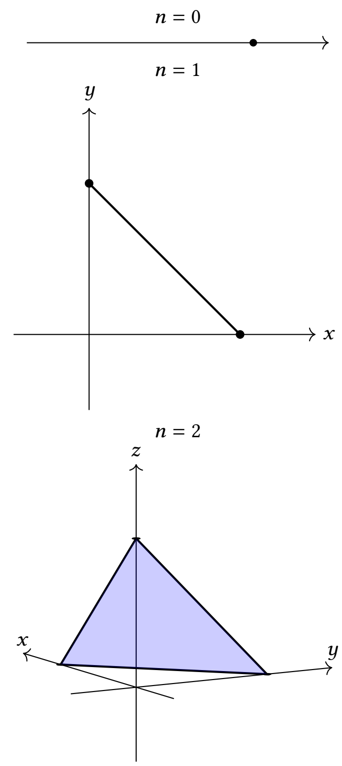
1

Simplicial sets

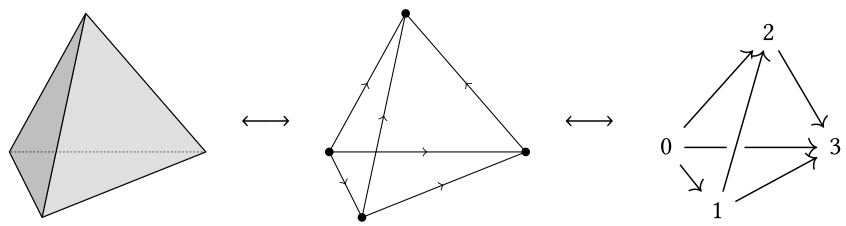
We can easily draw the first few standard n-simplices as subsets of $\mathbb{R}^{n+1}$:



The standard 3-simplex can be identified with a tetrahedron, as in the main text. However, for obvious reasons, we cannot draw it as a subset of $\mathbb{R}^4$.

1 *Simplices as topological spaces and categories*

One of the main themes in the study of higher category theory is the interplay between ordinary topology, simplicial topology, and category theory.



The fundamental topological object we will study is the *geometrical n-simplex.*

Definition 1.1 (geometrical n-simplex)**.** For any $n \geq 0$, the geometrical n-simplex, denoted $|\Delta^n|$, is the set (together with the subspace topology)

$$|\Delta^n| = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

In simplicial topology, one studies spaces glued out of geometrical simplices by replacing the topological simplices by combinatorial models of simplices. These live in a category.

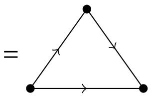
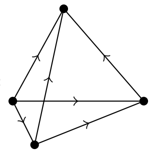
Definition 1.2 (simplex category)**.** Denote by Δ the category whose objects are linearly ordered sets

$$[n] = \{0, \ldots, n\}$$

and whose morphisms $[n] \to [m]$ are weakly monotonic[1] maps.

[1] i.e. nondecreasing in the sense that $i \geq j \implies f(i) \geq f(j)$

The objects $[n]$ of Δ are to be interpreted as simplices with ordered vertices. For example,

$[0] = \bullet$, $[1] =$, $[2] =$ , $[3] =$ 

We can connect our simplex category to geometric simplices using a functor.

Definition 1.3 (realization functor)**.** The realization functor

$$\rho \colon \Delta \longrightarrow \mathbf{Top}$$

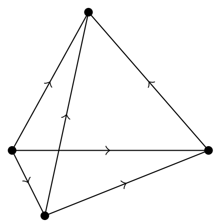
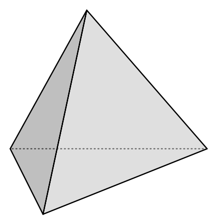
is defined on objects by $[n] \mapsto |\Delta^n|$ and on morphisms $[f] \colon [m] \to [n]$ by

$$f_* \colon |\Delta^m| \longrightarrow |\Delta^n|; \qquad (t_0, \ldots, t_m) \mapsto (s_1, \ldots, s_m),$$

where

$$s_i = \begin{cases} 0, & f^{-1}(i) = \emptyset \\ \sum_{j \in J^{-1}(i)} t_j & \text{otherwise.} \end{cases}$$

That is, ρ assigns to each combinatorial n-simplex its topological counterpart.

$\rho :$  $\longmapsto$ 

One can also think of these combinatorial simplices categorically. Again, this connection is given by a functor.

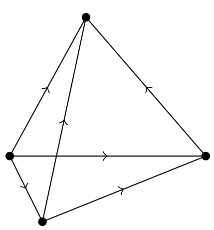
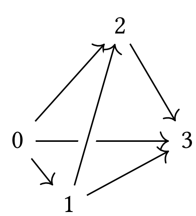
Definition 1.4 (categorification functor)**.** The categorification functor[2] is the functor

$$\chi \colon \Delta \longrightarrow \mathbf{Cat}$$

which sends the object $[n]$ to the poset category on $[n]$. Note that functors $[m] \to [n]$ are the same thing as weakly monotonic maps $[m] \to [n]$.

[2] This terminology is highly-nonstandard. What we call the "decategorification functor" here is a restriction to a full subcategory of a functor usually referred to in the literature as the "fundamental category" functor.

That is, χ assigns to each combinatorial n-simplex its categorical counterpart.

$\chi :$  $\longmapsto$ 

These categories and functors may seem too simple to be of any real use, but one can build astonishingly complicated structures out of them.

2 *Simplicial sets*

A simplicial set is thought of as a collection of standard simplices of various degrees, glued together along their faces. That is, a simplicial set consists of, for each $n \in \mathbb{N}$, a collection of simplices of degree n, together with data specifying which faces should be identified with other faces, and how. This data can be specified efficiently in the following way.

Definition 1.5 (simplicial set)**.** A simplicial set is a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$. The category of simplicial sets is the category of such functors, i.e.

$$\mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set}).$$

We denote this category by $\mathbf{Set}_\Delta$.

As discussed briefly in Section **??**, one thinks of the image under the Yoneda embedding of the object $[n]$ as an n-simplex, called the *standard n-simplex*. One imagines more general simplicial sets as being built by gluing standard simplices together along their faces. Specifically, for any simplicial set K we should interpret $K([n])$ as the set of n-simplices of K. The gluing conditions come from functoriality. We will investigate this in Section 3.

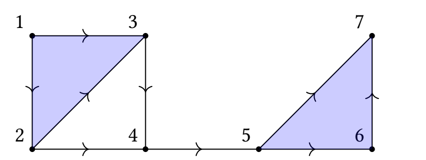
Definition 1.6 (standard simplex)**.** Let $n \geq 0$. The simplicial set

$$\Delta(-, [n]) = よ([n])$$

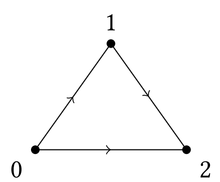
is denoted Δ^n, and called the *standard n-simplex*.

The claim is that the standard n-simplex can be interpreted geometrically, as precisely the sort of picture we have been drawing.[3] To see this, it will be helpful to explicitly analyze some cases in which n is small. Take $n = 2$. To justify the name, the standard 2-simplex had better look something like the 2-simplex we know and love.

[3] More generally, simplicial sets, when interpreted geometrically, can be viewed as one of the many ways to "build topological spaces out of triangles". The simplicial complexes common in introductory courses in (algebraic) topology are another such. However, as we will see, simplicial sets are often more flexible and easier to work with. Moreover, every an *ordered* simplicial complex C, where every 1-simplex has been equipped with a choice of orientation, as in the figure



can be canonically viewed as a simplicial set. Here we view the blue triangles as being filled by a 2-simplex, and the white ones as empty "holes".



Specifically, we should interpret $\Delta^2([n])$ as the set of n-simplices contained in Δ^2. Naïvely, we would expect three 0-simplices, three 1-simplices, and one 2-simplex.

However, the only way to find out what we actually have is to begin calculating. First, we calculate $\Delta^2([0]) = \mathbf{Set}_\Delta([0], [2])$:

$$\Delta^2([0]) = \{0 \mapsto 0, 0 \mapsto 1, 0 \mapsto 2\}.$$

To avoid formulae whose lengths quickly get out of hand, we will drop the functional notation, denoting the above by $\{0, 1, 2\}$.

This is good. We said that $\Delta^2([0])$ should be interpreted as the set of 0-simplices, and indeed $\Delta^2([0])$ has three elements, which represent in an obvious way the vertices of the familiar 2-simplex. Now calculating $\Delta^2([1])$, we find

$$\Delta^2([1]) = \{\{0,1\} \mapsto \{0,1\}, \{0,2\}, \{1,2\}, \{0,0\}, \{1,1\}, \{2,2\}\}.$$

This is not so good; we can interpret the three edges $\{0,1\}$, $\{0,2\}$, and $\{1,2\}$ as the edges between 0 and 1, 0 and 2, and 1 and 2 that we were expecting, but we also have three more. We interpret these as 'degenerate edges,' which are all bunched up at the vertices. This is one difference between simplicial complexes and simplicial sets: simplicial sets can have degenerate edges.[4]

Continuing, we find

$$\Delta^2([2]) = \{\{0,0,0\}, \{0,0,1\}, \{0,0,2\}, \ldots\}.$$

We recognize these as degenerate faces. The only nondegenerate face will be $\{0,1,2\}$.

Note that contrary to what you might expect, $\Delta^n([k])$ is not empty for $k > n$; it is simply filled with an enormous number of degenerate simplices.[5]

Example 1.7. Let X be a set. Denote the power set of X by $\mathcal{P}(X)$. Let $\mathcal{J} \subseteq \mathcal{P}(\{0,1,\ldots,n\})$. There is a simplicial set $\Delta^{\mathcal{J}}$ whose n-simplices are defined to be

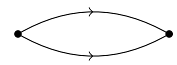
$$\Delta^{\mathcal{J}}([n]) \subseteq \Delta([m],[n])$$

consisting of those f such that $\mathrm{im}(f) \subset J$ for some $J \in \mathcal{J}$.

Here are some specific examples of $\mathcal{J}$ which will be useful to us.

- If $\mathcal{J} = \mathcal{P}(\{0,\ldots,n\})$, then
$$\Delta^{\mathcal{J}}([n]) = \Delta^n,$$
the standard n-simplex.
- If
$$\mathcal{J} = \mathcal{P}(\{0,1,\ldots,n\}) \setminus \{\{0,1,\ldots,n\}\},$$
then the simplicial set we obtain is called the *boundary* of Δ^n, and denoted $\partial\Delta^n$.
The sets $\Delta^n([i])$ and $\partial\Delta^n([i])$ are the same for $i < n$.
- Fix some i such that $0 \le i \le n$. Then with
$$\mathcal{J} = \mathcal{P}(\{0,1,\ldots,n\}) \setminus \{\{0,1,\ldots,n\}, \{0,1,\ldots,i-1,i+1,\ldots,n\}\},$$
we obtain the *ith horn* of Δ^n, which we denote Λ^n_i.
For example, the 2-simplices in the horn Λ^2_1 consists of the subset of monotone functions $[2] \to [2]$ whose range lies entirely within one of the sets $\{0,1,2,01,12,22\}$. That is
$$\{000, 001, 011, 111, 112, 122, 222\}.$$

[4] Another difference is that an n-simplex need not be specified by its set of vertices. For example, the following picture is not a simplicial complex, but it is a simplicial set.



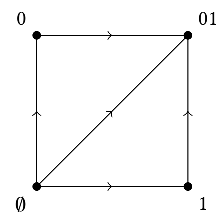
the reason for this is that, in a simplicial complex, there is at most one n-simplex with a given set of vertices. In the drawing above, however, both 1-simplices have the same set of vertices.

[5] To be precise, $\binom{n+k+1}{k}$ simplices.

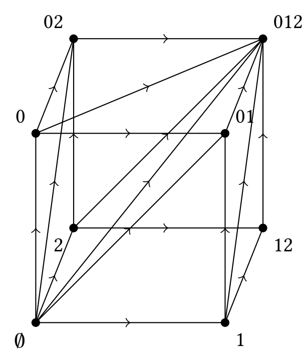
We can construct another interesting simplicial set from the power set $\mathcal{P}([n])$. The power set itself has the structure of a poset — its objects are ordered by inclusion of subsets. We can define a simplicial set $C^{[n]}$ by setting

$$C^{[n]}([k]) = \{f : [k] \to [n] \mid f \text{ order-preserving}\}$$

Drawing only non-degenerate simplices, C^1 will look like



Drawing only non-degenerate 1-simplices, C^2 looks like



Where the 2- and 3-simplices then fill the cube.

We can draw Λ^2_i, for $i = 0, 1, 2$, as follows.

$$\Lambda^2_0 = \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}, \qquad \Lambda^2_1 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}, \qquad \Lambda^2_2 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

Definition 1.8 (inner, outer horn)**.** If $i = 0$ or $i = n$, then we say that Λ^n_i is an outer horn. Otherwise, if $0 < i < n$, we say that Λ^n_i is an inner horn.

Definition 1.9 (simplicial complex)**.** A set $\mathcal{K} \subset \mathcal{P}(\{0, 1, \dots, n\})$ is called an simplicial complex if for every $\mathcal{J} \in \mathcal{K}$ and $I \subseteq \mathcal{J}$ with $I \neq \emptyset$, $I \in \mathcal{K}$.

Example 1.10. Let $\mathcal{K}$ be any simplicial complex. We can interpret $\mathcal{K}$ as a simplicial set by associating

$$\mathcal{K} \mapsto \Delta^{\mathcal{K}}.$$

What would have been called the simplices of $\mathcal{K}$ correspond to the nondegenerate elements of the simplicial set $\Delta^{\mathcal{K}}$.

Note that there are many simplicial sets which are not obtained from simplicial complexes in this way. That is, the theory of simplicial sets is a strict generalization of that of simplicial complexes.

Simplicial sets are functors, but the functorial notation is often unnecessarily burdensome. Therefore, it is standard to denote the n-simplices of a simplicial set K by K_n rather than $K([n])$. This is the notation that we will use from now on.

3 *Which data does a simplicial set give us?*

3.1 *Face and degeneracy maps*

Let K be a simplicial set. By definition evaluating K on the object $[n]$ gives a set K_n, which we interpret as the set of n-simplices making up K. We have claimed that the functoriality of K provides instructions for how these simplices should be glued together. In this section we will shine some light on this claim.

First, we note that we can understand how K acts on morphisms $\phi\colon [n] \to [m]$ by understanding how it behaves on simpler functions, known as the *face* and *degeneracy functions.*

Definition 1.11 (face function)**.** Let $n \geq 0$. For any k with $0 \leq k < n$, we the ith face function,

$$\partial_i\colon [n-1] \to [n]; \qquad j \to \begin{cases} j, & j < i \\ j+1, & j \geq i \end{cases}.$$

$$
\sigma_k \quad = \quad
\begin{tikzcd}[row sep=small]
\vdots & \vdots \\
k-2 \arrow[r, mapsto] & k-2 \\
k-1 \arrow[r, mapsto] & k-1 \\
k \arrow[r, mapsto] & k \\
k+1 \arrow[dr, mapsto] & k+1 \\
k+2 \arrow[dr, mapsto] & k+2 \\
\vdots & \vdots
\end{tikzcd}
$$

Applying a simplicial set K to the ith face function gives a map from K_n to K_{n-1}.

Definition 1.12 (face map)**.** Let $X\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. Corresponding to the face function $\partial_i\colon [n] \to [n-1]$ (see Definition 1.11), there is a map

$$d_i\colon X_n \longrightarrow X_{n-1}.$$

This map is also known as the ith face map.

If we were being pedantic, we would denote the face functions by ∂_i^n instead of ∂_i, but keeping track of these indices turns out to be more confusing than elucidating.

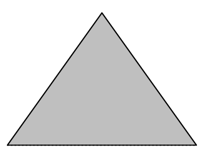
Acting on an n-simplex σ with a face map gives one of the $(n-1)$-simplices making up the boundary of σ. Recall that our combinatorial models for simplices have ordered vertices—the vertices of an n-simplex run from 0 to n. The ith face map takes an n-simplex to its ith face; that is, the face opposite the vertex labelled by i.

Example 1.13. Consider the simplicial set $K = \Delta^2$.

Take $\sigma \in \Delta^2_2$ given by the identity

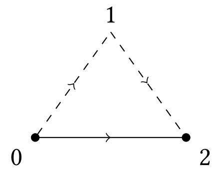
$$
\sigma\colon \quad
\begin{tikzcd}[row sep=small]
0 \arrow[r, mapsto] & 0 \\
1 \arrow[r, mapsto] & 1 \\
2 \arrow[r, mapsto] & 2
\end{tikzcd}
\cdot
$$

This corresponds to the single nondegenerate 2-simplex in Δ^2, which is what we intuitively think of as the 'whole' 2-simplex.



Applying d_1 corresponds to pulling back by ∂_1 as follows.

$$
d_1\sigma = \sigma \circ \partial_1 = \quad
\begin{tikzcd}[row sep=small]
0 \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0 \\
1 \arrow[dr, mapsto] & 1 \arrow[r, mapsto] & 1 \\
 & 2 \arrow[r, mapsto] & 2
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[row sep=small]
0 \arrow[r, mapsto] & 0 \\
1 \arrow[dr, mapsto] & 1 \\
 & 2
\end{tikzcd}
$$

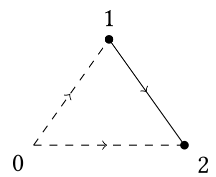


Similarly, d_0 takes σ to the edge between 1 and 2, and d_2 takes σ to the edge between 0 and 1.

Let $f \in \Delta^2_1 = \Delta([1], [2])$ be one of the six 1-simplices. For example, take

$$f\colon \begin{tikzcd}[row sep=small] 0 \arrow[dr, mapsto] & 0 \\ 1 \arrow[dr, mapsto] & 1 \\ & 2 \end{tikzcd}$$

corresponding to the solid edge from 1 to 2.



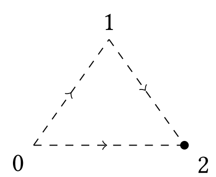
Again we compute. The face map d_0 comes from the face function ∂_0, which we can draw like this.

$$\partial_0\colon \begin{tikzcd}[row sep=small] 0 \arrow[dr, mapsto] & 0 \\ & 1 \end{tikzcd}$$

The map d_0 acts by precomposing with ∂_0.

$$d_0 f = f \circ \partial_0 = \begin{tikzcd}[row sep=small] 0 \arrow[dr, mapsto] & 0 \arrow[dr, mapsto] & 0 \\ & 1 \arrow[dr, mapsto] & 1 \\ & & 2 \end{tikzcd} = \begin{tikzcd}[row sep=small] 0 \arrow[ddr, mapsto] & 0 \\ & 1 \\ & 2 \end{tikzcd}$$

Therefore, d_0 takes the edge between 1 and 2 to the vertex 2.



Put more concisely, and as should now be clear from the form of ∂_0, we can see that d_0 takes any 1-simplex from a to b to the zero-simplex b. That is, taking the vertices of a 1-simplex as numbered in order, d_0 *deletes the zeroth vertex.vertex*

This is a general feature: the ith face d_i map acts on an n-simplex by deleting the ith vertex, returning the $(n-1)$-simplex opposite.

So we understand how the face maps work, at least on n-simplices.

Definition 1.14 (degeneracy function)**.** Let $n > 0$. For every i with $0 \leq i \leq n$, there is a function

$$\sigma_k \colon [n] \longrightarrow [n-1]; \qquad j \mapsto \begin{cases} j, & j \leq k \\ j-1, & j > k \end{cases}$$

called the ith degeneracy function.

$$\sigma_k \quad = \quad \begin{tikzcd} \vdots & \vdots \\ k-2 \arrow[r, mapsto] & k-2 \\ k-1 \arrow[r, mapsto] & k-1 \\ k \arrow[r, mapsto] & k \\ k+1 \arrow[ur, mapsto] & k+1 \\ k+2 \arrow[ur, mapsto] & k+2 \\ \vdots & \vdots \end{tikzcd}$$

Definition 1.15 (degeneracy map)**.** Let $X \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a simplicial set. Corresponding to the degeneracy function $\sigma_k \colon [n] \longrightarrow [n-1]$ there is a corresponding map

$$s_k \colon X_{n-1} \longrightarrow X_n.$$

This map is also known as the kth degeneracy map.

Up until now, most of the simplices we have been talking about have been *nondegenerate.* However, simplicial sets also contain degenerate simplices. Recall, for example, that the standard 2-simplex actually contains six 1-simplices, three of which correspond to the edges of the triangle, and three of which correspond to edges which are 'bunched up' at the vertices. The nondegenerate edges correspond to the injective maps $[1] \longrightarrow [2]$, and the degenerate edges correspond to the constant maps.

The ith degeneracy map takes an $(n-1)$-simplex to a degenerate n-simplex

The face and degeneracy functions are interesting because they generate all morphisms; that is, any morphism $\phi \colon [n] \longrightarrow [m]$ can be written as a composition of face and degeneracy functions.

Proposition 1.16. Let $\phi \colon [n] \longrightarrow [m]$ be a weakly monotonic function. Then ϕ can be written as a composition of face and degeneracy maps.

Sketch of proof. Since ϕ is by definition nondecreasing, we can specify it by, starting from 0, a sequence $\square$

Theorem 1.17 (simplicial identites)**.** The face maps and degeneracy maps satisfy the following conditions, known as the *simplicial identities.*

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i, && i < j \\ d_i \circ s_j &= s_{j-1} \circ d_i, && i < j \\ d_j \circ s_j &= 1 = d_{j+1} \circ s_j && \\ d_i \circ s_j &= s_j \circ d_{i-1}, && i > j+1 \\ s_i \circ s_j &= s_{j+1} \circ s_i, && i \leq j. \end{aligned}$$

Fact 1.18. A simplicial set X is equivalently the following data.

- The sets X_n, $n \geq 0$.
- The face and degeneracy maps satisfying the simplicial identities.

3.2 *Morphisms of simplicial sets; simplices and the Yoneda lemma*

So far, we have talked a lot about simplicial sets, i.e. functors $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$. From category theory, one feels that the natural notion of a morphism of simplicial sets should be that of a natural transformation. However, simplicial sets are geometric objects, and we have some idea, on a geometrical level, of what we would like morphisms between them to look like. Fortunately, these notions coincide.

Consulting the definition of a simplicial set, one sees that a morphism between simplicial sets $f : X \longrightarrow Y$ should consist of, for each $n \geq 0$, a map $f_n : X_n \longrightarrow Y_n$, compatible with the morphisms in Δ. In fact, because of Proposition 1.16, it suffices to demand that the f_i should be compatible with face and degeneracy maps.

What am I trying to say?

1. Morphims of simplicial sets. These are what you'd expect; n-simplices are sent to n-simplices in such a way that
2. Fully faithfulness: restricting to maps between standard simplices gives a copy of Δ in $\mathbf{Set}_\Delta$.
3. The n-simplices of a simplicial set X are in natural bijection to maps of the standard n-simplex into X.

4 *Basic properties of the category of simplicial sets*

Any functor category inherits a lot of structure from it codomain category. In the case of $\mathbf{Set}_\Delta = \mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$, we are especially lucky, as $\mathbf{Set}$ has an enormous amount of structure. In this section we note some of the structure $\mathbf{Set}_\Delta$ inherits from $\mathbf{Set}$.

4.1 *Existence of limits and colimits*

Proposition 1.19. The category $\mathbf{Set}_\Delta$ has small limits and colimits. These are computed level-wise, i.e. n-simplicices of a (co)limit of a diagram of simplicial sets are given by the the limit of the diagram's restriction to n-simplices.

Proof. The category **Set** has small limits and colimits, which implies by Theorem **??** that any functor category $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$ does.

Limits and colimits are computed pointwise thanks to Corollary **??**. □

Example 1.20 (products)**.** By Corollary **??**, we can compute this limit pointwise; that is

$$(X \times Y)_n = X_n \times Y_n.$$

Let us understand this this by studying $\Delta^1 \times \Delta^1$ in some detail.

The 0-simplices of $\Delta^1 \times \Delta^1$ are ordered pairs $\Delta^1_0 \times \Delta^1_0 = \{0, 1\} \times \{0, 1\}$. We can visualize this like so.

$$\begin{tikzcd} (0,1) \bullet & \bullet\, (1,1) \\ (0,0) \bullet & \bullet\, (1,0) \end{tikzcd}$$

Similarly, the 1-simplices $(a, b) \longrightarrow (c, d)$ are pairs of a 1-simplex $a \longrightarrow b$ and a 1-simplex $c \longrightarrow d$. There are 3 1-simplices in Δ^1_1: $\{a = \{0, 0\}, b = \{0, 1\}, c = \{1, 1\}\}$. Thus, we have the following 1-simplices in $\Delta^1 \times \Delta^1$ (where we do not draw degenerate simplices).

$$\begin{tikzcd} (0,1) \arrow[r] & (1,1) \\ (0,0) \arrow[u] \arrow[r] \arrow[ur] & (1,0) \arrow[u] \end{tikzcd}$$

Continuing in the same way, one finds that there are two non-degenerate 2-simplices, filling the above triangles. Thus, as one might expect, the product of two intervals is a square.

Example 1.21 (pushouts)**.** Corollary **??** also immediately implies that $\mathbf{Set}_\Delta$ has pushouts, which we can also compute poinwise. Recall that we think of pushouts as given by gluing, at least when

The fact that limits and colimits are computed pointwise implies immediately that morphisms of simplicial sets are computed pointwise.

Lemma 1.22. We have the following.

1. A morphism of simplicial sets is a monomorphism if and only if for each n, the component f_n is a monomorphism (i.e. injection) of sets.
2. A morphism of simplicial sets is a epimorphism if and only if for each n, the component f_n is a epimorphism (i.e. surjection) of sets.

Proof. Consequence of Corollary **??** □

4.2 Cartesian closure

Note 1.23. At the moment, the order of a lot of Cartesian products in this section is a bit out of whack, since I'm straddling two different conventions. I'll fix this if I have time.

Thanks to Proposition 1.19, the category $\mathbf{Set}_\Delta$ has products. It turns out that it also has exponential objects (i.e. an internal hom with respect to the Cartesian product). This makes it into a *Cartesian closed category.*

Before we prove that such an internal hom exists, let us assume that $\mathbf{Set}_\Delta$ admits an internal hom $\mathrm{Maps}(A, B)$, and see where we end up.

Lemma 1.24. Let B, C be simplicial sets. Should it exist, the internal hom $\mathrm{Maps}(B, C)$ must be given, up to natural isomorphism, by the functor

$$\mathbf{Set}_\Delta(B \times \Delta^\bullet, C),$$

where $\Delta^\bullet$ denotes the Yoneda embedding $\Delta \hookrightarrow \mathbf{Set}_\Delta$.

Proof. By definition, $\mathrm{Maps}(B, C)$ needs to be part of a hom-set adjunction

$$\mathbf{Set}_\Delta(A \times B, C) \cong \mathbf{Set}_\Delta(A, \mathrm{Maps}(B, C)).$$

In particular, when $A = \Delta^n$, we have that

$$\mathbf{Set}_\Delta(\Delta^n \times B, C) \cong \mathbf{Set}_\Delta(\Delta^n, \mathrm{Maps}(B, C)).$$

However, by the Yoneda lemma, there is an isomorphism

$$\mathbf{Set}_\Delta(\Delta^n, \mathrm{Maps}(B, C)) \cong \mathrm{Maps}(B, C)_n,$$

natural in n. Thus the internal hom, should it exist, must be given level-wise by

$$\mathrm{Maps}(B, C)_n \cong \mathbf{Set}_\Delta(B \times \Delta^n, C).$$

The naturality of the Yoneda embedding now implies that this level-wise isomorphism must really be a natural isomorphism, i.e. that

$$\mathrm{Maps}(B, C) \cong \mathbf{Set}_\Delta(B \times \Delta^\bullet, C).$$

□

Theorem 1.25. For simplicial sets B and C, the simplicial set $\mathrm{Maps}(B, C)$ defined by the formula

$$\mathrm{Maps}(B, C) = \mathbf{Set}_\Delta(B \times \Delta^\bullet, C),$$

is a model for the internal hom $\mathrm{Maps}(B, C)$. That is, for each simplicial set B, there is a bijection

$$\mathbf{Set}_\Delta(A \times B, C) \cong \mathbf{Set}_\Delta(A, \mathrm{Maps}(B, C)),$$

natural in A and C.

Proof. We define two mutually inverse maps. First, let $f \colon A \times B \to C$. We define a map $\tilde{f} \colon A \to \mathrm{Maps}(B, C)$ level-wise as follows: for any $\sigma \in A_n$, define $\tilde{f}_n(\sigma) \colon \Delta^n \times B \to C$ to be the composition

$$\Delta^n \times B \xrightarrow{\sigma \times B} A \times B \xrightarrow{f} C.$$

Now we define a map in the other direction. First, we define a family of morphisms which turn out to be the components of the counit. Consider a map

$$\mathrm{ev} \colon \mathrm{Maps}(B, C) \times B \to C$$

defined level-wise by

$$\mathrm{ev}_n \colon \mathbf{Set}_\Delta(\Delta^n \times B, C) \times B_n \to C_n \qquad (f, b) \mapsto f_n(\mathrm{id}_{\Delta^n}, b).$$

The, given $\tilde{f} \colon A \to \mathrm{Maps}(B, C)$, we define $f \colon A \times B \to C$ to be the composition

$$A \times B \xrightarrow{\tilde{f} \times B} \mathrm{Maps}(B, C) \times B \xrightarrow{\mathrm{ev}} C. \tag{1.1}$$

It is not *too* hard to show that these are really mutually inverse and check naturality.

□

Note 1.26. We have

$$\mathrm{Maps}(K, S)_0 = \mathbf{Set}_\Delta(K \times \Delta^0, S) = \mathbf{Set}_\Delta(K, S),$$

i.e. maps $K \to S$ correspond to vertices of $\mathrm{Maps}(K, S)$. The 1-simplices of $\mathrm{Maps}(K, S)$ corrspond to maps $\Delta^1 \times K \to S$, i.e. homotopies between maps, and so on.

5 *Constructing simplicial sets from cosimplicial objects*

So far, all the concrete examples of simplicial sets we have seen have been of the form $\Delta^{\mathcal{K}}$ for some $\mathcal{K} \subset \mathcal{P}(\{0, \ldots, n\})$. These examples are useful, but not very interesting in their own right.

In this section, we define two ways of creating a simplicial set out of existing mathematical data; in one case, a category, and in the other, a topological space. We will do this using *cosimplicial objects*, i.e. functors

$$\alpha \colon \Delta \to \mathcal{C}$$

where $\mathcal{C}$ is some locally small category.

The idea is as follows. Given a cosimplicial object $\alpha\colon \Delta \to \mathcal{C}$ and any object $C \in \mathcal{C}$, we can define a corresponding simplicial set $\mathcal{N}_\alpha(C)$ by first Yoneda embedding, and then composing the resulting hom functor with α:

$$C \mapsto \mathcal{N}_\alpha(C) = \mathcal{C}(\alpha(-), C).$$

That $\mathcal{N}_\alpha(C)$ is a simplicial set follows immediately from the contravariance of the first slot of the hom functor.

The n-simplices of $\mathcal{N}_\alpha(C)$ are then given by maps of the $\mathcal{C}$-model of the n-simplex into C:

$$\mathcal{N}_\alpha(C)_n = \mathcal{C}(\alpha([n]), C).$$

In fact, by the functoriality of hom this construction gives us a functor $\mathcal{C} \to \mathbf{Set}_\Delta$ taking $C \mapsto \mathcal{N}_\alpha(C)$.

The power of defining simplicial sets via cosimplicial objects comes from Lemma **??**, which provides us immediately with a left adjoint to the functor $\mathcal{N}_\alpha$. This left adjoint provides a weak inverse, i.e. a way of constructing from any simplicial set K an object of the category $\mathcal{C}$. More specifically, Lemma **??** guarantees us an adjunction

$$\mathcal{Y}_!\alpha : \mathbf{Set}_\Delta \longleftrightarrow \mathcal{C} : \mathcal{N}_\alpha,$$

where $\mathcal{Y}_!\alpha$ is the Yoneda extension of α.

The alert reader will remember that we have already defined two cosimplicial objects: the *realization functor* $\rho\colon \Delta \to \mathbf{Top}$ (Definition 1.3) and the *categorification functor* $\chi\colon \Delta \to \mathbf{Cat}$. (Definition 1.4). We said that they would provide a connection between the theory of simplicial sets and the theory of topological spaces and categories respectively. Indeed, these are the cosimplicial objects we will use.

5.1 Nerves

Definition 1.27 (nerve of a category)**.** Let $\mathcal{C}$ be a small category. By composing the Yoneda embedding with the categorification functor (Definition 1.4)

$$\chi\colon \Delta \to \mathbf{Cat},$$

we obtain a simplicial set called the <u>nerve</u> of $\mathcal{C}$ and denoted $N(\mathcal{C})$. That is,

$$N(\mathcal{C}) = \mathbf{Fun}(\chi(-), \mathcal{C}).$$

Let us examine this construction in some detail. We have

$$N(\mathcal{C})_0 = \mathbf{Fun}([0], \mathcal{C}).$$

Since $[0]$ is nothing else but the category with one object and one morphism, $\mathbf{Fun}([0], \mathcal{C})$ simply picks out the objects of $\mathcal{C}$. That is, the zero-simplices of $N(\mathcal{C})$ simply consist of the objects of $\mathcal{C}$.

Similarly, the category $[1]$ consists of two object and a morphism between them, so the functors in $\mathbf{Fun}([1], \mathcal{C})$ pick out diagrams of the form

$$X_0 \longrightarrow X_1$$

which is tantamount to picking out morphisms in $\mathcal{C}$. Functors in $\mathbf{Fun}([2], \mathcal{C})$ pick out commuting triangles.

$$\begin{tikzcd} & X_1 \arrow[dr] & \\ X_0 \arrow[ur] \arrow[rr] & & X_2 \end{tikzcd}$$

In general, $N(\mathcal{C})([n])$ picks out chains of morphisms,

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n \ ,$$

together with all possible compositions. One should picture these as forming an n-dimensional simplex.[6]

[6] We have already encountered a nerve in a previous sidebar. One can view the power set $\mathcal{P}([n])$ as a poset, and thus as a category. The cubes C^n, which we defined above as simplicial sets, are nothing more than the nerves of the categories $\mathcal{P}([n])$.

To summarize, one pictures the construction of the nerve of a category $\mathcal{C}$ in the following way.

1. One imagines for each object of the category $\mathcal{C}$ a zero-simplex, i.e. a little dot.
2. One imagines for each morphism between objects x and y an edge between the corresponding zero-simplices. So far, we have constructed a multidigraph, i.e. a directed graph with multiple edges between vertices.
3. Our multidigraph has lots of little triangles formed by morphisms. We fill in each of the commuting triangles with a 2-simplex.
4. Now we proceed inductively. Given the set of $(n-1)$-simplices of $N(\mathcal{C})$, we add an n-simplex whenever we have all the $(n-1)$-simplices forming its boundary.

Note that the n-simplices for $n > 2$ don't tell us very much; we add them whenever we have their boundaries. This is known as *2-coskeletality,* and is a reflection of the fact that categories only have objects and morphisms, and no higher data. We will meet the notion of coskeletality in Section 7.

Proposition 1.28. The nerve is left is right adjoint to the functor $\mathcal{Y}_!\chi$, where $\chi \colon \Delta \to \mathbf{Cat}$ is the categorification functor (Definition 1.4), which assigns $[n]$ to itself considered as a poset category, and $\mathcal{Y}_!$ denotes the Yoneda extension.

$$\mathcal{Y}_!\chi : \mathbf{Set}_\Delta \leftrightarrow \mathbf{Cat} : N$$

Proof. The nerve is of the form $\mathcal{C} \mapsto \mathbf{Set}_\Delta(\chi(-), \mathcal{C})$. The result follows from Lemma **??**. □

The functor $\mathcal{Y}_!\chi$ provides a way of constructing, from any simplicial set, a category. We will not say too much about it now. Rather, we will come back to it (in a slightly weaker form) in Section **??**, when we have the tools to understand it properly.

5.2 Singular sets

Definition 1.29 (singular set of a topological space)**.** Let X be a topological space. The functor

$$\mathbf{Top}(\rho(-), X) \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$$

is called the singular set of X, and denoted $\mathbf{Sing}(X)$.

Unlike the nerve, which creates a reasonably faithful simplicial likeness of any category, the singular set creates a caricature. Consider, for example, the geometric 2-simplex $|\Delta^2|$. We would hope that the singular set of $|\Delta^2|$ would be something like Δ^2, but alas: rather than three 0-simplices, there are uncountably many, one for each continuous map of the point $|\Delta^0|$ into $|\Delta^2|$. Similarly, there is a 1-simplex for each continuous map of the interval into $|\Delta^2|$.

However, things are not as hopeless as they may appear. We will later see that these extra simplices are in some sense superficial, and that the simplicial set $\mathbf{Sing}(|\Delta^2|)$ is, in an appropriate sense, equivalent to Δ^2.

By Lemma **??**, the singular set functor is right adjoint to the Yoneda extension of the realization functor ρ, which we call the *geometric realization.*

Definition 1.30 (geometric realization)**.** Let K be an simplicial set. The geometric realization of K, denoted $|K|$, is the image

$$|K| = (よ_! \rho)(K).$$

We now have the tools to study the geometric realization in some detail. This will be the content of Section 6.

5.3 A colimit formula for simplicial sets

There is one more cosimplicial object we have met, which may seem almost too simple to count: the Yoneda embedding $よ\colon \Delta \to \mathbf{Set}_\Delta$ itself! Applying the same procedure as above, we find the functor

$$\mathbf{Set}_\Delta \to \mathbf{Set}_\Delta; \qquad K \mapsto \mathbf{Set}_\Delta(よ(-), K).$$

This is (by the Yoneda lemma) simply the identity functor.

However, Lemma **??** now implies a very useful formula.

Proposition 1.31. Let K be any simplicial set. Then K is the colimit of the functor

$$(\Delta \downarrow K) \to \Delta \xrightarrow{よ} \mathbf{Set}_\Delta\,.$$

Proof. Lemma **??**. □

The objects of the category $(\Delta \downarrow K)$ consist of all morphisms $\Delta^n \to K$ for any n. This is simply the set of all simplices of K. The morphisms between simplices are all ways of mapping simplices into each other which

This vindicates the assertion that we have been making all along: that a simplicial set should be viewed as being built by gluing together its simplices.

6 Geometric realization

In Subsection 5.2, we saw that we could turn any topological space X into a simplicial set using the singular set functor (Definition 1.29)

$$\mathbf{Sing}\colon \mathbf{Top} \to \mathbf{Set}_\Delta; \qquad \mathbf{Sing}(X) = \mathbf{Top}(\rho(-), X).$$

We then saw that **Sing** has a left adjoint, the geometric realization functor, which provides a weak inverse to **Sing**. The geometric realization has the following formula.

$$|K| = (\text{よ}_! \rho)(K).$$

As we will see in this section, the geometric realiztion takes a simplicial set K to a topological space which is created by assigning to each n-simplex in K to an actual n-simplex in **Top**, and gluing these together in the appropriate way.

The goal of this section is to derive and understand the following formula for the geometric realization.

Theorem 1.32. Let $K\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. The geometric realization of K can be computed by the formula

$$|K| = \left(\coprod_{n\geq 0} K_n \times |\Delta^n| \right) /\sim,$$

where $\sim$ is the equivalence relation generated by the relation

$$(f^*(x), y) \sim (x, f_*(y)), \qquad \text{for all } f\colon [m] \to [n].$$

Proof. By the colimit formula for left Kan extensions (Fact **??**), we can write

$$|K| = \operatorname{colim}\left[(\Delta \downarrow K) \to \Delta \xrightarrow{\rho} \mathbf{Top} \right]. \tag{1.2}$$

Let f be a morphism $[m] \to [n]$. Then by the contravariance of any simplicial set K, f induces a morphism $f^*\colon K_n \to K_m$. Similarly, by the covariance of the geometric realization functor $\rho\colon \Delta \to \mathbf{Top}$, f induces a map $f_*\colon |\Delta^m| \to |\Delta^n|$.

By the coequalizer formula for colimits (Fact **??**), we can express the colimit in Equation 1.2 as the following coequalizer.

$$\coprod_{\substack{f \in \mathrm{Morph}(\Delta/K) \\ f\colon [m]\to[n]}} |\Delta^m| \underset{B}{\overset{A}{\rightrightarrows}} \coprod_{\substack{([n],\alpha)\in \mathrm{ob}(\Delta/K) \\ \alpha:\Delta^n \to K}} |\Delta^n| \longrightarrow \mathrm{coeq}$$

Here, the morphisms A and B are defined via the univeral property for coproducts by their components

$$A_f = \iota_{|\Delta^m|}, \qquad B_f = \iota_{|\Delta^n|} \circ f_*.$$

Let us understand the first coproduct, taken over the set of morphisms in $(\Delta \downarrow K)$. Recall that a morphism $([m], f) \to ([n], f')$ in $(\Delta \downarrow K)$ is a morphism $g\colon [m] \to [n]$ in Δ making the following diagram commute.

$$g\colon [m] \to [n]; \qquad \begin{tikzcd} \Delta^m \arrow[rr, "\Delta^g"] \arrow[dr, "f"'] & & \Delta^n \arrow[dl, "f'"] \\ & K & \end{tikzcd}$$

This means that morphisms in $(\Delta \downarrow K)$ can be uniquely parametrized by the following pieces of data.

- A morphism $g\colon [m] \to [n]$.
- A morphism $f'\colon \Delta^n \to K$.

The other morphism f must be the composite $f' \circ \Delta^g$.

This means that we are really taking the coproduct over two things: morphisms $[m] \to [n]$, and morphisms $\Delta^n \to K$. But by the Yoneda lemma

$$\mathbf{Set}_\Delta(\Delta^n, K) \cong K_n,$$

so we write the first coproduct as

$$\coprod_{f\colon [m]\to[n]} \left(\coprod_{x\in K_n} |\Delta^m| \right)$$

which (by slight abuse of notation, taking K_n with the discrete topology) is simply

$$\coprod_{f\colon [m]\to[n]} K_n \times |\Delta^m|.$$

□

Let $K\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. The geometric realization of K is the topological space

$$|K| = \left(\coprod_{n\geq 0} K_n \times |\Delta^n| \right) /\sim,$$

where $\sim$ is the equivalence relation generated by

$$(f^*(x), y) \sim (x, f_*(y)), \qquad \text{for all } f\colon [m] \to [n].$$

Recall the face functions σ_i and degeneracy functions ∂_i. Applying a simplicial set to these gave us the face and degeneracy maps s_i and d_i respectively. Applying the Yoneda embedding gives maps $S_i\colon \Delta^n \to \Delta^{n-1}$ and $D_i\colon \Delta^n \to \Delta^{n+1}$. The factorization of any morphism in Δ into a composition of face and degeneracy maps immediately gives us the following.

Corollary 1.33. The geometric realization of a simplicial set X can be computed using the formula

$$|K| = \left(\coprod_{n \geq 0} K_n \times |\Delta^n| \right) /\sim,$$

where $\sim$ is the equivalence relation generated by

$$(d_i(x), y) \sim (x, D_i(y)), \qquad (s_i(x), y) \sim (x, S_i(y)),$$

for all i.

The first identification tells us that we should identify any two standard n-simplices which share a face along that face. The second tells us that we should identify degenerate simplices to their corresponding nondegenerate simplices; that is, that degenerate simplices don't add anything new.

Theorem 1.34. The geometric realization $|X|$ of any simplicial set X is a CW complex. The cells of $|X|$ correspond to the nondegenerate simplices of X.

The adjunction

$$|\cdot| : \mathbf{Set}_\Delta \leftrightarrow \mathbf{Top} : \mathbf{Sing}$$

gives us a way of translating between the language of simplicial sets and the language of simplices in topological spaces.

7 *Skeleta and coskeleta*

For any $n \geq 0$, denote by $\iota_{\leq n} \colon \Delta_{\leq n} \hookrightarrow \Delta$ the full subcategory inclusion on objects $[0]$, $[1], \ldots, [n]$. We can restrict any simplicial set along this functor, giving its so-called *n-truncation.* This extends to a so-called *n-truncation functor.*

Definition 1.35 (truncation)**.** Let $n \geq 0$. The n-truncation functor is the pullback

$$\iota_{\leq n}^* = \mathrm{tr}_{\leq n} \colon \mathbf{Set}_\Delta \to \mathbf{Set}_{\Delta_{\leq n}}.$$

One visualizes the n-truncation of a simplicial set as forgetting simplices of degree greater than n.

As usual, this is a forgetful functor, so left and right adjoints (should they exist) provide algoritmic ways of attempting to solve the impossible problem of recovering the lost simplices.

Definition 1.36 (skeleton, coskeleton)**.** Let $n \geq 0$.

- The left adjoint to the n-truncation functor is called the n-skeleton functor, and denoted

$$\mathbf{sk}_{\leq n} \colon \mathbf{Set}_{\Delta_{\leq n}} \to \mathbf{Set}_\Delta .$$

- The right adjoint to the n-truncation functor is called the n-coskeleton functor, and denoted
$$\mathbf{sk}_{\leq n} \colon \mathbf{Set}_{\Delta_{\leq n}} \to \mathbf{Set}_{\Delta}\,.$$

Note that the skeleton and coskeleton functors are simply the left and right Kan extension functors, i.e.
$$\mathbf{sk}_{\leq n} = (\iota_{\leq n})_!, \qquad \mathbf{cosk}_{\leq n} = (\iota_{\leq n})_*.$$

For any simplicial set K, we can define the n-skeleton of K to be the simplicial set built by first forgetting simplices of degree greater than n, then adding them back in using the n-skeleton functor. Since the skeleton functor is a Kan extension, the n-skeleton of a simplicial set is the left Kan extension along the restriction. That is,
$$\mathrm{sk}_{\leq n}(K) = (\mathbf{sk}_{\leq n} \circ \mathrm{tr}_{\leq n})(K).$$

Similarly, we define its n-coskeleton to be
$$\mathrm{cosk}_{\leq n}(K) = (\mathbf{cosk}_{\leq n} \circ \mathrm{tr}_{\leq n})(K).$$

Example 1.37. Let K be any simplicial set. Let us compute the n-skeleton of K. By definition of the left Kan extension

$$\begin{tikzcd}
\Delta^{\mathrm{op}}_{\leq n} \arrow[rr, "K \circ \iota^{\mathrm{op}}_{\leq n}"] \arrow[dr, "\iota^{\mathrm{op}}_{\leq n}"'] & & \mathbf{Set} \\
& \Delta^{\mathrm{op}} \arrow[ur, "(\iota^{\mathrm{op}}_{\leq n})_!(K \circ \iota^{\mathrm{op}}_{\leq n})"'] &
\end{tikzcd}$$

we have, by the colimit formula, the following expression for the m-simplices of $\mathrm{sk}_{\leq n}K$.
$$(\mathrm{sk}_{\leq n}K)_m = \operatorname{colim}\left[(\iota^{\mathrm{op}}_{\leq n} \downarrow [m]) \to \Delta^{\mathrm{op}}_{\leq n} \to \mathbf{Set}\right]$$

When $m \leq n$, then the category $(\iota^{\mathrm{op}}_{\leq n} \downarrow [m])$ has the terminal object $([m], \mathrm{id}_{[m]})$, so the colimit is
$$(K \circ \mathrm{tr}_{\leq n})([m]) = K_m.$$

Thus, the n-skeleton of K agrees with K on simplices of degree less than or equal to n.

Now let $m \geq n$. In this case, the category $(\mathrm{tr}_{\leq n} \downarrow [m])$ does not have a terminal object.

Do at some point: building simplicial set via pushouts as in saturated hull of boundary inclusion is monos.

8 Kan complexes

Definition 1.38 (Kan complex)**.** A Kan complex is a simplicial set such that for every $n > 0$ and every $0 \leq i \leq n$, for every map
$$\Lambda^n_i \xrightarrow{f} K$$

there exists a map $\bar{f}$ (possibly not unique!) making the following diagram commute.

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "f"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

The above condition is known as the *horn-filling condition,* and $\bar{f}$ is called the *horn filler.* Roughly, the horn-filling condition says that any horn in K can be completed to a simplex.

A priori, it is not clear that this condition is interesting. However, we will now show that every singular set (i.e. simplicial set of the form $\mathbf{Sing}(X)$ for some topological space X) is a Kan complex. If we will in the end be interested in translating between topological spaces and simplicial sets, it will be useful to have a necessary condition for a simplicial complex to be a singular set.

Theorem 1.39. For any topological space X, the set $\mathbf{Sing}(X)$ is a Kan complex.

Proof. The injection

$$j\colon |\Lambda_i^n| \to |\Delta^n|$$

admits a continuous right inverse

$$p\colon |\Delta^n| \to |\Lambda_i^n|.$$

That is,

$$p \circ j = \mathrm{id}_{|\Lambda_i^n|}.$$

By applying the functor $|\cdot|$, we can take any diagram

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "f"] \arrow[d, hook] & \mathbf{Sing}(X) \\ \Delta^n & \end{tikzcd}$$

to the diagram

$$\begin{tikzcd} |\Delta_i^n| \arrow[r, "|f|=f'"] \arrow[d, "j"'] & |\mathbf{Sing}(X)| \\ |\Delta^n| & \end{tikzcd},$$

now in **Top**. The counit gives us a morphism $\eta_X\colon |\mathbf{Sing}(X)| \to X$

$$\begin{tikzcd} |\Lambda_i^n| \arrow[r, "\eta_X \circ f'"] \arrow[d, "j"'] & X \\ |\Delta^n| & \end{tikzcd}.$$

But the map j has a left inverse, which we can compose with f' as follow.

$$\begin{tikzcd} {|\Lambda^n_i|} \arrow[r, "f'"] \arrow[d, shift left, "j"] & X \\ {|\Delta^n|} \arrow[u, bend left, "p"] \arrow[ur, swap, "f \circ p"] & \end{tikzcd}$$

Applying the **Sing** functor again, we find

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\epsilon_{\Lambda^n_i}"] \arrow[d, hook] & \mathbf{Sing}(|\Lambda^n_i|) \arrow[r] \arrow[d] & \mathbf{Sing}(X) \\ \Delta^n \arrow[r, "\epsilon_{\Delta^n}"] & \mathbf{Sing}(|\Delta^n|) \arrow[ur] & \end{tikzcd} .$$

The square commutes by the naturality of ϵ. The triangle commutes because it is the image of a commuting triangle. Thus, taking the composition gives us a morphism $\Delta^n \to \mathbf{Sing}(X)$. □

Proposition 1.40. Let $\mathcal{C}$ be a small category.

1. All inner horns of $N(\mathcal{C})$ have a filler; that is, any morphism

$$\Lambda^n_i \xrightarrow{f} N(\mathcal{C})$$

   where $0 < i < n$ can be extended to a commuting diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, swap, "\bar{f}"] & \end{tikzcd} .$$

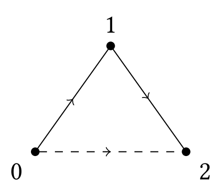
2. The simplicial set $N(\mathcal{C})$ has *all* horn fillers, i.e. is a Kan complex, if and only if $\mathcal{C}$ is a groupoid. Furthermore, this horn filler is unique for all $n \geq 2$.

Proof. There are no inner horns in degrees 0 or 1, so we start with a $(2, 1)$-horn.

Consider a horn

$$\Lambda^2_1 \longrightarrow N(\mathcal{C}) \ .$$

To build from this a map $\Delta^2 \to N(\mathcal{C})$, we have to fill in images of everything in Δ^2 missing from Λ^2_1, i.e. the dashed 1-simplex and the full 2-simplex.



When we map the horn Λ_1^2 into $N(\mathcal{C})$, we find two morphisms as follows.

$$\begin{tikzcd} & a \arrow[dr, "g"] & \\ b \arrow[ur, "f"] & & c \end{tikzcd}$$

A map of a 2-simplex into $N(\mathcal{C})$ consists of a commuting triangle in $\mathcal{C}$. Such a map making the diagram

$$\begin{tikzcd} \Lambda_1^2 \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

□

9 *Kan fibrations*

Definition 1.41 (Kan fibration)**.** Let $p\colon K \to S$ be a morphism of simplicial sets. We say that p is a Kan fibration if for every i with $0 \le i \le n$ and every diagram

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & K \arrow[d, "p"] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

there exists a (not necessarily unique!) morphism $\Delta^n \to K$ making the diagram

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & K \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & S \end{tikzcd}$$

commute.

This is a sort of a relative version of horn filling. Note that if we take $S = *$, the constant simplicial set (which is the terminal object in the category of simplicial sets), we recover something resembling notion of a horn filler. More precisely:

Corollary 1.42. A Kan fibration $K \to \Delta^0$ is the same as a Kan complex.

Proof. The bottom right triangle

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & K \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

contains no information. □