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Category-theoretic preliminaries

The theory of infinity categories rests on the shoulders of the theory of ordinary category theory. In this appendix, we recall some category-theoretic results that we will use. In general, we provide no proofs for these results; they are standard, and proofs can be found in any introductory text on category theory.

1 *Limits and colimits*

1.1 *Formulae for limits*

One can calculate limits using products and equalizers, and colimits using coproducts and coequalizers.

Fact 1.1. Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be locally small. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

- If $\mathcal{D}$ has small products and equalizers, then $\lim F$ is given by the following equalizer,

  $$\lim F \hookrightarrow \prod_{c \in \mathcal{C}} F(c) \underset{S}{\overset{R}{\rightrightarrows}} \prod_{\alpha \in \mathrm{Mor}(\mathcal{C})} F(\mathrm{codom}(\alpha))$$

  where the morphisms R and S are defined by their components

  $$R_\alpha = \pi_{F(\mathrm{codom}(\alpha))}\colon \prod_{c \in \mathcal{C}} F(c) \to F(\mathrm{codom}(\alpha))$$

  and

  $$S_\alpha = F(\alpha) \circ \pi_{\mathrm{dom}(\alpha)}\colon \prod_{c \in \mathcal{C}} F(c) \to F(\mathrm{codom}(\alpha))$$

- If $\mathcal{D}$ has small coproducts and coequalizers, then $\operatorname{colim} F$ is given by the following coequalizer,

  $$\coprod_{\alpha \in \mathrm{Mor}(\mathcal{C})} F(\mathrm{dom}(\alpha)) \underset{B}{\overset{A}{\rightrightarrows}} \coprod_{c \in \mathcal{C}} F(c) \twoheadrightarrow \operatorname{colim} F$$

where the morphisms A and B are defined by their components

$$A_\alpha = \iota_{F(\mathrm{dom}(\alpha))} \colon F(\mathrm{dom}(\alpha)) \to \coprod_{c \in \mathcal{C}} F(c)$$

and

$$B_\alpha = \iota_{\mathrm{codom}(\alpha)} \circ F(\alpha) \colon F(\mathrm{dom}(\alpha)) \to \coprod_{c \in \mathcal{C}} F(c)$$

1.2 Limits in functor categories

In this section, we review facts about limits and colimits in functor categories. Much of this text is devoted to one particular presheaf category $\mathbf{Set}_\Delta$, the category of functors $\Delta^{\mathrm{op}} \to \mathbf{Set}$,[1] and it will pay to understand categories of this form.

[1] Where Δ is the *simplex category*, Definition ??.

Fact 1.2. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

- If $\mathcal{C}$ has an initial object i, then $\lim F = F(i)$.
- If $\mathcal{C}$ has a terminal object i, then $\operatorname{colim} F = F(i)$.

Theorem 1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, with $\mathcal{D}$ small and $\mathcal{C}$ locally small.

- If $\mathcal{D}$ is complete, then so is $[\mathcal{C}, \mathcal{D}]$.
- If $\mathcal{D}$ is cocomplete, then so is $[\mathcal{C}, \mathcal{D}]$.

Corollary 1.4. Limits and colimits in functor categories are computed pointwise. That is, for any functor $F \colon J \to [\mathcal{C}, \mathcal{D}]$ and any $j \in J$,

$$(\lim F)(j) = \lim(Fj),$$

where the first limit is taken in $[\mathcal{C}, \mathcal{D}]$, and the second is taken in $\mathcal{D}$.

Corollary 1.5. A morphism $\eta \colon F \to G$ in a functor category is a monomorphism (resp. epimorphism) if and only if each component η_x is a monomorphism (resp. epimorphism).

Proof. A morphism $\eta \colon F \to G$ is a monomorphism if and only if the square

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & F \arrow[d, "\eta"] \\ F \arrow[r, "\eta"'] & G \end{tikzcd}$$

is a pullback square. But by Corollary 1.4, the above square is a pullback square if and only if its components are pullback squares, which in turn is true if and only if each component is a monomorphism.

The epimorphism case is dual. □

2 Kan extensions

2.1 Pointwise formulae for Kan extensions

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let $d \in \mathcal{D}$ be an object.

- We will denote by $\mathcal{C}_{/d}$ the category whose objects are pairs (c, f), where $c \in \mathcal{C}$ and $f\colon F(c) \to d$, and whose morphisms $(c, f) \to (c', f')$ are morphisms $h\colon c \to c'$ such that the triangle

$$\begin{tikzcd} F(c) \arrow[rr, "F(h)"] \arrow[dr, "f"'] & & F(c') \arrow[dl, "f'"] \\ & d & \end{tikzcd}$$

  commutes. Note that there is an obvious forgetful functor $\mathcal{C}_{/d} \to \mathcal{C}$.

- We will denote by $\mathcal{C}_{d/}$ the category whose objects are pairs (c, f), where $c \in \mathcal{C}$ and $f\colon d \to F(c)$, and whose morphisms $(c, f) \to (c', f')$ are morphisms $h\colon c \to c'$ such that the triangle

$$\begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F(c) \arrow[rr, "f(h)"] & & F(c') \end{tikzcd}$$

  commutes. Note that there is an obvious forgetful functor $\mathcal{C}_{d/} \to \mathcal{C}$.

The categories $\mathcal{C}_{/d}$ and $\mathcal{C}_{d/}$ are often referred to as (co)slice categories, comma categories, and under- and overcategories. There seems to be little agreement in the literature as to which of these terms refers to precisely which concept. We will call $\mathcal{C}_{d/}$ an *overcategory* and $\mathcal{C}_{/d}$ an *undercategory.*

The notatations used for these categories are as diverse as the names. What we are calling $\mathcal{C}_{/d}$ is often called $(F \downarrow d)$, for example.

Fact 1.6. Consider categories and functors as follows, where $\mathcal{C}$ is small.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "D"] \arrow[dr, "F"'] & & \mathcal{D} \\ & \mathcal{C}' & \end{tikzcd}$$

- If $\mathcal{D}$ has all small colimits, then the left Kan extension $F_! D$ exists and has values

$$(F_! D)(c') = \operatorname{colim}\left[\ \mathcal{C}_{c'/} \longrightarrow \mathcal{C} \xrightarrow{D} \mathcal{D}\ \right].$$

- If $\mathcal{D}$ has all small limits, then the right Kan extension $F_* D$ exists and has values

$$(F_* D)(c') = \lim\left[\ \mathcal{C}_{/c'} \longrightarrow \mathcal{C} \xrightarrow{D} \mathcal{D}\ \right].$$

2.2 Properties of Kan extensions

Theorem 1.7. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ locally small. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be fully faithful. If $\mathcal{D}$ has small colimits, then the unit

$$\epsilon\colon \mathrm{id}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})} \Rightarrow F^* \circ F_!$$

is a natural isomorphism.

Corollary 1.8. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ locally small. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be fully faithful.

- If $\mathcal{D}$ has small limits, then F_* is fully faithful.
- If $\mathcal{D}$ has small colimits, then $F_!$ fully faithful.

2.3 Yoneda extensions

For any locally small category $\mathcal{C}$, the *Yoneda embedding* is the functor

$$\mathcal{Y}\colon \mathcal{C} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) := \mathbf{Set}_{\mathcal{C}}; \qquad c \mapsto \mathrm{Hom}_{\mathcal{C}}(-, c).$$

The category $\mathbf{Set}_{\mathcal{C}}$ is the category of ($\mathbf{Set}$-valued) presheaves on $\mathcal{C}$. We will frequently be interested in Kan extensions along the Yoneda embedding; these are called *Yoneda extensions.* That is, for any functor $F\colon \mathcal{C} \to \mathcal{D}$, we will often want to compute the Kan extension $\mathcal{Y}_! F$.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\mathcal{Y}"'] & & \mathcal{D} \\ & \mathbf{Set}_{\mathcal{C}} \arrow[ur, "\mathcal{Y}_! F"'] & \end{tikzcd}$$

In this section, we prove some basic results about Yoneda extensions.

Lemma 1.9. Let $\mathcal{C}$ be a locally small category. For any functor $F \in \mathbf{Set}_{\mathcal{C}}$ we have an isomorphism

$$F \cong \operatorname{colim}\left[\mathcal{C}_{/F} \to \mathcal{C} \xrightarrow{\mathcal{Y}} \mathbf{Set}_{\mathcal{C}}\right].$$

Proof. By Theorem 1.7, the fully faithfulness of the Yoneda embedding, and the definition of the restriction functor, we have that

$$F \cong (\mathcal{Y}^* \circ \mathcal{Y}_!)(F) \cong (\mathcal{Y}_! \mathcal{Y})(F).$$

The functor $\mathcal{Y}_! \mathcal{Y}$ comes from the following Kan extension.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\mathcal{Y}"] \arrow[dr, "\mathcal{Y}"'] & & \mathbf{Set}_{\mathcal{C}} \\ & \mathbf{Set}_{\mathcal{C}} \arrow[ur, "\mathcal{Y}_! \mathcal{Y}"'] & \end{tikzcd}$$

The colimit formula for left Kan extensions (Fact 1.6) then tells us that

$$F \cong \operatorname{colim}\left[\mathcal{C}_{/F} \to \mathcal{C} \xrightarrow{\mathcal{Y}} \mathbf{Set}_{\mathcal{C}}\right]$$

as required. □

Lemma 1.10. Let $\mathcal{I}$ be a small category, and let $\mathcal{C}$ be locally small and cocomplete. Let F be a functor as below. Then the Yoneda extension $\mathcal{Y}_! F$ is left adjoint to the functor

$$G\colon \mathcal{C} \to \mathbf{Set}_{\mathcal{I}}; \qquad c \mapsto \mathcal{C}(F(-), c).$$

$$\begin{tikzcd} \mathcal{J} \arrow[rr, "F"] \arrow[dr, "\mathcal{Y}"'] & & \mathcal{C} \\ & \mathbf{Set}_{\mathcal{J}} \arrow[ur, "\mathcal{Y}_! F"'] & \end{tikzcd} \qquad \mathcal{Y}_! F : \mathbf{Set}_{\mathcal{J}} \leftrightarrow \mathcal{C} : G$$

Proof. Since $\mathcal{C}$ is cocomplete, we are entitled to use the colimit formula for left Kan extensions. That is, we can write

$$(\mathcal{Y}_! F)(H) = \operatorname{colim}\left[(\mathcal{Y}/H) \xrightarrow{\mathcal{U}} \mathcal{J} \xrightarrow{F} \mathcal{C}\right] = \operatorname{colim}\left[F \circ \mathcal{U}\right].$$

Thus, we have the following string of natural isomorphisms.

$$\begin{aligned} \mathcal{C}((\mathcal{Y}_! F)(H), c) &\cong \mathcal{C}(\operatorname{colim}[F \circ \mathcal{U}], c) \\ &\cong \lim \mathcal{C}(F(-), c) \circ \mathcal{U} \\ &\cong \lim G(c) \circ \mathcal{U} \\ &\cong \lim \mathbf{Set}_{\mathcal{J}}(\mathcal{Y}(-), G(c)) \circ \mathcal{U}. \\ &\cong \mathbf{Set}_{\mathcal{J}}(\operatorname{colim} \mathcal{Y} \circ \mathcal{U}, G(c)) \\ &\cong \mathbf{Set}_{\mathcal{J}}(H, G(c)). \end{aligned}$$

□

Corollary 1.11. Let $\mathcal{J}$ be a small category, and let $\mathcal{C}$ be locally small and cocomplete. Let $F \colon \mathcal{J} \to \mathcal{C}$. Then the Yoneda extension $\mathcal{Y}_! F$ preserves colimits.

Proof. By Lemma 1.10, $\mathcal{Y}_! F$ is a left adjoint. □

Theorem 1.12. Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be cocomplete. Let $F \colon \mathbf{Set}_{\mathcal{C}} \to \mathcal{D}$ be any functor. Then F preserves colimits if and only if it is the left Yoneda extension of its restriction $\mathcal{Y}^* F \colon \mathcal{C} \to \mathcal{D}$, i.e. if $F = (\mathcal{Y}_! \mathcal{Y}^*)(F)$.

Proof. Assume that $F = (\mathcal{Y}_! \mathcal{Y}^*)(F)$. By Lemma 1.10, $(\mathcal{Y}_! \mathcal{Y}^*)(F)$ is a left adjoint, hence preserves colimits.

Now assume that F preserves colimits. Pick some functor $G \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. By Lemma 1.9, we can write

$$\begin{aligned} F(G) &\cong F\left(\varinjlim\left[(\mathcal{Y}/G) \to \mathcal{C} \xrightarrow{\mathcal{Y}} \mathbf{Set}_{\mathcal{C}}\right]\right) \\ &\cong \varinjlim\left[(\mathcal{Y}/G) \to \mathcal{C} \xrightarrow{\mathcal{Y}} \mathbf{Set}_{\mathcal{C}} \xrightarrow{F} \mathcal{D}\right] \\ &\cong \varinjlim\left[(\mathcal{Y}/G) \to \mathcal{C} \xrightarrow{\mathcal{Y}^* F} \mathcal{D}\right] \\ &\cong \mathcal{Y}_!(\mathcal{Y}^* F)(G). \end{aligned}$$

□

Corollary 1.13. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ locally small. There is an equivalence of categories

$$[\mathbf{Set}_{\mathcal{C}}, \mathcal{D}]_{\text{cocontinuous}} \cong [\mathcal{C}, \mathcal{D}].$$

That is, colimit preserving functors $\mathbf{Set}_{\mathcal{C}} \to \mathcal{D}$ are equivalent to functors $\mathcal{C} \to \mathcal{D}$ via restriction along the Yoneda embedding.

Proof. We have an adjunction

$$\mathcal{Y}_! : [\mathcal{C}, \mathcal{D}] \leftrightarrow [\mathbf{Set}_{\mathcal{C}}, \mathcal{D}] : \mathcal{Y}^*.$$

By Corollary 1.11, the image of $\mathcal{Y}_!$ is completely contained in $[\mathbf{Set}_{\mathcal{C}}, \mathcal{D}]_{\text{cocontinuous}}$, so the above adjunction extends to an adjunction

$$\mathcal{Y}_! : [\mathcal{C}, \mathcal{D}] \leftrightarrow [\mathbf{Set}_{\mathcal{C}}, \mathcal{D}]_{\text{cocontinuous}} : \mathcal{Y}^*.$$

We know that the Yoneda embedding is fully faithful. Thus, by Theorem 1.7, we have that the unit is an isomorphism. Also, by Theorem 1.12, the counit is an isomorphism, so the above adjunction is in fact an equivalence of categories. □

2.4 Day convolution

Let $(\mathcal{C}, \otimes)$ be a monoidal category. The category $\mathbf{Set}_{\mathcal{C}}$ of functors from $\mathcal{C}^{\text{op}}$ to $\mathbf{Set}$ inherits a monoidal structure as follows.

Given functors $F, G \colon \mathcal{C} \to \mathbf{Set}$, we can define a functor

$$F \bar{\times} G \colon \mathcal{C}^{\text{op}} \times \mathcal{C}^{\text{op}} \to \mathbf{Set}; \qquad (c, d) \mapsto F(c) \times G(d).$$

Definition 1.14 (Day convolution)**.** The Day convolution of F and G, denoted $F \otimes G$, is given by the left Kan extension of $F \bar{\times} G$ along $\otimes^{\text{op}}$.

$$\begin{tikzcd}
\mathcal{C}^{\text{op}} \times \mathcal{C}^{\text{op}} \arrow[rr, "F \bar{\times} G"] \arrow[dr, "\otimes^{\text{op}}"'] & & \mathbf{Set} \\
& \mathcal{C}^{\text{op}} \arrow[ur, "F \otimes G"'] &
\end{tikzcd}$$

The colimit formula for left Kan extensions tells us that

$$(F \otimes G)(c) = \operatorname{colim} \left[(c \downarrow \otimes^{\text{op}}) \to \mathcal{C}^{\text{op}} \times \mathcal{C}^{\text{op}} \xrightarrow{F \bar{\times} G} \mathbf{Set} \right]$$

3 Enriched category theory

Definition 1.15 (enriched category)**.** Let $(\mathcal{M}, \otimes, 1_{\mathcal{M}}, \alpha, \lambda, \rho)$ be a monoidal category. A $\mathcal{M}$-enriched category $\mathcal{C}$ consists of the following data.

- A set $\text{ob}(\mathcal{C})$ of objects.

- For each pair x, y of objects, an object $\mathcal{C}(x,y) \in \mathcal{M}$ of morphisms.
- For each object $x \in \mathcal{C}$, a morphism
$$1_{\mathcal{M}} \to \mathcal{C}(x,x)$$
called the *unit*
- For objects x, y, and z, a morphism
$$\mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \to \mathcal{C}(x,z)$$
called the *composition.*

This composition must be unital and associative, i.e. the diagrams

$$\begin{tikzcd} & \mathcal{C}(x,y) \arrow[r] \arrow[d] \arrow[dr, "\mathrm{id}"] & \mathcal{C}(y,y) \otimes \mathcal{C}(x,y) \arrow[d] \\ & \mathcal{C}(x,y) \otimes \mathcal{C}(x,x) \arrow[r] & \mathcal{C}(x,y) \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \otimes \mathcal{C}(z,w) \arrow[r] \arrow[d] & \mathcal{C}(x,y) \otimes \mathcal{C}(y,w) \arrow[d] \\ \mathcal{C}(x,z) \otimes \mathcal{C}(z,w) \arrow[r] & \mathcal{C}(x,w) \end{tikzcd}$$

must commute (where unitors and associators are notationally supressed).

Lemma 1.16. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category, and let $F\colon \mathcal{M} \to \mathcal{N}$ be a monoidal functor with data
$$\Phi_{x,y}\colon F(x) \otimes_{\mathcal{N}} F(y) \to F(x \otimes_{\mathcal{M}} y), \qquad \phi\colon 1_{\mathcal{N}} \to F(1_{\mathcal{M}}).$$
This data allows us to build from $\mathcal{C}$ a category enriched over $\mathcal{N}$.

Proof.

- We apply F to each hom-object in $\mathcal{M}$ to get a new hom-object in $\mathcal{N}$;
$$\mathcal{C}(x,y)_{\mathcal{N}} = F(\mathcal{C}(x,y)_{\mathcal{M}}).$$
- The unit is the composition
$$\begin{tikzcd} 1_{\mathcal{N}} \arrow[r] & F(1_{\mathcal{M}}) \arrow[r] & F(\mathcal{C}(x,y)_{\mathcal{M}}) \arrow[r, equal] & \mathcal{C}(x,y)_{\mathcal{N}} \end{tikzcd}$$
- The composition is given by the obvious.

□