Example 5.7). □

3.3 *Homotopy equivalents*

Fix objects A and B in a model category, and consider maps $A \to B$. In this section we will show that under certain conditions, namely when A and B are both fibrant-cofibrant, left and right homotopy are equivalent.

Lemma 5.23. Suppose A is cofibrant, and suppose that $f, g\colon A \to B$ are left homotopic. Then given any path object P for B, there is a right homotopy between f and g with underlying path object P.

Proof. Suppose we are given a left homotopy

$$\begin{tikzcd} A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

and suppose that P has the following factorization

$$\begin{tikzcd} & P \arrow[d, "p"] \\ B \arrow[ur, "s"] \arrow[r, "{(\mathrm{id},\mathrm{id})}"'] & B \times B \end{tikzcd}$$

Consider the following diagram.

$$\begin{tikzcd} A \arrow[r, "s\circ f"] \arrow[d, "i_0"'] & P \arrow[d, "p"] \\ C \arrow[r, "(f\circ\sigma, H)"'] & B\times B \end{tikzcd}$$

It commutes because

$$p \circ s \circ f = (\mathrm{id}, \mathrm{id}) \circ f = (f, f)$$

and

$$(f \circ \sigma, H) \circ i_0 = (f \circ \sigma \circ i_0, H \circ i_0) = (f \circ \mathrm{id}, f) = (f, f).$$

By definition, p is a fibration, and we have seen in Lemma 5.17 that i_0 is a trivial cofibration. Thus, we have a lift $h\colon C \to P$.

$$\begin{tikzcd} A \arrow[r, "s\circ f"] \arrow[d, "i_0"'] & P \arrow[d, "p"] \\ C \arrow[r, "(f\circ\sigma, H)"'] \arrow[ur, dashed, "h"] & B\times B \end{tikzcd}$$

Then we have

$$\begin{aligned} p \circ h \circ i_1 &= (f \circ \sigma, H) \circ i_1 \\ &= (f \circ \sigma \circ i_1, H \circ i_1) \\ &= (f, g), \end{aligned}$$

so the diagram

$$\begin{tikzcd} & P \arrow[d, "p"] \\ A \arrow[ur, "h\circ i_1"] \arrow[r, "(f,g)"'] & B\times B \end{tikzcd}$$

commutes. Thus, $h \circ i_1$ gives a right homotopy between f and g. □

Corollary 5.24. Let $f, g\colon A \to B$ be morphisms, and suppose that A is cofibrant and B is fibrant. Then the following are equivalent.

1. f and g are left homotopic.
2. For every path object P for B, there is a right homotopy between f and g with path object P.
3. f and g are right homotopic.
4. For every cylinder object C for A, there is a left homotopy between f and g with cylinder object C.

Proof.

$1 \Rightarrow 2$. Lemma 5.23.

$2 \Rightarrow 3$. Trivial.

$3 \Rightarrow 4$. Dual to Lemma 5.23.

$4 \Rightarrow 1$. Trivial.

□

The above result shows that when considering morphisms between a cofibrant object and a fibrant object, there is no difference between left and right fibrations. Therefore, we will not distinguish them.

Definition 5.25 (homotopy equivalence)**.** Let $f\colon X \to Y$ be a morphism with X and Y fibrant-cofibrant. We say that f is a homotopy equivalence if there exists a morphism $g\colon Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.

The next lemma shows that given a homotopy equivalence, the result of pre- or post-composing with any morphism is again a homotopy equivalence.

Lemma 5.26. Let $f, g\colon X \to Y$ be homotopy equivalent morphisms between fibrant-cofibrant objects, and let W and Z be fibrant-cofibrant. Then the following hold.

1. For any $h\colon Y \to Z$, $h \circ f \sim h \circ g$.
2. For any $h'\colon W \to X$, $f \circ h' \sim g \circ h'$.

Proof.

1. Suppose our homotopy is given by the following factorization.

$$\begin{tikzcd} X \amalg X \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & Y \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

Post-composing with h gives us the following

$$\begin{tikzcd} X \amalg X \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & Y \arrow[r, "h"] & Z \\ C \arrow[ur, "H"'] & & \end{tikzcd} \implies \begin{tikzcd} X \amalg X \arrow[r, "{(h\circ f, h\circ g)}"] \arrow[d, "i"'] & Z \\ C \arrow[ur, "h\circ H"'] & \end{tikzcd}$$

But the second diagram is precisely a homotopy $h \circ f \sim h \circ g$.

2. Now suppose our homotopy is given by the following factorization.

$$\begin{tikzcd} & P \arrow[d, "s"] \\ X \arrow[ur, "H'"] \arrow[r, "{(f,g)}"'] & Y \times Y \end{tikzcd}$$

Pre-composing with h yields the following.

$$\begin{tikzcd} & & P \arrow[d, "s"] \\ W \arrow[r, "h"] & X \arrow[ur, "H'"] \arrow[r, "{(f,g)}"'] & Y \times Y \end{tikzcd} \implies \begin{tikzcd} & P \arrow[d, "s"] \\ X \arrow[ur, "H'\circ h"] \arrow[r, "{(f\circ h, g\circ h)}"'] & Y \times Y \end{tikzcd}$$

This is preciesly a homotopy $f \circ h \sim g \circ f$.

□

Corollary 5.27. Let X, Y, and Z be fibrant-cofibrant objects, and let f and g be homotopy equivalences as follows.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Then $g \circ f$ is a homotopy equivalence.

Proof. By definition, f has a homotopy inverse f' such that $f' \circ f \sim \mathrm{id}$ and $f \circ f' \sim \mathrm{id}$. Similarly, there is a g' such that $g' \circ g \sim \mathrm{id}$ and $g \circ g' \sim \mathrm{id}$. But then

$$\begin{aligned} \mathrm{id} &\sim g' \circ g \\ &\sim f' \circ g' \circ g \\ &\sim f' \circ g' \circ g \circ f \end{aligned}$$

and

$$\begin{aligned} \mathrm{id} &\sim f' \circ f \\ &\sim g' \circ f' \circ f \circ g. \end{aligned}$$

so $f' \circ g'$ functions as a homotopy inverse to $g \circ f$. □

Lemma 5.28. Let X and Y be fibrant-cofibrant objects, and $f\colon X \to Y$ a weak equivalence.

1. If f is a fibration, then it is a homotopy equivalence.
2. If f is a cofibration, then it is a homotopy equivalence.

Proof.

1. We have a lift as follows.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

Thus, $f \circ g = \mathrm{id}_Y$.

Thus the diagram

$$\begin{tikzcd} X \amalg X \arrow[r, "(g\circ f, \mathrm{id})"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ C \arrow[r, "f\circ\sigma"'] & Y \end{tikzcd}$$

commutes, because

$$\begin{aligned} f \circ (g \circ f, \mathrm{id}) &= (f \circ g \circ f, f) \\ &= (f, f) \\ &= (f \circ \sigma \circ i_1, f \circ \sigma \circ i_2) \\ &= f \circ \sigma \circ i. \end{aligned}$$

Since f is a trivial fibration and i is a cofibration, we have by Axiom (M4) a lift as follows.

$$\begin{tikzcd} X \amalg X \arrow[r, "(g\circ f, \mathrm{id})"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ C \arrow[r, "f\circ\sigma"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

The upper triangle tells us that H is a left homotopy between $g \circ f$ and id_X.

Since $f \circ g = \mathrm{id}_Y$, the cylinder object C itself functions as a homotopy $f \circ g \sim \mathrm{id}_Y$. Thus, f is a homotopy equivalent.

2. The commutativity of the solid diagram

$$\begin{tikzcd} X \arrow[r, "s\circ f"] \arrow[d, "f"'] & P \arrow[d, "p"] \\ Y \arrow[r, "(\mathrm{id}_Y, f\circ h)"'] \arrow[ur, dashed] & Y \times Y \end{tikzcd}$$

together with the fact that f is a trivial cofibration and p is a fibration imply that there is a dashed lift. This exhibits a right homotopy $f \circ h \sim \mathrm{id}_Y$. But $h \circ f = \mathrm{id}_X$, so $h \circ f \sim \mathrm{id}_X$.

So far we have shown that the result holds if f is a trivial fibration or a trivial cofibration.

Now suppose f is merely a weak equivalence. Then we can factor

$$f = g \circ h,$$

where g is a fibration and h is a trivial cofibration. But by Axiom (M2), g is also a trivial cofibration.

∎

Theorem 5.29 (Whitehead)**.** Let $f: X \to Y$ be a weak equivalence, and suppose that X and Y are fibrant-cofibrant. Then f is a homotopy equivalence.

Proof. By Axiom (M5), we are allowed to factor $f = p \circ i$, where p is a fibration, i is a cofibration, and at least one of p and i is a weak equivalence. By Axiom (M2), both i and p are weak equivalences.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathcal{W}\cap\mathcal{C}\mathrm{of}\ni i"'] & & Y \\ & Z \arrow[ur, "p\in\mathcal{W}\cap\mathcal{F}\mathrm{ib}"'] & \end{tikzcd}$$

By assumption, X is cofibrant, i.e. the map $\emptyset \to X$ is a cofibration. The map $i: X \to Z$ is also a cofibration, so their composition must also be a cofibration. But this means that the map $\emptyset \to Z$ is also a cofibration, so Z is cofibrant

Dually, the map $Y \to *$ is a fibration. Precomposing by p, we have that Z is fibrant. Thus, i and p are homotopy equivalences by Lemma 5.28, parts 2 and 1 respectively. But by Lemma 5.26, their composition is also a homotopy equivalence. ◻

Definition 5.30 (homotopy category of a model category)**.** Let $\mathcal{C}$ be a model category. The homotopy category of $\mathcal{C}$, denoted $\mathrm{Ho}(\mathcal{C})$, is defined as follows.

- The objects of $\mathrm{Ho}(\mathcal{C})$ are the fibrant-cofibrant objects of $\mathcal{C}$.
- The set of morphisms $X \to Y$ is the set of homotopy classes of $\mathcal{C}(X, Y)$.
- Composition is given by the rule

$$[f] \circ [g] \equiv [f \circ g].$$

This is well-defined by Lemma 5.26.

Example 5.31. Consider the category Set_Δ together with the Kan model structure (Example 5.6). We have seen (in Example 5.10) that for any simplicial set S, the object $\Delta^1 \times S$ is a cylinder object for S. Following our proofs through, one finds that the homotopy theory in the model structure on Set_Δ agrees with the homotopy theory on Set_Δ we have worked out in Chapter 4.

In the Kan model structure Set_Δ, the fibrant objects are Kan complexes, and every object is cofibrant. Therefore, the fibrant-cofibrant objects are simply Kan complexes.

The homotopy category $\mathrm{Ho}(\mathrm{Set}_\Delta)$ therefore simply has objects given by Kan complex, and morphisms given by equivalence classes of morphisms under homotopy. We will meet this category later. It is called the *homotopy category of spaces.*

4 *Fibrant-Cofibrant replacement*

Let $\mathcal{C}$ be a model category.

In Subsection 4.1, we give a way of assigning to any object $X \in \mathcal{C}$ a (not necessarily unique) cofibrant object QX to which it is weakly equivalent. We call any choice of such at cofibrant object a *cofibrant replacement.*

We also define the notion of cofibrant replacement on morphisms, i.e. a way of assigning

$$(f\colon X \to Y) \mapsto (Qf\colon QX \to QY).$$

Again, this assignment is not uniquely defined, so cofibrant does not yield a functor, as one might hope.

Next we next prove that for any cofibrant replacements Qf of f and Qg of g,

$$f \overset{l}{\sim} g \implies Qf \overset{l}{\sim} Qg,$$

showing that the cofibrant replacement of a morphism is uniquely defined up to homotopy if we restrict our attention to morphisms out of cofibrant objects X. In this setting, we have the further nice property that cofibrant replacement respects composition in the sense that

$$Q(g \circ f) \overset{l}{\sim} Qg \circ Qf.$$

In Subsection 4.2, we study the dual notion, called *fibrant replacement,* providing a method by which one can replace any object X by a fibrant object RX to which it is weakly equivalent. We then again define a notion of cofibrant replacement on morphims, i.e. a way of assigning

$$(f\colon X \to Y) \mapsto (Rf\colon RX \to RY).$$

Then, restricting to morphisms whose domain Y is fibrant, and considering morphisms up to right homotopy, we find that fibrant replacement becomes well-defined in the sense that

$$f \overset{r}{\sim} f' \implies Rf \overset{r}{\sim} Rf',$$

and preserves composition in the sense that $R(g \circ f) = Rg \circ Rf$.

In Subsection 4.3, we prove two preliminary lemmas, which show the following.

- The fibrant replacement of a cofibrant object is both fibrant and cofibrant.

- Fibrant replacement takes left-homotopic morphisms $f, g\colon X \to Y$ with X cofibrant to morphisms which are right homotopic.

The first lemma implies that taking first the cofibrant and then fibrant replacement of any object X yields an object RQX which is both fibrant and cofibrant; one calls such objects *fibrant-cofibrant,* and the act of replacing an object by a weakly equivalent fibrant-cofibrant one *fibrant-cofibrant replacement.* The second lemma then implies that fibrant-cofibrant replacement gives a functor from $\mathcal{C}$ to $\mathrm{Ho}(\mathcal{C})$.

4.1 Cofibrant replacement

Definition 5.32 (cofibrant replacement)**.** Let $\mathcal{C}$ be a model category. For each $X \in \mathcal{C}$, a cofibrant replacement of X consists of a cofibrant object QX and a trivial fibration $p_X : QX \to X$, defined as follows.

- If X is cofibrant, define $QX = X$, and $p_X = \mathrm{id}_X$.
- If X is not cofibrant, pick a factorization of the unique morphism $\emptyset \to X$

  $$\emptyset \to QX \xrightarrow{p_X} X$$

  where the map $\emptyset \to Qx$ is a cofibration and p_X is a trivial fibration.

For a morphism $f : X \to Y$, a fibrant replacement of f is any solution Rf to the following lifting problem.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & QY \arrow[d, "p_Y"] \\
QX \arrow[r, "f \circ p_X"'] \arrow[ur, dashed, "Qf"] & Y
\end{tikzcd}$$

Example 5.33. Consider the category $\mathrm{Ch}_{\geq}(R)$ of bounded-below chain complexes of modules over a ring R. Let M be an R-module, considered as a chain complex concentrated in degree 0. A cofibrant replacement for M is a chain complex of projective modules P^M together with a trivial fibration $P^M \to M$. This is the same thing as a projective resolution of M.

Lemma 5.34. For any map $f : X \to Y$, any choice of cofibrant replacement $f \mapsto Qf$ respects left homotopy; that is, if $f \overset{l}{\sim} f'$, then $Qf \overset{l}{\sim} Qf'$.

Proof. Suppose we have lifts Qf and Qf' of f and f' respectively, and consider a left homotopy between them.

$$\begin{tikzcd}
X \amalg X \arrow[r, "(f,f')"] \arrow[d, "i"'] & Y \\
C \arrow[ur, "H"'] &
\end{tikzcd}$$

Consider the following solid lifting problem.

$$\begin{tikzcd}
QX \amalg QX \arrow[r, "(Qf,Qf')"] \arrow[d, "i"'] & QY \arrow[d, "p_Y"] \\
C \arrow[r, "f \circ p_X \circ H"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

The outer square commutes because

$$\begin{aligned}
p_Y \circ (Qf, Qf') &= (f \circ p_X, f' \circ p_X) \\
&= f \circ p_X \circ (f, f') \\
&= f \circ p_X \circ H \circ i.
\end{aligned}$$

The map i is a cofibration by definition, and p_Y is a trivial fibration. Thus, we get a dashed lift. But this is precisely a left homotopy between Qf and Qf'. □

Corollary 5.35. We have

$$Q(g \circ f) \overset{l}{\sim} Q(g) \circ Q(f).$$

Proof. Consider the following square.

$$\begin{tikzcd} * \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & QZ \arrow[d, "p_Z"] \\ * \arrow[r] \arrow[d] & QY \arrow[r, "g\circ p_Y"] \arrow[d, "p_Y"] & Z \arrow[d, "\mathrm{id}"] \\ QX \arrow[r, "f\circ p_X"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

The outer square is the lifting problem defining $Q(g \circ f)$, and the top right and bottom left squares define Qg and Qf respectively. Thus, $Qg \circ Qf \overset{l}{\sim} Q(g \circ f)$. □

There is a dual notion.

4.2 Fibrant replacement

Definition 5.36 (fibrant replacement)**.** Let $\mathcal{C}$ be a model category. For each $X \in \mathcal{C}$, a fibrant replacement of X consists of a fibrant object RX and a trivial cofibration $i_X : X \to RX$, defined as follows.

- If X is fibrant, define $RX = X$, and $i_X = \mathrm{id}_X$.
- If X is not fibrant, pick factorization of the unique morphism $X \to *$

$$X \xrightarrow{i_X} RX \to *$$

where the map $\emptyset \to Qx$ is a fibration and i_X is a trivial cofibration.

For a morphism $f : X \to Y$, a fibrant replacement of f is a solution Rf to the following lifting problem.

$$\begin{tikzcd} X \arrow[r, "i_Y\circ f"] \arrow[d, "i_X"'] & RY \arrow[d] \\ RX \arrow[r] \arrow[ur, dashed, "Rf"] & * \end{tikzcd}$$

Lemma 5.37. For any map $f : X \to Y$, fibrant replacement $f \mapsto Rf$ respects right homotopy; that is, if $f \overset{r}{\sim} f'$, then $Rf \overset{r}{\sim} Rf'$.

Proof. Dual to Lemma 5.34. □

Corollary 5.38. We have

$$R(g \circ f) \overset{r}{\sim} R(g) \circ R(f).$$

Proof. Dual to Corollary 5.35 □

4.3 Fibrant-cofibrant replacement

Lemma 5.39. Suppose $f, f' : X \to Y$ are left homotopic, and that X is cofibrant. Then the fibrant replacements Rf and Rf' are right homotopic.

Proof. Since f and f' are left homotopic, so are $i_Y \circ f$ and $i_Y \circ f'$. But since these are between a fibrant and a cofibrant object, they are right homotopic. Pick some right homotopy H as follows.

$$\begin{tikzcd} & P \arrow[d, "p"] \\ X \arrow[ur, "H"] \arrow[r, "(i_Y \circ f, i_Y \circ f')"'] & RY \times RY \end{tikzcd}$$

Consider the following solid lifting problem.

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[d, "i_X"'] & R \arrow[d, "p"] \\ RX \arrow[ur, dashed] \arrow[r, "(Rf, Rf')"'] & RY \times RY \end{tikzcd}$$

The outer square commutes because

$$p \circ H = (i_Y \circ f, i_Y \circ f') = (Rf \circ i_X, Rf' \circ i_X).$$

□

Theorem 5.40. There is a functor $\pi : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, defined on objects by $X \mapsto RQX$ and on morphisms by $f \mapsto [RQf]$.

Proof. First, note that the assignment on objects is well-defined since RQX is fibrant by definition and cofibrant because the composition

$$\emptyset \to QX \xrightarrow{i_X} RQX$$

is a cofibration.

It remains to check that π respects composition. To see this, note that

$$\begin{aligned} \pi(g \circ f) &= [RQ(g \circ f)] \\ &= [R(Qg \circ Qf)] \\ &= [RQ(g) \circ RQ(f)] \\ &= [RQ(g)] \circ [RQ(f)] \\ &= \pi(g) \circ \pi(f), \end{aligned}$$

where the second equality follows from Corollary 5.35 and Lemma 5.39 and the third follows from Lemma 5.37. □

Theorem 5.41. The homotopy category $\mathrm{Ho}(\mathcal{C})$, together with the map $\pi\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, is the localization of $\mathcal{C}$ at $\mathcal{W} = \mathcal{W}$, the set of weak equivalences. That is, there is an equivalence of categories $\mathcal{C}[\mathcal{W}^{-1}] \simeq \mathrm{Ho}(\mathcal{C})$.

Proof. Define a category $\overline{\mathcal{C}}$ as follows.

- $\mathrm{Obj}(\overline{\mathcal{C}}) = \mathrm{Obj}(\mathcal{C})$
- For each $X, Y \in \mathrm{Obj}(\overline{\mathcal{C}})$, $\overline{\mathcal{C}}(X, Y) = \mathrm{Ho}(\mathcal{C})(RQX, RQY)$.

We have the following commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\pi"] \arrow[dr, "\bar{\pi}"'] & & \mathrm{Ho}(\mathcal{C}) \\ & \overline{\mathcal{C}} \arrow[ur, "\xi"'] & \end{tikzcd}$$

where $\bar{\pi}$ is the identity on objects and maps $f \mapsto [RQf]$, and ξ acts on objects by taking $X \mapsto RQX$ and is the identity on morphisms.

Note that ξ is surjective on objects and fully faithful, hence an equivalence of categories. Hence, we only need to show that $\overline{\mathcal{C}}$ satisfies the universal property for the localization.

Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor which takes weak equivalences to isomorphisms in $\mathcal{D}$. If we can show that there exists a unique functor $\overline{F}\colon \overline{\mathcal{C}} \to \mathcal{D}$ such that $F = \overline{F} \circ \bar{\pi}$, then we are done.

Suppose $f \overset{r}{\sim} g$. Pick some path object P, and a homotopy through P.

$$\begin{tikzcd} & P \arrow[d, "p"] \\ X \arrow[ur, "s"] \arrow[r, "{(\mathrm{id},\mathrm{id})}"'] & X \times X \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[d, "p"] \\ X \arrow[ur, "H"] \arrow[r, "{(f,g)}"'] & Y \times Y \end{tikzcd}$$

Putting these diagrams back to back and unfolding them, we have the following.

$$\begin{tikzcd} & Y & \\ X \arrow[ur, "f"] \arrow[r, "H"] \arrow[dr, "g"'] & P \arrow[u, "p_0"] \arrow[d, "p_1"'] & Y \arrow[ul, "\mathrm{id}"'] \arrow[l, "s"'] \arrow[dl, "\mathrm{id}"] \\ & Y & \end{tikzcd}$$

Consider the image in $\mathcal{D}$. Since p_0 and p_1 are weak equivalences, Theorem 5.29 implies that they are sent to isomorphisms in $\mathcal{D}$. Thus, $F(p_0) = F(s)^{-1} = F(p_1)$, so

$$\begin{aligned} F(f) &= F(p_0) \circ F(H) \\ &= F(p_1) \circ F(H) \\ &= F(g). \end{aligned}$$

Let $[g]\colon RQX \to RQY$ be a morphism in $\overline{\mathcal{C}}$. Pick a representative g. Consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "g"] & Y \\
QX \arrow[u, "p_X"] \arrow[r, "Qg"] \arrow[d, "i_{QX}"'] & QY \arrow[u, "p_Y"'] \arrow[d, "i_{QY}"] \\
RQX \arrow[r, "RQg"] & RQY
\end{tikzcd}$$

Applying F, we find the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(g)"] & F(Y) \\
F(QX) \arrow[u, "F(p_X)"] \arrow[r, "F(Qg)"] \arrow[d, "F(i_{QX})"'] & F(QY) \arrow[u, "F(p_Y)"'] \arrow[d, "F(i_{QY})"] \\
F(RQX) \arrow[r, "F(RQg)"] & F(RQY)
\end{tikzcd}$$

But the vertical morphisms are isomorphisms, so

$$F(RQg) = F(i_{QY}) \circ F(p_Y)^{-1} \circ F(g) \circ F(p_X) \circ F(i_{QX})^{-1}.$$

In order to have $F = \bar{F} \circ \bar{\pi}$, we must have

$$\overline{F}([RQg]) = F(RQg),$$

which specifies $\bar{F}$ completely. □

5 *Quillen adjunctions*

Proposition 5.42. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and consider an adjunction

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G.$$

The following are equivalent.

1. F preserves cofibrations and trivial cofibrations.

2. G preserves fibrations and trivial fibrations.

Proof. Suppose 2. holds. By Lemma 5.4, a morphism is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations. Thus, in order to show that F preserves cofibrations, it suffices to show that if f is a cofibration in $\mathcal{C}$, then $F(f)$ has the left lifting property with respect to all trivial fibrations.

Consider solid adjunct diagrams of the following form, with the first in $\mathcal{D}$ and the second in $\mathcal{C}$.

$$\begin{tikzcd} F(c) \arrow[r] \arrow[d, "F(f)"'] & d \arrow[d, "g"] \\ F(c') \arrow[r] \arrow[ur, dashed] & d' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r] \arrow[d, "f"'] & G(d) \arrow[d, "G(g)"] \\ c' \arrow[r] \arrow[ur, dashed] & G(d') \end{tikzcd}$$

Suppose that f is a cofibration and that F preserves cofibrations. Then the left-hand diagram has a lift for any trivial fibration g. The adjunct of this lift is a lift for the diagram on the right. This means that $G(g) \in \mathcal{C}\mathrm{of}_\perp = \mathcal{F}\mathrm{ib} \cap \mathcal{W}$.

Similarly, if f is a trivial cofibration and F preserves trivial cofibrations, then the left-hand diagram has a lift for any fibration g, whose adjunct is a lift for the diagram on the RHS, implying that $G(d) \in (\mathcal{C}\mathrm{of} \cap \mathcal{W})_\perp = \mathcal{F}\mathrm{ib}$.

The other statement is dual. □

Definition 5.43 (Quillen adjunction)**.** An adjunction of model categories satisfying the equivalent conditions in Proposition 5.42 is known as a Quillen adjunction.

Lemma 5.44 (Ken Brown's Lemma)**.** Let $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ be a Quillen adjunction.

1. The left adjoint F preserves weak equivalences between cofibrant objects
2. The right adjoint G preserves weak equivalences between fibrant objects.

Proof. By Proposition 5.42. Let A and B be cofibrant objects, and $f: A \longrightarrow B$ be a weak equivalence.

Pick for the map $(f, \mathrm{id}): A \amalg B \to B$ a factorization

$$A \amalg B \xrightarrow{i} C \xrightarrow{p} B,$$

where $i \in \mathcal{C}\mathrm{of}$ and $p \in \mathcal{F}\mathrm{ib} \cap \mathcal{W}$.

Consider the following diagram.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d, "i_A"] \arrow[dr, bend left, "f"] & \\ B \arrow[r, "i_B"'] & A \amalg B \arrow[r, "{(f,\mathrm{id})}"] \arrow[d, "q"'] & B \\ & C \arrow[ur, "p"'] & \end{tikzcd}$$

Because A and B are cofibrant, i_A and i_B are cofibrations. Thus, $q \circ i_A$ is a cofibration, so C is cofibrant.

Note that by commutativity, we have

$$f = p \circ q \circ i_A, \qquad \mathrm{id}_B = p \circ q \circ i_B.$$

By Axiom (M2), we have that $q \circ i_A$ and $q \circ i_B$ are weak equivalences, hence trivial cofibrations.

Applying F, we find that

$$F(f) = F(p) \circ F(q \circ i_A), \qquad F(\mathrm{id}) = F(p) \circ F(q \circ i_B).$$

We know that $F(\mathrm{id}_B) = \mathrm{id}_{F(b)}$ is a trivial cofibration, hence certainly a weak equivalence, and that $F(q \circ i_B)$ and $F(q \circ i_A)$ are a weak equivalence because F preserves trivial cofibrations. Thus $F(p)$ is a weak equivalence, implying that

$$F(f) = F(p) \circ F(q \circ i_A)$$

is a weak equivalence. □

Given a Quillen adjunction

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G,$$

consider the full subcategory of all cofibrant objects $\mathcal{C}_c \subset \mathcal{C}$. Since the full subcategory of $\mathcal{C}$ on fibrant-cofibrant objects is the same as the full subcategory of $\mathcal{C}_c$ on fibrant-cofibrant objects, there is an equivalence of categories

$$\mathcal{C}_c[\mathcal{W}^{-1}] \simeq \mathcal{C}[\mathcal{W}^{-1}].$$

We will denote any weak inverse by

$$Q\colon \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}_c[\mathcal{W}^{-1}].$$

Dually, we get a functor

$$R\colon \mathcal{D}[\mathcal{W}^{-1}] \to \mathcal{D}_f[\mathcal{W}^{-1}],$$

where $\mathcal{D}_f$ is the full subcategory of $\mathcal{D}$ on fibrant objects.

Definition 5.45 (derived functor)**.** Let

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G$$

be a Quillen adjunction.

- The left derived functor LF is the composite

$$\mathcal{C}[\mathcal{W}^{-1}] \xrightarrow{Q} \mathcal{C}_c[\mathcal{W}^{-1}] \xrightarrow{F} \mathcal{D}[\mathcal{W}^{-1}].$$

- The right derived functor RG is the composite

$$\mathcal{D}[\mathcal{W}^{-1}] \xrightarrow{R} \mathcal{D}_f[\mathcal{W}^{-1}] \xrightarrow{G} \mathcal{C}[\mathcal{W}^{-1}].$$

Proposition 5.46. Let

$$\mathcal{C} : F \leftrightarrow \mathcal{G} : \mathcal{D}$$

be a Quillen adjunction. Then there is an adjunction of derived functors

$$LF : \mathcal{C}[\mathcal{W}^{-1}] \leftrightarrow \mathcal{D}[\mathcal{W}^{-1}] : RG.$$

Definition 5.47 (Quillen equivalence)**.** A Quillen adjunction is called a Quillen equivalence if LF, or equivalently RG, is an equivalence of localized categories.

Theorem 5.48. The adjunction

$$|\cdot| : \mathrm{Set}_{\Delta} \leftrightarrow \mathrm{Top} : \mathrm{Sing}$$

is a Quillen adjunction.

6

More on model categories

1 Combinatorial model categories

In general, it is difficult to construct model categories. However, in certain special cases we can leverage existing tools. One such tool is the notion of a combinatorial model category.

Definition 6.1 (presentable category)**.** Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is presentable if it satisfies the following conditions:

1. The category $\mathcal{C}$ admits all (small) colimits.
2. There exists some set S of objects of $\mathcal{C}$ which generates $\mathcal{C}$ under small colimits in the sense that any object of $\mathcal{C}$ is a colimit of a diagram taking values in S.
3. Every object in $\mathcal{C}$ is small, i.e. compact in the sense of Definition 3.13 for some small regular cardinal κ.
4. Each hom-set $\mathrm{Hom}(X, Y)$ in $\mathcal{C}$ is small; that is, $\mathcal{C}$ is locally small.

Note 6.2. Condition 3. is equivalent to demanding that only the objects of S be compact.

Definition 6.3 (combinatorial model category)**.** Let $\mathcal{A}$ be a model category. We say that $\mathcal{A}$ is combinatorial if the following conditions are satisfied.

1. The category $\mathcal{A}$ is presentable.
2. There exists a set I of *generating cofibrations* such that $\mathcal{C}\mathrm{of} = \overline{I}$, the saturated hull of I.
3. There exists a set J of *generating trivial cofibrations* such that $\mathcal{C}\mathrm{of} \cap \mathcal{W} = \overline{J}$, the saturated hull of J.

Definition 6.4 (perfect class of morphisms)**.** Let $\mathcal{C}$ be a presentable category, and W a class of morphisms of $\mathcal{C}$. We say that W is perfect if the following conditions are satisfied.

1. The class W contains all isomorphisms.

2. The class W satisfies the 2/3 law.

3. The class W is stable under filtered colimits, in the sense that for each filtered poset I and functors $F, G \colon I \to \mathcal{C}$ such that each morphism in I

2 Homotopy limits and colimits

7

Infinity categories

In Subsection 5.1, we constructed a functor

$$N\colon \mathrm{Cat} \to \mathrm{Set}_{\Delta}$$

which sends any category to a simplicial likeness.

Lemma 7.1. Let K be a simplicial set. Then K is the nerve of some category if and only if K has unique inner horn fillers. That is, the following are equivalent.

- There exists a category $\mathcal{C}$ and an isomorphism $K \cong N(\mathcal{C})$.
- Every lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

  has a unique solution.

Proof. First, suppose that $K = N(\mathcal{C})$ for some category $\mathcal{C}$. The first inner horn for which we have to provide a unique lift is

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

□

1 *Basic definitition*

Definition 7.2 (infinity category)**.** A simplicial set $\mathcal{C}$ is called an <u>∞-category</u> if each inner horn

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \qquad 0 < i < n$$

has a filler.

Example 7.3. For any category $\mathcal{C}$, we have seen in Proposition 2.40 the nerve $N(\mathcal{C})$ has *unique* innner horn fillers, thus certainly has inner horn fillers.

Example 7.4. For any topological space X, the singular set $\mathrm{Sing}(X)$ has *all* horn fillers, hence certainly inner horn fillers.

In the previous chapter, we mainly worked with Kan complexes, so we used the Kan model structure on Set_Δ. There is, however, a different model structure on Set_Δ, which is better suited to infinity-categorical thinking.

2 *The homotopy category of an infinity category*

We would like to be able to view infinity-categories as a generalization of categories. This means that we would like to be able to throw away data from an infinity category to get a regular category. This throwing-away process creates from any infinity category $\mathcal{C}$ the so-called *homotopy category* $\mathrm{h}\mathcal{C}$. The homotopy category inherits a lot of information from the infinity category from whence it came.

Definition 7.5 (homotopy category)**.** Let $\mathcal{C}$ be an ∞-category. Define a category $\mathrm{h}\mathcal{C}$, called the homotopy category of $\mathcal{C}$, as follows.

- The objects $\mathrm{ob}(\mathcal{C})$ are the vertices $\mathcal{C}_0$.
- For objects x and y, the set $\mathcal{C}(x, y)$ consists of all edges f, g with $d_0 f = d_0 g = y$ and $d_1 f = d_1 g = x$, modulo the equivalence relation

$$f \sim g \iff \exists \sigma \in \mathcal{C}_2 \quad \text{with} \quad d_1\sigma = g, \quad d_2\sigma = f, \quad \text{and} \quad d_0\sigma = s_0 y.$$

$$\sigma = \begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

- We have already seen in Theorem 4.4 that this is an equivalence relation.
- Composition is induced by inner horn fillings $\Lambda^2_1 \to \Delta^2$; we define the composition $[f] \circ [g]$ to be $[h]$, where h is some edge given to us by filling the following solid horn.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "h"'] & & z \end{tikzcd}$$

To see that this is well-defined, i.e. respects equivalence classes, consider two fillings as follows.

$$\sigma = \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}, \qquad \sigma' = \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h'"'] & & z \end{tikzcd}.$$

We need to show that $h \sim h'$. We get this by filling the following horn.

$$
\begin{tikzcd}
y \arrow[r, "g"] & z \arrow[d, "s_0 z"] \\
x \arrow[u, "f"] \arrow[ur, "h"] \arrow[r, "h'"'] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[r, "g"] \arrow[dr, "f"] & z \arrow[d, "s_0 z"] \\
x \arrow[u, "f"] \arrow[r, "h'"'] & z
\end{tikzcd}
$$

First suppose that $g_1 \sim g_2$. Then filling the horn

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "h_2"'] \arrow[dr, "h_1"] & y \arrow[d, "g_1"] \\
z \arrow[r, "s_0 z"'] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "h_2"'] & y \arrow[d, "g_1"] \arrow[dl, "g_2"'] \\
z \arrow[r, "s_0 z"'] & z
\end{tikzcd}
$$

tells us that $h_1 \sim h_2$.

Now suppose that $f_1 \sim f_2$. In this case we prove something stronger, i.e. that if h is a filler for $g \circ f_1$, then it is also a filler for $g \circ f_2$. We find this from the following commutative diagram.

$$
\begin{tikzcd}
x \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, "h"] & y \arrow[d, "g"] \\
z \arrow[r, "g"'] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "h_2"'] & y \arrow[d, "g_1"] \arrow[dl, "g_2"'] \\
z \arrow[r, "s_0 z"'] & z
\end{tikzcd}
$$

This gives us a well-defined composition law. For any $x \in \mathcal{C}_0$, the identity on x is $s_0 x$; it is a left and right identity because the solid diagrams

$$
\begin{tikzcd}
& y \arrow[dr, "s_0 y"] & \\
x \arrow[ur, "f"] \arrow[rr, dashed, "f"'] & & y
\end{tikzcd}
\qquad
\begin{tikzcd}
& x \arrow[dr, "g"] & \\
x \arrow[ur, "s_0 x"] \arrow[rr, dashed, "g"'] & & y
\end{tikzcd}
$$

can be filled by $s_1 f$ and $s_0 g$ respectively.

To see that composition is associative consider the horn

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "g \circ f"'] & y \arrow[d, "h \circ g"] \arrow[dl, "g"'] \\
z \arrow[r, "h"'] & w
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "g \circ f"'] \arrow[dr, "(h \circ g) \circ f"] & y \arrow[d, "h \circ g"] \\
z \arrow[r, "h"'] & w
\end{tikzcd}
$$

where the bottom triangle on the right is unfilled. Filling it gives a homotopy

$$[(f \circ g) \circ h] = [f \circ (g \circ h)].$$

Lemma 7.6. The following are properties of $\mathrm{h}\mathcal{C}$.

1. Let $\mathcal{C}$ be a small category. Then $\mathrm{h}N(\mathcal{C}) \cong \mathcal{C}$.

2. For every topological space X,
$$\mathrm{hSing}(X) \cong \pi_{\geq 1}(X)$$

3. A simplicial set X is the nerve of a category if and only if it has unique inner horn fillers.

4. For every Kan complex K, the infinity-category $\mathrm{h}K$ is a groupoid.

Proof.

1. The 0-simplices of $N(\mathcal{C})$ are the objects of $\mathcal{C}$, and these are defined to be the objects of $\mathrm{h}N(\mathcal{C})$.

   The 1-simplices of $N(\mathcal{C})$ are precisely the morphisms of $\mathcal{C}$. In particular, identity morphisms correspond to degenerate simplices. In passing to $\mathrm{h}N(\mathcal{C})$, two morphisms are identified if there is a 2-simplex between them in the sense described above. But this says precisely that the only identifications are of the form $f \circ \mathrm{id} = f$ for all f.

   Simplices of degree 2 or higher are discarded.

2. The zero-simplices of both of these are the points of X, and the morphisms of both of these correspond to paths modulo homotopy.

3. Let f be an edge. By filling an outer horn, we find a left inverse g. By filling again, we find a left inverse h to this. But h has to be f, so we are done.

$\square$

Theorem 7.7. Denote by qCat the full subcategory of Set_Δ on ∞-categories.

1. There is a functor $\mathrm{h}\colon \mathrm{qCat} \mapsto \mathrm{Cat}$ which sends $\mathcal{C} \to \mathrm{h}\mathcal{C}$.

2. There is an adjunction
$$\mathrm{h} : \mathrm{qCat} \leftrightarrow \mathrm{Cat} : N,$$
   where N denotes the nerve (Definition 2.27).

Proof.

1. Let $f\colon X \longrightarrow Y$ be a map of ∞-categories. We need to define a map $\mathrm{h}f\colon \mathrm{h}X \longrightarrow \mathrm{h}Y$. On objects, this is easy, since $f_0\colon X_0 \longrightarrow Y_0$. On morphisms, we get a map $f_1\colon X_1 \longrightarrow Y_1$ which sends morphisms $x \longrightarrow y$ to morphisms $f_0(x) \longrightarrow f_0(y)$, as can be checked by substituting d_0 and d_1 into the downward arrows below.

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd}$$

We still have to check that the map f_1 respects equivalence classes. To this end, suppose $g \sim g' : x \to y$, i.e. there is a simplex as follows.

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "g"] \arrow[rr, "g'"'] & & y \end{tikzcd}$$

Then we have a simplex as follows

$$\begin{tikzcd} & f_0(y) \arrow[dr, "s_0(f_0(y))=f_1(s_0(y))"] & \\ f_0(x) \arrow[ur, "f_1(g)"] \arrow[rr, "f_1(g')"'] & & f_0(y) \end{tikzcd}$$

so $f_1(f) \sim f_1(g)$.

To see functoriality of h, suppose that if $F : X \to Y$ and $G : Y \to Z$ are morphisms of simplicial sets. We need to check that $\mathrm{h}(G \circ F) = \mathrm{h}G \circ \mathrm{h}F$. This is clearly true on objects, and on morphisms we have

$$\mathrm{h}(G \circ F)([f]) = [(G \circ F)(f)] = [G(F(f))] = \mathrm{h}G([F(f)]) = (\mathrm{h}G \circ \mathrm{h}F)([f]).$$

2. We exhibit a unit-counit adjunction. We first define a natural transformation ϵ with components
$$\epsilon_X : X \to N(\mathrm{h}X).$$

□

Note that we have made good on our promise, made in Section 5, to give a partial description of the left adjoint to the nerve N given by the Yoneda extension $よ_! \chi$: when restricted to ∞-categories, it is the given by the functor h.

3 *The coherent nerve*

3.1 *Topologically enriched categories*

Definition 7.8 (compactly generated Hausdorff space)**.** A topological space X is known as a compactly generated Hausdorff space if for every subset $U \subset X$, the following are equivalent.

- U is closed.
- For each compact subset $C \subset X$, $C \cap U$ is closed

We denote the full subcategory of Top on compactly generated Hausdorff spaces by CGHaus.

Our reason for working in the category CGHaus rather than Top is a techical one: the category Top is not Cartesian closed, and therefore is not a good candidate for an enriching category. The other reason is that the geometric realization of any simplicial set is a compactly generated Hausdorff, and therefore we are entitled to

Definition 7.9 (topological category)**.** A category enriched over CGHaus (with the Cartesian monoidal structure) is known as a topological category.

Example 7.10. Consider the category $\mathcal{C}$, whose objects are CW complexes, and whose hom-spaces $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ consist of the set of continuous functions $X \to Y$ taken with the compact-open topology.

3.2 *Simplicially enriched categories*

Definition 7.11 (simplicial category)**.** A category enriched over Set_Δ (with the Cartesian monoidal structure) is called a simplicial category.[1] Denote the category whose objects are simplicial categories and whose morphisms are enriched functors by Cat_Δ.

[1] This terminology, while common, is somewhat misleading. Generally, the phrase *simplicial foo* is used to refer to a functor from Δ^{op} to the category of foos. However, in this case, *simplicial category* refers to a category enriched over simplicial sets. While this is an abuse of terminology, it is not as extreme as it may appear. After unwinding the definitions, one finds that a simplicial object in Cat which is *constant on objects* is equivalent to a Set_Δ-enriched category.

Theorem 7.12. The category

3.3 *Topological and simplicial enrichment*

Lemma 7.13. The functor $\mathrm{Sing}\colon \mathrm{CGHaus} \to \mathrm{Set}_\Delta$ is strong monoidal.

Proof. To see this, we build a map

$$\Phi_{X,Y} \colon \mathrm{Sing}(X \times Y) \to \mathrm{Sing}(X) \times \mathrm{Sing}(Y)$$

as follows.

$$\begin{aligned}\mathrm{Sing}(X \times Y) &\cong \mathrm{Top}(|\Delta^\bullet|, X \times Y)\\ &\cong \mathrm{Top}(|\Delta^\bullet|, X) \times \mathrm{Top}(|\Delta^\bullet|, Y)\\ &\cong \mathrm{Sing}(X) \times \mathrm{Sing}(Y).\end{aligned}$$

The singular set $\mathrm{Sing}(|\Delta^0|) \cong \Delta^0$, so we don't have any choice in defining ϕ. □

Definition 7.14 (singular nerve)**.** Let $\mathcal{T}$ be a topological category. By Lemma A.16, we can use $\mathrm{Sing}\colon \mathrm{Top} \to \mathrm{Set}_\Delta$ to turn $\mathcal{T}$ into a simplicial category. We will call this the singular nerve of $\mathcal{T}$ and denote it by $\mathrm{Sing}(\mathcal{T})$.

Definition 7.15 (topological nerve)**.** Let $\mathcal{T}$ be a topological category. The topological nerve of $\mathcal{T}$, denoted $N_{\mathrm{top}}(\mathcal{T})$, is the infinity category

$$N_\Delta \circ \mathrm{Sing}(\mathcal{T}).$$

3.4 Rigidification

Intuitively, an ∞-category should be something like a category $\mathcal{C}$ with spaces $\mathrm{Map}_{\mathcal{C}}(x, y)$ instead of sets. However, the composition, associativity, and unitality of this category should not be strict, but rather should only hold up to a coherently chosen set of homotopies.[2]

However, as in the world of 2-categories and bicategories, we might hope that we can "strictify" an infinity category $\mathcal{C}$ so as to give a strict composition operation which is coherently homotopy equivalent to the non-strict composition operation. The ideal way to express such a "strictification" is as a simplicially enriched category (remember that simplicial sets can be used as a model for spaces). More precisely, we would like to define a "strictification functor"

$$\mathfrak{C} : \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta.$$

Which we will call the *rigidification*[3]. In fact, we are going to find that there is a Quillen adjunction

$$\mathfrak{C} : \mathrm{Set}_\Delta \leftrightarrow \mathrm{Cat}_\Delta : N_\Delta$$

Fortunately for us, we already have a very good technique for generating left-adjoint functors out of Set_Δ — cosimplicial objects! So we want to describe a cosimplicial object

$$\mathfrak{C} : \Delta \to \mathrm{Cat}_\Delta.$$

To this end, lets think a little bit about how we interpret a simplicial set as a category, and what we would want from a strictification. Fairly obviously we want to think of the 0-simplices as objects, and the 1-simplices as morphisms. In this light, how do we think of a 2-simplex

$$\sigma = \begin{tikzcd} & y \arrow[dr, "f"] & \\ x \arrow[ur, "g"] \arrow[rr, "h"'] & & z \end{tikzcd} \quad ?$$

Following our discussion above, we would generically say that σ "displays h as a composite of f with g." However, this is no good for rigidification. When we rigidify, we want to define a *strict* composite of f with g, rather than many possible, equivalent composites.

To make this possible, we simply define the strict composite $f \circ g = fg$ to be the *orieted path* of 1-simplices which first traces g and then f. We can then interpret our 2-simplex σ as displaying a *specific* 2-isomorphism $h \Rightarrow fg$. Our mapping space can thus be seen as the simplicial set

$$\Delta^1 = \overset{h}{\bullet} \longrightarrow \overset{fg}{\bullet} \quad \cong \mathrm{Map}(x, z)$$

[2] We leave aside, for the moment, what precisely we mean by "coherent". A quasi-categorical avatar of the non-uniqueness of composition is the fact that, in a quasi-category $\mathcal{C}$, each horn Λ^2_1 representing a pair of composable morphisms may have multiple different fillings to a simplex Δ^2, which yields multiple possible composites.

[3] Following the terminology of [**Dugger-Spivak**]. Unfortunately, the functor $\mathfrak{C}$ is still fairly commonly referred to as "The left adjoint to the homotopy coherent nerve" in the literature. Cumbersome, to say the least.

So how do we interpret a 3-simplex

$$\sigma = \begin{tikzcd} & & 2 \arrow[dr, "f"] & \\ 0 \arrow[urr, "v"] \arrow[rrr, "w"] \arrow[dr, "h"'] & & & 3 \\ & 1 \arrow[uur, "g"] \arrow[urr, "u"'] & & \end{tikzcd} \qquad ?$$

Well, let's think about the mapping space from 0 to 3. Counting the "strict composites" we've added, we have four morphisms from 0 to 3: w, uh, fv, and fgh. The 2-simplices give us 2-isomorphisms

$$\alpha : w \Rightarrow fv$$
$$\beta : w \Rightarrow uh$$
$$\gamma : uh \Rightarrow fgh$$
$$\delta : fv \Rightarrow fgh$$

As with our interpretation of the 2-simplex, the 3-simplex itself is then interpreted as displaying a specific 3-isomorphism

$$\gamma \circ \beta \Rrightarrow \delta \circ \alpha$$

Viewed as a simplicial set, this is then

$$\begin{tikzcd} fv \arrow[r] & fgh \\ w \arrow[u] \arrow[r] \arrow[ur] & uh \arrow[u] \end{tikzcd} \quad = \mathrm{Map}(0,3)$$

Notice that, to express the isomorphism between $\delta \circ \alpha$ and $\gamma \circ \beta$, we had to define a new 2-morphism $w \Rightarrow fgh$. So we really interpret the 3-simplex as this 2-isomorphism $\mu : w \Rightarrow fgh$ together with two 2-isomorphisms $\mu \Rrightarrow \delta \circ \alpha$ and $\mu \Rrightarrow \gamma \circ \beta$.

We then notice that we can do this more cleverly. Instead of giving the 1-simplices names, we can identify each string of 1-simplices in Δ^n with the set of vertices it passes through.[4] Relabelling our mapping space, we get

[4] Note that this doesn't work in a generic simplicial set, but does work in the nerve of a poset.

$$\begin{tikzcd} 023 \arrow[r] & 0123 \\ 03 \arrow[u] \arrow[r] \arrow[ur] & 013 \arrow[u] \end{tikzcd} \quad = \mathrm{Map}(0,3)$$

But this is just the nerve of the poset $P_{0,3}$ whose objects are subsets of $[3]$ containing both 1 and 3. With this in mind, we can make a preliminary definition our cosimplicial object $\mathfrak{C}$.

It's worth commenting here on why we don't simply use the cosimplicial object $[n] \mapsto [n]$, where $[n]$ is viewed as a simplicial category with discrete mapping spaces. The reason is that this collapses precisely the coherence data that we described above, and in more complicated infinity categories, will destroy much of the coherent data encoded in quasi-categories. There is, however, a morphism

$$\mathfrak{C}[\Delta^n] \to [n]$$

which is an isomorphism on objects, and induces a weak homotopy equivalence on mapping spaces. We will later see that these functors are weak equivalences in the relevant model structure on Cat_Δ.

Definition 7.16 (Rigidification)**.** We define a cosimplicial object called the *rigidification* to be the functor

$$\mathfrak{C}\colon \Delta \to \mathrm{Cat}_\Delta$$

defined as follows.

It sends objects $[n]$ to the simplicially enriched category $\mathfrak{C}[\Delta^n]$ defined as follows.

- $\mathrm{Obj}(\mathfrak{C}[\Delta^n]) = \{0, \ldots, n\}$.
- For $i, j \in \{0, \ldots, n\}$
$$\mathfrak{C}[\Delta^n](i, j) = N(P_{i,j}),$$
where $P_{i,j}$ is the poset of subsets of $\{i, \ldots, j\}$ containing i and j.
- To define composition, we need a morphism of simplicial sets
$$N(P_{i,j}) \times N(P_{j,k}) \to N(P_{i,k}).$$
We have a multiplication $P_{i,j} \times P_{j,k} \to P_{i,k}$ given by $(I, J) \mapsto I \cup J$. This induces the map we need.

The poset $P_{i,j}$ can be equivalently described as the power set of the set $L_{i,j} := \{k \in [n] \mid i < k < j\}$, and is thus, geometrically, a cube.

A morphism $f\colon [m] \to [n]$ induces a functor $\mathfrak{C}[\Delta^m] \to \mathfrak{C}[\Delta^n]$ on objects by $i \mapsto f(i)$, and on hom-simplicial-sets $N(P_{i,j})$ as follows. On the underlying posets, we send

$$P_{i,j} \ni \{i, k_1, \ldots, k_{r-1}, j\} \mapsto \{f(i), f(k_1), \ldots, f(k_{r-1}), f(j)\} \in P_{f(i),f(j)}.$$

This extends to a functor of poset categories, hence a morphism of nerves.

Taking the Yoneda extension of the rigidification, we find a functor $\mathfrak{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$, also called the rigidification.

By the usual theory of cosimplicial objects (explained in Section 5) we have the following results.

Lemma 7.17. The rigidification $\mathfrak{C}$ preserves colimits.

Definition 7.18 (homotopy-coherent nerve)**.** Let $\mathcal{C}$ be a simplicial category. The homotopy-coherent nerve of $\mathcal{C}$, denoted N_Δ, is the functor

$$N_\Delta(\mathcal{C}) = \mathrm{Cat}_\Delta(\mathfrak{C}(-), \mathcal{C})\colon \Delta^{\mathrm{op}} \to \mathrm{Set}_\Delta.$$

By Theorem A.10, we have an adjunction

$$\mathfrak{C} : \mathrm{Set}_\Delta \leftrightarrow \mathrm{Cat}_\Delta : N_\Delta.$$

The right adjoint of our adjunction fits into a commutative triangle of right Quillen functors

$$\begin{tikzcd} & \mathrm{Cat}_\Delta \arrow[dr, "N_\Delta"] & \\ \mathrm{Cat} \arrow[ur, "\mathrm{disc}"] \arrow[rr, "N"'] & & \mathrm{Set}_\Delta \end{tikzcd}$$

where disc includes categories into Cat_Δ as simplicial categories with discrete mapping spaces.

While it would go far beyond the scope of this document to prove them, we will state and use the following theorems.

Theorem 7.19 (Bergner)**.** There is a model structure on the category Cat_Δ with

(WE) weak equivalences given by those functors $F : \mathcal{C} \to \mathcal{D}$ which are essentially surjective and induce weak homotopy equivalences of simplicial sets

$$F_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(x,y)$$

for every $x, y \in \mathcal{C}$

(F) Fibrations given by those functors $F : \mathcal{C} \to \mathcal{D}$ which are isofibrations and induce Kan fibrations

$$F_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(x,y)$$

for every $x, y \in \mathcal{C}$.

Theorem 7.20 ([???])**.** The adjoint functors

$$\mathfrak{C} : \mathrm{Set}_\Delta \leftrightarrow \mathrm{Cat}_\Delta : N_\Delta$$

define a Quillen equivalence between the Joyal model structure and the Bergner model structure. A map $X \to Y$ of simplicial sets is a Joyal equivalence if and only if the induced functor $\mathfrak{C}[X] \to \mathfrak{C}[Y]$ is a Bergner equivalence.[5]

[5] The second statement here can be viewed as an ∞-categorical version of a very familiar theorem: A functor between 1-categories is an equivalence of categories if and only if it is essentially surjective and fully faithful (induces bijections on mapping spaces). Our version says that a functor $F : \mathcal{C} \to \mathcal{D}$ of quasi-categories is an equivalence if and only if it is essentially surjective and *homotopy fully faithful*, i.e. induces homotopy equivalences on all mapping spaces.

We will later see some exemplar computations to show that N_Δ preserves, e.g, fibrant objects.

Before we do that, though, let's go back and formalize some of our intuition about mapping spaces in $\mathfrak{C}[X]$. The k-simplices in $\mathfrak{C}[\Delta^n](i,j)$ have the form

$$\sigma := S_0 \subset S_1 \subset \cdots \subset S_k$$

where each $S_\ell \subset [n]$ such that $i, j \in S_\ell$ and, for all $s \in S_\ell$, $i \leq s \leq j$. We will reformulate the data involved in σ to obtain a new interpretation of the mapping spaces. We first write

$$S_0 = \{i = a_1 \leq a_2 \leq \cdots \leq a_r = j\}$$

and then define

$$S_k^\ell := \{s \in S_k \mid a_\ell \leq a_{\ell+1}\}.$$

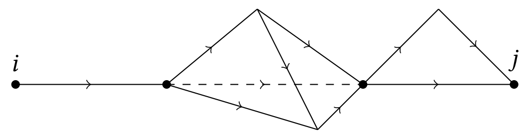
We can think of S_k^ℓ as an n-simplex in Δ^n with initial vertex a_ℓ and final vertex $a_{\ell+1}$. We can thus completely encode the information of S_0 and S_k in the string of simplices

$$S_k^1, S_k^2, \cdots, S_k^{r-1}$$

in Δ^n. Note that

- the final vertex of S_k^ℓ is the intitial vertex of $S_k^{\ell+1}$
- the initial vertex of S_k^1 is i, and
- the final vertex of S_k^{r-1} is j.

We can visualize this information as a *necklace* of n-simplices:



Definition 7.21. The *necklace* of shape $k_1, \dots, k_r$ is the simplicial set

$$\mathcal{N} = \Delta^{k_1} \coprod_{\Delta^0} \Delta^{k_2} \coprod_{\Delta^0} \cdots \coprod_{\Delta^0} \Delta^{k_r} =: \Delta^{k_1} \vee \Delta^{k_2} \vee \cdots \vee \Delta^{k_r}$$

The images in $\mathcal{N}$ of the vertices $0, k_\ell \in \Delta^{k_\ell}$ are called the *joints* of $\mathcal{N}$ and the set of joints of $\mathcal{N}$ is denoted by $J_{\mathcal{N}}$. The set of vertices of $\mathcal{N}$ is denoted by $V_{\mathcal{N}}$.

The category Nec of necklaces is the subcategory of Set_Δ whose objects are necklaces, and whose morphisms are maps of simplicial sets $f : \mathcal{N} \to \mathcal{M}$ preserving the minimal and maximal joints.

We have thus shown

Lemma 7.22. A k-simplex in $\mathfrak{C}[\Delta^n](i, j)$ is equivalently the following data:

- A necklace $\mathcal{N}$.
- A collection of subsets
$$J_{\mathcal{N}} = S_0 \subset S_1 \subset \cdots \subset S_k = V_{\mathcal{N}}$$
- A map $f : \mathcal{N}_{\vec{k}} \to \Delta^n$ sending the lowest joint to i and the highest joint to j.

the simplex is degenerate if and only if there exists $0 \leq \ell \leq k$ such that $S_\ell = S_{\ell+1}$.

This may seem like we have unnecessarily complicated the definition of the rigidification. However this reformulation will allow us to give a generic description of mapping spaces in the rigidification of a simplicial sets.

Given a simplicial set K and vertices $x, y \in K$, we define $(\mathrm{Nec}_{/K})_{x,y}$ as the full subcategory of $\mathrm{Nec}_{/K}$ on those maps $\mathcal{N} \to K$ which send the initial joint to x and the final joint to y. By the functoriality of the rigidification, given a map $\mathcal{N} \to K$ in $(\mathrm{Nec}_{/K})_{x,y}$, we get a map

$$\mathfrak{C}[\mathcal{N}](x, y) \to \mathfrak{C}[K](x, y)$$

where we abuse notation by denoting by x the lowest joint of $\mathcal{N}$ and by y the highest joint of $\mathcal{N}$. These piece together to form a cone, so that we have a canonical map

$$\operatorname*{colim}_{(\mathrm{Nec}_{/K})_{x,y}} \mathfrak{C}[\mathcal{N}](x, y) \to \mathfrak{C}[K](x, y)$$

We can define a simplicial category E_K whose objects are the vertices of K, and with

$$E_K(x, y) := \operatorname*{colim}_{(\mathrm{Nec}_{/K})_{x,y}} \mathfrak{C}[\mathcal{N}](x, y).$$

The composition operation is given by "gluing necklaces", i.e. given $f : \mathcal{N} \to K$ in $(\mathrm{Nec}_{/K})_{a,b}$ and $g : \mathcal{M} \to K$ in $(\mathrm{Nec}_{/K})_{b,c}$ we can construct a new map

$$f \vee g : \mathcal{N} \vee \mathcal{M} := \mathcal{N} \coprod_{\Delta^{\{b\}}} \mathcal{M} \to K$$

in $(\mathrm{Nec}_{/K})_{a,c}$. It is a matter of unwinding the definitions to show that the canonical maps above piece together into a canonical functor

$$E_K \to \mathfrak{C}[K].$$

Note 7.23. The construction $K \mapsto E_K$ extends to a functor

$$E_{(-)} : \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta.$$

The simplicial functors $E_K \to \mathfrak{C}[K]$ assemble to a natural transformation $\mu : E_{(-)} \Rightarrow [C]$. Each of the functors μ_K is bijective on objects.

Before characterizing mapping spaces in general, we need some more facts about necklaces.

Definition 7.24. We define a functor

$$C_{(-)} : \mathrm{Nec} \to \mathrm{Cat}_\Delta$$

Which sends $\mathcal{N}$ to the simplicially enriched category $C_\mathcal{N}$ with $C_\mathcal{N}(x, y) = N(P_{x,y})$ where $P_{x,y}$ is the poset whose objects are totally ordered subsets of $V_\mathcal{N}$ starting at x and ending at y, and containing $J_\mathcal{N}$.

Lemma 7.25. Let $\mathcal{N}$ be a necklace. Then there is an isomorphism

$$\mathfrak{C}[\mathcal{N}] \cong C_\mathcal{N}$$

of simplicial categories.

Proof. This is definitional if $\mathcal{N}$ is a simplex. We now show that, for two necklaces $\mathcal{N}$ and $\mathcal{M}$, we have

$$C_{\mathcal{N}\vee\mathcal{M}} \cong C_\mathcal{N} \coprod_{C_{\Delta^0}} C_\mathcal{M}.$$

The functor in question is clearly bijective on objects (i.e., that $C_{(-)}$ preserves pushouts). On mapping spaces, This follows by careful examination of the universal property of pushout.

It then follows that, for any necklace

$$\mathcal{N} = \Delta^{k_1} \vee \cdots \vee \Delta^{k_r}.$$

we have

$$\begin{aligned} \mathfrak{C}[\mathcal{N}] &\cong \mathfrak{C}\Delta^{k_1} \coprod_{\mathfrak{C}[\Delta^0]} \cdots \coprod_{\mathfrak{C}[\Delta^0]} \mathfrak{C}\Delta^{k_r} \\ &\cong C_{\Delta^{k_1}} \coprod_{C_{\Delta^0}} \cdots \coprod_{C_{\Delta^0}} C_{\Delta^{k_r}} \\ &\cong C_\mathcal{N}, \end{aligned}$$

proving the lemma. □

Corollary 7.26. Suppose K is a necklace. Then the functor

$$\mu_K : E_K \to \mathfrak{C}[K]$$

is an isomorphism of simplicial categories.

Proof. Since μ_K is bijective on objects, it will suffice to check that it is bijective on mapping spaces. Given $x, y \in K$, let $K_{x,y}$ be the full simplicial subset on those vertices $s \in K$ with $x \leq s \leq y$. Then $K_{x,y}$ is also necklace, and the inclusion $K_{x,y} \to K$ is a terminal object in $(\mathrm{Nec}_{/K})_{x,y}$, so that $E_K(x, y) = \mathfrak{C}[K_{x,y}](x, y)$ and

$$\mu_K : \mathfrak{C}[K_{x,y}](x, y) \to \mathfrak{C}[K](x, y)$$

is the inclusion. However, it follows immediately from 7.25 that this is an isomorphism of simplicial sets. ◻

Now, given a simplicial set K, the colimit cone exhibiting K as the colimit over its category of simplices $\Delta_{/K}$ gives rise to a canonical functor

$$\operatorname*{colim}_{\Delta_{/K}} E_{(\Delta^n)} \to E_K.$$

Lemma 7.27. Let K be a simplicial set, and $x, y \in K$. Then the map

$$\phi : \left(\operatorname*{colim}_{\Delta_{/K}} E_{(\Delta^k)}\right)(x, y) \to E_K(x, y)$$

is surjective.

Proof. Given an n-simplex $\sigma : \Delta^n \to E_K(x, y)$, there is by definition a necklace $\mathcal{N}$, a map $f : \mathcal{N} \to K$, and an n-simplex $\gamma : \Delta^n \to \mathfrak{C}[\mathcal{N}](x, y) \cong E_{\mathcal{N}}(x, y)$ representing σ in the colimit.

We thus have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\left(\operatorname{colim}_{\Delta_{/\mathcal{N}}} E_{\Delta^k}\right)(x, y) \arrow[r, "\cong"] \arrow[d, "f"'] & E_{\mathcal{N}}(x, y) \arrow[d, "f"] & \gamma \arrow[d, mapsto] \\
\left(\operatorname{colim}_{\Delta_{/K}} E_{(\Delta^k)}\right)(x, y) \arrow[r, "\phi"'] & E_K(x, y) & \sigma
\end{tikzcd}$$

where the top map is an isomorphism by 7.26. We thus see that, by commutativity, there exists an n-simplex in

$$\left(\operatorname*{colim}_{\Delta_{/K}} E_{(\Delta^n)}\right)$$

which is sent to σ under ϕ. ◻

Proposition 7.28 (Dugger-Spivak)**.** For any simplicial set K, the canonical functor

$$E_K \to \mathfrak{C}[K]$$

is an isomorphism of simplicial categories.

Proof. The functor in question is clearly bijective on objects, so it will suffices to show that it is an isomorphism on mapping spaces. For $x, y \in K$, we obtain a commutative diagram

$$\begin{tikzcd} \left(\operatorname{colim}_{\Delta_{/K}} E_{(\Delta^l)}\right)(x,y) \arrow[r, "\phi"] \arrow[d, "\cong"] & E_K(x,y) \arrow[d, "\mu_K"] \\ \left(\operatorname{colim}_{\Delta_{/K}} \mathfrak{C}[\Delta^k]\right)(x,y) \arrow[r, equal] & \mathfrak{C}[K](x,y) \end{tikzcd}$$

where the left-hand map is an isomorphism by 7.26. This thus implies that ϕ is injective. However, by 7.27, ϕ is also surjective, and thus an isomorphism, and thus

$$\mu_K : E_K(x,y) \to \mathfrak{C}[K](x,y)$$

is an isomorphism. □

Note 7.29. This proposition thus allows us to represent simplices of the mapping spaces $\mathfrak{C}[K](x,y)$ explicitly in terms of

- an element $f : \mathcal{N} \to K$ of $(\mathrm{Nec}_{/K})_{x,y}$
- an n-simplex in $\mathfrak{C}[\mathcal{N}](x,y)$

modulo equivalence relations. In the particular case where K is a simplicial subset of the nerve of a poset, this simplifies substantially.

Corollary 7.30 (Dugger-Spivak)**.** Let $X = N(Q)$ be the nerve of a poset Q. For $x, y \in Q$ define a poset $P_{x,y}$ whose objects are totally ordered chains

$$x \leq z_1 \leq z_2 \leq \ldots \leq z_k \leq y$$

ordered by inclusion. Then

$$\mathfrak{C}[X](x,y) \cong N(P_{x,y})$$

Proof. This follows from unravelling the relevant colimit. □

Note 7.31. For a simplicial subset $Y \subset N(Q)$ of the nerve of a poset, the mapping space $\mathfrak{C}[Y](x,y) \subset \mathfrak{C}[N(Q)](x,y)$ consists of precisely those simplices such that the corresponding necklace in $N(Q)$ factors through Y.

Using this characterization, we can prove that certain morphisms of simplicial sets induce Bergner equivalences, and thus are themselves equivalences of simplicial sets.

Definition 7.32. Let Δ^n be an n-simplex. The simplicial subset

$$Z_n := \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod_{\Delta^{\{2\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subset \Delta^n$$

is referred to as the spine of Δ^n.

Lemma 7.33. The inclusion $i_n : Z_n \to \Delta^n$ is a Joyal equivalence of simplicial sets.

Proof. It suffices to show that $\mathfrak{C}[i_n] : \mathfrak{C}[Z_n] \to \mathfrak{C}[\Delta^n]$ is a Bergner equivalence. It is clearly essentially surjective (indeed, surjective on vertices).

However, we know that $\mathfrak{C}[\Delta^n](i, j)$ is contractible if $i \le j$ and empty otherwise. It is immediate from the above theorem that $\mathfrak{C}[Z_n](i, j)$ is a 1-point space if $i \le j$ and empty otherwise. Thus $\mathfrak{C}[i_n]$ induces homotopy equivalences on mapping spaces. □

We can also use this characterization to prove that N_Δ preserves fibrant objects (which is also an implication of the Quillen equivalence above).

Proposition 7.34. Let $\mathcal{C}$ be a simplicial category such that for all x, y in $\mathcal{C}$, $\mathcal{C}(x, y)$ is a Kan complex. Then $N_\Delta(\mathcal{C})$ is an ∞-category.

Proof. We need to show that we can solve the following lifting problem.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & N_\Delta(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Passing to adjuncts, we find the following lifting problem.

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \mathfrak{C}[\Delta^n] \arrow[ur, dashed] & \end{tikzcd}$$

We can view the simplicial category $\mathfrak{C}[\Lambda^n_i]$ as a subcategory of $\mathfrak{C}[\Delta^n]$, where the only data missing is the mapping space

$$\mathfrak{C}[\Delta^n](0, n).$$

Therefore, a solution to the above lifting problem need only fill in this information. This means we need only solve the corresponding lifting problem.

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i](0, n) \arrow[r] \arrow[d, hook] & \mathcal{C}(0, n) \\ \mathfrak{C}[\Delta^n](0, n) \arrow[ur, dashed] & \end{tikzcd}$$

A k-simplex of $\mathfrak{C}[\Lambda^n_i](0, n)$ is a chain

$$S_0 \subset S_1 \subset \ldots \subset S_k$$

of subsets of $[n]$ containing 0 and n, such that, for each consecutive $s, t \in S_0$, the subset $S_k^{s,t} := \{q \in S_k \mid s \le q \le t\}$ defines a simplex in Λ^n_i. More precise, there exists $0 \le j \le n$ with $j \ne i$ such that $S_k^{s,t} \subset \Delta^{\{0,1,\ldots,\hat{j},\ldots,n\}}$.

Let $\mathfrak{C}[\Lambda^n_i](0, n)^i$ and $\mathfrak{C}[\Delta^n](0, n)^i$ be the full sub-simplicial sets (in the latter case, subposet) on those objects which contain i. We can define a map of posets

$$\mathfrak{C}[\Delta^n](0, n) \to \mathfrak{C}[\Delta^n](0, n)^i$$

sending $S \mapsto S \cup \{i\}$. This is clearly homotopic to the identity, and one can check that both the map and the homotopy descend to

$$\mathfrak{C}[\Lambda^n_i](0,n) \to \mathfrak{C}[\Lambda^n_i](0,n)^i$$

We thus have a commutative diagram

$$\begin{tikzcd}
\mathfrak{C}[\Lambda^n_i](0,n) \arrow[r, "\simeq"] \arrow[d] & \mathfrak{C}[\Lambda^n_i](0,n)^i \arrow[d] \\
\mathfrak{C}[\Delta^n](0,n) \arrow[r, "\simeq"] & \mathfrak{C}[\Delta^n](0,n)^i
\end{tikzcd}$$

However, by definition, $\mathfrak{C}[\Delta^n](0,n)^i$ is the image of the composition map

$$\mathfrak{C}[\Delta^n](0,i) \times \mathfrak{C}[\Delta^n](i,n) \to \mathfrak{C}[\Delta^n](0,n).$$

We thus can extend the commutative diagram above to

$$\begin{tikzcd}
\mathfrak{C}[\Lambda^n_i](0,n) \arrow[r, "\simeq"] \arrow[d] & \mathfrak{C}[\Lambda^n_i](0,n)^i \arrow[d] & \mathfrak{C}[\Lambda^n_i](0,i) \times \mathfrak{C}[\Lambda^n_i](i,n) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
\mathfrak{C}[\Delta^n](0,n) \arrow[r, "\simeq"] & \mathfrak{C}[\Delta^n](0,n)^i & \mathfrak{C}[\Delta^n](0,i) \times \mathfrak{C}[\Delta^n](i,n) \arrow[l, "\cong"']
\end{tikzcd}$$

By 2-out-of-3, $\mathfrak{C}[\Lambda^n_i](0,n) \to \mathfrak{C}[\Delta^n](0,n)$ is a weak homotopy equivalence. □

Example 7.35. Let $\mathrm{Kan} \subset \mathrm{Set}_\Delta$ be the full subcategory on Kan complexes. We may consider Kan to be simplicially enriched by taking $\mathrm{Kan}(K,S) = \mathrm{Maps}(K,S)$. We have seen in Corollary 3.31 that each mapping space is a Kan complex. Thus, the simplicial nerve $N_\Delta(\mathbf{Kan})$ is an ∞-category.

Definition 7.36 (category of spaces)**.** The <u>∞-category of spaces</u> is the category

$$\mathcal{S} = N_\Delta(\mathbf{Kan}).$$

4 Mapping spaces

Definition 7.37 (mapping spaces of a simplicial set)**.** Let S be any simplicial set. For any $x, y \in S_0$, define the mapping space

$$\mathrm{Maps}_S(x,y)$$

to be the object

$$\mathrm{h}\left(\mathfrak{C}[S](x,y)\right)$$

in the homotopy category of spaces $\mathcal{H}$.

5 Infinity-groupoids

Definition 7.38 (equivalence)**.** A morphism $f : x \to y$ in an infinity category is said to be an equivalence if the corresponding morphism in the homotopy category (Example 7.5) is an isomorphism.

Definition 7.39 (infinity-groupoid)**.** An ∞-category is said to be an ∞-groupoid if every morphism is an equivalence.

Theorem 7.40. The following are equivalent.

1. $\mathcal{C}$ is a Kan complex.
2. $\mathcal{C}$ is an infinity-groupoid.

Proof. The direction $1 \implies 2$ is obvious by Lemma 7.6. We will see why the other direction holds later. □

Definition 7.41 (maximal infinity groupoid)**.** Let $\mathcal{C}$ be an infinity category. The maximal infinity groupoid in $\mathcal{C}$, denoted $\mathcal{C}^{\simeq} \subset \mathcal{C}$, is the simplicial subset of $\mathcal{C}$ consisting of simplices all of whose edges are equivalences.

6 Functors between infinity categories

In this section, we build up the notions necessary to work with functors between infinity-categories.

In Subsection 6.1, we define functors between infinity-categories.

In Subsection 6.3, we define a monoidal product called the *join* on the category of categories, and show that it reproduces cones and cocones. We then generalize the construction to the category of simpicial sets, and show[6] that the two notions are compatible in the sense that the nerve is a monoidal functor between them; that is we have a natural isomorphism

$$N(\mathcal{C} \star \mathcal{D}) \cong N(\mathcal{C}) \star N(\mathcal{D}).$$

[6] Well, if there's time.

In Subsection 6.2, we generalize the notion of the *opposite category* to the context of simplicial sets, and note that it defines an *opposite functor* on the category of simplicial sets, when one extends simplex category Δ in an appropriate way.

In Subsection 6.4, we see that joins allow us to define the notion of a cone in an infinity-categorical context. We also define the infinity-categorical generalization of comma categories, and note that there is a duality between the join and the construction of over- and undercategories. We then show that over- and undercategories are 'topologically' very simple—they are contractible Kan complexes.

In Subsection 6.5 and Subsection 6.6, we define the notion of initial (resp. terminal) object in an infinity-category $\mathcal{C}$. We set up our definitions in such a way that initial objects in an infinity category are initia objects in the homotopy category, and they are unique in the sense that their 'convex hull' is a contractible Kan complex.

In Subsection 6.7, we define when an object in an infinity category is a limit or colimit of a functor.

6.1 Functors and diagrams

Definition 7.42 (infinity-functor)**.** An infinity-functor (hereafter simply called *functor*) between infinity-categories $\mathcal{C}$ and $\mathcal{D}$ is simply a map $\mathcal{C} \to \mathcal{D}$ of simplicial sets.

Here we list some theorems which we will prove later, after we have some more advanced notions.

Proposition 7.43. Let K be a simplicial set and let $\mathcal{C}$ be an infinity category. Then the simplicial set $\mathrm{Maps}(K, \mathcal{C})$ is an infinity-category.

Proof. Later. □

Definition 7.44 (homotopy coherent diagram)**.** Let $\mathcal{T}$ be a topological category and I an ordinary category. Let $\mathcal{C} = N_{\mathrm{top}}(\mathcal{T})$.

6.2 Opposite categories

Henceforth, we have been considering the domain of simplicial sets to be the simplex category Δ. However, we have sometimes relaxed this when it was convenient to do so. For example, we have notated the subsimplex of Δ^n missing the 0th vertex by $\Delta^{\{1,\dots,n\}}$, despite the fact that $\{1, \dots, n\}$ is not an object in Δ, so it is not clear how to make sense of this; in particular, it does not make any sense to write

$$\Delta^{\{1,\dots,n\}} = \Delta(-, \{1, \dots, n\}).$$

One way of atoning for this transgression is to extend the simplex category Δ to the category of finite, non-empty linearly ordered sets, denoted $\mathrm{FinLinOrd}$; the category Δ is then the skeleton of $\mathrm{FinLinOrd}$. The inclusion functor $\Delta \hookrightarrow \mathrm{FinLinOrd}$ is an equivalence, so passing from Δ to $\mathrm{FinLinOrd}$ is just a shift of perspective.

On $\mathrm{FinLinOrd}$, there is a functor Swap, which reverses the linear order. That is, it sends the set

$$[n] = \{0, \dots, n\} \mapsto \{n, \dots, 0\} = [\bar{n}].$$

Let us try to understand the behavior of Swap. Any map $f\colon [n] \to [m]$, say

$$f\colon [3] \to [4],$$

can be drawn diagrammatically as follows

$$
\begin{tikzcd}
0 \arrow[dr, mapsto] & 0 \\
1 \arrow[dr, mapsto] & 1 \\
2 & 2 \\
3 \arrow[ur, mapsto] & 3 \\
& 4
\end{tikzcd}
$$

The condition that the map f be weakly monotonic is equivalent to the condition that none of the arrows in the above diagram cross. The swap operation turns the above diagram on its head, taking uncrossed arrows to uncrossed arrows. Thus, morphisms $f\colon [n] \to [m]$ are taken to morphisms $f^{\mathrm{op}}\colon [\bar{n}] \to [\bar{m}]$. Furthermore, composition is respected because composition is done by stacking diagrams left to right.

This discussion means that Swap is an endofunctor on $\mathrm{FinLinOrd}$, and further that it is an equivalence of categories, since $\mathrm{Swap}^2 = \mathrm{id}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be regular categories. Any endofunctor $E\colon \mathcal{C} \to \mathcal{C}$ gives an endofunctor on the functor category $[\mathcal{C}, \mathcal{D}]$ as follows.

- On objects F, it is defined via the pullback.

$$F \mapsto F \circ E$$

- On morphisms η, it is defined via whiskering.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "E"] & \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "F'"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\eta"] & \mathcal{D} & \mapsto & \mathcal{C} \arrow[r, bend left, "F\circ E"{name=U2}] \arrow[r, bend right, "F\circ E'"'{name=D2}] \arrow[from=U2, to=D2, Rightarrow, "E^*(\eta)"] & \mathcal{D}
\end{tikzcd}
$$

Definition 7.45 (opposite functor)**.** We call the pullback by Swap the opposite functor, and denote it by $(-)^{\mathrm{op}}$.

Theorem 7.46. For any simplicial sets K and S, we have

$$(K \star S)^{\mathrm{op}} = S^{\mathrm{op}} \star K^{\mathrm{op}}.$$

Proof. □

6.3 Joins

In Subsection 6.2, we expanded the simplex category Δ to the category $\mathrm{FinLinOrd}$, which allowed us to

Definition 7.47 (join of categories)**.** Let $\mathcal{C}$ and $\mathcal{C}'$ be ordinary categories. The join of $\mathcal{C}$ and $\mathcal{C}'$, denoted $\mathcal{C} \star \mathcal{C}'$, is the category with objects

$$\mathrm{ob}(\mathcal{C} \star \mathcal{C}) = \mathrm{ob}(\mathcal{C}) \amalg \mathrm{ob}(\mathcal{C}')$$

and morphisms

$$(\mathcal{C} \star \mathcal{C}')(x, y) = \begin{cases} \mathcal{C}(x, y), & x, y \in \mathcal{C} \\ \mathcal{C}'(x, y), & x, y \in \mathcal{C}' \\ \{*\}, & x \in \mathcal{C}, y \in \mathcal{C}' \\ \emptyset, & x \in \mathcal{C}', y \in \mathcal{C} \end{cases}.$$

One imagines the join of two categories as follows.

Example 7.48. Here is an alternative definition for the category of cones over a functor $F\colon \mathcal{C} \to \mathcal{D}$, given in terms of the join: it is the category

$$\mathrm{Fun}_F(\Delta^0 \star \mathcal{C}, \mathcal{D}),$$

i.e. the subcategory of $\mathrm{Fun}(\Delta^0 \star \mathcal{C}, \mathcal{D})$ whose objects are functors which agree with F when restricted to $\mathcal{C}$, and whose morphisms are natural transformations

Example 7.49. Interpreting the linearly ordered set $[n]$ as a category, we have

$$[m] \star [n] = [m + n + 1].$$

Thus, on Δ the join agrees with the ordinal sum.

This looks tantalizingly like a monoidal structure on the simplex category Δ. Unfortunately, there is no unit object. This can be remedied by adding an object $[-1] = \emptyset$ to Δ. We will denote this category by $\hat{\Delta}$. This gives us a monoidal category $(\hat{\Delta}, \star)$.

The Day convolution (Definition A.14) allows us to extend $\star$ from $\hat{\Delta}$ to $\mathrm{Set}_{\hat{\Delta}}$.

Definition 7.50 (join of simplicial sets)**.** Let K and S be simplicial sets. The join of K and S, denoted $K \star S$, is the Day convolution of K and S.

Of course, this gives a functor $\hat{\Delta}^{\mathrm{op}} \to \mathrm{Set}$; to get a simplicial set, one simply forgets the data of the empty set of (-1)-simplices.

Theorem 7.51. We can compute the join by the following formula.

$$(K \star K')_n = \coprod_{[i] \oplus [j] \cong [n]} K_i \times K'_j,$$

Proof. The colimit formula for Kan extensions tells us that

$$(K \star S)_n = \operatorname{colim}\left[([n] \downarrow \star^{\mathrm{op}}) \to \hat{\Delta}^{\mathrm{op}} \times \hat{\Delta}^{\mathrm{op}} \to \mathrm{Set}\right]$$

□

and which acts on a morphism $\phi\colon [m] \to [n]$ by sending a simplex

$$\sigma = (\alpha \in K_i, \beta \in K_j, [i] \oplus [j] \cong [n]) \in (K \star K')_n$$

to a simplex

$$\phi^*\sigma = (\psi_1^*\alpha, \psi_2^*\beta, \phi^{-1}([i]) \oplus \phi^{-1}([j]) \cong [m]) \in (K \star K')_m,$$

where

$$\psi_1 = \phi|_{\phi^{-1}([i])}, \qquad \text{and} \qquad \phi_2 = \phi|_{\phi^{-1}([j])}.$$

This is functorial since

Proposition 7.52. The join of two ∞-categories is again an ∞-category.

Proposition 7.53. The join is functorial in each argument separately.

Proof. Given a map $f\colon K \to K'$ of simplicial sets, we need to specify a map

$$(f, \mathrm{id}_S)\colon K \star S \to K' \star S.$$

By the universal property of coproducts, we can do this by defining maps

$$K_i \times S_j \to K_i' \times S_j$$

for each $[i] \oplus [j] \cong [n]$. We have a map $f_i\colon K_i \to K_i'$, which gives a map $f_i \times \mathrm{id}$ as required. We need only check that this is a morphism of simplicial sets, i.e. makes the following square commute for each $\phi\colon [n] \to [m]$.

$$\begin{tikzcd}
(K \star S)_m \arrow[r] \arrow[d] & (K \star S)_n \arrow[d] \\
(K' \star S)_m \arrow[r] & (K' \star S)_n
\end{tikzcd}$$

Let $\sigma = (\alpha, \beta, [i] \oplus [j] \cong [n]) \in (K \star S)_m$. We follow it around.

$$\begin{tikzcd}
(\alpha, \beta) \arrow[r, mapsto] \arrow[d, mapsto] & (\psi_i^*\alpha, \psi_j^*\beta) \arrow[d, mapsto] \\
(f_i(\alpha), \beta) \arrow[r, mapsto] & (f_{\phi^{-1}([i])}\psi_i^*\alpha \overset{!}{=} \psi_i^*(f_i\alpha), \beta)
\end{tikzcd}$$

The condition that $f_{\phi^{-1}([i])}\psi_i^*\alpha = \psi_i^*(f_i\alpha)$ is preciesely the condition that the following diagram commute, which it does by naturality of f.

$$\begin{tikzcd}
K_i \arrow[r, "\psi_i^*"] \arrow[d, "f_i"'] & K_{\phi^{-1}([i])} \arrow[d, "f_{\phi^{-1}([i])}"] \\
K_i' \arrow[r, "\psi_i^*"'] & K'_{\phi^{-1}([i])}
\end{tikzcd}$$

Functoriality in the second argument is exactly analogous. $\square$

6.4 Cones and cocones

Definition 7.54 (left cone, right cone)**.** let K be a simplicial set. The left cone of K is the simplicial set

- The left cone of K is the simplicial set
$$K^{\triangleleft} = \Delta^0 \star K.$$
- The right cone of K is the simplicial set
$$K^{\triangleright} = K \star \Delta^0.$$

Definition 7.55 (overcategory, undercategory)**.** Let K and S be simplicial sets and let $p\colon K \longrightarrow S$ be a morphism.

- The overcategory $S_{p/}$ is the category whose n-simplices are
$$(S_{p/})_n = \{\text{maps } f\colon K \star \Delta^n \longrightarrow \mathcal{C} \text{ satisfying } F|_K = p\}.$$
- The undercategory $S_{/p}$ is the category whose n-simplices are
$$(S_{/p})_n = \{\text{maps } f\colon \Delta^n \star K \longrightarrow \mathcal{C} \text{ satisfying } F|_K = p\}.$$

Theorem 7.56. There is a pair of almost adjoint functors

$$K \star - : \mathrm{Set}_\Delta \leftrightarrow (\mathrm{Set}_\Delta)_{/p} : /p$$

More explicitly, given a map $p\colon X \longrightarrow \mathcal{C}$, there is a natural bijection

$$\mathrm{Hom}_p(K \star X, \mathcal{C}) \cong \mathrm{Hom}(K, \mathcal{C}_{/p}),$$

where Hom_p denotes the set of maps $f\colon K \star X \longrightarrow \mathcal{C}$ such that $f|_K = p$.

Corollary 7.57. A map

For regular categories, it is possible to define colimits as limits in opposite categories; the two notions are equivalent. In infinity categories, this is also the case.

Lemma 7.58. Let $\mathcal{C}$ be a Kan complex, and $x \in \mathcal{C}$. Then $\mathcal{C}_{/x}$ and $\mathcal{C}_{x/}$ are contractible Kan complexes.

Proof. We need to show that we can solve lifting problems of the following form.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{/x} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \{*\} \end{tikzcd}$$

Consider □

6.5 Initial objects

We would like initial objects in an infinity category to be objects that have an edge to each other object, and we would like this edge to be unique in some sense. Demanding uniqueness on the nose, however, would go agains the spirit of the theory of infinity categories. The right notion, not obviously, turns out to be the following.

Definition 7.59 (initial object)**.** Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$ an object. We say that x is an initial object if the map

$$\mathcal{C}_{x/} \to \mathcal{C}$$

is a trivial Kan fibration. More specifically, this map takes an n-simplex

$$\Delta^n \to \mathcal{C}_{x/},$$

i.e. a map

$$\Delta^0 \star \Delta^n \to \mathcal{C},$$

to the pullback

$$\Delta^n \hookrightarrow \Delta^0 \star \Delta^n \to \mathcal{C}.$$

Proposition 7.60. Let $\mathcal{C}$ be an ∞-category, and let x be an initial object in $\mathcal{C}$. Then for each other object $y \in \mathcal{C}$, there is an edge $x \to y$.

Furthermore, this edge is unique up to homotopy.

Proof. Consider the following solid lifting problem.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^0 \arrow[r, "y"] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

The left-hand arrow is a boundary inclusion $\partial\Delta^0 \to \Delta^0$, so we can find a dashed lift, giving us a zero-simplex in $\mathcal{C}_{x/}$. But by definition, a zero simplex in $\mathcal{C}_{x/}$ is a 1-simplex $\sigma\colon \Delta^0 \star \Delta^0 \to \mathcal{C}$ such that $\sigma|_{\text{first factor}} = x$. Furthermore, the commutativity of the lower triangle tells us that $\sigma|_{\text{second factor}} = y$, i.e. σ is an edge $x \to y$.

Now suppose we have two edges $f, g\colon x \to y$. These give us the following solid lifting problem.

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "{(f,g)}"] \arrow[d, hook] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^1 \arrow[r, "\mathrm{id}_y"'] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

Again, this is a boundary inclusion, giving us a dashed lift, i.e. a 2-simplex α in $\mathcal{C}$ whose 0th vertex is x. The commutativity of the upper square tells us that $d_2\alpha = f$ and

$d_0\alpha = g$, and the commutativity of the lower square tells us that $d_1\alpha = \mathrm{id}_y$. Thus, we have the following simplex in $\mathcal{C}$.

$$\alpha = \begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

But this provides precisely a homotopy $f \to g$. □

Corollary 7.61. Let $\mathcal{C}$ be an ∞-category. If x is an initial object in $\mathcal{C}$, then x is also initial in $\mathrm{h}\mathcal{C}$.

One might expect that these conditions were equivalent. However, that is not the case; an object can be initial in $\mathrm{h}\mathcal{C}$ without being initial in $\mathcal{C}$.

Example 7.62. Take any simply connected space X such that there exists a point $x \in X$ with $\pi_2(X, x)$ non-trivial. Then there is a 2-simplex $\alpha\colon \Delta^2 \to \mathrm{Sing}(X)$ with boundary $\partial\alpha = x$ which is not homotopy equivalent to x.

Since a 2-simplex with its zeroth vertex x is equivalently a map $\Delta^1 \to \mathcal{C}_{x/}$, this data organizes into the following lifting problem.

$$\begin{tikzcd} \partial\Delta^2 \arrow[r, "{(x,x,\alpha)}"] \arrow[d] & \mathrm{Sing}(X)_{x/} \arrow[d] \\ \Delta^2 \arrow[r, "x"'] \arrow[ur, dashed] & \mathrm{Sing}(X) \end{tikzcd}$$

The solution to this lifting problem would give us a homotopy from α to x. However, by assumption, there is no such homotopy. Thus, x is not initial in $\mathrm{Sing}(X)$. However, every point $x \in X$ is initial in $\mathrm{hSing}(X)$ because $\mathrm{hSing}(X) \cong \pi_{\geq 1}(X)$, and there is exactly one homotopy class of paths between any two points of X because X is simply connected.

We have now shown the following.

- Given two initial objects x, y in $\mathcal{C}$, that we can find an edge $x \to y$.
- Given two such edges, we can find a homotopy betweem them.
- Given two such homotopies, we can find a higher homotopy between them.
- Etc.

Thinking of the set of all initial objects as forming the vertices of some polyhedron, the filling conditions above can intuitively be interpreted as telling us that the polyhedron in question can be filled in with higher data, such that the resulting object has no holes; that is, that it is homotopy equivalent to the point. This intuition turns out to be correct.

Theorem 7.63. Let $\mathcal{C}$ be a simplicial set with at least one initial object, and let $\mathcal{C}'$ be the simplicial subset consisting of all simplices all of whose vertices are initial objects in $\mathcal{C}$. Then $\mathcal{C}'$ is a contractible Kan complex.

Proof. The statment that $\mathcal{C}'$ is contractible means that the map $\mathcal{C}' \to *$ is a weak equivalence. The fact that it is a Kan complex means precisely that the map $\mathcal{C}' \to *$ is a fibration. Therefore, we need to check that the map $\mathcal{C}' \to *$ is a trivial fibration.

In the Kan model structure, trivial fibrations are Kan fibrations which are also homotopy equivalences. However, we have immediately that $\mathcal{W} \cap \mathcal{F}\mathrm{ib} = \mathcal{C}\mathrm{of}_{\perp}$, i.e. trivial Kan fibrations are precisely those morphisms of simplicial sets that have the right lifting property with respect to monomorphisms. However, since the set of monomorphisms is the saturated hull of the set of boundary inclusions, this is equivalent to having the left lifting property with respect to boundary inclusions

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathcal{C}' \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} .$$

For $n = 0$, the lifting problem demands that there exists at least one initial object, which is true by assumption.

For $n \geq 1$, consider an n-boundary $\beta\colon \partial\Delta^n \to \mathcal{C}'$. Denote by x the 0th vertex; by assumption, this is an initial object. By forgetting the 0th face, we can consider this as a boundary inclusion

$$\partial\Delta^{\{1,\dots,n\}} \to \mathcal{C}_{x/}$$

making the following diagram commute.

$$\begin{tikzcd} \partial\Delta^{\{1,\dots,n\}} \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^{\{1,\dots,n\}} \arrow[r, "d_0\beta"'] & \mathcal{C} \end{tikzcd}$$

However, we can find a lift, i.e. a map $\Delta^{\{1,\dots,n\}} \to \mathcal{C}_{x/}$, i.e. an n-simplex in $\mathcal{C}$ which fills β. □

6.6 *Terminal objects*

Definition 7.64 (terminal object)**.** Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$ an object. We say that x is a terminal object if the map

$$\mathcal{C}_{/x} \to \mathcal{C}; \qquad \Delta^n \star \Delta^0 \mapsto \Delta^n$$

is a trivial Kan fibration.

The theory of terminal objects is completely dual to that of initial objects. For that reason, we list only results here.

Let $\mathcal{C}$ be an infinity category.

- If y is a terminal object in $\mathcal{C}$, for each object $x \in \mathcal{C}$ there is an edge $x \to y$, which is unique up to homotopy.

- These homotopies are in turn unique up to homotopy.
- Terminal objects in an ∞-category are always terminal in the homotopy category; however, the converse is not true.
- The simplicial subset of all simplices whose vertices are terminal is a contractible Kan complex.

6.7 Limits and colimits

Definition 7.65 (limit cone)**.** Let $\mathcal{C}$ be an infinity category, K a simplicial set, and $p\colon K \to \mathcal{C}$ a diagram.

- A colimit cone for p is an initial object in the category $\mathcal{C}_{p/}$.
- A limit cone for p is a terminal object in the category $\mathcal{C}_{/p}$.

Suppose that I and $\mathcal{D}$ are regular categories, and let $F\colon I \to \mathcal{D}$ be a functor. Passing to nerves, we get a functor $N(F)$ between infinity categories. Let us try and understand what a limit cone for $N(F)$ means in this context.

First, let us understand the zero-simplices of $N(\mathcal{D}_{/F})$. By definition, zero-simplices are given by maps

$$\sigma\colon \Delta^0 \star N(I) \to N(\mathcal{D}) \qquad \text{such that } \sigma|_{N(I)} = N(F).$$

By the monoidality of the nerve with respect to the join, such a map is the nerve of a functor

$$\tilde{\sigma}\colon \{*\} \star I \to \mathcal{D} \qquad \text{such that } \tilde{\sigma}|_I = F.$$

But this is precisely a natural transformation from the constant functor $\sigma|_{\{*\}} \to \mathcal{D}$ to F.

Similarly, a 1-simplex τ is a map

$$\tau\colon \Delta^1 \star N(I) \to N(\mathcal{D}) \qquad \text{such that } \tau|_{N(I)} = N(F).$$

This is the nerve of a functor

$$\tilde{\tau}\colon \{\bullet \to *\} \star I \to \mathcal{D} \qquad \text{such that } \tilde{\tau}|_I = F$$

Such functors are precisely morphisms of cones.

Corollary 7.66. Limit and colimit cones are unique up to contractible choice.

Proof. □

Theorem 7.67. Let S be a simpicial set, $\mathcal{C}$ an infinity category, and $p\colon K \to C$ a map of simplicial sets.

1. There is an equivalence of categories
$$(\mathcal{C}_{/p})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{p^{\mathrm{op}}/}.$$

2. The limit of p in $\mathcal{C}$ is the colimit of p^{op} in $\mathcal{C}^{\mathrm{op}}$.

Proof.

1. First we show the result on simplices. The n-simplices of $(\mathcal{C}^{\mathrm{op}})_{p^{\mathrm{op}}/}$ are

$$\{f\colon K^{\mathrm{op}} \star \Delta^n \to \mathcal{C}^{\mathrm{op}} \mid f|_{K^{\mathrm{op}}} = p^{\mathrm{op}}\} \cong \{f^{\mathrm{op}}\colon (\Delta^n)^{\mathrm{op}} \star K \to \mathcal{C} | f^{\mathrm{op}}|_K = p\}$$

   But these are precisely the n-simplices of $(\mathcal{C}_{/p})^{\mathrm{op}}$. Thus, the map on n-simplices is given by $f \mapsto f^{\mathrm{op}}$.

   Let $f\colon [m] \to [n]$. We need to check that the following naturality square commutes.

$$\begin{tikzcd}
(\otimes\mathcal{C}^{\mathrm{op}}_{p^{\mathrm{op}}/})_n \arrow[rrr, "\mathrm{op}"] \arrow[ddd] & & & (\mathcal{C}_{/p})^{\mathrm{op}}_n \arrow[ddd] \\
& f \arrow[r, mapsto] \arrow[d, mapsto] & f^{\mathrm{op}} \arrow[d, mapsto] & \\
& f \circ (\mathrm{id}_{K^{\mathrm{op}}} \star \phi) \arrow[r, mapsto] & f^{\mathrm{op}} \circ (\phi^{\mathrm{op}} \star \mathrm{id}_K) & \\
(\otimes\mathcal{C}^{\mathrm{op}}_{p^{\mathrm{op}}/})_m \arrow[rrr, "\mathrm{op}"'] & & & (\mathcal{C}_{/p})^{\mathrm{op}}_m
\end{tikzcd}$$

   But it does.

2. Let x be a colimit of p^{op} in $\mathcal{C}^{\mathrm{op}}$. Then the map

$$\mathcal{C}^{\mathrm{op}}_{x/} \to \mathcal{C}^{\mathrm{op}}$$

   is a trivial Kan fibration. Now consider the map

□

Let $\mathcal{C}$ be a category, and let $p\colon K \to \mathcal{C}$ be a diagram with a colimit $x \in \mathcal{C}_{/p}$. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map of ∞-categories. We say that f preserves the colimit of the diagram p if,

7 *Cofinal maps*

Definition 7.68 (cofinal map)**.** Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and let $p\colon K \to \mathcal{C}$ be a diagram. We say that p is cofinal if, for any right fibration $X \to T$, the induced map

$$\mathrm{Maps}_T(T, X) \to \mathrm{Maps}_T(S, X)$$

is a homotopy equivalence.

8

The fibration zoo

If the reader thought that the theory of Kan fibrations was vibrant, they are in for a treat.

Let $f : K \to S$ be a morphism of simplicial sets.

- We call the morphism f a Kan fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 \le i \le n$.
- We call the morphism f a left fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 \le i < n$.
- We call the morphism f a right fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 < i \le n$.
- We call the morphism f an inner fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 < i < n$.

We will use the following terminology.

- The morphism f is anodyne if it has the left-lifting property with respect to all Kan fibrations.
- The morphism f is left anodyne if it has the left-lifting property with respect to all left fibrations.
- The morphism f is right anodyne if it has the left-lifting property with respect to all right fibrations.
- The morphism f is inner anodyne if it has the left-lifting property with respect to all inner fibrations.

The story, roughly, will go as follows. We have seen that the fibers of a Kan fibration are groupoids. We will see that this is more data than is required the fibers of a left (dually, right) fibration are groupoids. Furthermore, an ∞-categorical version of the Grothendieck construction tells us that the fibers of a left fibration depend functorially on the base.

One might hope that the fibers of an inner fibration would be ∞-cateories, and indeed this is true, as follows easily from the stability of fibrations under pullbacks. However, the fibers do not depend functorially on the base. In order to get such functoriality, one needs to demand that inner fibrations satisfy an extra condition. Such an inner fibration is known as a (co)Cartesian fibration.

Definition 8.1 (Cartesian fibration)**.** Let $f\colon K \to S$ be a morphism of simplicial sets. The morphism f is a Cartesian fibration if the following conditions are satisfied.

- The morphism f is an inner fibration
- For every $h\colon x \to y$ in S and $\tilde{y} \in K_0$ there is a f-Cartesian morphism $\tilde{h}\colon \tilde{x} \to \tilde{y}$ such that $f(\tilde{h}) = h$.
- A f-Cartesian edge in K is an edge $f\colon x \to y$ such that the map

  $$K_{/f} \to K_{/y} \underset{S_{/f(y)}}{\times} S_{/p(f)}$$

  is a trivial Kan fibration.

Proposition 8.2. Let $p\colon K \to S$ be an inner fibration. A morphism $f\colon x \to y$ in K is p-Cartesian if and only if for every $n \geq 2$ and every solid commutative diagram

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[d, hook] \arrow[dr, "f"] & \\
\Lambda^n_n \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a dashed lift.

Proof. We show that the conditions are equivalent. An edge $f\colon x \to y$ is cartesian if and only if we can find lifts of the form.

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & K_{/f} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & K_{/y} \underset{S_{/f(y)}}{\times} S_{/p(f)}
\end{tikzcd}$$

Using the adjunction Theorem 7.56 and the reasoning of Lemma 3.25, we find that this is equivalent to the following lifting problem.

$$\begin{tikzcd}
\partial\Delta^n \star \Delta^1 \coprod_{\partial\Delta^n \star \Delta^{\{1\}}} \Delta^n \star \Delta^{\{1\}} \arrow[r] \arrow[d, hook] & K \arrow[d] \\
\Delta^n \star \Delta^1 \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where the component $\Delta^1 \to K$ is f. But, working out which simplices in Δ^{n+2} we have, we find that this is precisely what we needed to show. $\square$

Theorem 8.3. Let f be a trivial fibration. Then f is a Kan fibration.

Proof. Let f be a trivial fibration, and consider the following lifting problem.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

Restricting to the face opposite i, we find the lifting problem

$$\begin{tikzcd} \partial\Delta^{\{1,\ldots,n\}} \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^{\{1,\ldots,n\}} \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

which has a dashed solution. This allows us to turn our map $\Lambda^n_i \to K$ to a map $\partial\Delta^n \to K$ making the folowing diagram commute.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

which has a dashed lift. □

Note 8.4. A map $f\colon X \to Y$ is a left fibration if and only if $f^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is a a right fibration. Thus, it suffices to study left fibrations; the theory of right fibrations is dual.

Lemma 8.5. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a left fibration of infinity categories, and let $\tilde{\bar{f}}\colon \bar{x} \to p(y)$ be an equivalence in $\mathcal{D}$, and $y \in \mathcal{C}_0$. Then there exists $f\colon x \to y$ so that $p(f) = \bar{f}$.

Lemma 8.6. Consider morphisms of simplicial sets as follows

$$\bar{K} \hookrightarrow K \xrightarrow{p} X \xrightarrow{q} S$$

where q is an inner fibration. Let $r = p \circ q$, $\bar{p} = p|_{\bar{K}}$, $\bar{r} = r|_{\bar{K}}$. Then the induced map

$$S_{p/} \to X_{\bar{p}/} \underset{S_{\bar{p}/}}{\times} S_{r/}$$

is an inner fibration. If q is a left fibration, then the map is a left fibration.

Corollary 8.7. Let $\mathcal{C}$ be an infinity category, and let $p\colon K \to \mathcal{C}$ be a diagram in $\mathcal{C}$. Then the map $\mathcal{C}_{/p} \to \mathcal{C}$ is a left fibration, and $\mathcal{C}_{/p}$ is an infinity category.

Proof. Take $X = \mathcal{C}$, $S = \Delta^0$, and $\bar{K} = \emptyset$. This tells us that the map

$$\mathcal{C}_{p/} \to \mathcal{C}$$

is a left fibration. Thus, the composition

$$\mathcal{C}_{p/} \to \mathcal{C} \to *$$

is the composition of a left fibration and an inner fibration, so is itself an inner fibration. □

Corollary 8.8. Let $f\colon x \to y$ be a morphism in an ∞-category $\mathcal{C}$. Then f is an equivalence if and only if for every $n \geq 2$ and left horn $h\colon \Lambda^n_0 \to \mathcal{C}$, with $h|_{\{0,1\}} = f$, h admits a filler.

Theorem 8.9. The following are equivalent for an ∞-category $\mathcal{C}$.

- The homotopy category $\mathrm{h}\mathcal{C}$ is a groupoid.
- Every left horn in $\mathcal{C}$ has a filler.
- Every right horn has a filler.
- $\mathcal{C}$ is a Kan complex.

Corollary 8.10. All fibers of a left fibration are Kan complexes.

1 *The classical Grothendieck construction*

Definition 8.11 (Grothendieck construction)**.** Let $\mathcal{D}$ be a small category, and let $\chi\colon \mathcal{D} \to \mathrm{Grpd}$. The Grothendieck construction consists of the following data the following data.

- A category $\mathcal{C}_\chi$
- A functor $\pi\colon \mathcal{C}_\chi \to \mathcal{D}$

  The category $\mathcal{C}_\chi$ is as follows.

- **Objects:** Pairs (d, x), where $d \in \mathcal{D}$ and $x \in \chi(d)$.
- **Morphisms:** A morphism $(d, x) \to (d', x')$ consists of
  - A morphism $f\colon d \to d'$ in $\mathcal{D}$
  - An isomorphism $\alpha\colon \chi(f)(x) \cong x'$

  The functor π is the forgetful functor to $\mathcal{D}$.

The Grothendieck construction can be understood in the following way. The functor χ gives us a way of associating a groupoid to each object $d \in \mathcal{D}$. Each of these groupoids is itself a category, and has its own objects. The idea of the Grothendieck construction is to throw all of these objects together into one big category. To keep track of where each object x comes from, we think of x as ordered pairs (d, x), where the $x \in \chi(d)$. The functor π simply remembers where each x came from.

For two objects x and $\tilde{x}$ from the same $\chi(d)$, the morphisms $(d, x) \to (d, \tilde{x})$ are simply the morphisms between x and $\tilde{x}$. For an object x' from a different groupoid $\chi(d')$, the morphisms $(d, x) \to (d', x')$ come from translating, using χ and a morphism $d \to d'$, morphisms in $\chi(d)$ into morphisms in $\chi(d')$.

Example 8.12. Let G and H be a groups, and $\mathrm{B}G$ and $\mathrm{B}H$ their delooping groupoids. Let $\phi\colon G \to H$ be a group homomorphism, or equivalently a functor $\mathrm{B}G \to \mathrm{B}H$. This can also be taken to be a functor $\mathrm{B}G \to \mathrm{Grpd}$.

The Grothendieck construction consists of a category with one object, whose morphisms consist of an element $g \in G$

There is also a contravariant version of the Grothendieck construction, consisting of the following.

Definition 8.13 (contravariant Grothendieck construction)**.** For any functor $\mathcal{D}^{\mathrm{op}} \to \mathrm{Grpd}$, the contravariant Grothendieck construction consists of the following.

1. A category $\mathcal{C}^{\chi}$
2. A functor $\pi\colon \mathcal{C}^{\chi} \to \mathcal{D}$.

The category $\mathcal{C}^{\chi}$ is as follows.

- The objects of $\mathcal{C}^{\chi}$ are be pairs (d, x), with $d \in \mathcal{D}$ and $x \in \chi(d)$.
- The morphisms $(d', x') \to (d, x)$ are pairs (f, α), where $f\colon d' \to d$ and $\alpha\colon x' \to F(f)(x)$.

There is the following relationship between the covariant and contravariant Grothendieck constructions. $\mathcal{C}^{\chi} = \mathcal{C}_{\chi}^{\mathrm{op}}$.

Example 8.14. Let $X\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}$ be a diagram of sets, which we interpret as discrete groupoids. Then there is an equivalence of categories between the contravariant construction and the slice category $\mathcal{D}/X$.

The objects of $\mathcal{D}/X$ are pairs (d, α), where $d \in \mathcal{D}$ and $\alpha\colon \mathrm{Hom}_{\mathcal{D}}(-, d) \Rightarrow X$. The morphisms $(d, \alpha) \to (d', \beta)$ are morphisms $f\colon d \to d'$ such that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(-,d) \arrow[rr, Rightarrow, "f\circ -"] \arrow[dr, Rightarrow, "\alpha"'] & & \mathrm{Hom}_{\mathcal{D}}(-,d') \arrow[dl, Rightarrow, "\beta"] \\ & X & \end{tikzcd} \tag{8.1}$$

commutes.

Consider any such morphism f. The above diagram commutes if and only if it commutes component-wise, i.e. if for each c, the diagram below commutes.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(c,d) \arrow[rr, "f\circ -"] \arrow[dr, "\alpha_c"'] & & \mathrm{Hom}_{\mathcal{D}}(c,d') \arrow[dl, "\beta_c"] \\ & X(c) & \end{tikzcd} \tag{8.2}$$

In particular, it commutes when $c = d$. Following the identity id_d around, we find that morphisms f making the above diagram commute satisfy the following equality.

$$\begin{tikzcd}
\mathrm{id}_d \arrow[rr, mapsto] \arrow[dr, mapsto] & & f \arrow[dl, mapsto] \\
& \alpha_d(\mathrm{id}_d) = \beta_d(f) &
\end{tikzcd}$$

By the naturality of the Yoneda embedding, we have that

$$\beta_d(f) = X(f)(\beta_{d'}(\mathrm{id}_{d'})),$$

so

$$\alpha_d(\mathrm{id}_d) = X(f)(\beta_{d'}(\mathrm{id}_{d'})). \tag{8.3}$$

We can specify a functor $\mathcal{D}/X \to \mathcal{C}^X$ on objects by sending

$$(d, \alpha) \mapsto (d, \alpha_b(\mathrm{id}_b)),$$

and on morphisms $f\colon (d, \alpha) \to (d', \beta)$ commute by sending

$$f \mapsto (f, \mathrm{id}_{X(f)(\beta_{d'}(\mathrm{id}_{d'}))}).$$

We are allowed to use the above identity map preciesly because it is an isomorphism

$$\alpha_d(\mathrm{id}_d) \to X(f)(\beta_{d'}(d'))$$

as shown in Equation 8.3.

The fully faithfulness of the Yoneda embedding implies that the map between the objects of $\mathcal{D}/X$ and the objects of $\mathcal{C}^X$ is a bijection. We will now construct a functor $\mathcal{C}^X \to \mathcal{D}/X$, defined on objects to be the inverse of the one defined above, i.e. sending a pair (d, x) to the pair (d, α^x), where α^x is the natural transformation uniquely specified by $\alpha^x_d(\mathrm{id}_d) \mapsto x \in X(d)$.

Our inverse functor acts on any morphism (f, ϕ) by forgetting ϕ. It only remains to check that this provides a well-defined map of morphisms, as f has to make Diagram 8.1 commute. We can check that this is so by following a general morphism $g\colon c \to d$ around Diagram 8.2, finding the following condition.

$$\begin{tikzcd}
g \arrow[rr, mapsto] \arrow[dr, mapsto] & & f \circ g \arrow[dl, mapsto] \\
& \alpha_c(g) \overset{!}{=} \beta_c(f \circ g) &
\end{tikzcd}$$

By naturality of the Yoneda embedding, we have

$$\alpha_c(g) = (Xg)(\alpha_d(\mathrm{id}_d)), \qquad \beta_c(f \circ g) = X(f \circ g)(\beta_{d'}(\mathrm{id}_{d'})) = (Xg \circ Xf)(\beta_{d'}(\mathrm{id}_{d'})),$$

so our condition equivalently reads

$$Xg(\alpha_d(\mathrm{id}_d)) = Xg((Xf)(\beta_{d'}(\mathrm{id}_{d'}))).$$

This is true because by definition of f,

$$\alpha_d(\mathrm{id}_d) = (Xf)(\beta_{d'}(\mathrm{id}_{d'})).$$

The two functors defined in this way are manifestly each other's inverses, hence provide an isomorphism of categories, which is stronger than what we are looking for.

1.1 When can we invert the classical Grothendieck construction?

Suppose we are given a functor $F\colon \mathcal{C} \to \mathcal{D}$ exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. By Lemma 8.28, each of the fibers $\mathcal{C}_d$ is a groupoid. We may hope that we could reverse the Grothendieck construction by defining a functor $\mathcal{D} \to \mathrm{Grpd}$ defined on objects by $d \mapsto \mathcal{C}_d$.

We almost can. For any morphism $f\colon d \to d'$ in $\mathcal{D}$, we can build functor $f_!\colon \mathcal{C}_d \to \mathcal{C}_{d'}$ as follows.

Let $f\colon d \to d'$, and let $c \in \mathcal{C}_d$. By property 2 in Definition 8.18, we can find an object $c' \in \mathcal{C}_{d'}$ and a morphism $\bar{f}\colon c \to c'$ in $\mathcal{C}$ such that $F(\bar{f}) = f$. Define $f_!(c) = c'$, and $f_!()$

The above discussion means that the Grothendieck construction defines the following correspondence.

$$\left\{ \begin{matrix} F\colon \mathcal{C} \to \mathcal{D} \text{ exhibits } \mathcal{C} \\ \text{as cofibered in groupoids} \end{matrix} \right\} \xleftrightarrow{\simeq} \left\{ \begin{matrix} \text{pseudofunctors} \\ \mathcal{D} \to \mathrm{Grpd} \end{matrix} \right\}$$

1.2 Fibrance and cofibrance in groupoids

Example 8.15. The Grothendieck construction

$$\pi\colon \mathcal{C}_\chi \to \mathcal{D}$$

exhibits $\mathcal{D}$ as cofibered over groupoids. We simply transcribe the definition, making the necessary changes as we go along.

In ordered for π to exhibit $\mathcal{C}_\chi$ as cofibered in groupoids over $\mathcal{D}$, we have to check the following.

1. For every $(d, x) \in \mathcal{C}_\chi$ and every morphism

   $$\eta\colon d \to d' \qquad \text{in } \mathcal{D}$$

   there exists a lift

   $$\tilde{\eta} = (f, \alpha)\colon (d, x) \to (d', x') \qquad \text{in } \mathcal{C}_\chi$$

   such that

   $$\pi(\tilde{\eta}) = f\colon d \to d' = \eta.$$

2. For every morphism $\eta\colon (x, d) \to (x', d')$ in $\mathcal{C}$ and every object $(x'', d'') \in \mathcal{C}$, the map

   $$\mathcal{C}_\chi((d', x') \to (d'', x'')) \to \mathcal{C}_\chi((d, x), (d'', x'')) \underset{\mathcal{D}(d, d'')}{\times} \mathcal{D}(d', d'')$$

   is a bijection.

What do these mean?

1. We need to construct from our map $\eta\colon d \longrightarrow d'$ an isomorphism
$$\alpha\colon \chi(f)(x) \cong x'$$
for some x'. We can pick $x' = \chi(f)(x)$. Then $\tilde{\eta} = (f, \alpha)$ does the job. Indeed, $\pi(f, \alpha) = f$ as required.

2. Draw a picture. Given the data of
   - A morphism

Example 8.16. The contravariant Grothendieck construction exhibits $\mathcal{C}^{\chi} \longrightarrow \mathcal{D}$ as fibered in groupoids over $\mathcal{D}$.

To see this, we need to make the following checks.

- **Check:** For every $(d, x) \in \mathcal{C}^{\chi}$, and every morphism $\eta\colon d' \longrightarrow d$ in $\mathcal{D}$, we should be able to find some object $x' \in \chi(d')$ and a morphism $(f, \alpha)\colon (d', x') \longrightarrow (d, x)$.

  Solution: We can pick $f = \eta$, $x' = \chi(f)(x)$, and $\alpha = \mathrm{id}_{x'}$.

- **Check:** Fixing any diagram
$$\begin{tikzcd} & (d',x') \arrow[d, "(g,\alpha)"] \\ (d'',x'') \arrow[r, "(\eta,\beta)"'] & (d,x) \end{tikzcd}$$
in $\mathcal{C}^{\chi}$, for any morphism $f\colon d'' \longrightarrow d'$ making the below left diagram commute, we should be able to find a morphism $(f, \gamma)\colon (d'', x'') \longrightarrow (d', x')$ making the below right diagram commute.
$$\begin{tikzcd} & d' \arrow[dr, "g"] & \\ d'' \arrow[ur, dashed, "f"] \arrow[rr, "\eta"'] & & d \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & (d',x') \arrow[dr, "(g,\alpha)"] & \\ (d'',x'') \arrow[ur, dashed, "(f,\gamma)"] \arrow[rr, "(\eta,\beta)"'] & & (d,x) \end{tikzcd}$$

  Solution: We need a morphism $\gamma\colon x'' \longrightarrow \chi(f)(x')$ making the above diagram commute.

  Using the data available to us, we can create the following diagram in $\chi(d'')$.
$$\begin{tikzcd} & \chi(f)(x') \arrow[d, "\chi(f)(\alpha)"] \\ x'' \arrow[r, "\beta"'] & \chi(\eta)(x) \end{tikzcd}$$
  Note that we are entitled to draw $\chi(f)(\alpha)$ terminating at $\chi(\eta)(x)$ because
$$\chi(f)(\chi(g)(x)) = \chi(g \circ f)(x) = \chi(\eta)(x).$$
  Because $\chi(d'')$ is a groupoid, we get the morphism γ via
$$\gamma = (\chi(f)(a))^{-1} \circ \beta.$$

2 Left fibrations

Lemma 8.17. Consider simplicial sets and morphisms as follows.

$$\begin{tikzcd} K_0 \arrow[r, hook] \arrow[rrr, bend left, "r_0"] \arrow[rr, bend right, "p_0"'] & K \arrow[r, "p"] \arrow[rr, bend left, "r"] & X \arrow[r, "q"] & S \end{tikzcd}$$

We can translate a lifting problem of the first type into a lifting problem of the second.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X_{p/} \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & X_{p_0/} \times_{S_{r_0/}} S_{r/} \end{tikzcd} \qquad \begin{tikzcd} K \star A \coprod_{K_0 \star A} K_0 \star B \arrow[r] \arrow[d] & X \arrow[d] \\ K \star B \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

3 Left fibrations and cofibrancy in groupoids

Since left fibrations are of the form

$$\{\Lambda_i^n \hookrightarrow \Delta^n \mid n \geq 1, 0 \leq i < n\}_\perp,$$

Proposition 3.9 implies that they are closed under pullback.

Let $F\colon X \to Y$ be a left fibration of simplicial sets. For $x \in X$, the fiber

3.1 Cofibrancy in groupoids

Definition 8.18 (cofibrant in groupoids)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that f exhibits $\mathcal{C}$ as cofibrant in groupoids over $\mathcal{D}$ if the following conditions are satisfied.

1. For every $c \in \mathcal{C}$ and every morphism

$$\eta\colon F(c) \to d \qquad \text{in } \mathcal{D}$$

there exists a morphism

$$\tilde{\eta}\colon c \to \tilde{d} \qquad \text{in } \mathcal{C}$$

such that $F(\tilde{\eta}) = \eta$.

2. For every morphism $\eta\colon c \to c'$ in $\mathcal{C}$ and every object $c'' \in \mathcal{C}$, the map

$$\mathcal{C}(c', c'') \to \mathcal{C}(c, c'') \underset{\mathcal{D}(F(c), F(c''))}{\times} \mathcal{D}(F(c'), F(c'')).$$

is a bijection.

Let us unravel what the second condition means. The second set is the following pullback.

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \mathcal{D}(F(c'), F(c'')) \arrow[d, "\eta"] \\ \mathcal{C}(c, c'') \arrow[r, "F"'] & \mathcal{D}(F(c), F(c'')) \end{tikzcd}$$

Its elements are pairs

$$(f, \bar{g}), \qquad \text{where} \quad f\colon c \to c'' \quad \text{and} \quad \bar{g}\colon F(c') \to F(c'')$$

such that

$$F(f) = \bar{g} \circ F(\eta).$$

The bijection above is defined as follows. Starting on the LHS with a morphism $h\colon c' \to c''$, build the following commutative diagram.

$$\begin{tikzcd} & c' \arrow[dr, "h"] & \\ c \arrow[ur, "\eta"] \arrow[rr, "h \circ \eta"'] & & c'' \end{tikzcd}$$

We can read off a map $f = h \circ \eta\colon c \to c''$ and a map $\bar{g} = F(h)\colon F(c') \to F(c'')$, and indeed $F(f) = \bar{g} \circ F(\eta)$, giving us an element of the RHS.

The fact that this is a bijection means that the above operation is invertible. Fixing a morphism $\eta\colon c \to c'$ and an object c'', whenever we have the following data:

- A morphism $g\colon c \to c''$ as follows

$$\begin{tikzcd} & c' & \\ c \arrow[ur, "\eta"] \arrow[rr, "\bar{g}"'] & & c'' \end{tikzcd}$$

- A map $\bar{h}\colon F(c') \to F(c'')$ making the following diagram commute

$$\begin{tikzcd} & F(c') \arrow[dr, "\bar{h}"] & \\ F(c) \arrow[ur, "F(\eta)"] \arrow[rr, "F(g)"'] & & F(c'') \end{tikzcd}$$

we can lift $\bar{h}$ to a morphism h in $\mathcal{C}$ making the following diagram commute.

$$\begin{tikzcd} & c' \arrow[dr, "h"] & \\ c \arrow[ur, "\eta"] \arrow[rr, "g"'] & & c'' \end{tikzcd}$$

Lemma 8.19. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Let f be a morphism in $\mathcal{C}$ such that $F(f) = \mathrm{id}$. Then f is an isomorphism.

Proof. Let $f\colon c \to c'$ in $\mathcal{C}_d$ be a morphism such that $F(f) = \mathrm{id}_d$. Then we can form the following commutative diagram.

$$
\begin{tikzcd}
 & c' & \\
c \arrow[ur, "f"] \arrow[rr, "\mathrm{id}"'] & & c
\end{tikzcd}
$$

When mapped to $\mathcal{D}$, we find the following very simple diagram, with a dashed filler.

$$
\begin{tikzcd}
 & d \arrow[dr, dashed, "\mathrm{id}"] & \\
d \arrow[ur, "\mathrm{id}"] \arrow[rr, "\mathrm{id}"'] & & d
\end{tikzcd}
$$

Lifting this gives us a left inverse $\tilde{f}$ to f. Pulling the same trick with $\tilde{f}$ gives us a left inverse $\hat{f}$ to $\tilde{f}$. Then

$$
\begin{aligned}
f \circ \tilde{f} &= \mathrm{id} \\
\hat{f} \circ f \circ \tilde{f} &= \hat{f} \\
\hat{f} &= f,
\end{aligned}
$$

so $\tilde{f}$ is both a left and a right inverse for f, i.e. f is an isomorphism. □

Corollary 8.20. Let $F\colon \mathcal{C} \to \mathcal{D}$ exhibit $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Then the fibers of F are groupoids.

Proof. The morphisms in the fiber $\mathcal{C}_d$ are precisely those which are mapped to the identity id_d. □

Lemma 8.21. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Let f be a morphism in $\mathcal{C}$ such that $F(f)$ is an equivalence. Then f is an equivalence.

Definition 8.22 (coCartesian morphism)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of small categories. A morphism $f\colon c \to c'$ is said to be <u>F-coCartesian</u> if, for every $c'' \in \mathcal{C}$, the map

$$\mathcal{C}(c', c'') \to \mathcal{C}(c, c'') \times_{\mathcal{D}(F(c), F(c''))} \mathcal{D}(F(c'), F(c''))$$

is a bijection.

Definition 8.23 (coCartesian fibration)**.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be a <u>coCartesian fibration</u> if, for each $x \in \mathcal{C}$ and every morphism $\tilde{f}\colon F(x) \to y$ in $\mathcal{D}$, there exists a coCartesian morphism $f\colon x \to \tilde{y}$ such that $F(f) = \tilde{f}$.

The reader has probably guessed the next sentence: “There is a dual notion.”

Definition 8.24 (fibrant in groupoids)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that f <u>exhibits $\mathcal{C}$ as fibrant in groupoids over $\mathcal{D}$</u> if the following conditions are satisfied.

1. For every $c' \in \mathcal{C}$ and every morphism

$$\eta \colon d \to F(c') \qquad \text{in } \mathcal{D}$$

   there exists a morphism

$$\tilde{\eta} \colon \tilde{d} \to c' \qquad \text{in } \mathcal{C}$$

   such that $F(\tilde{\eta}) = \eta$.

2. For every morphism $\eta \colon c \to c'$ in $\mathcal{C}$ and every object $c'' \in \mathcal{C}$, the map

$$\mathcal{C}(c', c'') \to \mathcal{C}(c, c'') \underset{\mathcal{D}(F(c), F(c''))}{\times} \mathcal{D}(F(c'), F(c'')).$$

   is a bijection.

4 *The starred smash*

Definition 8.25 (starred smash)**.** Let $f \colon A \to A'$, $g \colon B \to B'$ be two morphisms of simplicial sets. Define a morphism $f \overset{\star}{\wedge} g$ using the universal property for pushouts as follows.

$$\begin{tikzcd}
A \star B \arrow[r, "f \star \mathrm{id}"] \arrow[d, "\mathrm{id} \star g"'] & A' \star B \arrow[d, hook] \arrow[ddr, bend left, "\mathrm{id} \star g"] & \\
A \star B' \arrow[r, hook] \arrow[drr, bend right, "f \star \mathrm{id}"'] & A' \star B \coprod_{A \star B} A \star B' \arrow[dr, dashed, "f \overset{\star}{\wedge} g"] & \\
& & A' \star B'
\end{tikzcd}$$

Proposition 8.26. We have the following facts.

1. If f is right anodyne, then $f \overset{\star}{\wedge} g$ and $f \wedge g$ are inner anodyne.
2. If f is left anodyne, then $f \overset{\star}{\wedge} g$ and $f \wedge g$ are left anodyne.
3. If f is anodyne, then $f \overset{\star}{\wedge} g$ is a Kan fibration.

Proof. Consider the set

$$\mathcal{S} = \{f \mid f \overset{\star}{\wedge} g \text{ is inner anodyne}\}.$$

That is, f consists of precisely those maps such that the lifting

$$\begin{tikzcd}
A_0 \star B \coprod_{A_0 \star B_0} A \star B_0 \arrow[r] \arrow[d, "f \overset{\star}{\wedge} g"'] & X \arrow[d, "\text{inner fibration}"] \\
A \star B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

has a solution

By a result analogous to Odious Lemma 3.25, we have immediately that such a lifting problem is equivalent to the following.

□

We have the following 'perpendicular statements' (whatever that means).

Fact 8.27. Consider morphisms of simplicial sets as follows

$$\bar{K} \hookrightarrow K \xrightarrow{p} X \xrightarrow{q} S$$

where q is an inner fibration. Let $r = p \circ q$, $\bar{p} = p|_{\bar{K}}$, $\bar{r} = r|_{\bar{K}}$. Denote the

$$S_{/p} \to X_{/\bar{p}} \underset{S_{/\bar{p}}}{\times} S_{/r}.$$

4.1 Cofibration in groupoids

We would like to find the correct ∞-categorical notion of being cofibered in groupoids.

Lemma 8.28. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Then for each $d \in \mathcal{D}$, the fiber $\mathcal{C}_d$ defined by the pullback below is a groupoid.

$$\begin{tikzcd} \mathcal{C}_d \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ \{d\} \arrow[r, hook] & \mathcal{D} \end{tikzcd}$$

Proof. Every morphism in $\mathcal{C}_d$ is by definition mapped to id_d, so by □

Theorem 8.29. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F exhibits $\mathcal{C}$ as cofibered in groupoids over D if and only if $N(F)\colon N(\mathcal{C}) \to N(\mathcal{D})$ is a left fibration.

Proof. Suppose F is a left fibration. We examine some horn filling conditions in small degrees. First, consider a horn filling

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d] & N(\mathcal{C}) \arrow[d] \\ \Delta^1 \arrow[r] & N(\mathcal{D}) \end{tikzcd}$$

The commutativity conditions tell us that we have the following solid diagrams in $\mathcal{C}$ and $\mathcal{D}$ respectively.

$$x \qquad \xrightarrow{F} \qquad F(x) \xrightarrow{\bar{f}} \bar{y}$$

A dashed lift

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d] & N(\mathcal{C}) \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & N(\mathcal{D}) \end{tikzcd}$$

gives us the following dashed data.

$$x \overset{f}{\dashrightarrow} y \qquad \xrightarrow{F} \qquad F(x) \xrightarrow{\bar{f}} \bar{y}$$

Thus, the first condition of Definition 8.18 is satisfied.

Similarly, a horn filling problem $\Lambda^2_0 \to \Delta^2$ gives us the following solid data, while a lift gives us the dashed data.

$$\begin{tikzcd} & y \arrow[dr, dashed, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \qquad \begin{tikzcd} & F(y) \arrow[dr, "\bar{g}"] & \\ F(x) \arrow[ur, "F(f)"] \arrow[rr, "F(h)"'] & & F(z) \end{tikzcd}$$

This says that there exists a lift as required, but not that it is unique.

Now consider two possible lifts $\bar{g}$ and $\bar{g}'$, arranged into a $(3, 0)$-horn as follows.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, "h"] & y \arrow[d, "g"] \\ y \arrow[r, "g'"'] & z \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "f"'] & y \arrow[d, "g"] \\ y \arrow[r, "g'"'] \arrow[ur, "\mathrm{id}_y"] & z \end{tikzcd}$$

These correspond to a 3-simplex in $\mathcal{D}$ as follows.

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "F(f)"'] \arrow[dr, "F(h)"] & F(y) \arrow[d, "\bar{g}'"] \\ F(y) \arrow[r, "\bar{g}"'] & F(z) \end{tikzcd} \qquad \begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "F(f)"'] & F(y) \arrow[d, "\bar{g}'"] \\ F(y) \arrow[r, "\bar{g}"'] \arrow[ur, "\mathrm{id}_{F(y)}"] & F(z) \end{tikzcd}$$

The lifting properties say that we can fill the above horn, telling us that $g = g'$, i.e. that the lift is unique. □

5 *The infinity-categorical Grothendieck construction*

Definition 8.30 (fiber over a point)**.** Let $f : X \to S$ be a map of simplicial sets, and let $s \in S$. The fiber over s, denoted X_s is the following pullback.

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{s\} \arrow[r, hook] & S \end{tikzcd}$$

Theorem 8.31. Let $p : X \to S$ be a left fibration of simplicial sets. Then the assignment

$$s \mapsto X_s$$

extends to a functor $\mathrm{h}S \to \mathrm{h}\mathcal{S}$, where $\mathcal{S}$ is the category of spaces (Definition 7.36).

Proof. By Corollary 8.10, each fiber X_s is a Kan complex.

Now, let $f\colon s \to s'$ be any morphism in S. Consider the following lifting problem.

$$\begin{tikzcd}
\{0\} \times X_s \arrow[rr, hook] \arrow[d, hook] & & X \arrow[d, "p"] \\
\Delta^1 \times X_s \arrow[r] \arrow[urr, dashed, "Lf"] & \Delta^1 \arrow[r, "f"] & S
\end{tikzcd}$$

The left-hand morphism is left anodyne because it is the smash product

$$(\{0\} \hookrightarrow \Delta^1) \overset{\star}{\wedge} (\emptyset \hookrightarrow X_s),$$

so we can find a dashed lift. This gives us a map $Lf\colon \Delta^1 \times X_s \to X$ such that $Lf_{\{0\}} = \mathrm{id}_{X_s}$. Restricting to $\{1\} \times X_s$, we get a map $Lf_{\{1\}\times X_s}\colon \{1\} \times X_s \to X_{s'}$, giving us a map

$$\tilde{f}\colon X_s \to X_{s'}.$$

The functor $\mathrm{h}S \to \mathrm{h}\mathcal{S}$ sends $[f] \mapsto [\tilde{f}]$. It remains to check that this is well-defined, i.e. respects $\square$

6 *More model structures*

Let S be a simplicial set.

- Denote by $(\mathrm{Set}_\Delta)_{/S}$ the category of simplicial sets over S.
- Denote by $\mathrm{Set}_\Delta{}^{\mathfrak{C}(S)}$ the category of functors of simplicial categories from $\mathfrak{C}(S) \to \mathrm{Set}_\Delta$.

Theorem 8.32. There is a Quillen equivalence

$$(\mathrm{Set}_\Delta)_{/S} \leftrightarrow \mathrm{Set}_\Delta{}^{\mathfrak{C}(S)}$$

where $(\mathrm{Set}_\Delta)_{/S}$ is taken to have the covariant model structure, and $\mathrm{Set}_\Delta{}^{\mathfrak{C}(S)}$ is taken to have the projective model structure.

7 *Some correspondences*

$$\left\{\begin{matrix}\text{functors } \mathcal{C}^{\mathcal{X}} \to \mathcal{D} \\ \text{fibered in groupoids}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{pseudofunctors} \\ \mathcal{D} \to \mathrm{Grpd}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{functors } \mathcal{C} \to \mathcal{D} \\ \text{cofibered in groupoids}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{coCartesian fibrations} \\ \mathcal{C}^{\mathcal{X}} \to \mathcal{D}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{pseudofunctors} \\ \mathcal{D} \to \mathrm{Grpd}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{coCartesian fibrations} \\ \mathcal{C}^{\mathcal{X}} \to \mathcal{D}\end{matrix}\right\}$$

Appendices

A

Category-theoretic preliminaries

The theory of infinity categories rests on the shoulders of the theory of ordinary category theory. In this appendix, we recall some category-theoretic results that we will use. In general, we provide no proofs for these results; they are standard, and proofs can be found in any introductory text on category theory.

1 Limits and colimits

1.1 Formulae for limits

One can calculate limits using products and equalizers, and colimits using coproducts and coequalizers.

Fact A.1. Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be locally small. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

- If $\mathcal{D}$ has small products and equalizers, then $\lim F$ is given by the following equalizer,

$$\lim F \hookrightarrow \prod_{c \in \mathcal{C}} F(c) \overset{R}{\underset{S}{\rightrightarrows}} \prod_{\alpha \in \mathrm{Mor}(\mathcal{C})} F(\mathrm{codom}(\alpha))$$

  where the morphisms R and S are defined by their components

$$R_\alpha = \pi_{F(\mathrm{codom}(\alpha))} \colon \prod_{c \in \mathcal{C}} F(c) \to F(\mathrm{codom}(\alpha))$$

  and

$$S_\alpha = F(\alpha) \circ \pi_{\mathrm{dom}(\alpha)} \colon \prod_{c \in \mathcal{C}} F(c) \to F(\mathrm{codom}(\alpha))$$

- If $\mathcal{D}$ has small coproducts and coequalizers, then $\operatorname{colim} F$ is given by the following coequalizer,

$$\coprod_{\alpha \in \mathrm{Mor}(\mathcal{C})} F(\mathrm{dom}(\alpha)) \overset{A}{\underset{B}{\rightrightarrows}} \coprod_{c \in \mathcal{C}} F(c) \twoheadrightarrow \operatorname{colim} F$$

where the morphisms A and B are defined by their components

$$A_\alpha = \iota_{F(\mathrm{dom}(\alpha))} : F(\mathrm{dom}(\alpha)) \to \coprod_{c \in \mathcal{C}} F(c)$$

and

$$B_\alpha = \iota_{\mathrm{codom}(\alpha)} \circ F(\alpha) \colon F(\mathrm{dom}(\alpha)) \to \coprod_{c \in \mathcal{C}} F(c)$$

1.2 *Limits in functor categories*

In this section, we review facts about limits and colimits in functor categories. Much of this text is devoted to one particular presheaf category Set_Δ, the category of functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$,[1] and it will pay to understand categories of this form.

[1] Where Δ is the *simplex category*, Definition 2.2.

Fact A.2. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

- If $\mathcal{C}$ has an initial object i, then $\lim F = F(i)$.
- If $\mathcal{C}$ has a terminal object i, then $\operatorname{colim} F = F(i)$.

Theorem A.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, with $\mathcal{D}$ small and $\mathcal{C}$ locally small.

- If $\mathcal{D}$ is complete, then so is $[\mathcal{C}, \mathcal{D}]$.
- If $\mathcal{D}$ is cocomplete, then so is $[\mathcal{C}, \mathcal{D}]$.

Corollary A.4. Limits and colimits in functor categories are computed pointwise. That is, for any functor $F \colon J \to [\mathcal{C}, \mathcal{D}]$ and any $j \in J$,

$$(\lim F)(j) = \lim(Fj),$$

where the first limit is taken in $[\mathcal{C}, \mathcal{D}]$, and the second is taken in $\mathcal{D}$.

Corollary A.5. A morphism $\eta \colon F \to G$ in a functor category is a monomorphism (resp. epimorphism) if and only if each component η_x is a monomorphism (resp. epimorphism).

Proof. A morphism $\eta \colon F \to G$ is a monomorphism if and only if the square

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & F \arrow[d, "\eta"] \\ F \arrow[r, "\eta"'] & G \end{tikzcd}$$

is a pullback square. But by Corollary A.4, the above square is a pullback square if and only if its components are pullback squares, which in turn is true if and only if each component is a monomorphism.

The epimorphism case is dual. □

2 *Kan extensions*

2.1 *Pointwise formulae for Kan extensions*

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let $d \in \mathcal{D}$ be an object.

- We will denote by $\mathcal{C}_{/d}$ the category whose objects are pairs (c, f), where $c \in \mathcal{C}$ and $f\colon F(c) \to d$, and whose morphisms $(c, f) \to (c', f')$ are morphisms $h\colon c \to c'$ such that the triangle

$$\begin{tikzcd} F(c) \arrow[rr, "F(h)"] \arrow[dr, "f"'] & & F(c') \arrow[dl, "f'"] \\ & d & \end{tikzcd}$$

  commutes. Note that there is an obvious forgetful functor $\mathcal{C}_{/d} \to \mathcal{C}$.

- We will denote by $\mathcal{C}_{d/}$ the category whose objects are pairs (c, f), where $c \in \mathcal{C}$ and $f\colon d \to F(c)$, and whose morphisms $(c, f) \to (c', f')$ are morphisms $h\colon c \to c'$ such that the triangle

$$\begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F(c) \arrow[rr, "f(h)"] & & F(c') \end{tikzcd}$$

  commutes. Note that there is an obvious forgetful functor $\mathcal{C}_{d/} \to \mathcal{C}$.

The categories $\mathcal{C}_{/d}$ and $\mathcal{C}_{d/}$ are often referred to as (co)slice categories, comma categories, and under- and overcategories. There seems to be little agreement in the literature as to which of these terms refers to precisely which concept. We will call $\mathcal{C}_{d/}$ an *overcategory* and $\mathcal{C}_{/d}$ an *undercategory*.

The notatations used for these categories are as diverse as the names. What we are calling $\mathcal{C}_{/d}$ is often called $(F \downarrow d)$, for example.

Fact A.6. Consider categories and functors as follows, where $\mathcal{C}$ is small.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "D"] \arrow[dr, "F"'] & & \mathcal{D} \\ & \mathcal{C}' & \end{tikzcd}$$

- If $\mathcal{D}$ has all small colimits, then the left Kan extension $F_!D$ exists and has values

$$(F_!D)(c') = \operatorname{colim}\left[\ \mathcal{C}_{c'/} \longrightarrow \mathcal{C} \xrightarrow{D} \mathcal{D}\ \right].$$

- If $\mathcal{D}$ has all small limits, then the right Kan extension F_*D exists and has values

$$(F_*D)(c') = \lim\left[\ \mathcal{C}_{/c'} \longrightarrow \mathcal{C} \xrightarrow{D} \mathcal{D}\ \right].$$

2.2 *Properties of Kan extensions*

Theorem A.7. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ locally small. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be fully faithful. If $\mathcal{D}$ has small colimits, then the unit

$$\epsilon\colon \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})} \Rightarrow F^* \circ F_!$$

is a natural isomorphism.

Corollary A.8. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ locally small. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be fully faithful.

- If $\mathcal{D}$ has small limits, then F_* is fully faithful.
- If $\mathcal{D}$ has small colimits, then $F_!$ fully faithful.

2.3 Yoneda extensions

For any locally small category $\mathcal{C}$, the *Yoneda embedding* is the functor

$$\mathcal{Y}\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) := \mathrm{Set}_{\mathcal{C}}; \qquad c \mapsto \mathrm{Hom}_{\mathcal{C}}(-, c).$$

The category $\mathrm{Set}_{\mathcal{C}}$ is the category of (Set-valued) presheaves on $\mathcal{C}$. We will frequently be interested in Kan extensions along the Yoneda embedding; these are called *Yoneda extensions.* That is, for any functor $F\colon \mathcal{C} \to \mathcal{D}$, we will often want to compute the Kan extension $\mathcal{Y}_! F$.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\mathcal{Y}"'] & & \mathcal{D} \\ & \mathrm{Set}_{\mathcal{C}} \arrow[ur, "\mathcal{Y}_! F"'] & \end{tikzcd}$$

In this section, we prove some basic results about Yoneda extensions.

Lemma A.9. Let $\mathcal{C}$ be a locally small category. For any functor $F \in \mathrm{Set}_{\mathcal{C}}$ we have an isomorphism

$$F \cong \mathrm{colim}\left[\mathcal{C}_{/F} \to \mathcal{C} \xrightarrow{\mathcal{Y}} \mathrm{Set}_{\mathcal{C}}\right].$$

Proof. By Theorem A.7, the fully faithfulness of the Yoneda embedding, and the definition of the restriction functor, we have that

$$F \cong (\mathcal{Y}^* \circ \mathcal{Y}_!)(F) \cong (\mathcal{Y}_! \mathcal{Y})(F).$$

The functor $\mathcal{Y}_! \mathcal{Y}$ comes from the following Kan extension.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\mathcal{Y}"] \arrow[dr, "\mathcal{Y}"'] & & \mathrm{Set}_{\mathcal{C}} \\ & \mathrm{Set}_{\mathcal{C}} \arrow[ur, "\mathcal{Y}_! \mathcal{Y}"'] & \end{tikzcd}$$

The colimit formula for left Kan extensions (Fact A.6) then tells us that

$$F \cong \mathrm{colim}\left[\mathcal{C}_{/F} \to \mathcal{C} \xrightarrow{\mathcal{Y}} \mathrm{Set}_{\mathcal{C}}\right]$$

as required. □

Lemma A.10. Let $\mathcal{I}$ be a small category, and let $\mathcal{C}$ be locally small and cocomplete. Let F be a functor as below. Then the Yoneda extension $\mathcal{Y}_! F$ is left adjoint to the functor

$$G\colon \mathcal{C} \to \mathrm{Set}_{\mathcal{I}}; \qquad c \mapsto \mathcal{C}(F(-), c).$$

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "F"] \arrow[dr, "\mathcal{Y}"'] & & \mathcal{C} \\ & \mathrm{Set}_{\mathcal{I}} \arrow[ur, "\mathcal{Y}_! F"'] & \end{tikzcd} \qquad \mathcal{Y}_! F : \mathrm{Set}_{\mathcal{I}} \leftrightarrow \mathcal{C} : G$$

Proof. Since $\mathcal{C}$ is cocomplete, we are entitled to use the colimit formula for left Kan extensions. That is, we can write

$$(\mathcal{Y}_! F)(H) = \operatorname{colim}\left[(\mathcal{Y}/H) \xrightarrow{\mathcal{U}} \mathcal{I} \xrightarrow{F} \mathcal{C}\right] = \operatorname{colim}\left[F \circ \mathcal{U}\right].$$

Thus, we have the following string of natural isomorphisms.

$$\begin{aligned} \mathcal{C}((\mathcal{Y}_! F)(H), c) &\cong \mathcal{C}(\operatorname{colim}[F \circ \mathcal{U}], c) \\ &\cong \lim \mathcal{C}(F(-), c) \circ \mathcal{U} \\ &\cong \lim G(c) \circ \mathcal{U} \\ &\cong \lim \mathrm{Set}_{\mathcal{I}}(\mathcal{Y}(-), G(c)) \circ \mathcal{U}. \\ &\cong \mathrm{Set}_{\mathcal{I}}(\operatorname{colim} \mathcal{Y} \circ \mathcal{U}, G(c)) \\ &\cong \mathrm{Set}_{\mathcal{I}}(H, G(c)). \end{aligned}$$

□

Corollary A.11. Let $\mathcal{I}$ be a small category, and let $\mathcal{C}$ be locally small and cocomplete. Let $F\colon \mathcal{I} \to \mathcal{C}$. Then the Yoneda extension $\mathcal{Y}_! F$ preserves colimits.

Proof. By Lemma A.10, $\mathcal{Y}_! F$ is a left adjoint. □

Theorem A.12. Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be cocomplete. Let $F\colon \mathrm{Set}_{\mathcal{C}} \to \mathcal{D}$ be any functor. Then F preserves colimits if and only if it is the left Yoneda extension of its restriction $\mathcal{Y}^* F\colon \mathcal{C} \to \mathcal{D}$, i.e. if $F = (\mathcal{Y}_! \mathcal{Y}^*)(F)$.

Proof. Assume that $F = (\mathcal{Y}_! \mathcal{Y}^*)(F)$. By Lemma A.10, $(\mathcal{Y}_! \mathcal{Y}^*)(F)$ is a left adjoint, hence preserves colimits.

Now assume that F preserves colimits. Pick some functor $G\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. By Lemma A.9, we can write

$$\begin{aligned} F(G) &\cong F\left(\varinjlim\left[(\mathcal{Y}/G) \to \mathcal{C} \xrightarrow{\mathcal{Y}} \mathrm{Set}_{\mathcal{C}}\right]\right) \\ &\cong \varinjlim\left[(\mathcal{Y}/G) \to \mathcal{C} \xrightarrow{\mathcal{Y}} \mathrm{Set}_{\mathcal{C}} \xrightarrow{F} \mathcal{D}\right] \\ &\cong \varinjlim\left[(\mathcal{Y}/G) \to \mathcal{C} \xrightarrow{\mathcal{Y}^* F} \mathcal{D}\right] \\ &\cong \mathcal{Y}_!(\mathcal{Y}^* F)(G). \end{aligned}$$

□

Corollary A.13. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ locally small. There is an equivalence of categories

$$[\mathrm{Set}_{\mathcal{C}}, \mathcal{D}]_{\text{cocontinuous}} \cong [\mathcal{C}, \mathcal{D}].$$

That is, colimit preserving functors $\mathrm{Set}_{\mathcal{C}} \to \mathcal{D}$ are equivalent to functors $\mathcal{C} \to \mathcal{D}$ via restriction along the Yoneda embedding.

Proof. We have an adjunction

$$\mathcal{Y}_! : [\mathcal{C}, \mathcal{D}] \leftrightarrow [\mathrm{Set}_{\mathcal{C}}, \mathcal{D}] : \mathcal{Y}^*.$$

By Corollary A.11, the image of $\mathcal{Y}_!$ is completely contained in $[\mathrm{Set}_{\mathcal{C}}, \mathcal{D}]_{\text{cocontinuous}}$, so the above adjunction extends to an adjunction

$$\mathcal{Y}_! : [\mathcal{C}, \mathcal{D}] \leftrightarrow [\mathrm{Set}_{\mathcal{C}}, \mathcal{D}]_{\text{cocontinuous}} : \mathcal{Y}^*.$$

We know that the Yoneda embedding is fully faithful. Thus, by Theorem A.7, we have that the unit is an isomorphism. Also, by Theorem A.12, the counit is an isomorphism, so the above adjunction is in fact an equivalence of categories. □

2.4 Day convolution

Let $(\mathcal{C}, \otimes)$ be a monoidal category. The category $\mathrm{Set}_{\mathcal{C}}$ of functors from $\mathcal{C}^{\mathrm{op}}$ to Set inherits a monoidal structure as follows.

Given functors $F, G \colon \mathcal{C} \to \mathrm{Set}$, we can define a functor

$$F \bar{\times} G \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}; \qquad (c, d) \mapsto F(c) \times G(d).$$

Definition A.14 (Day convolution)**.** The Day convolution of F and G, denoted $F \otimes G$, is given by the left Kan extension of $F \bar{\times} G$ along $\otimes^{\mathrm{op}}$.

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \arrow[rr, "F\bar{\times}G"] \arrow[dr, "\otimes^{\mathrm{op}}"'] & & \mathrm{Set} \\ & \mathcal{C}^{\mathrm{op}} \arrow[ur, "F\otimes G"'] & \end{tikzcd}$$

The colimit formula for left Kan extensions tells us that

$$(F \otimes G)(c) = \operatorname{colim}\left[(c \downarrow \otimes^{\mathrm{op}}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \xrightarrow{F\bar{\times}G} \mathrm{Set}\right]$$

3 Enriched category theory

Definition A.15 (enriched category)**.** Let $(\mathcal{M}, \otimes, 1_{\mathcal{M}}, \alpha, \lambda, \rho)$ be a monoidal category. A $\mathcal{M}$-enriched category $\mathcal{C}$ consists of the following data.

- A set $\mathrm{ob}(\mathcal{C})$ of objects.

- For each pair x, y of objects, an object $\mathcal{C}(x,y) \in \mathcal{M}$ of morphisms.
- For each object $x \in \mathcal{C}$, a morphism

$$1_{\mathcal{M}} \to \mathcal{C}(x,x)$$

  called the *unit*
- For objects x, y, and z, a morphism

$$\mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \to \mathcal{C}(x,z)$$

  called the *composition.*

This composition must be unital and associative, i.e. the diagrams

$$\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r] \arrow[d] \arrow[dr, "\mathrm{id}"] & \mathcal{C}(y,y) \otimes \mathcal{C}(x,y) \arrow[d] \\
\mathcal{C}(x,y) \otimes \mathcal{C}(x,x) \arrow[r] & \mathcal{C}(x,y)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \otimes \mathcal{C}(z,w) \arrow[r] \arrow[d] & \mathcal{C}(x,y) \otimes \mathcal{C}(y,w) \arrow[d] \\
\mathcal{C}(x,z) \otimes \mathcal{C}(z,w) \arrow[r] & \mathcal{C}(x,w)
\end{tikzcd}$$

must commute (where unitors and associators are notationally supressed).

Lemma A.16. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category, and let $F\colon \mathcal{M} \to \mathcal{N}$ be a monoidal functor with data

$$\Phi_{x,y}\colon F(x) \otimes_{\mathcal{N}} F(y) \to F(x \otimes_{\mathcal{M}} y), \qquad \phi\colon 1_{\mathcal{N}} \to F(1_{\mathcal{M}}).$$

This data allows us to build from $\mathcal{C}$ a category enriched over $\mathcal{N}$.

Proof.

- We apply F to each hom-object in $\mathcal{M}$ to get a new hom-object in $\mathcal{N}$;

$$\mathcal{C}(x,y)_{\mathcal{N}} = F(\mathcal{C}(x,y)_{\mathcal{M}}).$$

- The unit is the composition

$$1_{\mathcal{N}} \longrightarrow F(1_{\mathcal{M}}) \longrightarrow F(\mathcal{C}(x,y)_{\mathcal{M}}) = \mathcal{C}(x,y)_{\mathcal{N}}$$

- The composition is given by the obvious.

$\square$