

Notes on Deligne's theorem

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Chapter 1

Algebra

Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any questions you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.

M. Atiyah, 2002

In this chapter we collect some basic definitions and corollaries so that we may refer to them later. The methods of proof are often non-standard, as we prefer to provide proofs which lend themselves to categorification.

1.1 Monoids

Definition 1.1.1 (monoid). A monoid is a set M together with a binary operation $\cdot : M \times M \rightarrow M$ such that

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, c , and
- There exists an element $1 \in M$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in M$.

Lemma 1.1.1. The above definition is equivalent to the following.

Let M be a set, \cdot an associative binary operation on M . Then we say M is a monoid if there is an element $1 \in M$ such that $1 \cdot 1 = 1$ and the maps $M \rightarrow M$, $a \mapsto 1 \cdot a$ and $a \mapsto a \cdot 1$ are bijections.

Proof. If $1 \cdot a = a$ for all a , then the maps $a \mapsto 1 \cdot a$ and $a \mapsto a \cdot 1$ are trivially bijections.

Now, if the map $a \mapsto 1 \cdot a$ is a bijection, then every element a of M can be written $1 \cdot a'$ for some a' in M . Then

$$\begin{aligned} 1 \cdot a &= 1 \cdot (1 \cdot a') \\ &= (1 \cdot 1) \cdot a' \\ &= 1 \cdot a' \\ &= a. \end{aligned}$$

The proof for right-multiplication by 1 is identical. □

1.2 Groups

Definition 1.2.1 (group). A group (G, \cdot) is a monoid with inverses, i.e. a set G with a function $\cdot : G \times G \rightarrow G$ which is

1. Associative: $(ab)c = a(bc)$
2. Unital: There is an element $e \in G$ such that $eg = ge$ for all $g \in G$
3. Invertible: for all $g \in G$, there is an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$

Lemma 1.2.1. The above definition is equivalent to the following.

A group is a monoid such that for all $g \in G$, the maps $G \rightarrow G; h \mapsto gh$ and $h \mapsto hg$ are bijections.

Proof. If all elements of G are invertible, left-multiplication is obviously bijective. Now suppose left-multiplication by g is a bijection. Then for any $h \in G$, there exists an element $h' \in G$ such that $g \cdot h' = h$. But this is certainly true in particular when $h = e$, so all elements of G are invertible.

The right-multiplication case is identical. □

Definition 1.2.2 (abelian group). A group G is abelian if for all $a, b \in G$, $ab = ba$.

Definition 1.2.3 (free abelian group). Let E be a set. The free abelian group generated by E is the group whose elements are formal finite sums of elements of E .

Definition 1.2.4 (direct sum of abelian groups). Let A, B be abelian groups. Their direct sum, denote $A \oplus B$, is the group whose

1. underlying set is $A \times B$, the cartesian product of the underlying sets of A and B , and whose
2. multiplication is given component-wise.

1.3 Rings

Definition 1.3.1. (ring) A ring $(R, +, \cdot)$ is a set R with two binary operations $+$ and \cdot such that

0. R is closed under $+$ and \cdot .
1. R is an Abelian group with respect to $+$.
2. \cdot is associative: $(xy)z = x(yz)$.
3. \cdot is distributive from the left and from the right: $x(y + z) = xy + xz$, $(x + y)z = xz + yz$.

Further,

- R is a commutative ring if $a \cdot b = b \cdot a$ for all $a, b \in R$.
- R is a ring with identity if R contains an element 1_R such that $1_R \cdot x = x \cdot 1_R$ for all $x \in R$.

Theorem 1.3.1 (elementary properties of rings). Let R be a ring. Then

1. $0a = a0 = 0$ for all $a \in R$
2. $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$.
3. $(-a)(-b) = ab$ for all $a, b \in R$.
4. $(na)b = a(nb) = n(ab)$ for all $n \in \mathbb{Z}$, $a, b \in R$.
- 5.

$$\left(\sum_{i=1}^n a_i \right) \left(\sum_{j=1}^m b_j \right) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j.$$

Proof.

1. $0a = (0 + 0)a = 0a + 0a \implies 0a = 0$.
2. $0 = 0b = (a - a)b = ab + (-a)b \implies ab + (-a)b = 0 \implies (-a)b = -(ab)$. The other side is similar.
3. By the previous problem, $(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$.
4. Suppose it is true for n . Then $((n + 1)a)b = (na)b + ab = a(nb) + ab = a((n + 1)b)$, and similarly for the other.
5. Two uses of induction.

□

Definition 1.3.2 (zero-divisor). A nonzero element a in a ring R is said to be a left (right) zero divisor if there exists $b \neq 0$ such that $ab = 0$ ($ba = 0$). A zero divisor is an element of R which is both a left and a right zero divisor.

Definition 1.3.3 (invertible element). An element a in a ring R is said to be left- (right-)invertible if there exists $b \in R$ such that $ab = 1$ ($ba = 1$). An element $a \in R$ which is both left- and right-invertible is said to be invertible, or a unit.

Definition 1.3.4 (integral domain). A commutative ring R with identity $1_R \neq 0$ and no zero divisors is called a division ring.

Definition 1.3.5 (division ring). A ring D with identity $1_D \neq 0$ in which every nonzero element is a unit is called a division ring.

Definition 1.3.6 (field). A field is a commutative ring.

Example 1.3.1 (important example). Let X be a topological space. The set

$$C^1(X) = \{f: X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$$

is a commutative ring with identity, with addition and multiplication defined pointwise. To see this, we need to check the axioms in Definition 1.3.1:

0. The sum of two continuous functions is continuous; the product of two continuous functions is continuous.
1. $C^1(X)$ inherit its abelian group structure from the real numbers; the additive identity is the function which maps all $x \in X$ to $0 \in \mathbb{R}$.
2. Associativity is inherited from the real numbers
3. Distributivity is inherited from the real numbers.

Definition 1.3.7 (ring homomorphism). Let R and S be rings. A function $f: R \rightarrow S$ is a ring homomorphism if

$$f(a + b) = f(a) + f(b), \quad \text{and} \quad f(ab) = f(a)f(b).$$

Definition 1.3.8 (ring isomorphism). A ring isomorphism is a bijective ring homomorphism.

Definition 1.3.9 (ideal). Let R be a ring and I a nonempty subset of R . I is called a left (right) ideal if

1. I is closed under addition:

$$a, b \in I \implies a + b \in I.$$

2. I absorbs elements of R under multiplication:

$$r \in R \quad \text{and} \quad x \in I \implies rx \in I \quad (xr \in I).$$

If I is both a left and a right ideal, it is called an ideal.

Theorem 1.3.2. Let R be a ring, and let $\{A_i \mid i \in I\}$ be a set of [left] ideals of X . Then

$$A = \bigcap_{i \in I} A_i$$

is an ideal.

Proof. According to Definition 1.3.9, we need to check two things:

1. Closure under addition: let $a, b \in A$. Then $a + b$ is in each of the A_i , so it must be in their intersection.
2. Absorption: for any $a \in A$ and any $r \in R$, ra must be in A_i for all i ; hence it must be in their intersection.

□

Definition 1.3.10 (ideal generated by a subset). Let X be a subset of a ring R . Let $\{A_i \mid i \in I\}$ be the family of all [left] ideals in R which contain X . Then

$$\bigcap_{i \in I} A_i$$

is called the [left] ideal generated by X , and is denoted (X) . The elements of X are called the generators of (X) .

Theorem 1.3.3. Let R be a ring. Let $a \in R$ and $X \subseteq R$. We have the following.

1. The principal ideal (a) consists of all elements of the form

$$ra + as + na + \sum_{i=1}^m r_i a s_i; \quad r, s, r_i, s_i \in R, \quad M \in \mathbb{N}^*, \quad n \in \mathbb{Z}.$$

2. If R has an identity, then

$$(a) = \left\{ \sum_{i=1}^n r_i a s_i \mid r_i, s_i \in R, \quad n \in \mathbb{N}^* \right\}.$$

3. If a is the center of R , then

$$(a) = \{ra + na \mid r \in R; \quad n \in \mathbb{N}^*\}.$$

4. $Ra = \{ra \mid r \in R\}$ is a left ideal in R . If R has identity, then $a \in Ra$.

5. If R has an identity and a is in the center of R then $Ra = (a)$.

6. If R has an identity and X is in the center of R , then the ideal (X) consists of all finite sums

$$\sum_i r_i a_i, \quad n \in \mathbb{N}^*, \quad r_i \in R, \quad a_i \in X.$$

Proof.

1. We prove that any ideal which contains a must contain each term individually; since ideals must be closed under addition, the ideal must contain arbitrary sums of all such terms. Next we prove that the prescription given for (a) is in fact an ideal.
 - Since by definition $a \in (a)$ and (a) is closed under left- and right-multiplication by arbitrary elements of r , for any $r \in R$, ra must be in (a) .
 - Similarly, as must be in the

□

Definition 1.3.11 (principal ideal). An ideal generated by a single element is called principal.

Definition 1.3.12 (principal ideal ring). A principal ideal ring is a ring in which every ideal is principal.

Definition 1.3.13 (principal ideal domain). An integral domain (Definition 1.3.4) is

1.4 Modules

Definition 1.4.1 (module). Let R be a ring. A (left) R -module is an additive abelian group A together with a function $*$: $R \times A \rightarrow A$ such that for all $r, s \in R$ and $a, b \in A$,

1. $r * (a + b) = r * a + r * b$
2. $(r + s) * a = r * a + s * a$
3. $(rs) * a = r * (s * a)$.

If R has an identity element 1_R and $1_R * a = a$ for all $a \in A$, then A is said to be a unitary R -module.

Note 1.4.1. Right R -modules are defined in the obvious way.

Note 1.4.2. We will now stop notationally differentiating between R -multiplication and A -multiplication, using juxtaposition for both.

Definition 1.4.2 (bimodule). A R - S -bimodule is an Abelian group A which is a left R -module and a right S -module, such that

$$(ra)s = r(as) \quad \text{for all } r \in R, \quad s \in S, \quad a \in A.$$

We say that the left-multiplication and the right multiplication are *consistent*.

Theorem 1.4.1. If R is a commutative ring and A is a left R -module, then A can be canonically transformed into a right module or a bimodule.

Proof. With $ar \equiv ra$, A is both a right R -module and an R -bimodule. The axioms in Definition 1.4.1 are immediate. \square

Definition 1.4.3 (nilpotent). Let R be a ring. An element $r \in R$ is said to be nilpotent if

$$r^n = 0 \quad \text{for some } n \in \mathbb{N}^+.$$

Definition 1.4.4 (commutator). Let R be a ring, $a, b \in R$. The commutator of a and b is

$$[a, b] \equiv ab - ba.$$

Theorem 1.4.2. Let R be a ring, $a, b \in R$. The commutator satisfies the following properties.

- $[a, b] = -[b, a]$
- $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

Further, if A is a ring and B is an A -algebra (in the sense of Definition 1.6.1), then for any $a \in A$, $b, c \in B$, the following identity holds.

$$a[b, c] = [ab, c] = [b, ac] = [b, ca] = [b, a]c.$$

Proof.

- $[a, b] = ab - ba = -(ba - ab) = -[b, a]$.
- We have

$$\begin{aligned} [a, [b, c]] &= a(bc - cb) - (bc - cb)a \\ [b, [c, a]] &= b(ca - ac) - (ca - ac)b \\ [c, [a, b]] &= c(ab - ba) - (ab - ba)c, \end{aligned}$$

so

$$\begin{aligned} [a, [b, c]] + [b, [c, a]] + [c, [a, b]] &= abc - acb - bca + cba \\ &\quad + bca - bac - cab + acb \\ &\quad + cab - cba - abc + bac. \end{aligned}$$

Terms cancel in pairs.

- $a[b, c] = a(bc - cb) = (ab)c - c(ab) = [ab, c] = b(ac) - (ac)b = [b, ac] = (bc - cb)a = [b, c]a.$

□

Definition 1.4.5 (free module). Let E be a set, R be a ring. The free left module of E over R , denoted $R^{(E)}$, is the module whose elements are finite formal R -linear combinations of the elements of E .

Example 1.4.1. The free abelian group over a set E is also a free \mathbb{Z} -module over E with the identification

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = na.$$

1.5 Vector spaces

Definition 1.5.1 (graded vector space). Let M be a monoid. An M -graded vector space is a

1.6 Algebras

Definition 1.6.1 (algebra over a ring). Let R be a commutative ring. An R -algebra is a ring A which is also an R -module (with left-multiplication $*$: $R \times A \rightarrow A$) such that the multiplication map \cdot : $A \times A \rightarrow A$ is R -bilinear, i.e.

$$r * (a \cdot b) = (r * a) \cdot b = a \cdot (r * b), \quad \text{for any } a, b \in A, \quad r \in R.$$

Theorem 1.6.1. Let A, R be unital rings, R commutative, and let $f: R \rightarrow A$ be a unital ring homomorphism. Then A naturally has the structure of an R -module. Furthermore, if the image of R is in the center of A , then A naturally has the structure of an R -algebra.

Proof. Define the action

$$*: R \times A \rightarrow A; \quad (r, a) \mapsto f(r) \cdot a.$$

The verification that this makes A into an R -module is trivial; we need only check left and right distributivity, associativity, and that $1_R * a = a$. We omit this.

To see that what we have is even an algebra, we need to check that $*$ is bilinear, i.e.

$$r * (a \cdot b) = f(r) \cdot (a \cdot b) = (f(r) \cdot a) \cdot b = (r * a) \cdot b$$

and

$$r * (a \cdot b) = f(r) \cdot a \cdot b = a \cdot f(r) \cdot b = a \cdot (r * b).$$

□

Often we are interested specifically in algebras over a field. In that case we have the following.

Definition 1.6.2 (algebra over a field). An algebra A over a field k is a k -vector space with a k -bilinear product $A \times A \rightarrow A$.

1.7 Tensor products

In this section we give a brief overview of the construction of the tensor product.

Definition 1.7.1 (middle-linear map). Let R be a ring, M be a right R -module, N a left R -module, and G an abelian group. A map $\varphi: M \times N \rightarrow G$ is said to be R -middle-linear if for all $m, m' \in M$, $n, n' \in N$, and $r \in R$ we have

1. $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$
2. $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$

$$3. \varphi(m \cdot r, n) = \varphi(m, r \cdot n).$$

Definition 1.7.2 (tensor product of modules). Let R be a ring, M be a right R -module, N a left R -module. The tensor product of M and N , denoted $M \otimes_R N$, is constructed as follows.

Let F be the free abelian group generated by $M \times N$ (Definition 1.2.3). Let K be the subgroup of F generated by elements of the form

- $(m, n + n') - (m, n) - (m, n')$
- $(m + m', n) - (m, n) - (m', n)$
- $(m \cdot r, n) - (m, r \cdot n)$.

The tensor product of M and N is then

$$M \otimes_R N = F/K.$$

Definition 1.7.3 (tensor algebra over a module). The tensor algebra

Note 1.7.1. If R is commutative, then we can canonically make M and N into bimodules. In this case we don't need to distinguish between left and right multiplication. If R is a field, we recover the notion of the 'standard' tensor product, which is usually given by the following definition.

Definition 1.7.4 (tensor product of vector spaces). Let k be a field, V and W k -vector spaces. The tensor product $V \otimes W$ is the vector space defined as follows.

Denote by $\mathcal{F}(V \times W)$ the free vector space over $V \times W$. Consider the vector subspace K of $\mathcal{F}(V \times W)$ generated by elements of the forms

- $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$
- $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$
- $(\alpha v, w) - \alpha(v, w)$
- $(v, \alpha w) - \alpha(v, w)$

for all $v_1, v_2 \in V, w_1, w_2 \in W, \alpha \in k$. Then define

$$V \otimes W = \mathcal{F}(V \times W)/K.$$

Pairs $(v, w) \in V \times W$ are called representing tuples.

Definition 1.7.5 (tensor product of linear maps). Let V, W, X , and Y be k -vector spaces, and let $S: V \rightarrow X, T: W \rightarrow Y$. Then the tensor product of S and T is the map

$$S \otimes T: V \otimes W \rightarrow X \otimes Y; \quad (v \otimes w) \mapsto S(v) \otimes T(w).$$

Of course, one must show that this is well-defined by showing that it vanishes on the subspace by which one quotients in the definition of the tensor product. But this is trivial.

Chapter 2

Categories

In this chapter (and to some extent the following chapters), I will do my best to stick to the conventions used over at the nlab ([20]).

2.1 Structure internal to categories

2.1.1 Basic definitions

Definition 2.1.1 (category). A category \mathbf{C} consists of

- a class $\text{Obj}(\mathbf{C})$ of *objects*, and
- for every two objects $A, B \in \text{Obj}(\mathbf{C})$, a class $\text{Hom}(A, B)$ of *morphisms* with the following properties.
 1. For $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(B, C)$, there is an associated morphism

$$g \circ f \in \text{Hom}(A, C),$$

called the *composition* of f and g .

2. This composition is associative.
3. For every $A \in \text{Obj}(\mathbf{C})$, there is at least one morphism 1_A , called the *identity morphism* which functions as both a left and right identity with respect to the composition of morphisms.

Note 2.1.1. There is a reason we say that the objects and morphisms of a category are a *class* rather than a set. It may be that there may be ‘too many’ objects to be contained in a set. For example, we will see that there is a category of sets, but there is no set of all sets. Categories whose objects and/or morphisms *are* small enough to be contained in a set will play an especially important role.

Notation 2.1.1. Following Aluffi ([10]), we will use the sans-serif font **mathsf** to denote categories. For example \mathbf{C} , \mathbf{Set} .

If ever it is potentially unclear which category we are talking about, we will add a subscript to Hom , writing for example $\text{Hom}_{\mathbf{C}}(A, B)$ instead of $\text{Hom}(A, B)$.

Example 2.1.1. The prototypical category is \mathbf{Set} , the category whose objects are sets and whose morphisms are set functions.

Example 2.1.2 (category with one object). The category $\mathbf{1}$, where $\text{Obj}(\mathbf{1})$ is the singleton $\{*\}$, and the only morphism is the identity morphism $\text{id}_*: * \rightarrow *$.

Example 2.1.3. Pretty much all standard algebraic constructions naturally live in categories. For example, we have

- the category \mathbf{Grp} , whose objects are groups and whose morphisms are group homomorphisms;

- the category **Ab**, whose objects are abelian groups and whose morphisms are group homomorphisms;
- the category **Ring**, whose objects are rings and whose morphisms are ring homomorphisms;
- the category **$R\text{-Mod}$** , whose objects are modules over a ring R and whose morphisms are module homomorphisms;
- the category **Vect_k** , whose objects are vector spaces over a field k and whose morphisms are linear maps;
- the category **$k\text{-Alg}$** , whose objects are algebras over a field k and whose morphisms are algebra homomorphisms

Example 2.1.4. In addition to *algebraic* structures between sets, categories help to formulate geometric structures, such as

- the category **Top**, whose objects are topological spaces and whose morphisms are continuous maps;
- the category **Met**, whose objects are metric spaces and whose morphisms are metric maps;
- the category **Man^p** , whose objects are manifolds of class C^p and whose morphisms are p -times differentiable functions;

and many more.

Definition 2.1.2 (opposite category). Let \mathbf{C} be a category. The opposite category \mathbf{C}^{op} is the category whose objects are the same as the objects $\text{Obj}(\mathbf{C})$ and whose morphisms $f \in \text{Hom}_{\mathbf{C}^{\text{op}}}(A, B)$ are defined to be the morphisms $\text{Hom}_{\mathbf{C}}(B, A)$.

That is to say, the opposite category is the category one gets by formally reversing all the arrows in a category. If $f \in \text{Hom}_{\mathbf{C}}(A, B)$, i.e. $f: A \rightarrow B$, then in \mathbf{C}^{op} , $f: B \rightarrow A$.

Definition 2.1.3 (subcategory). Let \mathbf{C} be a category. A subcategory \mathbf{S} of \mathbf{C} is defined as follows.

- The objects $\text{Obj}(\mathbf{S})$ of \mathbf{S} are a subcollection of the objects of \mathbf{C}
- For $S, T \in \text{Obj}(\mathbf{S})$, the morphisms $\text{Hom}_{\mathbf{S}}(S, T)$ are a subcollection of the morphisms $\text{Hom}_{\mathbf{C}}(S, T)$ such that
 - for every $S \in \text{Obj}(\mathbf{S})$, the identity $1_S \in \text{Hom}_{\mathbf{S}}(S, S)$.
 - for all $f \in \text{Hom}_{\mathbf{S}}(S, T)$ and $g \in \text{Hom}_{\mathbf{S}}(T, U)$, the composite $g \circ f \in \text{Hom}_{\mathbf{S}}(S, U)$.

If \mathbf{S} is a subcategory of \mathbf{C} , we will write $\mathbf{S} \subseteq \mathbf{C}$.

Definition 2.1.4 (full subcategory). Let \mathbf{C} be a category, $\mathbf{S} \subseteq \mathbf{C}$ a subcategory. We say that \mathbf{S} is full in \mathbf{C} if for every $S, T \in \text{Obj}(\mathbf{S})$, $\text{Hom}_{\mathbf{S}}(S, T) = \text{Hom}_{\mathbf{C}}(S, T)$.

Definition 2.1.5 (isomorphism). Let \mathbf{C} be a category, $A, B \in \text{Obj}(\mathbf{C})$. A morphism $f \in \text{Hom}(A, B)$ is said to be an isomorphism if there exists a morphism $g \in \text{Hom}(B, A)$ such that

$$g \circ f = 1_A, \quad \text{and} \quad f \circ g = 1_B.$$

Definition 2.1.6 (monomorphism). Let \mathbf{C} be a category, $A, B \in \text{Obj}(\mathbf{C})$. A morphism $f: A \rightarrow B$ is said to be a monomorphism if for all $Z \in \text{Obj}(\mathbf{C})$ and all $g_1, g_2: Z \rightarrow A$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

$$Z \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} A \xrightarrow{f} B$$

Theorem 2.1.1. In **Set**, monomorphisms are precisely injections.

Proof. Suppose $f: A \rightarrow B$ is a monomorphism. Then for any set Z and any maps $g_1, g_2: Z \rightarrow A$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. In particular, take $Z = \{*\}$ and suppose $g_1: * \mapsto a_1$ and $g_2: * \mapsto a_2$. Then $(f \circ g_1)(*) = f(a_1)$ and $(f \circ g_2)(*) = f(a_2)$, so

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

But this is exactly the definition of injectivity.

Now suppose that f is injective. Then for any Z and g_1, g_2 as above,

$$f \circ g_1(z) = f \circ g_2(z) \implies g_1(z) = g_2(z) \quad \text{for all } z \in Z.$$

But this means that $g_1 = g_2$, so f is mono. □

Example 2.1.5. In \mathbf{Vect}_k , monomorphisms are injective linear maps.

Definition 2.1.7 (epimorphism). Let \mathbf{C} be a category, $A, B \in \mathbf{Obj}(\mathbf{C})$. A morphism $f: A \rightarrow B$ is said to be a epimorphism if for all $Z \in \mathbf{Obj}(\mathbf{C})$ and all $g_1, g_2: B \rightarrow Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

$$A \xrightarrow{f} B \xrightarrow[g_2]{g_1} Z$$

Example 2.1.6. In \mathbf{Set} , epimorphisms are surjections.

Example 2.1.7. In \mathbf{Vect}_k , epimorphism are surjective linear maps.

Definition 2.1.8 (small, locally small category, hom-set). A category \mathbf{C} is

- small if $\mathbf{Obj}(\mathbf{C})$ is a set.
- locally small if for all $A, B \in \mathbf{Obj}(\mathbf{C})$, $\mathbf{Hom}_{\mathbf{C}}(A, B)$ is a set. In this case, we call $\mathbf{Hom}_{\mathbf{C}}(A, B)$ the hom-set. (Actually, terminology is often abused, and $\mathbf{Hom}_{\mathbf{C}}(A, B)$ is called a hom-set even if it is not a set.)

2.1.2 Functors

Just as morphisms connect objects in the same category, functors allow us to connect different categories. In fact, functors can be viewed as morphisms in a ‘category of categories.’

Definition 2.1.9 (functor). Let \mathbf{C} and \mathbf{D} be categories. A functor \mathcal{F} from \mathbf{C} to \mathbf{D} is a mapping which associates

- to each object $X \in \mathbf{Obj}(\mathbf{C})$ an object $\mathcal{F}(X) \in \mathbf{Obj}(\mathbf{D})$.
- to each morphism $f \in \mathbf{Hom}(X, Y)$ a morphism $\mathcal{F}(f)$ such that $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ for all X , and one of the two following properties are satisfied.
 - The morphism $f \in \mathbf{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$ and if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, then

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

In this case we say that \mathcal{F} is covariant.

- The morphism $f \in \mathbf{Hom}(\mathcal{F}(Y), \mathcal{F}(X))$, and if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, then

$$\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g).$$

In this case, we say that \mathcal{F} is contravariant.

Note 2.1.2. One can also define a contravariant $\mathbf{C} \rightsquigarrow \mathbf{D}$ functor as a covariant functor $\mathbf{C}^{\text{op}} \rightsquigarrow \mathbf{D}$.

Note 2.1.3. When the adjective is unspecified, a *functor* will mean a *covariant functor*.

Notation 2.1.2. We will typeset functors with calligraphic letters using the font `eucal`, and notate them with squiggly arrows. For example, if \mathbf{C} and \mathbf{D} are categories and \mathcal{F} is a functor from \mathbf{C} to \mathbf{D} , then we would write

$$\mathcal{F}: \mathbf{C} \rightsquigarrow \mathbf{D}.$$

Example 2.1.8. Let $\mathbf{1}$ be the category with one object (Example 2.1.2). Let \mathbf{C} be any category, let $X \in \mathbf{Obj}(\mathbf{C})$. Then we have the functor

$$\mathcal{F}_X: \mathbf{1} \rightsquigarrow \mathbf{C}; \quad \mathcal{F}(*) = X, \quad \mathcal{F}(\text{id}_*) = \text{id}_X.$$

One of the nice things about functors is that they map commutative diagrams to commutative diagrams. For example, if \mathcal{F} is a contravariant functor, then one might see the following.

$$\begin{array}{ccc} A & \xrightarrow{h=g \circ f} & B \\ & \searrow f & \nearrow g \\ & C & \end{array} \quad \xrightarrow{\mathcal{F}} \quad \begin{array}{ccc} \mathcal{F}(A) & \xleftarrow{\mathcal{F}(h)=\mathcal{F}(f) \circ \mathcal{F}(g)} & \mathcal{F}(B) \\ & \nwarrow \mathcal{F}(f) & \swarrow \mathcal{F}(g) \\ & \mathcal{F}(C) & \end{array}$$

Definition 2.1.10 (full functor, faithful functor). Let \mathbf{C} and \mathbf{D} be locally small categories (Definition 2.1.8), and let $\mathcal{F} : \mathbf{C} \rightsquigarrow \mathbf{D}$. Then \mathcal{F} induces a family of set-functions

$$\mathcal{F}_{X,Y} : \text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{D}}(\mathcal{F}(X), \mathcal{F}(Y)).$$

We say that \mathcal{F} is

- full if $\mathcal{F}_{X,Y}$ is surjective for all $X, Y \in \text{Obj}(\mathbf{C})$
- faithful if $\mathcal{F}_{X,Y}$ is injective for all $X, Y \in \text{Obj}(\mathbf{C})$,
- fully faithful if \mathcal{F} is full and faithful.

Note 2.1.4. Fullness and faithfulness are *not* the functorial analogs of surjectivity and injectivity. A functor between small categories can be full (faithful) without being surjective (injective) on objects. Instead, we have the following result.

Lemma 2.1.1. A fully faithful functor is injective on objects up to isomorphism. That is, if $\mathcal{F} : \mathbf{C} \rightsquigarrow \mathbf{D}$ is a fully faithful functor and $\mathcal{F}(A) \simeq \mathcal{F}(B)$, then $A \simeq B$.

Proof. Let $\mathcal{F} : \mathbf{C} \rightsquigarrow \mathbf{D}$ be a fully faithful functor, and suppose that $\mathcal{F}(A) \simeq \mathcal{F}(B)$. Then there exist $f' : \mathcal{F}(A) \rightarrow \mathcal{F}(B)$ and $g' : \mathcal{F}(B) \rightarrow \mathcal{F}(A)$ such that $f' \circ g' = 1_{\mathcal{F}(B)}$ and $g' \circ f' = 1_{\mathcal{F}(A)}$. Because the function $\mathcal{F}_{A,B}$ is bijective it is invertible, so there is a unique morphism $f \in \text{Hom}_{\mathbf{C}}(A, B)$ such that $\mathcal{F}(f) = f'$, and similarly there is a unique $g \in \text{Hom}_{\mathbf{C}}(B, A)$ such that $\mathcal{F}(g) = g'$.

Now,

$$1_{\mathcal{F}(A)} = g' \circ f' = \mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f),$$

and since \mathcal{F} is injective, we must have $g \circ f = 1_A$. Identical logic shows that we must also have $f \circ g = 1_B$. Thus $A \simeq B$. \square

Definition 2.1.11 (essentially surjective). A functor $\mathcal{F} : \mathbf{C} \rightsquigarrow \mathbf{D}$ is essentially surjective if for every $A' \in \text{Obj}(\mathbf{D})$, there exists $A \in \text{Obj}(\mathbf{C})$ such that $A' \simeq \mathcal{F}(A)$.

Example 2.1.9 (diagonal functor). Let \mathbf{C} be a category, $\mathbf{C} \times \mathbf{C}$ the product category of \mathbf{C} with itself. The diagonal functor $\Delta : \mathbf{C} \rightsquigarrow \mathbf{C} \times \mathbf{C}$ is the functor which sends

- each object $A \in \text{Obj}(\mathbf{C})$ to the pair $(A, A) \in \text{Obj}(\mathbf{C} \times \mathbf{C})$, and
- each morphism $f : A \rightarrow B$ to the ordered pair

$$(f, f) \in \text{Hom}_{\mathbf{C} \times \mathbf{C}}(A \times B, A \times B).$$

Definition 2.1.12 (bifunctor). A bifunctor is a functor whose domain is a product category.

2.1.3 Natural transformations

Definition 2.1.13 (natural transformation). Let \mathbf{C} and \mathbf{D} be categories, and let \mathcal{F} and \mathcal{G} be covariant functors from \mathbf{C} to \mathbf{D} . A natural transformation η between \mathcal{F} and \mathcal{G} consists of

- for each object $A \in \text{Obj}(\mathbf{C})$ a morphism $\eta_A : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$, such that

- for all $A, B \in \text{Obj}(C)$, for each morphism $f \in \text{Hom}(A, B)$, the diagram

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \\ \eta_A \downarrow & & \downarrow \eta_B \\ \mathcal{G}(A) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(B) \end{array}$$

commutes.

If the functors \mathcal{F} and \mathcal{G} are contravariant, the changes needed to make the definition make sense are obvious: basically, the arrows need to be reversed. However, this is one of those times where it's easier to just pretend that, in line with Note 2.1.2, we have a covariant functor from the opposite category.

In English: a natural transformation is a map between two functors.

Notation 2.1.3. We will use double-shafted arrows to denote natural transformations: if \mathcal{F} and \mathcal{G} are functors and η is a natural transformation from \mathcal{F} to \mathcal{G} , we will write

$$\eta: \mathcal{F} \Rightarrow \mathcal{G}.$$

Definition 2.1.14 (set of all natural transformations). Let \mathcal{C} and \mathcal{D} be categories, \mathcal{F} and \mathcal{G} functors $\mathcal{C} \rightsquigarrow \mathcal{D}$. The set of all natural transformations $\mathcal{F} \Rightarrow \mathcal{G}$ is denoted $\text{Nat}(\mathcal{F}, \mathcal{G})$.

Example 2.1.10. Recall that Grp is the category of groups. Denote by CRing the category of commutative rings.

We can construct the following functors $\text{CRing} \rightsquigarrow \text{Grp}$.

- The functor GL_n , which assigns to each commutative ring the group of all $n \times n$ matrices with nonzero determinant, and to each morphism $f: K \rightarrow K'$ the homomorphism $\text{GL}_n(K) \rightarrow \text{GL}_n(K')$ which maps a matrix with entries in K to a matrix with entries in K' by mapping each entry individually.
- The functor $(\cdot)^*$, which maps each commutative ring K to its group of units K^* , and each morphism $K \rightarrow K'$ to its restriction to K^* .

Denote by $\det_K M$ the determinant of a matrix M with its entries in a commutative ring K . Then the determinant is a map

$$\det_K: \text{GL}_n(K) \rightarrow K^*.$$

Because the determinant is defined by the same formula for each K , the action of f commutes with \det_K : it doesn't matter whether we map the entries of M with f first and then take the determinant, or take the determinant first and then feed the result to f .

That is to say, the following diagram commutes.

$$\begin{array}{ccc} \text{GL}_n(K) & \xrightarrow{\det_K} & K^* \\ \text{GL}_n f \downarrow & & \downarrow f^* \\ \text{GL}_n K' & \xrightarrow{\det_{K'}} & K'^* \end{array}$$

But this means that \det is a natural transformation $\text{GL}_n \Rightarrow (\cdot)^*$.

We can compose natural transformations in the obvious way.

Lemma 2.1.2. Let \mathcal{C} and \mathcal{D} be categories, $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be functors, and Φ and Ψ be natural transformations as follows.

$$\begin{array}{ccc} & \mathcal{F} & \\ & \downarrow \Phi & \\ \mathcal{C} & \xrightarrow{\mathcal{G}} & \mathcal{D} \\ & \downarrow \Psi & \\ & \mathcal{H} & \end{array}$$

This induces a natural transformation $\mathcal{F} \Rightarrow \mathcal{H}$.

Proof. For each object $A \in \text{Obj}(\mathcal{C})$, the composition $\Psi_A \circ \Phi_A$ exists and maps $\mathcal{F}(A) \rightarrow \mathcal{H}(A)$. Let's write

$$\Psi_A \circ \Phi_A = (\Psi \circ \Phi)_A.$$

We have to show that these are the components of a natural transformation, i.e. that they make the following diagram commute for all $A, B \in \text{Obj}(\mathcal{C})$, all $f : A \rightarrow B$.

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \\ (\Phi \circ \Psi)_A \downarrow & & \downarrow (\Phi \circ \Psi)_B \\ \mathcal{H}(A) & \xrightarrow{\mathcal{H}(f)} & \mathcal{H}(B) \end{array}$$

We can do this by adding a middle row.

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \\ \Phi_A \downarrow & & \downarrow \Phi_B \\ \mathcal{G}(A) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(B) \\ \Psi_A \downarrow & & \downarrow \Psi_B \\ \mathcal{H}(A) & \xrightarrow{\mathcal{H}(f)} & \mathcal{H}(B) \end{array}$$

The top and bottom squares are the naturality squares for Φ and Ψ respectively. The outside square is the one we want to commute, and it manifestly does because each of the inside squares does. \square

Definition 2.1.15 (vertical composition). The above composition $\Psi \circ \Phi$ is called vertical composition.

We can also compose natural transformations in a not so obvious way.

Lemma 2.1.3. Consider the following arrangement of categories, functors, and natural transformations.

$$\begin{array}{ccccc} & \mathcal{F} & & \mathcal{G} & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Phi \\ \curvearrowleft \end{array} & \mathcal{D} & \begin{array}{c} \curvearrowright \\ \Psi \\ \curvearrowleft \end{array} & \mathcal{E} \\ & \mathcal{F}' & & \mathcal{G}' & \end{array}$$

This induces a natural transformation $\mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$.

Proof. By definition, Φ and Ψ make the diagrams

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \\ \Phi_A \downarrow & & \downarrow \Phi_B \\ \mathcal{F}'(A) & \xrightarrow{\mathcal{F}'(f)} & \mathcal{F}'(B) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{G}(A) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(B) \\ \Psi_A \downarrow & & \downarrow \Psi_B \\ \mathcal{G}'(A) & \xrightarrow{\mathcal{G}'(f)} & \mathcal{G}'(B) \end{array}$$

commute, respectively. Since functors take commutative diagrams to commutative diagrams, we can map everything in the first diagram to \mathcal{E} with \mathcal{G} .

$$\begin{array}{ccc} (\mathcal{G} \circ \mathcal{F})(A) & \xrightarrow{(\mathcal{G} \circ \mathcal{F})(f)} & (\mathcal{G} \circ \mathcal{F})(B) \\ \mathcal{G}(\Phi_A) \downarrow & & \downarrow \mathcal{G}(\Phi_B) \\ (\mathcal{G} \circ \mathcal{F}')(A) & \xrightarrow{(\mathcal{G} \circ \mathcal{F}')(f)} & (\mathcal{G} \circ \mathcal{F}')(B) \end{array}$$

Also, since $\mathcal{F}'(f) : \mathcal{F}'(A) \rightarrow \mathcal{F}'(B)$ is a morphism in \mathcal{D} and Ψ is a natural transformation, the following diagram commutes.

$$\begin{array}{ccc} (\mathcal{G} \circ \mathcal{F}')(A) & \xrightarrow{(\mathcal{G} \circ \mathcal{F}')(f)} & (\mathcal{G} \circ \mathcal{F}')(B) \\ \Psi_{\mathcal{F}'(A)} \downarrow & & \downarrow \Psi_{\mathcal{F}'(B)} \\ (\mathcal{G}' \circ \mathcal{F}')(A) & \xrightarrow{(\mathcal{G}' \circ \mathcal{F}')(f)} & (\mathcal{G}' \circ \mathcal{F}')(B) \end{array}$$

Sticking these two diagrams on top of each other gives a new commutative diagram.

$$\begin{array}{ccc}
 (\mathcal{G} \circ \mathcal{F})(A) & \xrightarrow{(\mathcal{G} \circ \mathcal{F})(f)} & (\mathcal{G} \circ \mathcal{F})(B) \\
 \mathcal{G}(\Phi_A) \downarrow & & \downarrow \mathcal{G}(\Phi_B) \\
 (\mathcal{G} \circ \mathcal{F}')(A) & \xrightarrow{(\mathcal{G} \circ \mathcal{F}')(f)} & (\mathcal{G} \circ \mathcal{F}')(B) \\
 \Psi_{\mathcal{F}'(A)} \downarrow & & \downarrow \Psi_{\mathcal{F}'(B)} \\
 (\mathcal{G}' \circ \mathcal{F}')(A) & \xrightarrow{(\mathcal{G}' \circ \mathcal{F}')(f)} & (\mathcal{G}' \circ \mathcal{F}')(B)
 \end{array}$$

The outside rectangle is nothing else but the commuting square for a natural transformation

$$(\Psi * \Phi): \mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$$

with components $(\Psi * \Phi)_A = \Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$. □

Definition 2.1.16 (horizontal composition). The natural transformation $\Psi * \Phi$ defined above is called the horizontal composition of Φ and Ψ .

Note 2.1.5. Really, the above definition of the horizontal composition is lopsided and ugly. It becomes less so if we notice the following. The first step in our construction of $\Psi * \Phi$ was to apply the functor \mathcal{G} to the commutative diagram

$$\begin{array}{ccc}
 \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \\
 \Phi_A \downarrow & & \downarrow \Phi_B \\
 \mathcal{F}'(A) & \xrightarrow{\mathcal{F}'(f)} & \mathcal{F}'(B)
 \end{array}$$

We could instead have applied the functor \mathcal{G}' , giving us the following.

$$\begin{array}{ccc}
 (\mathcal{G}' \circ \mathcal{F})(A) & \xrightarrow{(\mathcal{G}' \circ \mathcal{F})(f)} & (\mathcal{G}' \circ \mathcal{F})(B) \\
 \mathcal{G}'(\Phi_A) \downarrow & & \downarrow \mathcal{G}'(\Phi_B) \\
 (\mathcal{G}' \circ \mathcal{F}')(A) & \xrightarrow{(\mathcal{G}' \circ \mathcal{F}')(f)} & (\mathcal{G}' \circ \mathcal{F}')(B)
 \end{array}$$

Then we could have glued to it the bottom of the commuting square

$$\begin{array}{ccc}
 (\mathcal{G} \circ \mathcal{F})(A) & \xrightarrow{(\mathcal{G} \circ \mathcal{F})(f)} & (\mathcal{G} \circ \mathcal{F})(B) \\
 \Psi_{\mathcal{F}(A)} \downarrow & & \downarrow \Psi_{\mathcal{F}(B)} \\
 (\mathcal{G}' \circ \mathcal{F})(A) & \xrightarrow{(\mathcal{G}' \circ \mathcal{F})(f)} & (\mathcal{G}' \circ \mathcal{F})(B)
 \end{array}$$

if you do this you get *another* natural transformation, with components $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$. Why did we use the first definition rather than this one?

It turns out that $\Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$ and $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$ are equal. To see this, note that from the morphism

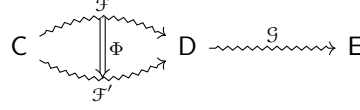
$$\Phi_A: \mathcal{F}(A) \rightarrow \mathcal{F}'(A),$$

the natural transformation Ψ gives us a commuting square

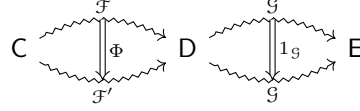
$$\begin{array}{ccc}
 (\mathcal{G} \circ \mathcal{F})(A) & \xrightarrow{\mathcal{G}(\Phi_A)} & (\mathcal{G} \circ \mathcal{F}')(A) \\
 \Psi_{\mathcal{F}(A)} \downarrow & & \downarrow \Psi_{\mathcal{F}'(A)} \\
 (\mathcal{G}' \circ \mathcal{F})(A) & \xrightarrow{\mathcal{G}'(\Phi_A)} & (\mathcal{G}' \circ \mathcal{F}')(A)
 \end{array}$$

the two ways of going from top left to bottom right are nothing else but $\Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$ and $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$.

Example 2.1.11 (whiskering). Consider the following assemblage of categories, functors, and natural transformations.



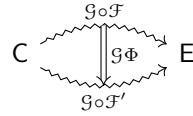
The horizontal composition allows us to take Φ to a natural transformation $\mathcal{G} \circ \mathcal{F}$ to $\mathcal{G} \circ \mathcal{F}'$ as follows. First, augment the diagram as follows.



We can then take the horizontal composition of Φ and $1_{\mathcal{G}}$ to get a natural transformation from $\mathcal{G} \circ \mathcal{F}$ to $\mathcal{G} \circ \mathcal{F}'$ with components

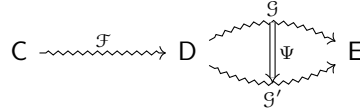
$$(1_{\mathcal{G}} * \Phi)_A = \mathcal{G}(\Phi_A).$$

This natural transformation is called the *right whiskering* of Φ with \mathcal{G} , and is denoted $\mathcal{G}\Phi$. That is to say, $(\mathcal{G}\Phi)_A = \mathcal{G}(\Phi_A)$.



The reason for the name is clear: we removed a whisker from the RHS of our diagram.

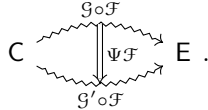
We can also remove a whisker from the LHS. Given this:



we can build a natural transformation (denoted $\Psi\mathcal{F}$) with components

$$(\Psi\mathcal{F})_A = \Psi_{\mathcal{F}(A)},$$

making this:



This is called the *left whiskering* of Ψ with \mathcal{F}

Definition 2.1.17 (natural isomorphism). A natural isomorphism $\eta: \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation such that each η_A is an isomorphism.

Definition 2.1.18 (functor category). Let \mathcal{C} and \mathcal{D} be categories. The functor category $\mathcal{D}^{\mathcal{C}}$ (or $[\mathcal{C} : \mathcal{D}]$) is the category whose objects are functors $\mathcal{C} \rightsquigarrow \mathcal{D}$, and whose morphisms are natural transformations between them.

Definition 2.1.19 (product category). Let \mathcal{C} and \mathcal{D} be categories. The product category $\mathcal{C} \times \mathcal{D}$ is the category whose

- objects are ordered pairs (C, D) , where $C \in \text{Obj}(\mathcal{C})$, whose
- morphisms are ordered pairs (f, g) , where $f: C_1 \rightarrow C_2 \in \text{Hom}_{\mathcal{C}}(C_1, C_2)$ and $g: D_1 \rightarrow D_2 \in \text{Hom}_{\mathcal{D}}(D_1, D_2)$, in which
- composition is taken componentwise, so that

$$(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ f_2, g_1 \circ g_2),$$

and

- the identity morphisms are given in the obvious way:

$$1_{(C,D)} = (1_C, 1_D).$$

We would like to be able to express when two categories are the ‘same.’ The correct notion of sameness is provided by the following definition.

Definition 2.1.20 (categorical equivalence). Let C and D be categories. We say that C and D are equivalent if there is a pair of functors

$$C \begin{matrix} \xrightarrow{\mathcal{F}} \\ \xleftarrow{\mathcal{G}} \end{matrix} D$$

and natural isomorphisms $R: \mathcal{F} \Rightarrow \mathcal{G}$ and $S: \mathcal{G} \Rightarrow \mathcal{F}$ such that $S \circ R = 1_{\mathcal{F}}$ and $R \circ S = 1_{\mathcal{G}}$.

Note 2.1.6. The above definition of categorical equivalence is equivalent to the following: C and D are equivalent if there is a functor $\mathcal{F}: C \rightsquigarrow D$ which is fully faithful (Definition 2.1.10) and essentially surjective (Definition 2.1.11). That is to say, if the equivalence \mathcal{F} is ‘bijective up to isomorphism.’

2.1.4 Special categories

Definition 2.1.21 (comma category). Let A, B, C be categories, \mathcal{S} and \mathcal{T} functors as follows.

$$A \begin{matrix} \xrightarrow{\mathcal{S}} \\ \xleftarrow{\mathcal{T}} \end{matrix} C \begin{matrix} \xrightarrow{\mathcal{T}} \\ \xleftarrow{\mathcal{S}} \end{matrix} B$$

The comma category $(\mathcal{S} \downarrow \mathcal{T})$ is the category whose

- objects are triples (α, β, f) where $\alpha \in \text{Obj}(A)$, $\beta \in \text{Obj}(B)$, and $f \in \text{Hom}_C(\mathcal{S}(\alpha), \mathcal{T}(\beta))$, and whose
- morphisms $(\alpha, \beta, f) \rightarrow (\alpha', \beta', f')$ are all pairs (g, h) , where $g: \alpha \rightarrow \alpha'$ and $h: \beta \rightarrow \beta'$, such that the diagram

$$\begin{array}{ccc} \mathcal{S}(\alpha) & \xrightarrow{\mathcal{S}(g)} & \mathcal{S}(\alpha') \\ f \downarrow & & \downarrow f' \\ \mathcal{T}(\beta) & \xrightarrow{\mathcal{T}(h)} & \mathcal{T}(\beta') \end{array}$$

commutes.

Note 2.1.7. We will often specify the comma category $(\mathcal{S} \downarrow \mathcal{T})$ by simply writing down the diagram

$$A \begin{matrix} \xrightarrow{\mathcal{S}} \\ \xleftarrow{\mathcal{T}} \end{matrix} C \begin{matrix} \xrightarrow{\mathcal{T}} \\ \xleftarrow{\mathcal{S}} \end{matrix} B .$$

Let us check in some detail that a comma category really is a category. To do so, we need to check the three properties listed in Definition 2.1.1

1. We must be able to compose morphisms, i.e. we must have the following diagram.

$$\begin{array}{ccccc} & & (g', h') \circ (g, h) & & \\ & \searrow & \downarrow & \swarrow & \\ (\alpha, \beta, f) & \xrightarrow{(g, h)} & (\alpha', \beta', f') & \xrightarrow{(g', h')} & (\alpha'', \beta'', f'') \end{array}$$

But we certainly do, since by definition, each square of the following diagram commutes,

$$\begin{array}{ccccc} \mathcal{S}(\alpha) & \xrightarrow{\mathcal{S}(g)} & \mathcal{S}(\alpha') & \xrightarrow{\mathcal{S}(g')} & \mathcal{S}(\alpha'') \\ f \downarrow & & f' \downarrow & & \downarrow f'' \\ \mathcal{T}(\alpha) & \xrightarrow{\mathcal{T}(h)} & \mathcal{T}(\alpha') & \xrightarrow{\mathcal{T}(h')} & \mathcal{T}(\alpha'') \end{array}$$

so the square formed by taking the outside rectangle

$$\begin{array}{ccc} \mathcal{S}(\alpha) & \xrightarrow{\mathcal{S}(g') \circ \mathcal{S}(g)} & \mathcal{S}(\alpha'') \\ f \downarrow & & \downarrow f'' \\ \mathcal{T}(\alpha) & \xrightarrow{\mathcal{T}(h') \circ \mathcal{T}(h)} & \mathcal{T}(\alpha'') \end{array}$$

commutes. But \mathcal{S} and \mathcal{T} are functors, so

$$\mathcal{S}(g') \circ \mathcal{S}(g) = \mathcal{S}(g' \circ g),$$

and similarly for $\mathcal{T}(h' \circ h)$. Thus, the composition of morphisms is given via

$$(g', h') \circ (g, h) = (g' \circ g, h' \circ h).$$

2. We can see from this definition that associativity in $(\mathcal{S} \downarrow \mathcal{T})$ follows from associativity in the underlying categories \mathbf{A} and \mathbf{B} .
3. The identity morphism is the pair $(1_{\mathcal{S}(\alpha)}, 1_{\mathcal{T}(\beta)})$. It is trivial from the definition of the composition of morphisms that this morphism functions as the identity morphism.

A special case of a comma category, the so-called *slice category*, occurs when $\mathbf{C} = \mathbf{A}$, \mathcal{S} is the identity functor, and $\mathbf{B} = \mathbf{1}$, the category with one object and one morphism (Example 2.1.2).

Definition 2.1.22 (slice category). A slice category is a comma category

$$\mathbf{A} \xrightarrow{\text{id}_A} \mathbf{A} \xleftarrow{\mathcal{T}} \mathbf{1}.$$

Let us unpack this prescription. Taking the definition literally, the objects in our category are triples (α, β, f) , where $\alpha \in \text{Obj}(\mathbf{A})$, $\beta \in \text{Obj}(\mathbf{1})$, and $f \in \text{Hom}_{\mathbf{A}}(\text{id}_A(\alpha), \mathcal{T}(\beta))$.

There's a lot of extraneous information here, and our definition can be consolidated considerably. Since the functor \mathcal{T} is given and $\mathbf{1}$ has only one object (call it $*$), the object $\mathcal{T}(\beta)$ (call it X) is singled out in \mathbf{A} . We can think of \mathcal{T} as \mathcal{F}_X . Similarly, since the identity morphism doesn't do anything interesting, we can collapse the LHS of the diagram.

The objects of a slice category therefore consist of pairs (α, f) , where $\alpha \in \text{Obj}(\mathbf{A})$ and

$$f : \alpha \rightarrow X;$$

the morphisms $(\alpha, f) \rightarrow (\alpha', f')$ consist of maps $g : \alpha \rightarrow \alpha'$. This allows us to define a slice category more neatly.

Let \mathbf{A} be a category, $X \in \text{Obj}(\mathbf{A})$. The slice category $(\mathbf{A} \downarrow X)$ is the category whose objects are pairs (α, f) , where $\alpha \in \text{Obj}(\mathbf{A})$ and $f : \alpha \rightarrow X$, and whose morphisms $(\alpha, f) \rightarrow (\alpha', f')$ are maps $g : \alpha \rightarrow \alpha'$ such that the diagram

$$\begin{array}{ccc} \alpha & \xrightarrow{g} & \alpha' \\ & \searrow f & \swarrow f' \\ & X & \end{array}$$

commutes.

One can also define a coslice category, which is what you get when you take a slice category and turn the arrows around.

Definition 2.1.23 (coslice category). Let \mathbf{A} be a category, $X \in \text{Obj}(\mathbf{A})$. The coslice category $(X \downarrow \mathbf{A})$ is the comma category given by the diagram

$$\mathbf{1} \xrightarrow{\mathcal{F}_X} \mathbf{A} \xleftarrow{\text{id}_A} \mathbf{A}.$$

Its objects are ordered pairs (α, f) , where $\alpha \in \text{Obj}(\mathbf{A})$ and $f : X \rightarrow \alpha$, and whose morphisms $(\alpha, f) \rightarrow (\alpha', f')$ are \mathbf{A} -morphisms $g : \alpha \rightarrow \alpha'$ such that the diagram

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow f' \\ \alpha & \xrightarrow{g} & \alpha' \end{array}$$

commutes.

2.1.5 Universal properties

In general, the ‘nicest’ definition of a structure uses only the category-theoretic information about the category in which the structure lives; the definition of a product (of sets, for example) is best given without making use of hand-wavy statements like “ordered pairs of an element here and an element there.” The idea is similar to the situation in linear algebra, where one avoids using introducing a basis every time one needs to prove something, preferring to use properties intrinsic to the vector space itself.

The easiest way to do this in general is by using the idea of a *universal property*.

Definition 2.1.24 (initial objects, final objects, zero objects). Let \mathcal{C} be a category. An object $A \in \text{Obj}(\mathcal{C})$ is said to be an

- initial object if $\text{Hom}(A, B)$ has exactly one element for all $B \in \text{Obj}(\mathcal{C})$, i.e. if there is exactly one arrow from A to every object in \mathcal{C} .
- final object if $\text{Hom}(B, A)$ has exactly one element for all $B \in \text{Obj}(\mathcal{C})$, i.e. if there is exactly one arrow from every object in \mathcal{C} to A .
- zero object if it is both initial and final.

Theorem 2.1.2. Let \mathcal{C} be a category, let I and I' be two initial objects in \mathcal{C} . Then there exists a unique isomorphism between I and I' .

Proof. Since I is initial, there exists exactly one morphism from I to *any* object, including I itself. By Definition 2.1.1, Part 3 I must have at least one map to itself: the identity morphism 1_I . Thus, the only morphism from I to itself is the identity morphism. Similarly, the only morphism from I' to itself is also the identity morphism.

But since I is initial, there exists a unique morphism from I to I' ; call it f . Similarly, there exists a unique morphism from I' to I ; call it g . By Definition 2.1.1, Part 1, we can take the composition $g \circ f$ to get a morphism $I \rightarrow I$.

But there is only one isomorphism $I \rightarrow I$: the identity morphism! Thus

$$g \circ f = 1_I.$$

Similarly,

$$f \circ g = 1_{I'}.$$

This means that, by Definition 2.1.5, f and g are isomorphisms. They are clearly unique because of the uniqueness condition in the definition of an initial object. Thus, between any two initial objects there is a unique isomorphism, and we are done.

Here is a bad picture of this proof.

$$1_I = g \circ f \hookrightarrow I \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} I' \begin{array}{c} \xrightarrow{1_{I'} = f \circ g} \\ \xleftarrow{1_I = g \circ f} \end{array}$$

□

Definition 2.1.25 (category of morphisms from an object to a functor). Let \mathcal{C}, \mathcal{D} be categories, let $\mathcal{U}: \mathcal{D} \rightsquigarrow \mathcal{C}$ be a functor. Further let $X \in \text{Obj}(\mathcal{C})$. The category of morphisms $(X \downarrow \mathcal{U})$ is the following comma category (see Definition 2.1.21):

$$1 \rightsquigarrow \mathcal{F}_X \rightsquigarrow \mathcal{C} \rightsquigarrow \mathcal{U} \rightsquigarrow \mathcal{D}.$$

Just as for (co)slice categories (Definitions 2.1.22 and 2.1.23), there is some unpacking to be done. In fact, the unpacking is very similar to that of coslice categories. The LHS of the diagram collapses because the functor \mathcal{F}_X picks out a single element X ; therefore, the objects of $(X \downarrow \mathcal{U})$ are ordered pairs

$$(\alpha, f) : \quad \alpha \in \text{Obj}(\mathcal{D}), \quad f: X \rightarrow \mathcal{U}(\alpha),$$

and the morphisms $(\alpha, f) \rightarrow (\alpha', f')$ are morphisms $g: \alpha \rightarrow \alpha'$ such that the diagram

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow f' \\ \mathcal{U}(\alpha) & \xrightarrow{\mathcal{U}(g)} & \mathcal{U}(\alpha') \end{array}$$

commutes.

Just as slice categories are the opposite to coslice categories, we can take the opposite of the previous definition.

Definition 2.1.26 (category of morphisms from a functor to an object). Let \mathcal{C}, \mathcal{D} be categories, let $\mathcal{U}: \mathcal{D} \rightsquigarrow \mathcal{C}$ be a functor. The category of morphisms from a functor to an object $(\mathcal{U} \downarrow X)$ is the comma category

$$\mathcal{D} \xrightarrow{\mathcal{U}} \mathcal{C} \xleftarrow{\mathcal{F}_X} 1.$$

The objects in this category are pairs (α, f) , where $\alpha \in \text{Obj}(\mathcal{D})$ and $f: \mathcal{U}(\alpha) \rightarrow X$. The morphisms $(\alpha, f) \rightarrow (\alpha', f')$ are morphisms $g: \alpha \rightarrow \alpha'$ such that the diagram

$$\begin{array}{ccc} \mathcal{U}(\alpha) & \xrightarrow{\mathcal{U}(g)} & \mathcal{U}(\alpha') \\ f \searrow & & \swarrow f' \\ & X & \end{array}$$

commutes.

Definition 2.1.27 (initial morphism). Let \mathcal{C}, \mathcal{D} be categories, let $\mathcal{U}: \mathcal{D} \rightsquigarrow \mathcal{C}$ be a functor, and let $X \in \text{Obj}(\mathcal{C})$. An initial morphism (called a *universal arrow* in [17]) is an initial object in the category $(X \downarrow \mathcal{U})$, i.e. the comma category which has the diagram

$$1 \xrightarrow{\mathcal{F}_X} \mathcal{C} \xleftarrow{\mathcal{U}} \mathcal{D}$$

This is not by any stretch of the imagination a transparent definition, but decoding it will be good practice.

The definition tells us that an initial morphism is an object in $(X \downarrow \mathcal{U})$, i.e. a pair (I, φ) for $I \in \text{Obj}(\mathcal{D})$ and $\varphi: X \rightarrow \mathcal{U}(I)$. But it is not just any object; it is an initial object. This means that for any other object (α, f) , there exists a unique morphism $(I, \varphi) \rightarrow (\alpha, f)$. But morphisms are simply maps $g: I \rightarrow \alpha$ such that the diagram

$$\begin{array}{ccc} & X & \\ \varphi \swarrow & & \searrow f \\ \mathcal{U}(I) & \xrightarrow{\mathcal{U}(g)} & \mathcal{U}(\alpha) \end{array}$$

commutes.

We can express this schematically via the following diagram (which is essentially the above diagram, rotated to agree with the literature).

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathcal{U}(I) \\ & \searrow f & \downarrow \mathcal{U}(g) \\ & & \mathcal{U}(\alpha) \end{array} \qquad \begin{array}{c} I \\ \downarrow \exists! g \\ \alpha \end{array}$$

As always, there is a dual notion.

Definition 2.1.28 (terminal morphism). Let \mathbf{C}, \mathbf{D} be categories, let $\mathcal{U}: \mathbf{D} \rightsquigarrow \mathbf{C}$ be a functor, and let $X \in \text{Obj}(\mathbf{C})$. A terminal morphism is a terminal object in the category $(\mathcal{U} \downarrow X)$.

This time “terminal object” means a pair (I, φ) such that there is a unique morphism $g: \alpha \rightarrow I$ such that the diagram

$$\begin{array}{ccc} \mathcal{U}(\alpha) & \xrightarrow{\mathcal{U}(g)} & \mathcal{U}(I) \\ & \searrow f & \swarrow \varphi \\ & X & \end{array}$$

commutes.

Again, with the diagram helpfully rotated, we have the following.

$$\begin{array}{ccc} \mathcal{U}(\alpha) & & \alpha \\ \mathcal{U}(g) \downarrow & \searrow f & \downarrow \exists! g \\ \mathcal{U}(I) & \xrightarrow{\varphi} & X \end{array} \quad \begin{array}{ccc} & & I \end{array}$$

Definition 2.1.29 (universal property). Strictly speaking, an object satisfying a universal property is just an object which is initial or terminal. However, enough interesting universal properties are constructed via comma categories that it will pay to study them in some detail.

One often hand-wavily says that an object I satisfies a universal property if (I, φ) is an initial or terminal morphism. This is actually rather annoying; one has to remember that when one states a universal property, one is defining not only an object I but also a morphism φ , which is often left implicit.

Example 2.1.12 (tensor algebra). One often sees some variation of the following universal characterization of the tensor algebra, which was taken (almost) verbatim from Wikipedia. We will try to stretch it to fit our definition, following the logic through in some detail.

Let V be a vector space over a field k , and let A be an algebra over k . The tensor algebra $T(V)$ satisfies the following universal property.

Any linear transformation $f: V \rightarrow A$ from V to A can be uniquely extended to an algebra homomorphism $T(V) \rightarrow A$ as indicated by the following commutative diagram.

$$\begin{array}{ccc} V & \xrightarrow{i} & T(V) \\ & \searrow f & \downarrow \tilde{f} \\ & & A \end{array}$$

As it turns out, it will take rather a lot of stretching.

Let $\mathcal{U}: k\text{-Alg} \rightsquigarrow \text{Vect}_k$ be the forgetful functor which assigns to each algebra over a field k its underlying vector space. Pick some k -vector space V . We consider the category $(V \downarrow \mathcal{U})$, which is given by the following diagram.

$$1 \rightsquigarrow_{\mathcal{F}_V} \text{Vect}_k \leftarrow_{\mathcal{U}} k\text{-Alg}$$

By Definition 2.1.27, the objects of $(V \downarrow \mathcal{U})$ are pairs (A, L) , where A is a k -algebra and L is a linear map $V \rightarrow \mathcal{U}(A)$. The morphisms are algebra homomorphisms $\rho: A \rightarrow A'$ such that the diagram

$$\begin{array}{ccc} & V & \\ L \swarrow & & \searrow L' \\ \mathcal{U}(A) & \xrightarrow{\mathcal{U}(\rho)} & \mathcal{U}(A') \end{array}$$

commutes. An object $(T(V), i)$ is initial if for any object (A, f) there exists a unique morphism $g: T(V) \rightarrow A$ such that the diagram

$$\begin{array}{ccc}
 V & \xrightarrow{i} & \mathcal{U}(T(V)) \\
 & \searrow L & \downarrow \mathcal{U}(g) \\
 & & \mathcal{U}(A)
 \end{array}
 \qquad
 \begin{array}{ccc}
 T(V) & & \\
 \downarrow \exists! g & & \\
 A & &
 \end{array}$$

commutes.

Thus, the pair $(i, T(V))$ is the initial object in the category $(V \downarrow \mathcal{U})$. We called $T(V)$ the *tensor algebra* over V .

But what is i ? Notice that in the Wikipedia definition above, the map i is from V to $T(V)$, but in the diagram above, it is from V to $\mathcal{U}(T(V))$. What gives?

The answer is that the diagram in Wikipedia's definition does not take place in a specific category. Instead, it implicitly treats $T(V)$ only as a vector space. But this is exactly what the functor \mathcal{U} does.

Example 2.1.13 (tensor product). According to the excellent book [3], the tensor product satisfies the following universal property.

Let V_1 and V_2 be vector spaces. Then we say that a vector space V_3 together with a bilinear map $\iota: V_1 \times V_2 \rightarrow V_3$ has the *universal property* provided that for any bilinear map $B: V_1 \times V_2 \rightarrow W$, where W is also a vector space, there exists a unique linear map $L: V_3 \rightarrow W$ such that $B = L\iota$. Here is a diagram describing this 'factorization' of B through ι :

$$\begin{array}{ccc}
 V_1 \times V_2 & \xrightarrow{\iota} & V_3 \\
 & \searrow B & \downarrow L \\
 & & W
 \end{array}$$

It turns out that the tensor product defined in this way is neither an initial or final morphism. This is because in the category \mathbf{Vect}_k , there is no way of making sense of a bilinear map.

We will discover the correct way of dealing with tensor products a bit later, after we have defined a *symmetric monoidal category*.

Example 2.1.14 (categorical product). Here is the universal property for a product, taken verbatim from Wikipedia ([21]).

Let \mathcal{C} be a category with some objects X_1 and X_2 . A product of X_1 and X_2 is an object X (often denoted $X_1 \times X_2$) together with a pair of morphisms $\pi_1: X \rightarrow X_1$ and $\pi_2: X \rightarrow X_2$ such that for every object Y and pair of morphisms $f_1: Y \rightarrow X_1$, $f_2: Y \rightarrow X_2$, there exists a unique morphism $f: Y \rightarrow X$ such that the following diagram commutes.

$$\begin{array}{ccccc}
 & & Y & & \\
 & \swarrow f_1 & \downarrow f & \searrow f_2 & \\
 X_1 & \xleftarrow{\pi_1} & X_1 \times X_2 & \xrightarrow{\pi_2} & X_2
 \end{array}$$

Consider the comma category $(\Delta \downarrow (X_1, X_2))$ (where Δ is the diagonal functor, see Example 2.1.9) given by the following diagram.

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C} \xleftarrow{\mathcal{F}_{(X_1, X_2)}} 1$$

The objects of this category are pairs $(A, (s, t))$, where $A \in \mathbf{Obj}(\mathcal{C})$ and

$$(s, t): \Delta(A) = (A, A) \rightarrow (X_1, X_2).$$

The morphisms $(A, (s, t)) \rightarrow (B, (u, v))$ are morphisms $r: A \rightarrow B$ such that the diagram

$$\begin{array}{ccc}
 (A, A) & \xrightarrow{(r, r)} & (B, B) \\
 \downarrow (s, t) & & \downarrow (u, v) \\
 (X_1, X_2) & &
 \end{array}$$

commutes.

An object $(X_1 \times X_2, (\pi_1, \pi_2))$ is final if for any other object $(Y, (f_1, f_2))$, there exists a unique morphism $f: Y \rightarrow X_1 \times X_2$ such that the diagram

$$\begin{array}{ccc} (Y, Y) & & Y \\ (f, f) \downarrow & \searrow (f_1, f_2) & \downarrow \exists! f \\ (X_1 \times X_2, X_1 \times X_2) & \xrightarrow{(\pi_1, \pi_2)} & (X_1, X_2) \\ & & \downarrow \\ & & X_1 \times X_2 \end{array}$$

commutes. If we re-arrange our diagram a bit, it is not too hard to see that it is equivalent to the one given above. Thus, we can say: a product of two sets X_1 and X_2 is a final object in the category $(\Delta \downarrow (X_1, X_2))$

2.1.6 Less basic definitions

In what follows, \mathcal{C} will be a category and A, B , etc. objects in $\text{Obj}(\mathcal{C})$.

Definition 2.1.30 (pullback). Let f, g be morphisms as follows.

$$\begin{array}{ccc} & B & \\ & \downarrow g & \\ A & \xrightarrow{f} & C \end{array}$$

A pullback of f along g (also called a pullback of g over f , sometimes notated $A \times_C B$) is a commuting square

$$\begin{array}{ccc} U & \xrightarrow{q} & B \\ p \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

such that for any other commuting square

$$\begin{array}{ccc} V & \xrightarrow{t} & B \\ s \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

there is a unique morphism $h: V \rightarrow U$ such that the diagram

$$\begin{array}{ccccc} V & & & & \\ & \searrow \exists! h & & \searrow t & \\ & & U & \xrightarrow{q} & B \\ & & p \downarrow & & \downarrow g \\ & & A & \xrightarrow{f} & C \\ & \searrow s & & & \end{array}$$

commutes.

Example 2.1.15. In Set , U is given (up to unique isomorphism) by

$$U = \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

The morphisms p and q are given by the projections $p(a, b) = a, q(a, b) = b$.

To see that this really does satisfy the universal property, consider any other set V and functions $s: V \rightarrow A$ and $t: V \rightarrow B$ making the above diagram commute. Then for all $v \in V$, $f(s(v)) = t(g(v))$.

Now consider the map $V \rightarrow U$ sending v to $(s(v), t(v))$. This certainly makes the above diagram commute; furthermore, any other map from V to U would not make the diagram commute. Thus U and h together satisfy the universal property.

Definition 2.1.31 (kernel of a morphism). Let \mathbf{C} be a category with an initial object (Definition 2.1.24) 0 and pullbacks. The kernel $\ker(f)$ of a morphism $f: A \rightarrow B$ is the pullback along f of the unique morphism $0 \rightarrow B$

$$\begin{array}{ccc} \ker(f) & \longrightarrow & 0 \\ \downarrow \iota & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

Example 2.1.16. In \mathbf{Vect}_k , the initial object is the zero vector space $\{0\}$. By definition, for any vector spaces V and W and any linear map $f: V \rightarrow W$, the kernel of f is the vector space

$$\ker(f) = \{v \in V \mid f(v) = 0\}.$$

Pushouts are the dual notion to pullbacks.

Definition 2.1.32 (pushouts). Let f, g be morphisms as follows.

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ f \downarrow & & \\ A & & \end{array}$$

The pushout of f along g (or g along f) is a commuting square

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ f \downarrow & & \downarrow q \\ A & \xrightarrow{p} & U \end{array}$$

such that for any other commuting square

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ f \downarrow & & \downarrow t \\ A & \xrightarrow{s} & V \end{array}$$

there exists a unique morphism $h: U \rightarrow V$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ f \downarrow & & \downarrow q \\ A & \xrightarrow{p} & U \end{array} \quad \begin{array}{c} \searrow t \\ \exists! h \\ \downarrow \\ V \end{array}$$

s

commutes.

Example 2.1.17. Let us construct the pushout in \mathbf{Set} . If we ignore the object C and the morphisms f and g , we discover that U must satisfy the universal property of the coproduct of A and B . Let us therefore make the ansatz that $U = A \amalg B = A \sqcup B$ and see what happens when we add C , f , and g back in.

In doing so, we find that the square A - C - B - U must also commute, i.e. we must have that $(q \circ g)(c) = (p \circ f)(c)$ for all $c \in C$. Since p and q are just inclusions, we see that

$$U = A \amalg B / \sim,$$

where \sim is the equivalence relation generated by the relations $f(c) \sim g(c)$ for all $c \in C$.

Definition 2.1.33 (cokernel of a morphism). Let \mathbf{C} be a category with terminal object 1 . The cokernel of a morphism $f: A \rightarrow B$ is the pushout of f along the unique morphism $A \rightarrow 1$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \pi \\ 1 & \longrightarrow & \operatorname{coker}(f) \end{array}$$

Example 2.1.18. In \mathbf{Vect}_k , the terminal object is the vector space $\{0\}$. If V and W are k -vector spaces and f is a linear map $V \rightarrow W$, then $\text{coker}(f)$ is

$$W/\sim,$$

where \sim is the relation generated by $f(v) \sim 0$ for all $v \in V$. But this relation is exactly the one which mods out by $\text{im}(f)$, so $\text{coker}(f) = W/\text{im}(f)$.

Definition 2.1.34 (normal monomorphism). A monomorphism (Definition 2.1.6) $f: A \rightarrow B$ is normal if it is the kernel of some morphism. To put it more plainly, f is normal if there exists an object C and a morphism $g: B \rightarrow C$ such that $B = \ker(g)$.

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Example 2.1.19. In \mathbf{Vect}_k , monomorphisms are injective linear maps (Example 2.1.5). If f is injective then sequence

$$\{0\} \longrightarrow V \xrightarrow{f} W \xrightarrow{\pi} W/\text{im}(f) \longrightarrow \{0\}$$

is exact, and we always have that $\text{im}(f) = \ker(\pi)$. Thus in \mathbf{Vect}_k , every monomorphism is exact.

Definition 2.1.35 (conormal epimorphism). An epimorphism $f: A \rightarrow B$ is conormal if it is the image of some morphism.

Example 2.1.20. In \mathbf{Vect}_k , epimorphisms are surjective linear maps. If $f: V \rightarrow W$ is a surjective linear map, then the sequence

$$\{0\} \longrightarrow \ker(f) \xrightarrow{\iota} V \xrightarrow{f} W \longrightarrow \{0\}$$

is exact. But then $\text{im}(\iota) = \ker(f)$, so f is conormal. Thus in \mathbf{Vect}_k , every epimorphism is conormal.

Note 2.1.8. To show that in our proofs that \mathbf{Vect}_k monomorphisms were normal and epimorphisms were conormal, we showed that monomorphisms were the kernels of their cokernels, and epimorphisms were the cokernels of their kernels. This will be a general feature of Abelian categories.

Definition 2.1.36 (binormal category). A category is binormal if all monomorphisms are normal and all epimorphisms are conormal.

Example 2.1.21. As we have seen, \mathbf{Vect}_k is binormal.

2.2 Structures on categories

2.2.1 Direct sums

Products

We saw in Example 2.1.14 the definition for a categorical product. In some categories, we can naturally take the product of any two objects; one generally says that such a category *has products*. We formalize that in the following.

Definition 2.2.1 (category with products). Let \mathbf{C} be a category such that for every two objects $A, B \in \text{Obj}(\mathbf{C})$, the object $A \times B$ which satisfies the universal product Example 2.1.14 exists. Then we say that \mathbf{C} has products.

Note 2.2.1. Sometimes people call a category with products a *Cartesian category*, but some people also use this terminology to mean a category with all finite limits. We will see later that any category with both products and equalizers has all finite limits.

Theorem 2.2.1. Let \mathbf{C} be a category with products. Then the Cartesian product can be extended to a bifunctor (Definition 2.1.12) $\mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$.

Proof. Let $X, Y \in \text{Obj}(\mathbf{C})$. We need to check that the assignment $(X, Y) \rightarrow \times(X, Y) \equiv X \times Y$ is functorial, i.e. that \times assigns

- to each pair $(X, Y) \in \text{Obj}(\mathbf{C} \times \mathbf{C})$ an object $X \times Y \in \text{Obj}(\mathbf{C})$ and
- to each pair of morphisms $(f, g): (X, Y) \rightarrow (X', Y')$ a morphism

$$\times(f, g) = f \times g: X \times Y \rightarrow X' \times Y'$$

such that

- $1_X \times 1_Y = 1_{X \times Y}$, and
- \times respects composition as follows.

$$\begin{array}{ccccc} & & (f', g') \circ (f, g) = (f' \circ f, g' \circ g) & & \\ & \nearrow & & \searrow & \\ (X, Y) & \xrightarrow{(f, g)} & (X', Y') & \xrightarrow{(f', g')} & (X'', Y'') \\ & \downarrow \wr \times & & & \\ X \times Y & \xrightarrow{f \times g} & X' \times Y' & \xrightarrow{f' \times g'} & X'' \times Y'' \\ & \searrow & & \nearrow & \\ & & f' \times g' \circ f \times g = (f' \circ f) \times (g' \circ g) & & \end{array}$$

We know how \times assigns objects in $\mathbf{C} \times \mathbf{C}$ to objects in \mathbf{C} . We need to study how \times assigns to a morphism (f, g) a morphism $f \times g$. We do this by diagram chasing.

Suppose we are given two maps $f: X_1 \rightarrow X_2$ and $g: Y_1 \rightarrow Y_2$. This induces a morphism (f, g) in $\mathbf{C} \times \mathbf{C}$.

Recall the universal property for products: a product $X_1 \times Y_1$ is a final object in the category $(\Delta \downarrow (X_1, Y_1))$. Objects in this category can be thought of as diagrams in $\mathbf{C} \times \mathbf{C}$.

$$(X_1 \times Y_1, X_1 \times Y_1) \xrightarrow{(\pi_1, \pi_2)} (X_1, Y_1) .$$

By assumption, we can take the product of both X_1 and Y_1 , and X_2 and Y_2 . This gives us two diagrams living in $\mathbf{C} \times \mathbf{C}$, which we can put next to each other.

$$(X_1 \times Y_1, X_1 \times Y_1) \xrightarrow{(\pi_1, \pi_2)} (X_1, Y_1)$$

$$(X_2 \times Y_2, X_2 \times Y_2) \xrightarrow{(\rho_1, \rho_2)} (X_2, Y_2)$$

We can draw in our morphism (f, g) , and take its composition with (π_1, π_2) .

$$\begin{array}{ccc} (X_1 \times Y_1, X_1 \times Y_1) & \xrightarrow{(\pi_1, \pi_2)} & (X_1, Y_1) \\ & \searrow (f \circ \pi_1, g \circ \pi_2) & \downarrow (f, g) \\ (X_2 \times Y_2, X_2 \times Y_2) & \xrightarrow{(\rho_1, \rho_2)} & (X_2, Y_2) \end{array}$$

Now forget about the top right of the diagram. I'll erase it to make this easier.

$$\begin{array}{ccc} (X_1 \times Y_1, X_1 \times Y_1) & & \\ & \searrow (f \circ \pi_1, g \circ \pi_2) & \\ (X_2 \times Y_2, X_2 \times Y_2) & \xrightarrow{(\rho_1, \rho_2)} & (X_2, Y_2) \end{array}$$

The universal property for products says that there exists a unique map $h: X_1 \times Y_1 \rightarrow X_2 \times Y_2$ such that the diagram below commutes.

$$\begin{array}{ccc}
 (X_1 \times Y_1, X_1 \times Y_1) & & X_1 \times Y_1 \\
 \downarrow (h, h) & \searrow (f \circ \pi_1, g \circ \pi_2) & \downarrow \exists! h \\
 (X_2 \times Y_2, X_2 \times Y_2) & \xrightarrow{(\rho_1, \rho_2)} & (X_1, X_2) \\
 & & \downarrow \\
 & & X_2 \times Y_2
 \end{array}$$

And h is what we will use for the product $f \times g$.

Of course, we must also check that h behaves appropriately. Draw two copies of the diagram for the terminal object in $(\Delta \downarrow (X, Y))$ and identity arrows between them.

$$\begin{array}{ccc}
 (X \times Y, X \times Y) & \xrightarrow{(\pi_1, \pi_2)} & (X, Y) \\
 \downarrow (1_{X \times Y}, 1_{X \times Y}) & & \downarrow (1_X, 1_Y) \\
 (X \times Y, X \times Y) & \xrightarrow{(\pi_1, \pi_2)} & (X, Y)
 \end{array}$$

Just as before, we compose the morphisms to and from the top right to draw a diagonal arrow, and then erase the top right object and the arrows to and from it.

$$\begin{array}{ccc}
 (X \times Y, X \times Y) & & X \times Y \\
 \downarrow (1_{X \times Y}, 1_{X \times Y}) = (1_X \times 1_Y, 1_X \times 1_Y) & \searrow (1_X \circ \pi_1, 1_Y \circ \pi_2) & \downarrow 1_X \times 1_Y = 1_X \times 1_Y \\
 (X \times Y, X \times Y) & \xrightarrow{(\pi_1, \pi_2)} & (X, Y) \\
 & & \downarrow \\
 & & X \times Y
 \end{array}$$

By definition, the arrow on the right is $1_X \times 1_Y$. It is also $1_{X \times Y}$, and by the universal property it is unique. Therefore $1_{X \times Y} = 1_X \times 1_Y$.

Next, put three of these objects together. The proof of the last part is immediate.

$$\begin{array}{ccc}
 (X_1 \times Y_1, X_1 \times Y_1) & & (X_1, Y_1) \\
 \downarrow (f \times g, f \times g) & & \downarrow (f, g) \\
 (X_2 \times Y_2, X_2 \times Y_2) & & (X_2, Y_2) \\
 \downarrow (f' \times g', f' \times g') & & \downarrow (f', g') \\
 (X_3 \times Y_3, X_3 \times Y_3) & & (X_3, Y_3)
 \end{array}$$

$(f' \times g') \circ (f \times g) = (g' \circ g) \times (f' \circ f)$
 $(g' \circ g, f' \circ f)$

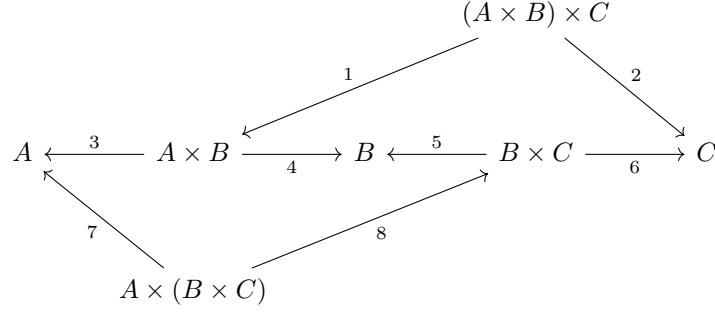
□

The meaning of this theorem is that in any category where you can take products of the objects, you can also take products of the morphisms.

Theorem 2.2.2. The categorical product is associative: for all objects A, B, C , there is an isomorphism

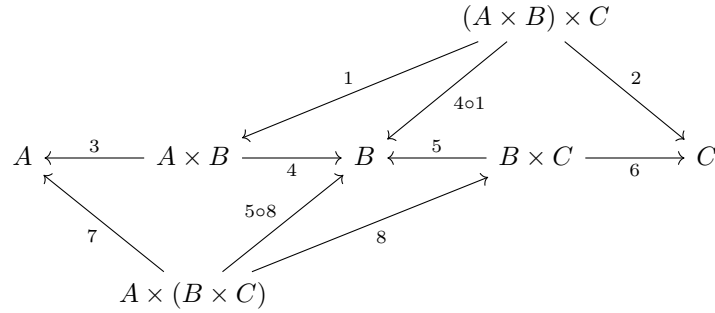
$$(A \times B) \times C \simeq A \times (B \times C).$$

Proof. Consider the following diagram, which consists of four interwoven copies of the canonical projections. To make things simpler, I'll just number the projection maps.

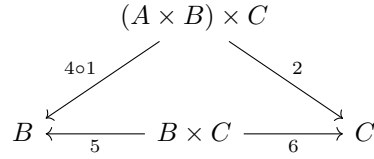


This diagram is commutative in a sort of trivial sense; there is at most one way of getting between any two objects.

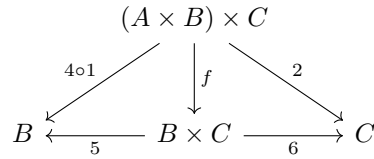
By composing projections, we can add two more arrows as follows: one from $(A \times B) \times C \rightarrow A \times B \rightarrow B$, and one from $A \times (B \times C) \rightarrow B \times C \rightarrow B$. This diagram is commutative since all we did is add compositions.



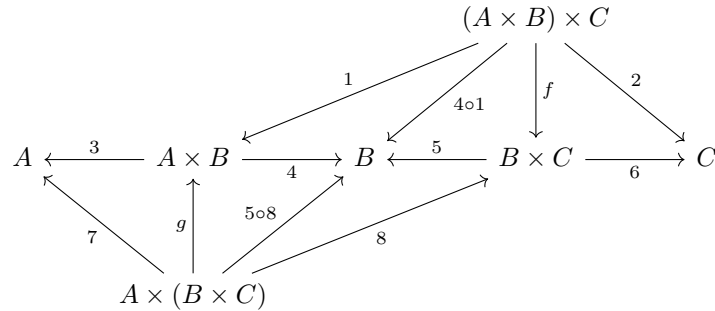
Now focus on the triangle in the top right.



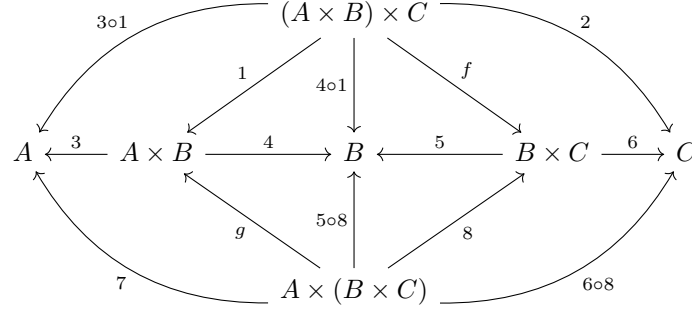
The universal property for products tells us that we get a unique morphism, $f: (A \times B) \times C \rightarrow B \times C$, which makes the triangle commute.



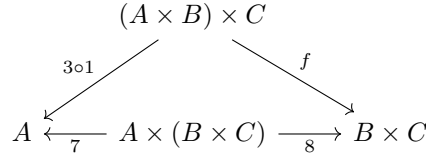
The same logic gives us another morphism $g: A \times (B \times C) \rightarrow A \times B$ such that the bottom left triangle commutes, so we might as well draw that in as well.



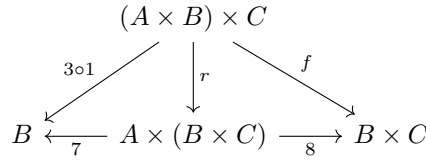
So far, everything commutes since all the arrows we've added have been either from compositions or the universal property of products. We can compose projections again to get morphisms $(A \times B) \times C \rightarrow A \times B \rightarrow A$ and $A \times (B \times C) \rightarrow B \times C \rightarrow C$. We can also shift stuff around a bit so it's easier to draw.



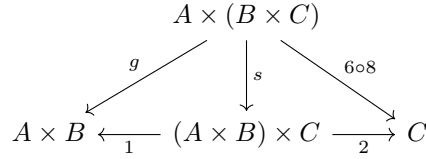
Buried inside the above diagram is the following triangle.



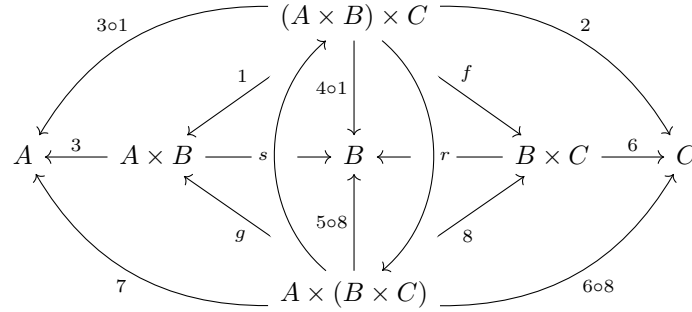
The universal property for products gives us a unique map r which makes the triangle commute.



Similarly, we have a unique map s which makes the triangle in the left of the diagram commute.



With great difficulty we can add these to our diagram.



Our diagram is still commutative.

Now TODO

□

Example 2.2.1. The category \mathbf{Vect}_k of vector spaces over a field k has the direct sum \oplus as a product.

Coproducts

Definition 2.2.2. Let \mathbf{C} be a category, X_1 and $X_2 \in \mathbf{Obj}(\mathbf{C})$. The coproduct of X_1 and X_2 , denoted $X_1 \amalg X_2$, is the initial object in the category $((X_1, X_2) \downarrow \Delta)$. In everyday language, we have the following.

An object $X_1 \amalg X_2$ is called the coproduct of X_1 and X_2 if

1. there exist morphisms $i_1: X_1 \rightarrow X_1 \amalg X_2$ and $i_2: X_2 \rightarrow X_1 \amalg X_2$ such that
2. for any object Y and morphisms $f_1: X_1 \rightarrow Y$ and $f_2: X_2 \rightarrow Y$ there exists a unique morphism $f: X_1 \amalg X_2 \rightarrow Y$ such that $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$, i.e. the following

diagram commutes.

$$\begin{array}{ccccc} & & Y & & \\ & f_1 \nearrow & \uparrow f & \nwarrow f_2 & \\ X_1 & \xrightarrow{i_1} & X_1 \amalg X_2 & \xleftarrow{i_2} & X_2 \end{array}$$

Definition 2.2.3 (category with coproducts). We say that a category \mathcal{C} has coproducts if for all $A, B \in \text{Obj}(\mathcal{C})$, the coproduct $A \amalg B$ is in $\text{Obj}(\mathcal{C})$.

We have the following analog of Theorem 2.2.1.

Theorem 2.2.3. Let \mathcal{C} be a category with coproducts. Then we have a functor $\amalg: \mathcal{C} \times \mathcal{C} \rightsquigarrow \mathcal{C}$.

Proof. We verify that \amalg allows us to define canonically a coproduct of morphisms. Let $X_1, X_2, Y_1, Y_2 \in \text{Obj}(\mathcal{C})$, and let $f: X_1 \rightarrow Y_1$ and $g: X_2 \rightarrow Y_2$. We have the following diagram.

$$\begin{array}{ccccc} X_1 & \xrightarrow{i_{1,X}} & X_1 \amalg X_2 & \xleftarrow{i_{2,X}} & X_2 \\ \downarrow f & \searrow i_{i,Y} \circ f & & \swarrow i_{2,Y} \circ g & \downarrow g \\ Y_1 & \xrightarrow{i_{1,Y}} & Y_1 \amalg Y_2 & \xleftarrow{i_{2,Y}} & Y_2 \end{array}$$

But by the universal property of coproducts, the diagonal morphisms induce a map $X_1 \amalg X_2 \rightarrow Y_1 \amalg Y_2$, which we define to be $f \amalg g$.

$$\begin{array}{ccccc} X_1 & \xrightarrow{i_{1,X}} & X_1 \amalg X_2 & \xleftarrow{i_{2,X}} & X_2 \\ & \searrow i_{i,Y} \circ f & \downarrow f \amalg g & \swarrow i_{2,Y} \circ g & \\ & & Y_1 \amalg Y_2 & & \end{array}$$

The rest of the verification that \amalg really is a functor is identical to that in Theorem 2.2.1. \square

Example 2.2.2. In Set , the coproduct is the disjoint union.

Direct sums

Definition 2.2.4 (category with direct sums). We say that a category \mathcal{C} has direct sums if

1. \mathcal{C} has products,
2. \mathcal{C} has coproducts, and
3. there is a natural isomorphism (Definition 2.1.17) between them.

We can cast this in the form of a commutative diagram.

$$\begin{array}{ccc} & \times & \\ \text{C} \times \text{C} & \begin{array}{c} \text{---} \text{wavy} \text{---} \\ \updownarrow \\ \text{---} \text{wavy} \text{---} \end{array} & \text{C} \\ & \amalg & \end{array}$$

Example 2.2.3. In the category Vect_k of vector spaces over a field k , the direct sum \oplus is a categorical direct sum.

Example 2.2.4. The category Ab of abelian groups has direct sums.

2.2.2 Additive categories

Recall that Ab is the category of abelian groups (Definition 1.2.2).

Definition 2.2.5 (Ab-enriched category). A category \mathcal{C} is Ab-enriched if

1. for all objects $A, B \in \text{Obj}(\mathcal{C})$, the hom-set $\text{Hom}_{\mathcal{C}}(A, B)$ has the structure of an abelian group (i.e. one can add morphisms), such that
2. the composition

$$\circ: \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$$

is additive in each slot: for any $f_1, f_2 \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, C)$, we must have

$$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g,$$

and similarly in the second slot.

Definition 2.2.6 (additive category). A category \mathcal{C} is additive if it has direct sums and is Ab-enriched.

Example 2.2.5. The category \mathbf{Ab} of Abelian groups is an additive category. We have already seen that it has direct sums. Given any two abelian groups A and B and morphisms $f, g: A \rightarrow B$, we can define the sum $f + g$ via

$$(f + g)(a) = f(a) + g(a) \quad \text{for all } a \in A.$$

Then for another abelian group C and a morphism $h: B \rightarrow C$, we have

$$[h \circ (f + g)](a) = h(f(a) + g(a)) = h(f(a)) + h(g(a)) = [h \circ f + h \circ g](a),$$

so

$$h \circ (f + g) = h \circ f + h \circ g,$$

and similarly in the other slot.

Example 2.2.6. The category \mathbf{Vect}_k is additive. Since vector spaces are in particular abelian groups under addition, it is naturally Ab-enriched,

2.2.3 Abelian categories, linear categories

Definition 2.2.7 (abelian category). A category \mathcal{C} is abelian if it is additive (Definition 2.2.6), has a zero object (Definition 2.1.24), and is binormal (Definition 2.1.36).

Example 2.2.7. As we have seen, the category \mathbf{Vect}_k is abelian; it is additive by

Lemma 2.2.1. Let \mathcal{C} be an abelian category. For each $A, B \in \text{Obj}(\mathcal{C})$ there is a unique morphism $0_{AB} \in \text{Hom}_{\mathcal{C}}(A, B)$ called the *zero morphism* such that

- for any object C and any morphism $f: B \rightarrow C$, we have the equality

$$f \circ 0_{AB} = 0_{AC}.$$

- for any object D and any morphism $g: D \rightarrow A$, we have the equality

$$0_{AB} \circ g = 0_{DA}$$

Proof. By definition, any abelian category has a zero object $0 \in \text{Obj}(\mathcal{C})$. For any two objects A and $B \in \text{Obj}(\mathcal{C})$, there is a unique morphism $A \rightarrow 0$ and a unique morphism $0 \rightarrow B$. Their composition gives us a morphism $A \rightarrow B$.

$$\begin{array}{ccccc} & & \curvearrowright & & \\ A & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & B \\ & \downarrow \text{!} & & \downarrow \text{!} & \end{array}$$

Further, for any morphism $f: B \rightarrow C$, the composition $0 \rightarrow B \rightarrow C$ gives us a morphism from 0 to C . But there is only one morphism from 0 to C . Hence, $f \circ 0_{AB}$ is the composition of a morphism $A \rightarrow 0$ and a morphism $0 \rightarrow C$, i.e.

$$f \circ 0_{AB} = 0_{AC}.$$

$$\begin{array}{ccccccc} & & \curvearrowright & & & & \\ A & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & B & \xrightarrow{f} & C \\ & & \curvearrowleft & & & & \end{array}$$

The proof of the second part is very similar. □

Theorem 2.2.4. Let \mathcal{C} be an abelian category. Then every morphism $f: A \rightarrow B$ has a canonical decomposition

$$A \xrightarrow{\iota} \text{coker}(\ker(f)) \xrightarrow{\bar{f}} \ker(\text{coker}(f)) \xrightarrow{\pi} B,$$

where ι is epi, \bar{f} is an isomorphism, and π is mono.

Proof.

□

Definition 2.2.8 (k -linear category). Let k be a field. A category \mathcal{C} is called k -linear if

1. it is an abelian category, and
2. its hom-sets are k -vector spaces with their additive structure given by the abelian groups from the category's abelian structure.

Example 2.2.8. The category Vect_k is k -linear.

Definition 2.2.9 (k -linear functor). let \mathcal{C} and \mathcal{D} be two k -linear categories, and $\mathcal{F}: \mathcal{C} \rightsquigarrow \mathcal{D}$ a functor. Suppose that for all objects $C, D \in \text{Obj}(\mathcal{C})$ all morphisms $f, g: C \rightarrow D$, and all $\alpha, \beta \in k$, we have

$$\mathcal{F}(\alpha f + \beta g) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g).$$

Then we say that \mathcal{F} is k -linear.

2.2.4 Monoidal categories

Monoidal categories are the first ingredient in the categorification of the tensor product. Roughly speaking, the tensor product allows us to multiply two vector spaces to produce a new vector space. There are natural isomorphisms making this multiplication associative, and an ‘identity vector space’ given by the ground field regarded as a one-dimensional vector space over itself.

Recall the definition of a monoid (Definition 1.1.1). Basically, a monoid is a group without inverses.

A monoidal category will be a category in which the objects have the structure of a monoid, i.e. there is a suitably defined ‘multiplication’ which is unital and associative. The prototypical example of categorical multiplication is the product (see Example 2.1.14), although it is far from the only one.

Definition 2.2.10 (monoidal category). A monoidal category is a category \mathcal{C} equipped with a monoidal structure. A monoidal structure is the following:

- A bifunctor (Definition 2.1.12) $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ called the *tensor product*,
- An object I called the *unit object*, and
- Three natural isomorphisms (Definition 2.1.17) subject to coherence conditions expressing the fact that the tensor product
 - is associative: there is a natural isomorphism α

$$\begin{array}{ccc} & (-\otimes-) \otimes - & \\ & \downarrow \alpha & \\ \mathcal{C} \times \mathcal{C} \times \mathcal{C} & & \mathcal{C} \\ & -\otimes(-\otimes-) & \end{array}$$

called the *associator*, with components

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$$

- has left and right identity: there are two natural isomorphisms λ

$$\begin{array}{ccc} & I \otimes - & \\ & \downarrow \lambda & \\ \mathcal{C} & & \mathcal{C} \\ & 1_{\mathcal{C}} & \end{array}$$

and ρ

respectively called the *left unitor* and *right unitor* with components

$$\lambda_A: I \otimes A \xrightarrow{\sim} A$$

and

$$\rho_A: A \otimes I \xrightarrow{\sim} A.$$

The coherence conditions are that the following diagrams commute for all A, B, C , and $D \in \text{Obj}(\mathcal{C})$.

- The *triangle diagram*

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{\alpha_{A,I,B}} & A \otimes (I \otimes B) \\ \rho_A \otimes 1_B \searrow & & \swarrow 1_A \otimes \lambda_B \\ & A \otimes B & \end{array}$$

- The *home plate diagram*.¹ (usually the *pentagon diagram*,² but what's life without an occasional nod to our national pastime?)

$$\begin{array}{ccccc} & & (A \otimes (B \otimes C)) \otimes D & & \\ & \nearrow \alpha_{A,B,C} \otimes 1_D & & \searrow \alpha_{A,B \otimes C,D} & \\ ((A \otimes B) \otimes C) \otimes D & & & & A \otimes ((B \otimes C) \otimes D) \\ \downarrow \alpha_{A \otimes B,C,D} & & & & \downarrow 1_A \otimes \alpha_{B,C,D} \\ (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A,B,C \otimes D}} & & & A \otimes (B \otimes (C \otimes D)) \end{array}$$

More succinctly, a monoidal structure on a category \mathcal{C} is a quintuple $(\otimes, 1, \alpha, \lambda, \rho)$.

Notation 2.2.1. This notation is prone to change. If the associator and unitors are not important or understood from context, for example, they are often left out. We will often say “Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category.”

Example 2.2.9. The simplest (though not the prototypical) example of a monoidal category is \mathbf{Set} with the Cartesian product. We check the axioms in Definition 2.2.10.

- The Cartesian product on \mathbf{Set} is a set-theoretic product, and can be naturally viewed, thanks to Theorem 2.2.1, as the bifunctor.
- Any set with one element $I = \{*\}$ functions as the unit object.
- For all sets A, B, C
 - The universal property of products guarantees us an isomorphism $\alpha_{A,B,C}: (A \times B) \times C \rightarrow A \times (B \times C)$ which sends $((a, b), c) \mapsto (a, (b, c))$.
 - Since $\{*\}$ is terminal, we get an isomorphism $\lambda_A: \{*\} \times A \rightarrow A$ which sends $(*, a) \mapsto a$.
 - Similarly, we get a map $\rho_A: A \times \{*\} \rightarrow A$ which sends $(a, *) \mapsto a$.

¹Unfortunately, the home plate, as drawn, is actually upside down.

²I have also seen it called the *permutoassociahedron*.

The pentagon and triangle diagram commute vacuously since the cartesian product is associative.

Lemma 2.2.2. The tensor product \otimes of vector spaces is a bifunctor $\mathbf{Vect}_k \times \mathbf{Vect}_k \rightarrow \mathbf{Vect}_k$.

Proof. This is easy to check. □

Example 2.2.10. The category \mathbf{Vect}_k is a monoidal category with

1. The bifunctor \otimes is given by the tensor product.
2. The unit 1 is given by the field k regarded as a 1-dimensional vector space over itself.
3. The associator is the map which sends $(v_1 \otimes v_2) \otimes v_3$ to $v_1 \otimes (v_2 \otimes v_3)$. It is not *a priori* obvious that this is well-defined, but it is also not difficult to check.
4. The left unitor is the map which sends $(x, v) \in k \times V$ to $xv \in V$.
5. The right unitor is the map which sends

$$(v, x) \mapsto xv.$$

Definition 2.2.11 (monoidal subcategory). Let $(\mathbf{C}, \otimes, 1)$ be a monoidal category. A monoidal subcategory of \mathbf{C} is a subcategory $\mathbf{S} \subseteq \mathbf{C}$ which is closed under the tensor product, and which contains 1 .

Lemma 2.2.3. Let \mathbf{C} be a category with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$. Then for all $X, Y \in \mathbf{Obj}(\mathbf{C})$, the diagram

$$\begin{array}{ccc} (1 \otimes X) \otimes Y & & \\ \alpha_{1, X, Y} \downarrow & \searrow \lambda_X \otimes 1_Y & \\ 1 \otimes (X \otimes Y) & \xrightarrow{\lambda_{X \otimes Y}} & X \otimes Y \end{array}$$

commutes.

Proof. □

Note 2.2.2. This insight is due to [14].

The triangle and pentagon diagram are the first in a long list of hulking coherence diagrams whose purpose is to pacify categorical structures into behaving like their algebraic counterparts. For a monoid, multiplication is associative by fiat, and the extension of this to n -fold products follows by a trivial application of induction. In monoidal categories, associators are isomorphisms rather than equalities, so we have to ensure n -ary by hand.

Having said that, these diagrams are not as unnatural as they appear. The pentagon diagram is exactly what you get when you write down all possible ways of bracketing the tensor product of four objects and draw arrows between the ones which differ by exactly one associator. Since the associators are isomorphisms, we are allowed to

3

One might wonder why one makes these definitions in particular. Why does one care about specifically the triangle diagram and the pentagon diagram?

Theorem 2.2.5 (Mac Lane's coherence theorem). Any two ways of freely composing unitors and associators to go from one expression to another coincide.

Sketch of proof. The proof involves showing that any two such natural transformation can be put in a canonical form in which they agree. □

³The number of ways to parenthesize a product involving n objects is the $(n-1)$ st Catalan number

$$C_{n-1} = \frac{1}{n-1} \binom{2n-4}{n-2}.$$

Definition 2.2.12 (invertible object). Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. An invertible object (sometimes called a *line object*) is an object $L \in \text{Obj}(\mathcal{C})$ such that both of the functors $\mathcal{C} \rightsquigarrow \mathcal{C}$

- $\ell_L: A \mapsto L \otimes A$
- $\varepsilon_L: A \mapsto A \otimes L$

are categorical equivalences.

Lemma 2.2.4. If $(\mathcal{C}, \otimes, 1)$ is a monoidal category and L is an invertible object, then there is an object, unique up to isomorphism, such that $L^{-1} \in \text{Obj}(\mathcal{C})$ such that $L \otimes L^{-1} \simeq L^{-1} \otimes L \simeq 1$. Furthermore, L is invertible only if there exists such an L^{-1} .

Proof. Suppose L is invertible. Then the functor $\ell_L: A \mapsto L \otimes A$ is bijective up to isomorphism, i.e. for any $A \in \text{Obj}(\mathcal{C})$, there is an object $A' \in \text{Obj}(\mathcal{C})$, unique up to isomorphism, such that

$$\ell_L(A') = L \otimes A' \simeq A.$$

But if this is true for *any* A , it must also be true for 1 , so there exists an object L^{-1} , unique up to isomorphism, such that

$$\ell_L(L^{-1}) = L \otimes L^{-1} \simeq 1.$$

The same logic tells us that there exists some other element L'^{-1} , such that

$$\varepsilon_L(L'^{-1}) = L'^{-1} \otimes L \simeq 1.$$

Now

$$1 = L'^{-1} \otimes L \simeq L'^{-1} \otimes (L \otimes L^{-1}) \otimes L \simeq (L'^{-1} \otimes L) \otimes (L^{-1} \otimes L) \simeq (L'^{-1} \otimes L) \otimes 1 \simeq L'^{-1} \otimes L,$$

so L'^{-1} is also a left inverse for L . But since ℓ_L is an equivalence of categories, L only has one left inverse up to isomorphism, so L^{-1} and L'^{-1} must be isomorphic. \square

Definition 2.2.13. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. Then the full subcategory (Definition 2.1.4) $\text{Line}(\mathcal{C}, \otimes, 1) \subseteq (\mathcal{C}, \otimes, 1)$ whose objects are the line objects in \mathcal{C} is called the line subcategory of $(\mathcal{C}, \otimes, 1)$. Since invertibility is closed under the tensor product and 1 is invertible ($1^{-1} \simeq 1$), $\text{Line}(\mathcal{C})$ is a monoidal subcategory of \mathcal{C} .

The following definition was taken *mutatis mutandis* from [13].

Definition 2.2.14 (monoidal functor). Let \mathcal{C} and \mathcal{C}' be monoidal categories. A functor $\mathcal{F}: \mathcal{C} \rightsquigarrow \mathcal{C}'$ is lax monoidal if it is equipped with

- a natural transformation $\Phi_{x,y}: \mathcal{F}(x) \otimes \mathcal{F}(y) \rightarrow \mathcal{F}(x \otimes y)$, and
- a morphism $\varphi: 1_{\mathcal{C}'} \rightarrow \mathcal{F}(1_{\mathcal{C}})$ such that
- the following diagrams commute for any $x, y, z \in \text{Obj}(\mathcal{C})$.

$$\begin{array}{ccccc} (\mathcal{F}(x) \otimes \mathcal{F}(y)) \otimes \mathcal{F}(z) & \xrightarrow{\Phi_{x,y} \otimes 1_{\mathcal{F}(z)}} & \mathcal{F}(x \otimes y) \otimes \mathcal{F}(z) & \xrightarrow{\Phi_{x \otimes y, z}} & \mathcal{F}((x \otimes y) \otimes z) \\ \downarrow \alpha_{\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)} & & & & \downarrow \mathcal{F}(\alpha_{x,y,z}) \\ \mathcal{F}(x) \otimes (\mathcal{F}(y) \otimes \mathcal{F}(z)) & \xrightarrow{1_{\mathcal{F}(x)} \otimes \Phi_{y,z}} & \mathcal{F}(x) \otimes \mathcal{F}(y \otimes z) & \xrightarrow{\varphi_{x,y \otimes z}} & \mathcal{F}(x \otimes (y \otimes z)) \\ & & & & \\ & & 1 \otimes \mathcal{F}(x) & \xrightarrow{\lambda_{\mathcal{F}(x)}} & \mathcal{F}(x) \\ & & \downarrow \varphi \otimes 1_{\mathcal{F}(x)} & & \uparrow \mathcal{F}(\lambda_x) \\ & & \mathcal{F}(1) \otimes \mathcal{F}(x) & \xrightarrow{\Phi_{1,x}} & \mathcal{F}(1 \otimes x) \end{array}$$

$$\begin{array}{ccc}
 \mathcal{F}(x) \otimes 1 & \xrightarrow{\rho_{\mathcal{F}(x)}} & \mathcal{F}(x) \\
 \downarrow 1_{\mathcal{F}(x)} \otimes \varphi & & \uparrow F(\lambda_x) \\
 \mathcal{F}(x) \otimes \mathcal{F}(1) & \xrightarrow{\Phi_{x,1}} & \mathcal{F}(x \otimes 1)
 \end{array}$$

If Φ is a natural isomorphism and φ is an isomorphism, then \mathcal{F} is called a strong monoidal functor.

We will denote the above monoidal functor by $(\mathcal{F}, \Phi, \varphi)$.

Note 2.2.3. The above diagrams above are exactly those necessary to ensure that the monoidal structure is preserved. They do this by demanding that the associator and the unitors be \mathcal{F} -equivariant.

Note 2.2.4. In much of the literature, a strong monoidal functor is simply called a monoidal functor.

Lemma 2.2.5. The composition of lax monoidal functors is lax monoidal.

Definition 2.2.15 (monoidal natural transformation). Let (F, Φ, φ) and (G, Γ, γ) be monoidal functors. A natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ is monoidal if the following diagrams commute.

$$\begin{array}{ccc}
 \mathcal{F}(X) \otimes \mathcal{F}(Y) & \xrightarrow{\eta_X \otimes \eta_Y} & \mathcal{G}(X) \otimes \mathcal{G}(Y) \\
 \Phi_{X,Y} \downarrow & & \downarrow \Gamma_{X,Y} \\
 \mathcal{F}(X \otimes Y) & \xrightarrow{\eta_{X \otimes Y}} & \mathcal{G}(X \otimes Y)
 \end{array}
 \quad
 \begin{array}{ccc}
 1 & & \\
 \varphi \downarrow & \searrow \gamma & \\
 \mathcal{F}(1) & \xrightarrow{\eta_1} & \mathcal{G}(1)
 \end{array}$$

2.2.5 Braided monoidal categories

Any insightful remarks in this section are due to [12].

Braided monoidal categories capture the idea that we should think of morphisms between tensor products spatially, as diagrams embedded in 3-space. This sounds odd, but it turns out to be the correct way of looking at a wide class of problems.

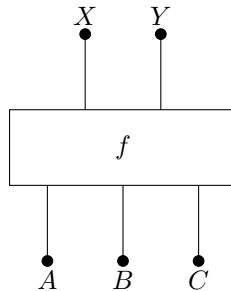
To be slightly more precise, we can think of the objects X , Y , etc. in any monoidal category as little dots.

$$\begin{array}{c}
 X \\
 \bullet
 \end{array}
 \qquad
 \begin{array}{c}
 Y \\
 \bullet
 \end{array}$$

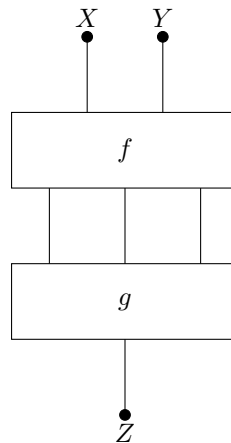
We can express the tensor product $X \otimes Y$ by putting the dots representing X and Y next to each other.

$$\begin{array}{cc}
 X & Y \\
 \bullet & \bullet
 \end{array}$$

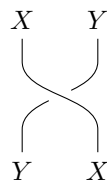
A morphism f between, say, $X \otimes Y$ and $A \otimes B \otimes C$ can be drawn as some lines and widgets connecting them.



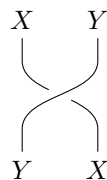
We can compose morphisms by concatenating their diagrams.



In a braided monoidal category, we require that for any two objects X and Y we have an isomorphism $\gamma_{XY}: X \otimes Y \rightarrow Y \otimes X$, which we draw like this.



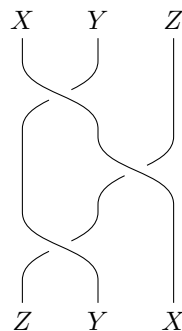
Since γ_{AB} is an isomorphism, it has an inverse γ_{AB}^{-1} (not necessarily equal to γ_{BA} !) which we draw like this.



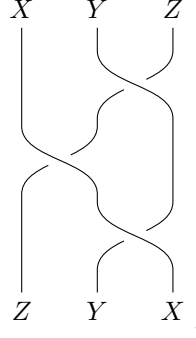
The idea of a braided monoidal category is that we want to take these pictures seriously: we want two expressions involving repeated applications of the $\gamma_{..}$ and their inverses to be equivalent if and only if the braid diagrams representing them are homotopic. Thus we want, for example,

$$\gamma_{XY} \circ \gamma_{YZ} \circ \gamma_{XZ} = \gamma_{YZ} \circ \gamma_{XY} \circ \gamma_{XZ}$$

since



is homotopic to



A digression into the theory of braid groups would take us too far afield. The punchline is that to guarantee that all such compositions involving the γ are identified in the correct way, we must define braided monoidal categories as follows.

Definition 2.2.16 (braided monoidal category). A category \mathcal{C} with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$ is braided if for every two objects A and $B \in \text{Obj}(\mathcal{C})$, there is an isomorphism $\gamma_{A,B}: A \otimes B \rightarrow B \otimes A$ such that the following *hexagon diagrams* commute.

$$\begin{array}{ccccc}
 & & A \otimes (B \otimes C) & \xrightarrow{\gamma_{A,B \otimes C}} & (B \otimes C) \otimes A \\
 & \nearrow \alpha_{ABC} & & & \searrow \alpha_{BCA} \\
 (A \otimes B) \otimes C & & & & B \otimes (C \otimes A) \\
 & \searrow \gamma_{AB \otimes 1} & & & \nearrow 1 \otimes \gamma \\
 & & (B \otimes A) \otimes C & \xrightarrow{\alpha_{BAC}} & (B \otimes A) \otimes C \\
 & & & & \\
 & \nearrow \alpha_{ABC}^{-1} & (A \otimes B) \otimes C & \xrightarrow{\gamma_{A \otimes B, C}} & C \otimes (A \otimes B) \\
 A \otimes (B \otimes C) & & & & \searrow \alpha_{CAB}^{-1} \\
 & \searrow 1 \otimes \gamma_{BC} & & & \nearrow \gamma_{AC \otimes 1} \\
 & & A \otimes (C \otimes B) & \xrightarrow{\alpha_{ACB}^{-1}} & (A \otimes C) \otimes B
 \end{array}$$

The collection of such γ form a natural isomorphism between the bifunctors

$$(A, B) \mapsto A \otimes B \quad \text{and} \quad (A, B) \mapsto B \otimes A,$$

and is called a braiding.

Definition 2.2.17 (braided monoidal functor). A lax monoidal functor $(\mathcal{F}, \Phi, \phi)$ (Definition 2.2.14) is braided monoidal if it makes the following diagram commute.

$$\begin{array}{ccc}
 \mathcal{F}(x) \otimes \mathcal{F}(y) & \xrightarrow{\gamma_{\mathcal{F}(x), \mathcal{F}(y)}} & \mathcal{F}(y) \otimes \mathcal{F}(x) \\
 \downarrow \Phi_{x,y} & & \downarrow \Phi_{x,y} \\
 \mathcal{F}(x \otimes y) & \xrightarrow{\mathcal{F}(\gamma_{x,y})} & \mathcal{F}(y \otimes x)
 \end{array}$$

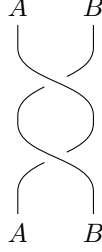
Note 2.2.5. There are no extra conditions imposed on a monoidal natural transformation to turn it into a braided natural transformation.

2.2.6 Symmetric monoidal categories

Until now, we have been calling the bifunctor \otimes in Definition 2.2.10 a tensor product. This has been an abuse of terminology: in general, one defines tensor products not to be those bifunctors which come from any monoidal category, but only those which come from symmetric monoidal categories. We will define these shortly.

Conceptually, passing from the definition of a braided monoidal category to that of a symmetric monoidal category is rather simple. One only requires that for any two objects A and B , $\gamma_{BA} = \gamma_{AB}^{-1}$, i.e. $\gamma_{BA} \circ \gamma_{AB} = 1_{A \otimes B}$.

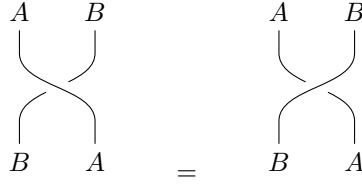
We can interpret this nicely in terms of our braid diagrams. We can draw $\gamma_{BA} \circ \gamma_{AB}$ like this.



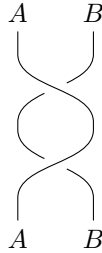
The requirement that this must be homotopic to the identity transformation



can be expressed by making the following rule: in a *symmetric* monoidal category, we don't care about the difference between undercrossings and overcrossings:



Then we can exchange the diagram representing $\gamma_{BA} \circ \gamma_{AB}$ for



which is clearly homotopic to the identity transformation on $A \otimes B$.

Definition 2.2.18 (symmetric monoidal category). Let \mathbf{C} be a braided monoidal category with braiding γ . We say that \mathbf{C} is a symmetric monoidal category if for all $A, B \in \text{Obj}(\mathbf{C})$, $\gamma_{BA} \circ \gamma_{AB} = 1_{A \otimes B}$. A braiding γ which satisfies such a condition is called symmetric.

Note 2.2.6. There are no extra conditions imposed on a monoidal natural transformation to turn it into a symmetric natural transformation.

2.2.7 Enriched categories

We have already defined an **Ab**-enriched category (Definition 2.2.5); this was a category whose morphisms had the structure of an abelian group rather than just a set. The notion of enrichment is much more general than this, and it will pay to study it.

The rough idea is this: we have a category \mathcal{C} , and we would like to treat the hom-sets $\text{Hom}_{\mathcal{C}}(A, B)$ as living in some category \mathbf{K} , not necessarily \mathbf{Set} . This is a common occurrence; often morphisms between structures inherit properties from the structures themselves.

- In \mathbf{Ab} , the endomorphisms $\text{End}_{\mathbf{Ab}}(G) = \text{Hom}_{\mathbf{Ab}}(G, G)$ naturally have a ring structure: we can define addition pointwise

$$(\alpha + \beta)(g) = \alpha(g) + \beta(g),$$

multiplication by composition

$$(\alpha \circ \beta)(g) = \alpha(\beta(g)).$$

These operations satisfy the ring axioms. Therefore, we would like to view the endomorphisms in \mathbf{Ab} as living not just in \mathbf{Set} , but in \mathbf{Ring} .

- In \mathbf{Vect}_k , the linear maps between vector spaces inherit the structure of a vector space: we can define addition and scalar multiplication by

$$(A + B)v = Av + Bv \quad \text{and} \quad (\lambda A)v = \lambda(Av).$$

Therefore, we would like to view the morphisms in \mathbf{Vect}_k as living in \mathbf{Vect}_k .

If we are to replace hom-sets with more general objects belonging to a generic enriching category, we need to make sure that the enriching category has enough structure to mimic the behavior of the hom-sets. In particular, we need a way to compose morphisms. This is not as trivial as it sounds; we are in general not allowed to talk about the elements of an object in a category, so we need to replace our definition of the composition of morphisms as a set map $\text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$ with something that doesn't know about the elements of the hom-sets. The correct way of doing this is with a monoidal structure.

Definition 2.2.19 (enriched category). Let \mathbf{K} be a monoidal category with monoidal structure $(\times, 1, \alpha, \lambda, \rho)$. A (small) \mathbf{K} -enriched category \mathcal{C} consists of the following.

- A set $\text{Obj}(\mathcal{C})$ of objects
- For each ordered pair (A, B) of objects of \mathcal{C} , an object $\text{Hom}_{\mathcal{C}}(A, B) \in \text{Obj}(\mathbf{K})$ (also written $\mathcal{C}(A, B)$) called the *hom-object*,
- For each ordered triple (A, B, C) of objects of \mathcal{C} a morphism $\circ_{A,B,C} : \text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, C)$ called the *composition morphism*
- for each object $A \in \text{Obj}(\mathcal{C})$ a morphism $j_A : 1 \rightarrow \text{Hom}_{\mathcal{C}}(A, A)$ called the identity element

such that the following diagrams commute.

$$\begin{array}{ccccc}
 (\mathcal{C}(C, D) \times \mathcal{C}(B, C)) \times \mathcal{C}(A, B) & \xrightarrow{\alpha} & \mathcal{C}(C, D) \times (\mathcal{C}(B, C) \times \mathcal{C}(A, B)) \\
 \circ \times 1 \downarrow & & \downarrow 1 \times \circ \\
 \mathcal{C}(B, D) \times \mathcal{C}(A, B) & \xrightarrow{\circ} \mathcal{C}(A, D) & \xleftarrow{\circ} \mathcal{C}(C, D) \times \mathcal{C}(A, C) \\
 & & \\
 \mathcal{C}(B, B) \times \mathcal{C}(A, B) & \xrightarrow{\circ} \mathcal{C}(A, B) & \xleftarrow{\circ} \mathcal{C}(A, B) \times \mathcal{C}(A, A) \\
 j \times 1 \uparrow & \nearrow \lambda & \nwarrow \rho & \uparrow 1 \times j \\
 1 \times \mathcal{C}(A, B) & & & \mathcal{C}(A, B) \times 1
 \end{array}$$

The first diagram enforces associativity; the second ensures that composition is unital.

2.3 Limits and colimits

One often gets categorical concepts by ‘categorifying’ concepts from set theory. One example of this is the notion of a *diagram*, which is the categorical generalization of an indexed family.

Definition 2.3.1 (indexed family). Let J and X be sets. A family of elements in X indexed by J is a function

$$x: J \rightarrow X; \quad j \mapsto x_j.$$

To categorify this, one considers a functor from one category J , called the *index category*, to another category C . Using a functor from a category instead of a function from a set allows us to index the morphisms as well as the objects.

Definition 2.3.2 (diagram). Let J and C be categories. A diagram of type J in C is a covariant functor

$$\mathcal{D}: J \rightsquigarrow C.$$

Definition 2.3.3 (cone). Let C be a category, J an index category, and $\mathcal{D}: J \rightsquigarrow C$ be a diagram. Let 1 be the category with one object and one morphism (Example 2.1.2) and \mathcal{F}_X the functor $1 \rightsquigarrow C$ which picks out $X \in \text{Obj}(C)$ (Example 2.1.8). Let \mathcal{K} be the unique functor $J \rightsquigarrow 1$.

$$\begin{array}{ccc} & 1 & \\ \mathcal{K} \uparrow & \rightsquigarrow & \mathcal{F}_X \\ & J & \rightsquigarrow \mathcal{D} \rightarrow C \end{array}$$

A cone of shape J from X is an object $X \in \text{Obj}(C)$ together with a natural transformation

$$\varepsilon: \mathcal{F}_X \circ \mathcal{K} \Rightarrow \mathcal{D}.$$

That is to say, a cone to J is an object $X \in \text{Obj}(C)$ together with a family of morphisms $\Phi_A: X \rightarrow \mathcal{F}(A)$ (one for each $A \in \text{Obj}(J)$) such that for all $A, B \in \text{Obj}(J)$ and all $f: A \rightarrow B$ the following diagram commutes.

$$\begin{array}{ccc} & X & \\ \Phi_A \swarrow & & \searrow \Phi_B \\ \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \end{array}$$

Note 2.3.1. Here is an alternate definition. Let Δ be the functor $C \rightsquigarrow [J, C]$ (the category of functors $J \rightsquigarrow C$, see TODO) which assigns to each object $X \in \text{Obj}(C)$ the constant functor $\Delta_X: J \rightsquigarrow C$, i.e. the functor which maps every object of J to X and every morphism to 1_X . A cone over \mathcal{D} is then an object in the comma category (Definition 2.1.21) $(\Delta \downarrow \mathcal{D})$ given by the diagram

$$C \rightsquigarrow \Delta \rightarrow [J, C] \xleftarrow{\mathcal{F}_{\mathcal{D}}} 1.$$

The objects of this category are pairs (X, f) , where $X \in \text{Obj}(C)$ and $f: \Delta(X) \rightarrow \mathcal{D}$; that is to say, f is a natural transformation $\Delta_X \Rightarrow \mathcal{D}$.

This allows us to make the following definition.

Definition 2.3.4 (category of cones over a diagram). Let C be a category, J an index category, and $\mathcal{D}: J \rightsquigarrow C$ a diagram. The category of cones over \mathcal{D} is the category $(\Delta \downarrow \mathcal{D})$.

Note 2.3.2. The previous definition not only reiterates what cones look like, but prescribes what morphisms between cones look like as well. Let (X, Φ) be a cone over a diagram $\mathcal{D}: J \rightsquigarrow C$, i.e.

- an object in the category $(\Delta \downarrow \mathcal{D})$, i.e.
- a pair (X, Φ) , where $X \in \text{Obj}(C)$ and $\Phi: \Delta_X \Rightarrow \mathcal{D}$ is a natural transformation, i.e.
- for each $J \in \text{Obj}(J)$ a morphism $\Phi_J: X \rightarrow \mathcal{D}(J)$ such that for any other object $J' \in J$ and any morphism $f: J \rightarrow J'$ the following diagram commutes.

$$\begin{array}{ccc} & X & \\ \Phi_J \swarrow & & \searrow \Phi_{J'} \\ \mathcal{D}(J) & \xrightarrow{\mathcal{D}(f)} & \mathcal{D}(J') \end{array}$$

This agrees with our previous definition of a cone.

Let (Y, Γ) be another cone over \mathcal{D} . Then a morphism $\Xi: (X, \Phi) \rightarrow (Y, \Gamma)$ is

- a morphism $\Xi \in \text{Hom}_{(\Delta \downarrow \mathcal{D})}((X, \Phi), (Y, \Gamma))$, i.e.
- a natural transformation $\Xi: \Delta_X \Rightarrow \Delta_Y$ (i.e. a morphism $\Delta_X \rightarrow \Delta_Y$ in the category $[\mathbf{J}, \mathbf{C}]$) such that the diagram

$$\begin{array}{ccc} \Delta_X & \xRightarrow{\Xi} & \Delta_Y \\ & \searrow \Phi \quad \swarrow \Gamma & \\ & \mathcal{D} & \end{array}$$

commutes, i.e.

- a morphism $\xi: X \rightarrow Y$ such that for each $J \in \text{Obj}(\mathbf{J})$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\xi} & Y \\ & \searrow \Phi_J \quad \swarrow \Gamma_J & \\ & \mathcal{D}(J) & \end{array}$$

commutes.

Cocones are the dual notion to cones. We make the following definition.

Definition 2.3.5 (cocone). A cocone over a diagram \mathcal{D} is an object in the comma category $(\mathcal{D} \downarrow \Delta)$.

Definition 2.3.6 (category of cocones). The category of cocones over a diagram \mathcal{D} is the category $(\mathcal{D} \downarrow \Delta)$.

The categorical definitions of cones and cocones allow us to define limits and colimits succinctly.

Definition 2.3.7 (limits, colimits). A limit of a diagram $\mathcal{D}: \mathbf{J} \rightsquigarrow \mathbf{C}$ is a final object in the category $(\Delta \downarrow \mathcal{D})$. A colimit is an initial object in the category $(\mathcal{D} \downarrow \Delta)$.

Note 2.3.3. The above definition of a limit unwraps as follows. The limit of a diagram $\mathcal{D}: \mathbf{J} \rightsquigarrow \mathbf{C}$ is a cone (X, Φ) over \mathcal{D} such that for any other cone (Y, Γ) over \mathcal{D} , there is a unique map $\xi: Y \rightarrow X$ such that for each $J \in \text{Obj}(\mathbf{J})$, the following diagram commutes.

$$\begin{array}{ccc} Y & \xrightarrow{\xi} & X \\ & \searrow \Gamma_J \quad \swarrow \Phi_J & \\ & \mathcal{D}(J) & \end{array}$$

2.4 The Yoneda lemma

2.4.1 The Hom functor

Given any locally small category (Definition 2.1.8) \mathbf{C} and two objects $A, B \in \text{Obj}(\mathbf{C})$, we have thus far notated the set of all morphisms $A \rightarrow B$ by $\text{Hom}_{\mathbf{C}}(A, B)$. This may have seemed a slightly odd notation, but there was a good reason for it: $\text{Hom}_{\mathbf{C}}$ is really a functor.

To be more explicit, we can view $\text{Hom}_{\mathbf{C}}$ as a functor $\mathbf{C}^{\text{op}} \times \mathbf{C} \rightsquigarrow \mathbf{Set}$, which sends each pair $(A', A) \in \text{Obj}(\mathbf{C}^{\text{op}} \times \mathbf{C})$ to the set $\text{Hom}_{\mathbf{C}}(A', A)$ of morphisms $A' \rightarrow A$.

This is fine, and probably the definition you were expecting, but we're not done: as a functor it also has to send the morphisms in $\mathbf{C} \times \mathbf{C}^{\text{op}}$ to \mathbf{Set} -arrows. How does *that* work?

A morphism in $\mathbf{C} \times \mathbf{C}^{\text{op}}$ between two objects (A', A) and (B', B) is an ordered pair (f', f) of two morphisms:

$$f': B' \rightarrow A' \quad \text{and} \quad f: A \rightarrow B,$$

where $A, B \in \text{Obj}(\mathbf{C})$ and $A', B' \in \text{Obj}(\mathbf{C}^{\text{op}})$. We can draw this as follows.

$$\begin{array}{ccc} A' & \xleftarrow{f'} & B' \\ A & \xrightarrow{f} & B \end{array}$$

We have to send this to a set-function $\text{Hom}_{\mathbf{C}}(A', A) \rightarrow \text{Hom}_{\mathbf{C}}(B', B)$. The way to do this is as follows: send each $m \in \text{Hom}(A', A)$ to the element

$$f \circ m \circ f' \in \text{Hom}_{\mathbf{C}}(B', B).$$

As the function $m \mapsto f \circ m \circ f'$ is the action of the functor $\text{Hom}_{\mathbf{C}}$ on the morphism (f', f) , it is natural to call it $\text{Hom}_{\mathbf{C}}(f', f)$.

Of course, we should check that this *really is* a functor, i.e. that it treats identities and compositions correctly. First, we need to show that $\text{Hom}_{\mathbf{C}}(1_{A'}, 1_A)$ is the identity function $1_{\text{Hom}_{\mathbf{C}}(A', A)}$. It is, since if $m \in \text{Hom}_{\mathbf{C}}(A', A)$,

$$\text{Hom}_{\mathbf{C}}(1_{A'}, 1_A): m \mapsto 1_A \circ m \circ 1_{A'} = m.$$

Next, we have to check that compositions work the way they're supposed to. Suppose we have objects and morphisms like this:

$$\begin{array}{ccccc} A' & \xleftarrow{f'} & B' & \xleftarrow{g'} & C' \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

where the primed stuff is in \mathbf{C}^{op} and the unprimed stuff is in \mathbf{C} . This can be viewed as three objects (A', A) , etc. in $\mathbf{C}^{\text{op}} \times \mathbf{C}$ and two morphisms (f', f) and (g', g) between them.

Okay, so we can compose (f', f) and (g', g) to get a morphism

$$(g', g) \circ (f', f) = (f' \circ g', g \circ f): (A', A) \rightarrow (C', C).$$

Note that the order of the composition in the first argument has been turned around: this is expected since the first argument lives in \mathbf{C}^{op} . To check that $\text{Hom}_{\mathbf{C}}$ handles compositions correctly, we need to verify that

$$\text{Hom}_{\mathbf{C}}(g, g') \circ \text{Hom}_{\mathbf{C}}(f, f') = \text{Hom}_{\mathbf{C}}(f' \circ g', g \circ f).$$

Let $m \in \text{Hom}_{\mathbf{C}}(A', A)$. We victoriously compute

$$\begin{aligned} [\text{Hom}_{\mathbf{C}}(g, g') \circ \text{Hom}_{\mathbf{C}}(f, f')](m) &= \text{Hom}_{\mathbf{C}}(g, g')(f \circ m \circ f') \\ &= g \circ f \circ m \circ f' \circ g' \\ &= \text{Hom}_{\mathbf{C}}(f' \circ g', g \circ f). \end{aligned}$$

Let's formalize this in a definition.

Definition 2.4.1 (hom functor). Let \mathbf{C} be a locally small category. The hom functor $\text{Hom}_{\mathbf{C}}$ is the functor

$$\mathbf{C}^{\text{op}} \times \mathbf{C} \rightsquigarrow \mathbf{Set}$$

which sends the object (A', A) to the set $\text{Hom}_{\mathbf{C}}(A', A)$ of morphisms $A' \rightarrow A$, and the morphism $(f', f): (A', A) \rightarrow (B', B)$ to the function

$$\text{Hom}_{\mathbf{C}}(f', f): \text{Hom}_{\mathbf{C}}(A', A) \rightarrow \text{Hom}_{\mathbf{C}}(B', B); \quad m \mapsto f \circ m \circ f'.$$

Now recall from computer science the concept of currying. Suppose we are given a function of two arguments, say $f(x, y)$. The idea of currying is this: if we like, we can view f as a family of functions of only one variable y , indexed by x :

$$h_x(y) = f(x, y).$$

We can even view h as a function which takes one argument x , and which returns a *function* h_x of one variable y .

This sets up a correspondence between functions $f: A \times B \rightarrow C$ and functions $h: A \rightarrow C^B$, where C^B is the set of functions $B \rightarrow C$. The map which replaces f by h is called *currying*. We can also go the other way (i.e. $h \mapsto f$), which is called *uncurrying*.

We have been intentionally vague about the nature of our function f , and what sort of arguments it might take; everything we have said also holds for, say, bifunctors.

In particular, it gives us two more ways to view our bifunctor $\text{Hom}_{\mathcal{C}}$. We can fix any $A \in \text{Obj}(\mathcal{C})$ and curry either argument. This gives us the following.

Definition 2.4.2 (curried hom functor). Let \mathcal{C} be a locally small category, $A \in \text{Obj}(\mathcal{C})$. We can construct from the hom functor $\text{Hom}_{\mathcal{C}}$

- a covariant functor $h^A: \mathcal{C} \rightsquigarrow \mathbf{Set}$ which maps
 - an object $B \in \text{Obj}(\mathcal{C})$ to the set $\text{Hom}_{\mathcal{C}}(A, B)$, and
 - a morphism $f: B \rightarrow B'$ to a \mathbf{Set} -function

$$h^A(f): \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, B'); \quad m \mapsto f \circ m.$$

- a contravariant functor $h_A: \mathcal{C} \rightsquigarrow \mathbf{Set}$ (or equivalently a covariant functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$) which maps
 - an object $B \in \text{Obj}(\mathcal{C})$ to the set $\text{Hom}_{\mathcal{C}}(B, A)$, and
 - a morphism $f: B \rightarrow B'$ to a \mathbf{Set} -function

$$h_A(f): \text{Hom}_{\mathcal{C}}(B', A) \rightarrow \text{Hom}_{\mathcal{C}}(B, A); \quad m \mapsto m \circ f.$$

2.4.2 Representable functors

Definition 2.4.3 (representable functor). Let \mathcal{C} be a category. A functor $\mathcal{F}: \mathcal{C} \rightsquigarrow \mathbf{Set}$ is representable if there is an object $A \in \text{Obj}(\mathcal{C})$ and a natural isomorphism (Definition 2.1.17)

$$\eta: \mathcal{F} \Rightarrow \begin{cases} h^A, & \text{if } \mathcal{F} \text{ is covariant} \\ h_A, & \text{if } \mathcal{F} \text{ is contravariant.} \end{cases}$$

Example 2.4.1. Consider the forgetful functor $\mathcal{U}: \mathbf{Grp} \rightsquigarrow \mathbf{Set}$ which sends a group to its underlying set, and a group homomorphism to its underlying function. This functor is represented by the group \mathbb{Z} .

To see this, we have to check that there is a natural isomorphism η between \mathcal{U} and $h^{\mathbb{Z}} = \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, -)$. That is to say, for each group G , there a \mathbf{Set} -isomorphism (i.e. a bijection)

$$\eta_G: \mathcal{U}(G) \rightarrow \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$$

satisfying the naturality conditions in Definition 2.1.13.

First, let's show that there's a bijection by providing an injection in both directions. Pick some $g \in G$. Then there is a unique group homomorphism which sends $1 \mapsto g$, so we have an injection $\mathcal{U}(G) \hookrightarrow \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$.

Now suppose we are given a group homomorphism $\mathbb{Z} \rightarrow G$. This sends 1 to some element $g \in G$, and this completely determines the rest of the homomorphism. Thus, we have an injection $\text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \hookrightarrow \mathcal{U}(G)$.

All that is left is to show that η satisfies the naturality condition. Let F and H be groups, and $f: G \rightarrow H$ a homomorphism. We need to show that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{U}(G) & \xrightarrow{\mathcal{U}(f)} & \mathcal{U}(H) \\ \eta_G \downarrow & & \downarrow \eta_H \\ \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) & \xrightarrow{h^{\mathbb{Z}}(f)} & \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, H) \end{array}$$

The upper path from top left to bottom right assigns to each $g \in G$ the function $\mathbb{Z} \rightarrow G$ which maps $1 \mapsto f(g)$. Walking down η_G from $\mathcal{U}(G)$ to $\text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$, g is mapped to the function $\mathbb{Z} \rightarrow G$ which maps $1 \mapsto g$. Walking right, we compose this function with f to get a new function $\mathbb{Z} \rightarrow H$ which sends $1 \mapsto f(g)$. This function is really the image of a homomorphism and is therefore unique, so the diagram commutes.

In fact, most forgetful functors to \mathbf{Set} are representable.

2.4.3 The Yoneda embedding

Theorem 2.4.1 (Yoneda lemma). Let \mathcal{F} be a functor from \mathbf{C} to \mathbf{Set} . Let $A \in \mathbf{Obj}(\mathbf{C})$. Then there is a bijection between the set $\mathbf{Nat}(h^A, \mathcal{F})$ of natural transformations $h^A \Rightarrow \mathcal{F}$ and the set $\mathcal{F}(A)$.

Proof. A natural transformation $\Phi: h^A \Rightarrow \mathcal{F}$ consists of a collection of \mathbf{Set} -morphisms (that is to say, functions) $\Phi_B: \mathbf{Hom}_{\mathbf{C}}(A, B) \rightarrow \mathcal{F}(B)$, one for each $B \in \mathbf{Obj}(\mathbf{C})$. We need to show that to each element of $\mathcal{F}(A)$ there corresponds exactly one Φ . We will do this by showing that any Φ is completely determined by where Φ_A sends the identity morphism $1_A \in \mathbf{Hom}_{\mathbf{C}}(A, A)$, so there is exactly one natural transformation for each place Φ can send 1_A .

The proof can be illustrated by the following commutative diagram.

$$\begin{array}{ccc}
 \mathbf{Hom}(A, A) & \xrightarrow{h^A(f)} & \mathbf{Hom}(A, B) \\
 \downarrow \Phi_A & & \downarrow \Phi_B \\
 \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B)
 \end{array}$$

$1_A \xrightarrow{\quad} f$
 $\downarrow \quad \downarrow$
 $a \xrightarrow{\quad} (\mathcal{F}f)(a) \stackrel{!}{=} \Phi_B(f)$

Any natural transformation Φ has a component Φ_A , and Φ_A has to send the identity transformation 1_A *somewhere*. It can send it to any element of $\mathcal{F}(A)$; let's call $\Phi_A(1_A) = a$.

Now the naturality conditions force our hand. For any $B \in \mathbf{Obj}(\mathbf{C})$ and any $f: A \rightarrow B$, $\Phi_B(f)$ *has to* be equal to $(\mathcal{F}f)(a)$. We get no choice in the matter.

But this completely determines Φ ! So we have shown that there is exactly one natural transformation for every element of $\mathcal{F}(A)$. We are done! \square

In the thread [19] on the sci.math google group, John Baez explains the importance of the Yoneda lemma in the following way.

Category theory can be viewed in many ways, but one is as a massive generalization of set theory. The category \mathbf{Set} has sets as objects and functions between them as morphisms. In practice, other categories are often built by taking as objects “sets with extra structure” and as morphisms “functions preserving the extra structure.” For example, \mathbf{Vect} is the category whose objects are vector spaces and whose morphisms are linear functions. The extra structure in this case is the linear structure. One can easily list hundreds more such examples.

However, an abstract category needn't have as its objects “sets with structure” – its objects are simply abstract thingamabobs! (I would have said “simply abstract objects”, but that would sound circular here, since it is, so I resorted to a more technical term.) At this point the Yoneda lemma leaps to our rescue by saying that all (small enough) categories can be embedded in categories in which the objects are sets with structure and the morphisms are structure-preserving functions between these.

Here's the basic idea of how it works, in watered-down form. To each object x , we associate the set $S(x)$ of all morphisms *to* x from all other objects. This set has a certain structure which one can work out, and a morphism $f: x \rightarrow y$ gives rise to a structure-preserving function $S(f)$ from $S(x)$ to $S(y)$ in the obvious way: given a morphism from something to x , just compose it with f to get a morphism to y .

So we have taken our original category and embedded in a category of “sets with structure.”

Definition 2.4.4 (yoneda embedding). Let \mathbf{C}, \mathbf{D} be locally small categories, $\mathcal{F}: \mathbf{C} \rightarrow \mathbf{D}$ be a functor. The Yoneda embedding is the induced functor

$$\mathcal{Y}: \mathbf{C} \rightarrow [\mathbf{C}^{\text{op}}, \mathbf{Set}],$$

where the notation $[\mathbf{C}^{\text{op}}, \mathbf{Set}]$ was defined in Definition 2.1.18.

Lemma 2.4.1. The Yoneda embedding is fully faithful (Definition 2.1.10).

Proof. We have to show that for all $A, B \in \text{Obj}(\mathbf{C})$, the map

$$\mathcal{Y}_{A,B}: \text{Hom}_{\mathbf{C}}(A, B) \rightarrow \text{Hom}_{[\mathbf{C}^{\text{op}}, \mathbf{Set}]}(h^A, h^B)$$

is a bijection.

Fix $B \in \text{Obj}(\mathbf{C})$, and consider the functor h^B . By the Yoneda lemma, there is a bijection between $\text{Nat}(h^A, h^B)$ and $h^B(A)$. By definition,

$$h^B(A) = \text{Hom}_{\mathbf{C}}(A, B).$$

But $\text{Nat}(h^A, h^B)$ is nothing else but $\text{Hom}_{[\mathbf{C}^{\text{op}}, \mathbf{Set}]}(h^A, h^B)$, so we are done. \square

2.5 Adjunctions

Consider the following functors:

- $\mathcal{U}: \mathbf{Grp} \rightsquigarrow \mathbf{Set}$, which sends a group to its underlying set, and
- $\mathcal{F}: \mathbf{Set} \rightsquigarrow \mathbf{Grp}$, which sends a set to the free group on it.

The functors \mathcal{U} and \mathcal{F} are dual in the following sense: \mathcal{U} is the most efficient way of moving from \mathbf{Grp} to \mathbf{Set} since all groups are in particular sets; \mathcal{F} might be thought of as providing the most efficient way of moving from \mathbf{Set} to \mathbf{Grp} . But how would one go about formalizing this?

Well, these functors have the following property. Let S be a set, G be a group, and let $f: S \rightarrow \mathcal{U}(G)$, $s \mapsto f(s)$ be a set-function. Then there is an associated group homomorphism $\tilde{f}: \mathcal{F}(S) \rightarrow G$, which sends $s_1 s_2 \dots s_n \mapsto f(s_1 s_2 \dots s_n) = f(s_1) \dots f(s_n)$. In fact, \tilde{f} is the unique homomorphism $\mathcal{F}(S) \rightarrow G$ such that $\tilde{f}(s) = f(s)$ for all $s \in S$.

Similarly, for every group homomorphism $g: \mathcal{F}(S) \rightarrow G$, there is an associated function $S \rightarrow \mathcal{U}(G)$ given by restricting g to S . In fact, this is the unique function $S \rightarrow \mathcal{U}(G)$ such that $g(s) = \tilde{f}(s)$ for all $s \in S$.

Thus for each $f \in \text{Hom}_{\mathbf{Grp}}(\mathcal{F}(S), G)$ we can construct an $\tilde{f} \in \text{Hom}_{\mathbf{Set}}(\mathcal{F}(S), G)$, and vice versa.

Let us add some mathematical scaffolding to the ideas explored above. We build two functors $\mathbf{Set}^{\text{op}} \times \mathbf{Grp} \rightsquigarrow \mathbf{Set}$ as follows.

1. Our first functor maps the object $(S, G) \in \text{Obj}(\mathbf{Set}^{\text{op}} \times \mathbf{Grp})$ to the hom-set $\text{Hom}_{\mathbf{Grp}}(\mathcal{F}(S), G)$, and a morphism $(\alpha, \beta): (S, G) \rightarrow (S', G')$ to a function

$$\text{Hom}_{\mathbf{Grp}}(\mathcal{F}(S), G) \rightarrow \text{Hom}_{\mathbf{Grp}}(\mathcal{F}(S'), G'); \quad m \mapsto \mathcal{F}(\alpha) \circ m \circ \beta$$

2. Our second functor maps (S, G) to $\text{Hom}_{\mathbf{Set}}(S, \mathcal{U}(G))$, and (α, β) to

$$m \mapsto \alpha \circ m \circ \mathcal{U}(\beta).$$

We can define a natural isomorphism Φ between these functors with components

$$\Phi_{S,G}: \text{Hom}_{\mathbf{Grp}}(\mathcal{F}(S), G) \rightarrow \text{Hom}_{\mathbf{Set}}(S, \mathcal{U}(G)); \quad f \mapsto \tilde{f}.$$

This mathematical structure turns out to be a recurring theme in the study of categories, called an *adjunction*.

Definition 2.5.1 (hom-set adjunction). Let \mathbf{C}, \mathbf{D} be categories and \mathcal{F}, \mathcal{G} functors as follows.

$$\begin{array}{ccc} \mathbf{C} & \xrightleftharpoons[\mathcal{G}]{\mathcal{F}} & \mathbf{D} \end{array}$$

We say that \mathcal{F} is left-adjoint to \mathcal{G} (or equivalently \mathcal{G} is right-adjoint to \mathcal{F}) and write $\mathcal{F} \dashv \mathcal{G}$ if there is a natural isomorphism

$$\Phi: \text{Hom}_{\mathcal{D}}(\mathcal{F}(-), -) \Rightarrow \text{Hom}_{\mathcal{C}}(-, \mathcal{G}(-)),$$

which fits between \mathcal{F} and \mathcal{G} like this.

$$\begin{array}{ccc} & \text{Hom}_{\mathcal{D}}(\mathcal{F}(-), -) & \\ \text{C}^{\text{op}} \times \mathcal{D} & \begin{array}{c} \Downarrow \Phi \\ \end{array} & \text{Set} \\ & \text{Hom}_{\mathcal{D}}(-, \mathcal{G}(-)) & \end{array}$$

The natural isomorphism amounts to a family of bijections

$$\Phi_{A,B}: \text{Hom}_{\mathcal{D}}(\mathcal{F}(A), B) \rightarrow \text{Hom}_{\mathcal{C}}(A, \mathcal{G}(B))$$

which satisfies the coherence conditions for a natural transformation.

Here are two equivalent definitions which are often used.

Definition 2.5.2 (unit-counit adjunction). We say that two functors $\mathcal{F}: \mathcal{C} \rightsquigarrow \mathcal{D}$ and $\mathcal{G}: \mathcal{D} \rightsquigarrow \mathcal{C}$ form a unit-counit adjunction if there are two natural transformations

$$\eta: 1_{\mathcal{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}, \quad \text{and} \quad \varepsilon: \mathcal{F} \circ \mathcal{G} \Rightarrow 1_{\mathcal{D}},$$

called the *unit* and *counit* respectively, which make the following so-called *triangle diagrams*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\mathcal{F}\eta} & \mathcal{F}\mathcal{G}\mathcal{F} \\ & \searrow 1_{\mathcal{F}} & \Downarrow \varepsilon_{\mathcal{F}} \\ & & \mathcal{F} \end{array} \quad \begin{array}{ccc} \mathcal{G} & \xrightarrow{\eta\mathcal{G}} & \mathcal{G}\mathcal{F}\mathcal{G} \\ & \searrow 1_{\mathcal{G}} & \Downarrow \mathcal{G}\varepsilon \\ & & \mathcal{G} \end{array}$$

commute.

The triangle diagrams take quite some explanation. The unit η is a natural transformation $1_{\mathcal{C}} \rightarrow \mathcal{G} \circ \mathcal{F}$. We can draw it like this.

$$\begin{array}{ccc} & 1_{\mathcal{C}} & \\ \mathcal{C} & \begin{array}{c} \Downarrow \eta \\ \end{array} & \mathcal{C} \\ & \mathcal{G} \circ \mathcal{F} & \end{array}$$

Analogously, we can draw ε like this.

$$\begin{array}{ccc} & \mathcal{F} \circ \mathcal{G} & \\ \mathcal{D} & \begin{array}{c} \Downarrow \varepsilon \\ \end{array} & \mathcal{D} \\ & 1_{\mathcal{D}} & \end{array}$$

Now consider the following diagram.

$$\begin{array}{ccccc} & & 1_{\mathcal{C}} & & \\ & & \Downarrow \eta & & \\ \mathcal{C} & \rightsquigarrow \mathcal{F} \rightsquigarrow & \mathcal{D} & \rightsquigarrow \mathcal{G} \rightsquigarrow & \mathcal{C} & \rightsquigarrow \mathcal{F} \rightsquigarrow & \mathcal{D} \\ & & \mathcal{F} \circ \mathcal{G} & & \\ & & \Downarrow \varepsilon & & \\ & & 1_{\mathcal{D}} & & \end{array}$$

We can whisker the η on top from the right, and the ε below from the left, to get the following diagram,

$$\begin{array}{ccccc} & & \mathcal{F} & & \\ & & \Downarrow \mathcal{F}\eta & & \\ \mathcal{C} & \rightsquigarrow \mathcal{F} \rightsquigarrow & \mathcal{D} & \rightsquigarrow \mathcal{G} \rightsquigarrow & \mathcal{C} & \rightsquigarrow \mathcal{F} \rightsquigarrow & \mathcal{D} \\ & & \mathcal{F} \circ \mathcal{G} \circ \mathcal{F} & & \\ & & \Downarrow \varepsilon_{\mathcal{F}} & & \\ & & \mathcal{F} & & \end{array}$$

then consolidate to get

$$\begin{array}{ccc}
 & \mathcal{F} & \\
 & \downarrow \mathcal{F}\eta & \\
 \mathcal{C} & \xrightarrow{\mathcal{F} \circ \mathcal{G} \circ \mathcal{F}} & \mathcal{D} \\
 & \downarrow \varepsilon \mathcal{F} & \\
 & \mathcal{F} &
 \end{array}$$

We can then take the composition $\varepsilon \mathcal{F} \circ \mathcal{F} \eta$ to get a natural transformation $\mathcal{F} \Rightarrow \mathcal{F}$

$$\begin{array}{ccc}
 & \mathcal{F} & \\
 & \downarrow \varepsilon \mathcal{F} \circ \mathcal{F} \eta & \\
 \mathcal{C} & \xrightarrow{\quad} & \mathcal{D} \\
 & \downarrow \mathcal{F} &
 \end{array}$$

the first triangle diagram says that this must be the same as the identity natural transformation $1_{\mathcal{F}}$.

The second triangle diagram is analogous.

Lemma 2.5.1. The functors \mathcal{F} and \mathcal{G} form a unit-counit adjunction if and only if they form a hom-set adjunction.

Proof. Suppose \mathcal{F} and \mathcal{G} form a hom-set adjunction with natural isomorphism Φ . Then for any $A \in \text{Obj}(\mathcal{C})$, we have $\mathcal{F}(A) \in \text{Obj}(\mathcal{D})$, so Φ give us a bijection

$$\Phi_{A, \mathcal{F}(A)}: \text{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(A)) \rightarrow \text{Hom}_{\mathcal{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$

We don't know much in general about $\text{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(A))$, but we know it always contains $1_{\mathcal{F}(A)}$. We can use $\Phi_{A, \mathcal{F}(A)}$ to map this to

$$\Phi_{A, \mathcal{F}(A)}(1_{\mathcal{F}(A)}): A \rightarrow (\mathcal{G} \circ \mathcal{F})(A).$$

Let's call $\Phi_{A, \mathcal{F}(A)}(1_{\mathcal{F}(A)}) = \eta_A$.

Similarly, if $B \in \text{Obj}(\mathcal{D})$, then $\mathcal{G}(B) \in \text{Obj}(\mathcal{C})$, so Φ gives us a bijection

$$\Phi_{\mathcal{G}(B), B}: \text{Hom}_{\mathcal{D}}((\mathcal{F} \circ \mathcal{G})(B), B) \rightarrow \text{Hom}_{\mathcal{C}}(\mathcal{G}(B), \mathcal{G}(B)).$$

Since $\Phi_{\mathcal{G}(B), B}$ is a bijection, it is invertible, and we can evaluate the inverse on $1_{\mathcal{G}(B)}$. Let's call

$$\Phi_{\mathcal{G}(B), B}^{-1}(1_{\mathcal{G}(B)}) = \varepsilon_B.$$

Clearly, η_A and ε_B are completely determined by Φ and Φ^{-1} respectively. It turns out that the converse is also true; in a matter reminiscent of the proof of the Yoneda lemma, we can express $\Phi_{A, B}$ in terms of η , and $\Phi_{A, B}^{-1}$ in terms of ε , for *any* A and B . Here's how this is done.

We use the naturality of Φ . We know that for any $A \in \text{Obj}(\mathcal{C})$, $B \in \text{Obj}(\mathcal{D})$, and $g: \mathcal{F}(A) \rightarrow B$, the following diagram has to commute.

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(A)) & \xrightarrow{g \circ (-)} & \text{Hom}_{\mathcal{D}}(\mathcal{F}(A), B) \\
 \downarrow \Phi_{A, \mathcal{F}(A)} & & \downarrow \Phi_{A, B} \\
 \text{Hom}_{\mathcal{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)) & \xrightarrow{\mathcal{G}(g) \circ (-)} & \text{Hom}_{\mathcal{C}}(A, \mathcal{G}(B)).
 \end{array}$$

Let's start at the top left with $1_{\mathcal{F}(A)}$ and see what happens. Taking the top road to the bottom right, we have $\Phi_{A, B}(g)$, and from the bottom road we have $\mathcal{G}(g) \circ \eta_A$. The diagram commutes, so we have

$$\Phi_{A, B}(g) = \mathcal{G}(g) \circ \eta_A.$$

Similarly, the commutativity of the diagram

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{D}}((\mathcal{F} \circ \mathcal{G})(B), B) & \xrightarrow{(-) \circ \mathcal{F}(f)} & \text{Hom}_{\mathcal{D}}(\mathcal{F}(A), B) \\
 \uparrow \Phi_{\mathcal{G}(B), B}^{-1} & & \uparrow \Phi_{A, B}^{-1} \\
 \text{Hom}_{\mathcal{C}}(\mathcal{G}(B), \mathcal{G}(B)) & \xrightarrow{(-) \circ f} & \text{Hom}_{\mathcal{C}}(A, \mathcal{G}(B))
 \end{array}$$

means that, for any $f: A \rightarrow \mathcal{G}(B)$,

$$\Phi_{A,B}^{-1}(f) = \varepsilon_B \circ \mathcal{F}(f)$$

To show that η and ε as defined here satisfy the triangle identities, we need to show that for all $A \in \text{Obj}(\mathcal{C})$ and all $B \in \text{Obj}(\mathcal{D})$,

$$(\varepsilon \mathcal{F})_A \circ (\mathcal{F} \eta)_A = (1_{\mathcal{F}})_A \quad \text{and} \quad (\mathcal{G} \varepsilon)_B \circ (\eta \mathcal{G})_B = (1_{\mathcal{G}})_B.$$

We have

$$(\varepsilon \mathcal{F})_A \circ (\mathcal{F} \eta)_A = \varepsilon_{\mathcal{F}(A)} \circ \mathcal{F}(\eta_A) = \Phi_{A,\mathcal{F}(A)}^{-1}(\eta_A) = 1_A = (1_{\mathcal{F}})_A$$

and

$$(\mathcal{G} \varepsilon)_B \circ (\eta \mathcal{G})_B = \mathcal{G}(\varepsilon_B) \circ \eta_{\mathcal{G}(B)} = \Phi_{\mathcal{G}(B),B}(\varepsilon_B) = 1_B = (1_{\mathcal{G}})_B.$$

□

Definition 2.5.3 (adjunct). Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction as follows.

$$\begin{array}{ccc} \mathcal{C} & \xrightleftharpoons[\mathcal{G}]{\mathcal{F}} & \mathcal{D} \end{array}$$

Then for each $A \in \text{Obj}(\mathcal{C})$ $B \in \text{Obj}(\mathcal{D})$, we have a natural isomorphism (i.e. a bijection)

$$\Phi_{A,B} : \text{Hom}_{\mathcal{D}}(\mathcal{F}(A), B) \rightarrow \text{Hom}_{\mathcal{C}}(A, \mathcal{G}(B)).$$

Thus, for each $f \in \text{Hom}_{\mathcal{D}}(\mathcal{F}(A), B)$ there is a corresponding element $\tilde{f} \in \text{Hom}_{\mathcal{C}}(A, \mathcal{G}(B))$, and vice versa. The morphism \tilde{f} is called the adjunct of f , and f is called the adjunct of \tilde{f} .

2.6 Internal hom functors

2.6.1 The internal hom functor

Recall the definition of the hom functor on a (locally small) category \mathcal{C} (Definition 2.4.1): it is the functor which maps two objects to the set of morphisms between them, so it is a functor

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \rightsquigarrow \mathbf{Set}.$$

If we take $\mathcal{C} = \mathbf{Set}$, then our hom functor never really leaves \mathbf{Set} ; it is *internal* to \mathbf{Set} . This is our first example of an *internal hom functor*. In fact, it is the prototypical internal hom functor, and we can learn a lot by studying its properties.

Let X and Y be sets. Denote the set of all functions $X \rightarrow Y$ by $[X, Y]$.

Let S be any other set, and consider a function $f: S \rightarrow [X, Y]$. For each element $s \in S$, f picks out a function $h_s: X \rightarrow Y$. But this is just a curried version of a function $S \times X \rightarrow Y$! So as we saw in Section 2.4.1, we have a bijection between the sets $[S, [X, Y]]$ and $[S \times X, Y]$. In fact, this is even a *natural* bijection, i.e. a natural transformation between the functors

$$[-, [-, -]] \quad \text{and} \quad [- \times -, -]: \mathbf{Set}^{\text{op}} \times \mathbf{Set}^{\text{op}} \times \mathbf{Set} \rightsquigarrow \mathbf{Set}.$$

Let's check this. First, we need to figure out how our functors act on functions. The first one is the most complicated, so let's get it out of the way. Suppose we have sets and functions like so.

$$\begin{array}{ccc} A'' & \xleftarrow{f''} & B'' \\ A' & \xleftarrow{f'} & B' \\ A & \xrightarrow{f} & B \end{array}$$

We want our functor to map

$$(A'', A', A) \mapsto [A'', [A', A]] = \text{Hom}_{\mathbf{Set}}(A'', \text{Hom}_{\mathbf{Set}}(A', A)),$$

so it should map (f'', f', f) to a function

$$\text{Hom}_{\text{Set}}(A'', \text{Hom}_{\text{Set}}(A', A)) \rightarrow \text{Hom}_{\text{Set}}(B'', \text{Hom}_{\text{Set}}(B', B)).$$

The notation $\text{Hom}_{\text{Set}}(-, -)$ is getting a little lengthy, so I'll stick to $[-, -]$ for now.

The way to do that is by sending $m \in [A'', [A', A]]$ to

$$[f', f] \circ m \circ f''.$$

The other one's not so tough. Our functor maps an object (A'', A', A) to $[A'' \times A', A]$. We need to map (f'', f', f) to a function

$$[f'' \times f', f]: [A'' \times A', A] \rightarrow [B'' \times B', B].$$

We do that by sending $m \in [A'' \times A', A]$ to

$$f \circ m \circ (f'', f') = [f'' \times f', f](m) \in [B'' \times B', B].$$

Checking that these really *are* functorial would be a bit much; each is just a few applications of the Yoneda embedding. We will however check that there is a natural isomorphism between them, which amounts to checking that the following diagram commutes.

$$\begin{array}{ccc}
 [A'' \times A', A] & \xrightarrow{[f'' \times f', f]} & [B'' \times B', B] \\
 \downarrow \Phi_{[A'' \times A', A]} & & \downarrow \Phi_{[B'' \times B', B]} \\
 & \begin{array}{ccc} m & \xrightarrow{\quad} & f \circ m \circ (f', f'') \\ \downarrow & & \downarrow \\ \Phi(m) & \xrightarrow{\quad} & [f', f] \circ \Phi(m) \circ f'' \stackrel{!}{=} \Phi(f \circ m \circ (f', f'')) \end{array} & \\
 [A'', [A', A]] & \xrightarrow{[f'', [f', f]]} & [B'', [B', B]]
 \end{array}$$

In other words, we have to show that

$$\Phi_{[A'' \times A', A]}(f \circ m \circ (f', f'')) = [f', f] \circ \Phi_{[B'' \times B', B]}(m) \circ f''.$$

So what is each of these? Well, $f \circ m \circ (f', f'')$ is a map $B'' \times B' \rightarrow B$, which maps (say) $(b'', b') \mapsto b$.

The natural transformation Φ tells us to curry this, i.e. turn it into a map $B'' \rightarrow [B', B]$. Not just any map, though: a map which when evaluated on b'' turns into a map which, when evaluated on b' , yields b .

We know that $f \circ m \circ (f', f''): (b'', b') \mapsto b$, i.e.

$$f(m(f''(b''), f'(b'))) = b.$$

If we can show that this is *also* what $[f', f] \circ \Phi_{[B'' \times B', B]}(m) \circ f''$ is equal to, we are done.

Well, let's go through what this definition means. First, we take b'' and feed it to f'' . Next, we let $\Phi_{[B'' \times B', B]}(m)$ act on the result, i.e. we fill the first argument of m with $f''(b'')$. What we get is the following:

$$m(f''(b''), -).$$

Then we are to precompose this with f' and stick the result into f :

$$f(m(f''(b''), f'(-))).$$

Indeed, when we evaluate this on b' , we get b , so the diagram commutes.

In other words, Φ is a natural bijection between the hom-sets $[-, [-, -]]$ and $[- \times -, -]$.

We picked this example because the collection $[A, B]$ of all functions between two sets A and B is itself a set. Therefore it makes sense to think of the hom-sets $\text{Hom}_{\mathcal{C}}(A, B)$ as living within the same category as A and B . However, this is far from peculiar to sets.

Definition 2.6.1 (internal hom functor). Let (\mathbf{C}, \otimes) be a monoidal category. An internal hom functor is a functor

$$[-, -]_{\mathbf{C}}: \mathbf{C}^{\text{op}} \times \mathbf{C} \rightsquigarrow \mathbf{C}$$

such that for every $X \in \text{Obj}(\mathbf{C})$ we have a pair of adjoint functors

$$(-) \otimes X \dashv [X, -]_{\mathbf{C}}.$$

Notation 2.6.1. The convention at the nLab is to denote the internal hom by square braces $[A, B]$, and this is for the most part what we will do. Unfortunately, we have already used this notation for the *regular* hom functor, so we will often add a subscript if the category in which the hom functor lives is not clear: $[-, -]_{\mathbf{C}}$ for a hom functor internal to \mathbf{C} , $[-, -]_{\mathbf{Set}}$ for the standard hom functor (or the hom functor internal to \mathbf{Set} , which amounts to the same). We will also occasionally denote the internal hom with a lower-case hom: $\text{hom}_{\mathbf{C}}(A, B)$ (a very common convention).

Many sources (for example DMOS [16]), we will distinguish the internal hom with an underline: $\underline{\text{Hom}}_{\mathbf{C}}(A, B)$.

Definition 2.6.2 (closed monoidal category). A monoidal category equipped with an internal hom functor is called a closed monoidal category.

Note 2.6.1. Here is another definition of $[X, Y]_{\mathbf{C}}$: it is the object representing (Definition 2.4.3) the functor

$$T \mapsto \text{Hom}_{\mathbf{C}}(T \otimes X, Y).$$

For the remainder of this chapter, let $(\mathbf{C}, \otimes, 1)$ be a closed monoidal category with internal hom functor $[-, -]_{\mathbf{C}}$.

2.6.2 The evaluation map

The internal hom functor gives us a way to talk about evaluating morphisms $f: X \rightarrow Y$ without mentioning elements of X .

Definition 2.6.3 (evaluation map). Let $X \in \text{Obj}(\mathbf{C})$. Then the adjunction

$$(-) \otimes X \dashv [X, -]_{\mathbf{C}}$$

gives us, for any $A, X, Y \in \text{Obj}(\mathbf{C})$, a natural bijection

$$\text{Hom}_{\mathbf{C}}(A \otimes X, Y) \xrightarrow{\sim} \text{Hom}_{\mathbf{C}}(A, [X, Y]_{\mathbf{C}}).$$

In particular, with $A = [X, Y]_{\mathbf{C}}$, we have a bijection

$$\text{Hom}_{\mathbf{C}}([X, Y]_{\mathbf{C}} \otimes X, Y) \xrightarrow{\sim} \text{Hom}_{\mathbf{C}}([X, Y]_{\mathbf{C}}, [X, Y]_{\mathbf{C}}).$$

The adjunct (Definition 2.5.3) of $1_{[X, Y]_{\mathbf{C}}} \in \text{Hom}_{\mathbf{C}}([X, Y]_{\mathbf{C}}, [X, Y]_{\mathbf{C}})$ is an object in

$$\text{Hom}_{\mathbf{C}}([X, Y]_{\mathbf{C}} \otimes X, Y),$$

denoted

$$\text{eval}_{X, Y}: [X, Y]_{\mathbf{C}} \otimes X \rightarrow Y,$$

and called the evaluation map.

Example 2.6.1. As we saw in Example 2.2.9, the category \mathbf{Set} is a monoidal category with bifunctor given by the cartesian product. The internal hom is simply the regular hom functor

$$\text{Hom}_{\mathbf{Set}}(-, -) = [-, -].$$

Let us explore the evaluation map on \mathbf{Set} . It is the adjunct of the identity map $1_{[X, Y]}$ under the adjunction

$$[[X, Y] \times X, Y] \dashv [[X, Y], [X, Y]].$$

Thus, it is a function

$$\text{eval}_{X, Y}: [X, Y] \times X \rightarrow Y; \quad (f, x) \mapsto \text{eval}_{X, Y}(f, x).$$

So far, we don't know how $\text{eval}_{X,Y}$ behaves.

The above adjunction is given by currying: we start on the LHS with a map $\text{eval}_{X,Y}$ with two arguments, and we turn it into a map which fills in only the first argument. Thus the map on the RHS corresponding to $\text{eval}_{X,Y}$ is given by

$$f \mapsto \text{eval}_{X,Y}(f, -).$$

If we want the map $f \mapsto \text{eval}_{X,Y}(f, -)$ to be the identity map, f and $\text{eval}_{X,Y}(f, -)$ must agree on all elements x , i.e.

$$f(x) = \text{eval}_{X,Y}(f, x) \quad \text{for all } x \in X.$$

Thus, the evaluation map is the map which sends $(f, x) \mapsto f(x)$.

2.6.3 The composition morphism

The evaluation map allows us to define composition of morphisms without talking about internal hom objects as if they have elements.

Definition 2.6.4 (composition morphism). For $X, Y, Z \in \text{Obj}(\mathbf{C})$, the composition morphism

$$\circ_{X,Y,Z}: [Y, Z]_{\mathbf{C}} \otimes [X, Y]_{\mathbf{C}} \rightarrow [X, Z]_{\mathbf{C}}$$

is the $(-) \otimes X \vdash [X, -]_{\mathbf{C}}$ -adjunct of the composition

$$[Y, Z]_{\mathbf{C}} \otimes [X, Y]_{\mathbf{C}} \otimes X \xrightarrow{(1_{[Y,Z]_{\mathbf{C}}}, \text{eval}_{X,Y})} [Y, Z]_{\mathbf{C}} \otimes Y \xrightarrow{\text{eval}_{Y,Z}} Z.$$

Example 2.6.2. In **Set**, the composition morphism $\circ_{X,Y,Z}$ also lives up to its name. Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$, and $x \in X$. The above composition goes as follows.

1. The map $(1_{[Y,Z]}, \text{eval}_{X,Y})$ turns the triple (g, f, x) into the pair $(g, f(x))$.
2. The map $\text{eval}_{Y,Z}$ turns $(g, f(x))$ into $g(f(x)) = (g \circ f)(x)$.

The evaluation morphism $\circ_{X,Y,Z}$ is the currying of this, i.e. it sends

$$(f, g) \mapsto (f \circ g)(-).$$

2.6.4 Dual objects

Recall that for any k -vector space V , there is a dual vector space

$$V^* = \{L: V \rightarrow k\}.$$

This definition generalizes to any closed monoidal category.

Definition 2.6.5 (dual object). Let $X \in \text{Obj}(\mathbf{C})$. The dual object to X , denoted X^* , is defined to be the object

$$[X, 1]_{\mathbf{C}}.$$

That is to say, X^* is the internal hom object modelling the hom set of morphisms from X to the identity object 1.

Notation 2.6.2. It is easy to see that the evaluation morphism has a component

$$\text{eval}_{X^*,X}: X^* \otimes X \rightarrow 1.$$

To clean things up a bit, we will write eval_X instead of $\text{eval}_{X^*,X}$.

Notation 2.6.3. In many sources, e.g. DMOS ([16]), the dual object to X is denoted X^\vee instead of X^* .

Lemma 2.6.1. There is a natural isomorphism between the functors

$$\mathrm{Hom}_{\mathcal{C}}(-, X^*) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{C}}((-) \otimes X, 1).$$

Proof. For any $X, T \in \mathrm{Obj}(\mathcal{C})$, the definition of the internal hom $[-, -]_{\mathcal{C}}$ gives us a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(T \otimes X, 1) \simeq \mathrm{Hom}_{\mathcal{C}}(T, [X, 1]_{\mathcal{C}}) = \mathrm{Hom}_{\mathcal{C}}(T, X^*).$$

□

Lemma 2.6.2. The map $X \mapsto X^*$ can be made into a contravariant functor.

Proof.

□

2.7 Higher categories

This section follows [18] closely.

One of the main benefits of categories is that they let us distinguish between different notions of sameness. In a set, two elements are either the same or different, and that is that. In a category, however, objects can be the same in certain ways without being identical: there are many n -dimensional real vector spaces, but they are all isomorphic. However this notion of ‘sameness without identity’ is not omnipresent in category theory; we still have to resort to the limited, set-theoretic sense of equivalence when talking about morphisms.

In higher category theory, this is remedied by introducing 2-morphisms, which are ‘morphisms between morphisms.’ We can then say that two 1-morphisms are isomorphic if there is a pair of mutually inverse 2-morphisms between them. If you take a category and add 2-morphisms, what you get is called a 2-category.

Of course, we now have no notion of isomorphism for 2-categories, so we must add 3-morphisms, etc. The result of the n th iteration of this procedure is called an n -category. If you keep going forever (or more rigorously, formally consider the object you'd get if you did), you get an ∞ -category.

You may have noticed that we have been using the same notation (\Rightarrow) for 2-morphisms and natural transformations. The reason for this is that natural transformations are the morphisms in the 2-category \mathbf{Cat} of (small) categories.

Chapter 3

Sheaves

3.1 Presheaves

Definition 3.1.1 (category of opens). Let (X, τ) be a topological space. The category of opens of X , $\text{Open}(X)$ is the category whose objects are

$$\text{Obj}(\text{Open}(X)) = \tau$$

and whose morphisms are, for $U, V \in \tau$,

$$\text{Hom}(U, V) = \begin{cases} r(V, U), & U \subseteq V \\ \emptyset, & \text{otherwise.} \end{cases}$$

To put it another way, if U and V are open sets in X , then there is exactly one morphism from U to V if $U \subseteq V$, and none otherwise.

Of course, we must check that $\text{Open}(X)$ really is a category.

1. If $U \subseteq V$ and $V \subseteq W$, then $U \subseteq W$, so we are forced to define the composition

$$r(W, V) \circ r(V, U) = r(W, U).$$

2. The composition \circ is associative since set inclusion is.
3. The identity morphism $r(U, U)$ is the identity with respect to composition: restricting a set to itself is the same thing as not restricting.

Definition 3.1.2 (presheaf). A presheaf on a topological space (X, τ) is a contravariant functor from $\text{Open}(X)$ to some category \mathcal{C} . Usually, \mathcal{C} will be the category **Ring** of rings.

Example 3.1.1 (important example). The prototypical example of a presheaf on a topological space X is the presheaf \mathcal{C}_X of real continuous functions on X . Since by Definition 3.1.2 \mathcal{C}_X must be a contravariant functor, we need to specify what \mathcal{C}_X does to the objects $\text{Obj}(\text{Open}(X))$ and the morphisms $\text{res}_{V,U}$.

For every open set $U \in \text{Obj}(\text{Open}(X))$, define

$$\mathcal{C}_X(U) = \{f: U \rightarrow \mathbb{R} \mid f \text{ continuous}\},$$

the ring of all real continuous functions $U \rightarrow \mathbb{R}$.

Recall that for two open sets U and $V \in \text{Obj}(\text{Open}(X))$, there is a unique morphism $r(V, U)$ from U to V if and only if $V \subseteq U$. Thus, we need to assign to each $r(V, U)$ a ring homomorphism from $\mathcal{C}_X(V)$ to $\mathcal{C}_X(U)$. We use the restriction homomorphism, which maps $f \in \mathcal{C}_X(V)$ to $f|_U$, its restriction to U . We denote this by

$$\text{res}_{V,U}(f) = f|_U.$$

In other words,

$$\mathcal{C}_X(r(V, U)) = \text{res}_{V,U}.$$

3.2 Sheaves

Definition 3.2.1 (sheaf). Let X be a topological space. A presheaf \mathcal{F} on X is called a sheaf if it satisfies the following:

1. **Identity:** Let $U \subseteq X$ be an open set, and let $\{U_i\}$ be an open cover of U . If $f_1, f_2 \in \mathcal{F}(U)$ such that

$$f_1|_{U_i} = f_2|_{U_i}$$

for all i , then $f_1 = f_2$. That is to say, if two sections of \mathcal{F} over U agree on every element of an open cover of U , then they must agree on U .

2. **Glueability:** Suppose $\{U_i\}$ is an open cover of U , and $f_i \in \mathcal{F}(U_i)$ is a collection of sections of \mathcal{F} such that if $U_i \cap U_j \neq \emptyset$ then

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

Then there exists some $f \in \mathcal{F}(U)$ such that

$$f|_{U_i} = f_i$$

for all i . In other words, If we have sections of an open cover of U which agree on overlaps, then we can glue them together to get a section on all of U .

Example 3.2.1 (important example continued). The presheaf \mathcal{C}_X from Example 3.1.1 is a sheaf.

Example 3.2.2 (a presheaf which is not a sheaf). Let $X = \mathbb{R}$, and define a presheaf \mathcal{F} on X via

$$\mathcal{F}(U) = \{f: U \rightarrow \mathbb{R} \mid f \text{ bounded}\}.$$

This is a presheaf since the restriction of a bounded map is bounded.

However, we do not have condition 2 of Definition 3.2.1: glueability. To see this, consider the following open cover

$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} U_n, \quad U_n = (n-1, n+1).$$

Clearly, the identity function 1_{U_n} on U_n is bounded for all n . However, the function $f: \mathbb{R} \rightarrow \mathbb{R}$ which agrees with 1_{U_n} on all the U_n is the identity function $1_{\mathbb{R}}$, which is unbounded.

Definition 3.2.2 (stalk). Let X be a topological space, $x \in X$, and let \mathcal{F} be a presheaf on X . The stalk of \mathcal{F} at x is

$$\mathcal{F}_x = \{(f, U) \mid x \in U, f \in \mathcal{F}(U)\} / \sim,$$

where

$$(f, U) \sim (g, V)$$

if there exists an open set $W \subseteq U \cap V$ with $x \in W$ such that $f|_W = g|_W$.

Lemma 3.2.1. If \mathcal{F} is a sheaf of objects with some algebraic structure, like rings, then the stalk \mathcal{F}_x inherits this algebraic structure.

Proof. For the sake of concreteness, consider the case in which \mathcal{F} is a presheaf of rings. Let $[(f, U)], [(g, V)] \in \mathcal{F}_x$. We need to define

$$[(f, U)] + [(g, V)] \quad \text{and} \quad [(f, U)] \cdot [(g, V)].$$

In each case, we can simply use representatives of the equivalence classes, letting

$$[(f, U)] + [(g, V)] = [(f + g, U \cap V)], \quad \text{and} \quad [(f, U)] \cdot [(g, V)] = [(f \cdot g, U \cap V)].$$

We need of course to prove well-definition: that if $(f, U) \sim (f', U')$ and $(g, V) \sim (g', V')$, then

$$[f + g] = [f' + g'],$$

and similarly for multiplication.

But if $(f, U) \sim (f', U')$, then there exists an open set $U'' \subseteq U \cap U'$ with $x \in U''$ such that $f|_{U''} = f'|_{U''}$, and similarly V'' . But since $x \in U''$ and $x \in V''$, we must also have that $x \in U'' \cap V''$. Since f and f' (and g and g') agree on $U'' \cap V''$, so must $f + g$ and $f' + g'$. Thus $[(f + g, U \cap V)]$ is well-defined.

The proof of the well-definition of multiplication is exactly analogous. \square

We now drop the open set in the notation of an element of a stalk, writing $[f]$ instead of $[(f, U)]$.

Example 3.2.3 (important example continued). What do the stalks of \mathcal{C}_X (Example 3.1.1) look like? Let $x \in X$, and define $\varphi: (\mathcal{C}_X)_x \rightarrow \mathbb{R}$ via

$$\varphi: [f] \mapsto f(x).$$

This is a ring homomorphism since

$$\varphi([f][g]) = \varphi([fg]) = (fg)(x) = f(x)g(x) = \varphi([f])\varphi([g]),$$

and similarly for addition. It is also surjective; to see this, we need only find a single function which maps to any $r \in \mathbb{R}$. (the constant function will do.)

Denote $\ker(\varphi) = \mathfrak{m}_x$. By the first isomorphism theorem,

$$(\mathcal{C}_X)_x / \mathfrak{m}_x \simeq \mathbb{R}.$$

But since \mathbb{R} is a field, \mathfrak{m}_x must be maximal.

If $\mathfrak{M} \neq \mathfrak{m}_x$ is another maximal ideal, then there must exist $[g] \in \mathfrak{M} \setminus \mathfrak{m}_x$ (since if $\mathfrak{M} \subseteq \mathfrak{m}_x$, \mathfrak{M} isn't maximal). Since $[g] \notin \ker(\varphi)$, $g(x) \neq 0$. But since g is continuous, there exists a neighborhood V of x such that $g \neq 0$ on V .

But then g is invertible on V , and hence is invertible in the stalk $(\mathcal{C}_X)_x$; thus, \mathfrak{M} contains an invertible element, hence contains the identity 1_R , hence is the whole ring; and we have a contradiction.

Thus each stalk of \mathcal{C}_X has a unique maximal ideal.

3.3 Ringed spaces

Definition 3.3.1 (ringed space). A ringed space is a double (X, \mathcal{O}_X) , where X is a topological space and \mathcal{O}_X is a sheaf of rings on X .

Definition 3.3.2 (local ring). A ring R is local if it has a unique maximal ideal.

Definition 3.3.3 (locally ringed space). A locally ringed space is a ringed space whose stalks are local rings.

Example 3.3.1 (important example continued). By the toil of Example 3.2.3, each stalk of \mathcal{C}_X is a local ring, hence \mathcal{C}_X is a locally ringed space.

Note 3.3.1. It is *not* necessary that each local section of a locally ringed space be a local ring; only the stalks must be local.

There are many ways to build sheaves. Here are two.

Definition 3.3.4 (restriction sheaf). Let X be a topological space, and \mathcal{F} a sheaf on X . Let U be an open subset of X . The restriction sheaf $\mathcal{F}|_U$ is the sheaf which maps open subsets $V \subseteq U$ to their images under \mathcal{F} :

$$\mathcal{F}|_U(V) = \mathcal{F}(V), \quad \text{for } V \subseteq U.$$

For open sets $V, W \subseteq U$, the restriction $r(W, V)$ is mapped to

$$\mathcal{F}|_U(r(W, V)) \equiv \mathcal{F}(r(W, V)) = \text{res}_{W, V},$$

the same restriction as in \mathcal{F} .

Definition 3.3.5 (pushforward sheaf). Let X and Y be topological spaces, $\pi: X \rightarrow Y$ continuous, \mathcal{F} be a sheaf on X . Define the pushforward of \mathcal{F} by π , denoted $\pi_*\mathcal{F}$, by

$$(\pi_*\mathcal{F})(V) = \mathcal{F}(\pi^{-1}(V)), \quad V \subseteq Y \text{ open},$$

and

$$(\pi_*\mathcal{F})(r(V, U)) = \text{res}_{\pi^{-1}(V), \pi^{-1}(U)}.$$

We can view sheaves as objects in a category. We can then talk about their morphisms, and find the correct definition of a sheaf isomorphism.

First, we need to define a homomorphism of presheaves. Since we defined a presheaf on a topological space X as a functor $\text{Open}(X) \rightarrow \mathbf{C}$ for some category \mathbf{C} (Definition 3.1.2), we can define a sheaf homomorphism as a morphism in the functor category $\mathbf{C}^{\text{Open}(X)}$, which is to say

Definition 3.3.6 (presheaf homomorphism). Let \mathcal{C}, \mathcal{D} be \mathbf{C} -presheaves on a topological space X . A presheaf homomorphism $\mathcal{C} \rightarrow \mathcal{D}$ is a morphism in the category $\mathbf{C}^{\text{Open}(X)}$, i.e. a natural transformation between \mathcal{C} and \mathcal{D} .

Definition 3.3.7 (sheaf homomorphism). Since sheaves are in particular presheaves, a sheaf homomorphism is simply a presheaf homomorphism between sheaves.

Definition 3.3.8 (sheaf isomorphism). A sheaf isomorphism is defined in the obvious way.

Definition 3.3.9 (locally isomorphic). Let X be a topological space, \mathcal{F} , and let \mathcal{G} be \mathbf{C} -sheaves on X . We say that \mathcal{F} is locally isomorphic to \mathcal{G} if for every $x \in X$, there exists a neighborhood U of x such that the restriction sheaf (Definition 3.3.4) $\mathcal{F}|_U$ is sheaf-isomorphic to \mathcal{G} .

Chapter 4

Spinors

4.1 A few more algebraic concepts

Definition 4.1.1 (graded algebra). Let A be an algebra, G be a monoid. We say that A is G -graded if A decomposes additively as

$$A = \bigoplus_{i \in G} A_i$$

with the property that $A_i \cdot A_j \subseteq A_{i+j}$, where the addition $i + j$ is understood to be taking place in G .

Definition 4.1.2 (filtered algebra). Let A be an algebra. A filtration on A is an increasing sequence

$$\{0\} \subset F_0 \subset F_2 \subset \cdots \subset A$$

of sub vector spaces of A such that

$$A = \bigcup_{i \geq 0} F_i,$$

and which are compatible with the multiplication law in the sense that

$$F_r \cdot F_s \subseteq F_{r+s}.$$

A filtered algebra is an algebra A together with a filtration $\{F_i\}_{i=0}^{\infty}$ of A .

Definition 4.1.3 (associated graded algebra). Let A be an algebra with a filtration $\{F_i\}_{i=1}^{\infty}$. The associated graded algebra can be defined as follows.

Define vector spaces

$$G_i = F_i/F_{i-1} \quad \text{for } i \geq 1, \quad G_0 = F_0.$$

Then for all $i \geq 1$, $F_i = G_i \oplus F_{i-1}$. Iterating this, we have

$$F_i = G_i \oplus G_{i-1} \oplus \cdots \oplus G_0,$$

and by induction or colimits or something

$$A = \bigoplus_{i=0}^{\infty} G_i,$$

where the equality is to be taken as equality of vector spaces.

In order to turn $\bigoplus_{i=0}^{\infty} G_i$ into a graded algebra we need to define multiplication on it in such a way that it is compatible with the grading, i.e. for $\alpha \in G_i$ and $\beta \in G_j$ we should have $\alpha \cdot \beta \in G_{i+j}$. Let $\alpha = \alpha_i + F_{i-1}$ be the equivalence class of α , and similarly $\beta = \beta_j + F_{j-1}$.

This can be accomplished by defining

$$\alpha\beta = (\alpha_i + F_{i-1})(\beta_j + F_{j-1}) = \alpha_i\beta_j + F_{i+j-1} \in G_{i+j}.$$

Note 4.1.1. Any graded algebra

$$A = \bigoplus_n A_n$$

is also a filtered algebra with

$$F_n = \bigoplus_{i=0}^n A_i.$$

Thus from any graded algebra one can, via the canonical filtration, arrive at an associated graded algebra. This is isomorphic to A as a vector space, but not in general as an algebra.

Definition 4.1.4 (tensor product of algebras). Let R be a ring, and A and B be R -algebras. Since in particular every algebra is an R -module if you forget about the multiplication, we can take the tensor product of A and B as R -modules:

$$A \otimes_R B.$$

We can then define a multiplication on $A \otimes_R B$ by defining

$$(a \otimes b) \cdot (a' \otimes b') = (a \cdot a') \otimes (b \cdot b')$$

and extending via linearity.

The module $A \otimes_R B$ together with the multiplication rule described above is called the tensor product of A and B .

Note 4.1.2. This is a perfectly good definition of a tensor product for general algebras, and is even a graded algebra with the grading

$$(A \otimes B)_n = \bigoplus_{i+j=n} a_i \otimes b_j.$$

However, as we will see in the next chapter, for graded algebras there is exactly one other choice. In the language of category theory, there are two choices of braiding that make the category of G -graded vector spaces into a symmetric monoidal category. It turns out that this other choice is the more interesting one.

Definition 4.1.5 (graded tensor product of algebras). Let A and B be \mathbb{Z}_r -graded algebras. Then we have a decomposition

$$A = \bigoplus_{i+j=n}^r A_i$$

and similarly for B . The graded tensor product, denoted $\hat{\otimes}$, of A and B is defined by the grading

$$(A \hat{\otimes} B)_n = \bigoplus_{i+j=n}^r A_i \otimes B_j$$

and the multiplication law

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b) \cdot \deg(c)} (ac) \otimes (bd),$$

where a, d are of pure degree.

Definition 4.1.6 (Koszul sign rule). The multiplication law

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b) \cdot \deg(c)} (ac) \otimes (bd),$$

is called the Koszul sign rule.

Definition 4.1.7 (\mathbb{Z}_2 -graded tensor product). In the case we are interested in, $d = 2$. In this case we denote the tensor product of two \mathbb{Z}_2 graded algebras A and B by $A \hat{\otimes} B$, whose grading is

$$(A \otimes B)_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1), \quad (A \otimes B)_1 = (A_0 \otimes B_1) \oplus (A_1 \otimes B_0).$$

4.2 Clifford algebras

Clifford algebras can be defined as universal objects in an appropriate category. We first define this category.

Definition 4.2.1 (quadratic vector space). A quadratic vector space is a pair (V, q) where V is a vector space over a field k and q is a (possibly degenerate) quadratic form on V .

Note 4.2.1. Quadratic forms and symmetric bilinear forms can be freely exchanged via a polarization; we will use the same notation for both, differentiating them only by the number of arguments they take. For example,

$$q(v) = q(v, v).$$

Definition 4.2.2 (category of quadratic vector spaces). Let \mathbf{QVec} be the category whose objects are quadratic vector spaces and whose morphisms $(V, q_V) \rightarrow (W, q_W)$ are linear maps $f: V \rightarrow W$ such that $f^*q_W = q_V$.

Definition 4.2.3 (Clifford map). Let (V, q) be a quadratic vector space and A be an associative k -algebra with unit 1_A . We say that a k -linear map $\varphi: V \rightarrow A$ is Clifford if for all $x \in V$,

$$\varphi(x)^2 = -q(x)1_A.$$

Definition 4.2.4 (category $\mathbf{Cliff}(V, q)$). $\mathbf{Cliff}(V, q)$ is the category whose objects are Clifford maps $f: V \rightarrow A$ to some associative unital k -algebra A and whose morphisms are algebra homomorphisms $\rho: A \rightarrow A'$ such that the diagram

$$\begin{array}{ccc} & V & \\ f \swarrow & & \searrow f' \\ A & \xrightarrow{\rho} & A' \end{array}$$

commutes.

Definition 4.2.5 (Clifford algebra). Let (V, Q) be a quadratic vector space. The Clifford algebra $\mathbf{Cl}(V, q)$ is the algebra from the initial object $(\iota, \mathbf{Cl}(V, q))$ in the category $\mathbf{Cliff}(V, q)$, if it exists.

Theorem 4.2.1. The Clifford algebra always exists. That is to say for any quadratic vector space (V, q) there is a pair $(\iota, \mathbf{Cl}(V, q))$ where ι is a Clifford map and $\mathbf{Cl}(V, q)$ is an associative unital algebra such that for any Clifford map $f: V \rightarrow A$, there is a unique algebra homomorphism $\mathbf{Cl}(f): \mathbf{Cl}(V) \rightarrow A$ such that the following diagram commutes.

$$\begin{array}{ccc} & V & \\ \iota \swarrow & & \searrow f \\ \mathbf{Cl}(V, q) & \xrightarrow{\mathbf{Cl}(f)} & A \end{array}$$

Proof. Denote by $T(V)$ the tensor algebra over V and by ϕ the canonical injection $V \hookrightarrow T(V)$.

The universal property of the tensor algebra tells us that any linear map $f: V \rightarrow A$ to an associative, unital algebra A can be extended to a unique algebra homomorphism $\bar{f}: T(V) \rightarrow A$ making the following diagram commute.

$$\begin{array}{ccc} & V & \\ \phi \swarrow & & \searrow f \\ T(V) & \xrightarrow{\bar{f}} & A \end{array}$$

Now consider the case in which V is taken to be a quadratic vector space (V, q) and f is Clifford. Then the algebra homomorphism \bar{f} has the property that for any $v \in V^{\otimes 1}$, $\bar{f}(v \otimes v + q(v)1) = 0$. Thus, all elements of $T(V)$ which can be written in the form

$$v \otimes v + q(v)1, \quad v \in V^{\otimes 1}$$

are in the kernel of \bar{f} , and so must be the ideal generated by such elements, which we will call \mathcal{I}_q . We can take the quotient $T(V)/\mathcal{I}_q$, calling the associated canonical projection $\pi_q: T(V) \rightarrow T(V)/\mathcal{I}_q$.

Since \mathcal{I}_q is in the kernel of f , we can replace f by a map $\tilde{f}: T(V)/\mathcal{I}_q \rightarrow A$. This allows us to fill in the diagram below.

$$\begin{array}{ccccc}
 T(V) & & \xrightarrow{\tilde{f}} & & A \\
 \downarrow \pi_q & \swarrow \phi & & \searrow f & \\
 & V & & & \\
 & \swarrow \pi_q \circ \phi & & \searrow & \\
 T(V)/\mathcal{I}_q & & \xrightarrow{\tilde{f}} & & A
 \end{array}$$

Notice that $\pi_q \circ \phi$ is Clifford.

The above discussion means that for any Clifford map $f: V \rightarrow A$, we have a unique algebra homomorphism $\tilde{f}: T(V)/\mathcal{I}_q \rightarrow A$ such that the diagram above commutes. But that means that the pair $(\pi_q \circ \phi, T(V)/\mathcal{I}_q)$ satisfies the universal property for Clifford algebras. Thus we must have that, up to a unique isomorphism, $\text{Cl}(V, q) = T(V)/\mathcal{I}_q$, and $\iota = \pi_q \circ \phi$. \square

Theorem 4.2.2. The Clifford algebra defines a functor Cl from \mathbf{QVec} to the category of associative unital algebras.

Proof. We need two pieces of information to determine that Cl is a functor: how it acts on the objects (V, q) of \mathbf{QVec} , and how it acts on the morphisms $f: V \rightarrow V'$. We have studied in some detail how it acts on objects by sending (V, q) to $\text{Cl}(V, q)$. Now we must see how it acts on morphisms.

By definition, the morphisms $(V, q) \rightarrow (V', q')$ in \mathbf{QVec} are linear maps $L: V \rightarrow V'$ which preserve the quadratic form. We can write down the following commutative diagram.

$$\begin{array}{ccc}
 (V, q) & \xrightarrow{L} & (V', q') \\
 \downarrow \iota_V & \searrow \iota_{V'} \circ L & \downarrow \iota_{V'} \\
 \text{Cl}(V, q) & & \text{Cl}(V', q')
 \end{array}$$

Notice that the map L preserves the quadratic form q and $\iota_{V'}$ is Clifford. Thus, we have

$$[(\iota_{V'} \circ L)(v)]^2 = [\iota_{V'}(L(v))]^2 = \iota_{V'}(L(v)^2) = \iota_{V'}(-q'(L(v))) = \iota_{V'}(-q(v)1) = -q(v)1,$$

so the map $\iota_{V'} \circ L$ is Clifford. But the universal property of Clifford algebras guarantees us that any Clifford map from (V, q) to an associative, unital algebra (of which $\text{Cl}(V', q')$ is an example) must factor through $\text{Cl}(V, q)$! This gives us, for any such L , a unique map $\text{Cl}(V, q) \rightarrow \text{Cl}(V', q')$ which we will call $\text{Cl}(L)$. This fills in the diagram as follows.

$$\begin{array}{ccc}
 (V, q) & \xrightarrow{L} & (V', q') \\
 \downarrow \iota_V & \searrow \iota_{V'} \circ L & \downarrow \iota_{V'} \\
 \text{Cl}(V, q) & \xrightarrow{\exists! \text{Cl}(L)} & \text{Cl}(V', q')
 \end{array}$$

Checking that the assignment $L \rightarrow \text{Cl}(L)$ really is functorial amounts to checking that it is covariant and that it is associative, both of which follow from juxtaposition of diagrams. \square

Note 4.2.2. The automorphism group of any quadratic vector space (V, q) is the group of linear transformations which preserve the quadratic form, and is called the *orthogonal group*, denoted

$$\text{O}(V) = \{M \in \text{GL}(V) \mid M^*q = q\}.$$

The functoriality of Cl means that any $M \in \text{O}(V)$ induces an algebra automorphism of $\text{Cl}(M)$, the unique map which makes the following diagram commute.

$$\begin{array}{ccc} V & \xrightarrow{M} & V \\ \downarrow \iota & & \downarrow \iota \\ \text{Cl}(V, q) & \xrightarrow{\text{Cl}(M)} & \text{Cl}(V, q) \end{array}$$

To put it another way, Theorem 4.2.2 ensures us of an embedding $\text{O}(V, q) \hookrightarrow \text{Aut}(\text{Cl}(V, q))$.

Theorem 4.2.3. There is a \mathbb{Z}_2 grading on $\text{Cl}(V, q)$.

Proof. Take $P \in \text{O}(V)$ to be the map which sends $v \mapsto -v$ for all $v \in V$. By the functoriality of Cl , this extends to an algebra automorphism $\text{Cl}(P) = \alpha$ such that $\alpha^2 = 1_{\text{Cl}(V, q)}$. This means that $\text{Cl}(V, q)$ splits, as a vector space, into two parts

$$\text{Cl}(V, q) = \text{Cl}^0(V, q) \oplus \text{Cl}^1(V, q).$$

It is not hard to check that the multiplication has the grading in Definition 4.1.7. □

Lemma 4.2.1. The Clifford algebra $\text{Cl}(V, q)$ inherits a filtration from the tensor algebra $T(V)$.

Proof. The tensor algebra has a natural filtration given by

$$F_r = \bigoplus_{n=0}^r V^{\otimes n};$$

we have

$$T(V) = \bigcup_{i=0}^{\infty} F_i$$

and

$$F_r \otimes F_s \subseteq F_{r+s}.$$

This makes $T(V)$ into a filtered algebra.

Now consider the image of each F_r under the canonical projection $\pi_q: T(V) \rightarrow T(V)/\mathcal{I}_q$, say

$$\mathcal{F}_r = \pi_q(F_r).$$

Certainly, this makes $\text{Cl}(V, q)$ into a filtered vector space. We must check that the filtration respects the algebraic structure. We have, for $\alpha \in F_r$ and $\beta \in F_s$, $\alpha \otimes \beta \in F_{r+s}$, so

$$\pi_q(\alpha \otimes \beta) \in \mathcal{F}_{r+s}.$$

However, π_q is an algebra homomorphism, so

$$\pi_q(\alpha \otimes \beta) = \pi_q(\alpha) \cdot \pi_q(\beta) \in \mathcal{F}_r \cdot \mathcal{F}_s.$$

□

Theorem 4.2.4. For any quadratic form q , the associated graded algebra to $\text{Cl}(V, q)$ is isomorphic (as a vector space) to the exterior algebra $\Lambda(V)$.

Proof. Very easy to see by writing everything down in a basis. I'd really like a categorical proof of this, however. □

Theorem 4.2.5. Let $V = V_1 \oplus V_2$ be a q -orthogonal decomposition of the vector space V . Then there is a natural isomorphism of Clifford algebras

$$\text{Cl}(V, q) \rightarrow \text{Cl}(V_1, q_1) \hat{\otimes} \text{Cl}(V_2, q_2).$$

Proof. Consider the map

$$f: V \rightarrow \text{Cl}(V_1, q_1) \hat{\otimes} \text{Cl}(V_2, q_2); \quad v \mapsto v_1 \otimes 1 + 1 \otimes v_2$$

where $v = v_1 + v_2$ is the q -orthogonal decomposition of v .

$$\begin{aligned} f(v) \cdot f(v) &= (v_1 \otimes 1 + 1 \otimes v_2) \cdot (v_1 \otimes 1 + 1 \otimes v_2) \\ &= (v_1 \cdot v_1) \otimes 1 + v_1 \otimes v_2 - v_1 \otimes v_2 + 1 \otimes (v_2 \cdot v_2) \\ &= (v_1^2 + v_2^2)1 \\ &= -(q(v_1) + q(v_2))1 \\ &= -q(v)1, \end{aligned}$$

where the minus sign on the second line comes from the fact that the image of V under the canonical inclusion is in $\text{Cl}^1(V, q)$ and the Koszul sign rule. Thus the map f is Clifford, so it extends to an algebra homomorphism $\tilde{f}: \text{Cl}(V, q) \rightarrow \text{Cl}(V_1, q_1) \hat{\otimes} \text{Cl}(V_2, q_2)$.

This is injective since □

Definition 4.2.6 (transpose). On any tensor power $V^{\otimes n}$ of a vector space V there is an action of the symmetric group S_n which sends

$$v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad \sigma \in S_n.$$

In particular, for any n we have the involution

$$v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_1.$$

This extends to an algebra involution on

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n},$$

called the transpose and denoted

$$(v_1 \otimes \cdots \otimes v_n)^t = (v_n \otimes \cdots \otimes v_1).$$

Any element I of the ideal \mathcal{I}_q can be written

$$I = \sum_i a_i \otimes (v_i \otimes v_i + q(v_i)1) \otimes b_i,$$

where a_i and b_i can be assumed to be of pure degree. Under the action of the transpose we have

$$I^t = \sum_i (b_i)^t \otimes (v_i \otimes v_i + q(v)1) \otimes (a_i)^t,$$

which is still in \mathcal{I}_q . Thus \mathcal{I}_q is invariant under the transpose and the transpose extends to an involution on $\text{Cl}(V, q) = T(V)/\mathcal{I}_q$, which we will also denote by $(\cdot)^t$.

Note 4.2.3. The transpose is an antiautomorphism, i.e. $(\psi\varphi)^t = \varphi^t\psi^t$.

4.3 Pin and Spin

Definition 4.3.1 (group of units of $\text{Cl}(V, q)$). Denote by $\text{Cl}^\times(V, q)$ the group of units of $\text{Cl}(V, q)$.

Lemma 4.3.1 (properties of $\text{Cl}^\times(V, q)$).

1. For all $v \in V$ with $q(v) \neq 0$, $v \in \text{Cl}^\times(V, q)$.
2. When $\dim(V) = n < \infty$ and $k = \mathbb{R}$ (or \mathbb{C}), $\text{Cl}^\times(V, q)$ is a Lie group of dimension 2^n .

3. The universal enveloping algebra of the Lie algebra $T_e(\mathrm{Cl}^\times(V, q))$ is isomorphic to $\mathrm{Cl}(V, q)$.

Proof.

1. Since $v \otimes v = -q(v)1$, $v \otimes (-v/q(v)) = 1$.

2. We specialize to the case $k = \mathbb{R}$; $k = \mathbb{C}$ is the same.

As a vector space, $\mathrm{Cl}(V, q) \simeq \Lambda(V) \simeq \mathbb{R}^{2^n}$. Thus $\mathrm{Cl}(V, q)$ can canonically be viewed both as topological manifold with a single chart a topological vector space. On a finite-dimensional topological vector space any (multi)linear map is continuous, so addition and multiplication are continuous. Thus the group of units is a C^0 -submanifold of $\mathrm{Cl}(V, q)$, and multiplication restricts to a continuous map on it.

But by a theorem of Gleason, Montgomery, and Zippin ([15]) any such topological group automatically has a compatible smooth atlas, and is therefore a Lie group.

3. This (or at least something equivalent to it) is just stated in [1]; I would have no idea how one would go about proving it.

□

Definition 4.3.2 (adjoint representation of $\mathrm{Cl}^\times(V, q)$). The adjoint representation of $\mathrm{Cl}^\times(V, q)$ is the map

$$\mathrm{Ad}: \mathrm{Cl}^\times(V, q) \rightarrow \mathrm{Aut}(\mathrm{Cl}(V, q)); \quad \varphi \mapsto \mathrm{Ad}_\varphi = \varphi(\cdot)\varphi^{-1}$$

i.e. $\mathrm{Ad}_\varphi(x) = \varphi x \varphi^{-1}$.

Lemma 4.3.2. For any v such that $q(v) \neq 0$, i.e. when v belongs to the set

$$\tilde{V} = \{v \in V \mid q(v) \neq 0\},$$

then Ad_v is an endomorphism on V . Furthermore, for any $w \in V$, $\mathrm{Ad}_v w$ has the explicit formula

$$-\mathrm{Ad}_v(w) = w - 2 \frac{q(v, w)}{q(v)} v.$$

Proof. For any $v \in \tilde{V}$, $v^{-1} = -v/q(v)$. Thus for any $w \in V$,

$$-q(v)\mathrm{Ad}_v(w) = -q(v)vwv^{-1} = v w v = -v^2 w - 2q(v, w)v = q(v)w - 2q(v, w)v.$$

Clearly, the assignment

$$w \mapsto w - 2 \frac{q(v, w)}{q(v)} v$$

is a linear map $V \rightarrow V$, hence an endomorphism. □

Lemma 4.3.3. For every $v \in \tilde{V}$, $\mathrm{Ad}_v \in \mathrm{O}(V)$.

Proof. Let $w \in V$.

$$\begin{aligned} q(\mathrm{Ad}_v(w)) &= q\left(w - 2 \frac{q(v, w)}{q(v)} v, w - 2 \frac{q(v, w)}{q(v)} v\right) \\ &= q(w) - 4 \frac{q(v, w)}{q(v)} q(v, w) + 4 \frac{q(v, w)^2}{q(v)^2} q(v) \\ &= q(w), \end{aligned}$$

so $\mathrm{Ad}_v^*(q) = q$. □

Definition 4.3.3 (pin group, spin group). Let $\mathrm{P}(V, q)$ denote the subgroup of $\mathrm{Cl}^\times(V, q)$ generated by \tilde{V} . This group is not in and of itself terribly important, but it has two important subgroups:

- $\text{Pin}(V, q)$, the subgroup of $P(V, q)$ generated by those $v \in V$ with $q(v) = \pm 1$. That is to say, each element of $\text{Pin}(V, q)$ can be written

$$v_1 v_2 \cdots v_r,$$

where each belongs to V , and $q(v_i) = \pm 1$ for all i .

- $\text{Spin}(V, q)$, defined by

$$\text{Spin}(V, q) = \text{Pin}(V, q) \cap \text{Cl}^0(V, q).$$

Notation 4.3.1. For $v \in V$, denote by R_v the reflection across the hyperplane

$$v^\perp = \{w \in V \mid q(v, w) = 0\}.$$

That is to say, R_v fixes v^\perp and maps $v \rightarrow -v$. Explicitly,

$$R_v w = w - 2 \frac{q(v, w)}{q(v)} v.$$

Theorem 4.3.1 (Cartan-Dieudonné). Any orthogonal transformation can be expressed as a finite number of reflections. That is, for any $A \in \text{O}(V, q)$, we have the decomposition

$$A = R_{v_1} R_{v_2} \cdots R_{v_n}.$$

for some $n \in \mathbb{N}$ and some $\{v_i\}_{i=1}^n \subseteq V$.

From Lemma 4.3.2, we have the following expression for $\text{Ad}_v(w)$ when $w \in \tilde{V}$:

$$-\text{Ad}_v(w) = w - 2 \frac{q(v, w)}{q(v)} v.$$

The RHS is simply the reflection of w over the hyperplane v^\perp , i.e.

$$-\text{Ad}_v = R_v.$$

However, there is a pesky minus sign in the LHS of the above. We can rectify this by modifying our definition of the adjoint slightly.

Definition 4.3.4 (twisted adjoint representation). For $\varphi \in \text{Cl}^\times(V, q)$, define the twisted adjoint representation

$$\widetilde{\text{Ad}}_\varphi: \text{Cl}(V, q) \rightarrow \text{Cl}(V, q); \quad w \mapsto \alpha(\varphi) w \varphi^{-1}.$$

The word representation is justified: we can view $\widetilde{\text{Ad}}$ as a map $\text{Cl}(V, q) \rightarrow \text{GL}(\text{Cl}(V, q))$, and this map is a homomorphism since for all $w \in \text{Cl}^\times(V, q)$

$$\widetilde{\text{Ad}}_{\varphi_1 \varphi_2}(w) = \alpha(\varphi_1 \varphi_2) \cdot w \cdot (\varphi_1 \varphi_2)^{-1} = \alpha(\varphi_1) (\alpha(\varphi_2) \cdot w \cdot \varphi_2^{-1}) \varphi_1^{-1} = (\widetilde{\text{Ad}}_{\varphi_1} \circ \widetilde{\text{Ad}}_{\varphi_2})(w).$$

Note 4.3.1. The twisted adjoint representation fixes the problem it was meant to solve, i.e. for $v \in \tilde{V}$, $w \in V$,

$$\widetilde{\text{Ad}}_v w = w - 2 \frac{q(v, w)}{q(v)} v.$$

This is easy to see: $v \in \tilde{V} \subseteq V^{\otimes 1}$, so $\alpha(v) = -v$ and

$$\widetilde{\text{Ad}}_v w = \alpha(v) w v^{-1} = -v w v^{-1} = -\text{Ad}_v w.$$

Definition 4.3.5 ($\tilde{P}(V, q)$). Let

$$\tilde{P}(V, q) = \left\{ \varphi \in \text{Cl}^\times(V, q) \mid \widetilde{\text{Ad}}_\varphi V = V \right\},$$

the set of all elements of $\text{Cl}^\times(V, q)$ which fix V under the twisted adjoint action.

Lemma 4.3.4. $P(V, q) \subseteq \tilde{P}(V, q)$.

Proof. Let $\varphi \in P(V, q)$. Then $\varphi = \varphi_1 \cdots \varphi_r$ for $\varphi_i \in \tilde{V}$. Since for any $v \in V$,

$$\widetilde{\text{Ad}}_{\varphi_i} v = v - 2 \frac{q(\varphi, v)}{q(\varphi)} \varphi \in V$$

and

$$\widetilde{\text{Ad}}_{\varphi}(v) = (\widetilde{\text{Ad}}_{\varphi_1} \circ \cdots \circ \widetilde{\text{Ad}}_{\varphi_r})(v),$$

the result follows by induction. \square

Theorem 4.3.2. Suppose V is finite-dimensional and q is nondegenerate. Then the kernel of the homomorphism

$$\widetilde{\text{Ad}}: \tilde{P}(V, q) \rightarrow \text{GL}(V)$$

is exactly the group k^\times of nonzero multiples of 1.

Proof. Choose an orthonormal basis for V , i.e. a basis $\{v_i\}_{i=1}^n$ for V such that $q(v_i, v_j) = \delta_{ij}$.

Suppose $\varphi \in \ker(\widetilde{\text{Ad}})$, i.e.

$$\widetilde{\text{Ad}}_{\varphi}(v) = v \quad \text{for all } v \in V.$$

Then

$$\alpha(\varphi)v = v\varphi.$$

One can continue by showing that φ must be independent of each v_i , and thus equal to a nonzero constant, but this is not very elegant. I think there is probably a better way. \square

Note 4.3.2. The image $\widetilde{\text{Ad}}(P(V, q))$ is exactly the subgroup of $O(V, q)$ generated by reflections. But by Theorem 4.3.1 (Cartan-Dieudonné), any element $A \in O(V, q)$ can be written as a product of reflections. Therefore the restriction $\widetilde{\text{Ad}}: P(V, q) \rightarrow O(V, q)$ is surjective.

Definition 4.3.6 (norm map). The norm map is the function

$$N: \text{Cl}(V, q) \rightarrow \text{Cl}(V, q); \quad N: \varphi \mapsto \varphi \cdot \alpha(\varphi^t).$$

Lemma 4.3.5. If V is finite-dimension and q is nondegenerate, then the restriction of N to $\tilde{P}(V, q)$ is a homomorphism

$$N: \tilde{P}(V, q) \rightarrow k^\times.$$

Proof. By definition, for any $\varphi \in \tilde{P}(V, q)$ we have

$$\alpha(\varphi) \cdot v \cdot \varphi^{-1} \in V.$$

The transpose $(\cdot)^t$ is the identity on V . Thus

$$\begin{aligned} \alpha(\varphi) \cdot v \cdot \varphi^{-1} &= (\alpha(\varphi) \cdot v \cdot \varphi^{-1})^t \\ &= (\varphi^{-1})^t \cdot v \cdot \alpha(\varphi^t). \end{aligned}$$

Thus

$$\begin{aligned} v &= \varphi^t \cdot \alpha(\varphi) \cdot v \cdot \varphi^{-1} \cdot \alpha(\varphi^t)^{-1} \\ &= [\varphi^t \alpha(\varphi)] v [\alpha(\varphi^t) \varphi]^{-1} \\ &= \alpha[\alpha(\varphi^t) \varphi] v [\alpha(\varphi^t) \varphi]^{-1} \\ &= \widetilde{\text{Ad}}_{\alpha(\varphi^t) \varphi}(v), \end{aligned}$$

so $\alpha(\varphi^t) \varphi \in k^\times$. Applying α we find that

$$N(\varphi) = \varphi^t \alpha(\varphi) \in k^\times$$

as desired.

To see that N is indeed a homomorphism and not just a function, it is enough to see that for any $\varphi, \psi \in \tilde{P}(V, q)$,

$$\begin{aligned} N(\varphi\psi) &= (\varphi\psi) \cdot \alpha(\varphi\psi)^t \\ &= \varphi(\psi \cdot \alpha(\psi)^t) \alpha(\varphi)^t \\ &= \varphi N(\psi) \alpha(\varphi)^t \\ &= \varphi \cdot \alpha(\varphi)^t N(\psi) \\ &= N(\varphi) \cdot N(\psi). \end{aligned}$$

□

Theorem 4.3.3. For each $\varphi \in \tilde{P}(V, q)$, the map $\widetilde{\text{Ad}}_\varphi: V \rightarrow V$ preserves the quadratic form q , and thus belongs to $O(V, q)$. This gives us a representation

$$\widetilde{\text{Ad}}: \tilde{P}(V, q) \rightarrow O(V, q).$$

Proof. First, note that $N(\alpha(\varphi)) = N(\varphi)$ since

$$N(\alpha(\varphi)) = \alpha(\varphi) \alpha(\alpha(\varphi)^t) = \alpha(\varphi \alpha(\varphi)^t) = \alpha(N(\varphi)) = N(\varphi).$$

Also, for all $v \in V$,

$$N(v) = v \alpha(v^t) = v \cdot (-v) = -v^2 = q(v).$$

Hence, for any $v \in \tilde{V}$ and any $\varphi \in \tilde{P}(V, q)$, we have

$$N(\widetilde{\text{Ad}}_\varphi(v)) = N(\varphi \cdot v \cdot \alpha(\varphi)^{-1}) = N(\varphi) \cdot N(v) \cdot N(\varphi^{-1}) = N(v) = q(v).$$

But by definition, $\widetilde{\text{Ad}}_\varphi(v) \in V$, so

$$N(\widetilde{\text{Ad}}_\varphi(v)) = q(\widetilde{\text{Ad}}_\varphi(v)).$$

Thus, $q(\widetilde{\text{Ad}}_\varphi(v)) = q(v)$, so $\widetilde{\text{Ad}}_\varphi^*(q) = q$.

□

Definition 4.3.7 ($\text{SP}(V, q)$). Let

$$\text{SP}(V, q) = P(V, q) \cap \text{Cl}^0(V, q),$$

i.e. the subset of $P(V, q)$ generated by products of the form

$$v_1 \cdot v_2 \cdots v_n, \quad n \text{ even.}$$

Lemma 4.3.6. The restriction

$$\widetilde{\text{Ad}}: \text{SP}(V, q) \rightarrow \text{SO}(V, q)$$

is surjective.

Proof. Since for any $v \in \tilde{V}$ the determinant $\det(R_v) = -1$, any element R of $\text{SO}(V, q)$ must be those expressible as an even number of reflections

$$R = R_{v_1} \circ R_{v_2} \cdots \circ R_{v_n} = R_{v_1 \cdot v_2 \cdots v_n}.$$

But since n is even $v_1 \cdot v_2 \cdots v_n \in \text{SP}(V, q)$ by definition, so R is in the image $\widetilde{\text{Ad}}(\text{SP}(V, q))$.

□

Note 4.3.3. We know that the maps

$$\widetilde{\text{Ad}}: P(V, q) \rightarrow O(V, q) \quad \text{and} \quad \widetilde{\text{Ad}}: \text{SP}(V, q) \rightarrow \text{SO}(V, q)$$

are surjective.

It is reasonable to wonder if the restrictions

$$\widetilde{\text{Ad}}: \text{Pin}(V, q) \rightarrow O(V, q) \quad \text{and} \quad \widetilde{\text{Ad}}: \text{Spin}(V, q) \rightarrow \text{SO}(V, q)$$

are also surjective. This seems likely because R_v is invariant under the scaling of v , i.e. $R_{tv} = R_v$ for $t \neq 0$, so one could try to normalize any v so that $q(v) = \pm 1$.

This sort of works. Since q is quadratic, for any $v \in V$, $t \in k$ we have $q(tv) = t^2 q(v)$. The above restrictions will be surjective only if for any $a \neq 0$, we can solve at least one of the equations $t^2 = \pm a$. This leads us to the following definition.

Definition 4.3.8 (spin field). A field k is called spin if for all $a \in k$ at least one of the equations

$$t^2 = a, \quad t^2 = -a$$

can be solved.

Example 4.3.1. The fields \mathbb{R} , \mathbb{C} , and \mathbb{Z}_p where p is a prime congruent to 3 (mod 4) are spin. The field \mathbb{Q} of rationals is not.

Lemma 4.3.7. If V is a k -vector space and k is spin, then the restrictions

$$\widetilde{\text{Ad}}: \text{Pin}(V, q) \rightarrow \text{O}(V, q) \quad \text{and} \quad \widetilde{\text{Ad}}: \text{Spin} \rightarrow \text{SO}(V, q)$$

are surjections.

Proof. We already sort of know how to prove this.

Suppose $O \in \text{O}(V, q)$. Then we can express O as a finite product of reflections

$$O = R_{v_1} \circ R_{v_2} \circ \cdots \circ R_{v_n}, \quad n \in \mathbb{N}.$$

But since k is by assumption spin we can find for each $i \in \{1, \dots, n\}$ a number t_i such that $q(t_i v_i) = \pm 1$. Since $R_{t_i v_i} = R_{v_i}$, we have

$$O = R_{t_1 v_1} \circ R_{t_2 v_2} \circ \cdots \circ R_{t_n v_n} = R_{t_1 v_1 \cdot t_2 v_2 \cdots t_n v_n},$$

where $t_1 v_1 \cdot t_2 v_2 \cdots t_n v_n \in \text{Pin}(V, q)$.

The proof that the second restriction is a surjection is the same except that n must be even. \square

Theorem 4.3.4. Let V be a finite-dimensional vector space over a spin field k , and suppose q is a non-degenerate quadratic form on V . Then there are short exact sequences

$$\begin{aligned} 0 &\longrightarrow F \longrightarrow \text{Spin}(V, q) \xrightarrow{\widetilde{\text{Ad}}} \text{SO}(V, q) \longrightarrow 1 \\ 0 &\longrightarrow F \longrightarrow \text{Pin}(V, q) \xrightarrow{\widetilde{\text{Ad}}} \text{O}(V, q) \longrightarrow 1 \end{aligned}$$

where

$$F = \{\varphi \in k \mid \varphi^2 = \pm 1\} = \begin{cases} \mathbb{Z}_2 = \{1, -1\} & \text{if } \sqrt{-1} \notin k \\ \mathbb{Z}_4 = \{\pm 1, \pm \sqrt{-1}\} & \text{otherwise.} \end{cases}$$

Proof. We already know that both $\text{Pin}(V, q)$ and $\text{Spin}(V, q)$ contain F so the maps from F are injective. We have just shown that $\widetilde{\text{Ad}}$ is surjective. Therefore, all we have to show is that in both cases,

$$\ker(\widetilde{\text{Ad}}) = F.$$

To this end, let $\varphi \in \text{Pin}(V, q) = v_1 \cdots v_n$ be in $\ker(\widetilde{\text{Ad}})$. Then

$$\varphi^2 = N(\varphi) = N(v_1) \cdots N(v_n) = v_1^2 \cdots v_n^2 = (\pm 1) \cdots (\pm 1) = \pm 1,$$

so $\varphi \in K$. In fact, this logic also holds for $\varphi \in \text{Spin}(V, q)$ if we imagine that n is even. But we already know that $K \subseteq \ker(\widetilde{\text{Ad}})$, so $K = \ker(\widetilde{\text{Ad}})$ and we are done. \square

Chapter 5

Superalgebra

In this chapter we give some basic definitions of various super-* structures. Categorical aspects will be left until the following chapter.

5.1 Super rings

Definition 5.1.1 (super ring). A super ring (or $\mathbb{Z}/2\mathbb{Z}$ graded ring) A is, additively, an abelian group A with a direct sum decomposition

$$A = A_0 \oplus A_1.$$

Elements $a \in A_0$ are said to be of degree zero, denoted $\tilde{a} = 0$; elements $b \in A_1$ are said to be of degree one, denoted $\tilde{b} = 1$.

Additionally, there is a multiplicative structure which is associative and distributive from the left and the right; the only change from the normal definition of a ring is that the multiplicative structure must obey the axiom

$$\tilde{a}\tilde{b} = \tilde{a} + \tilde{b},$$

where the addition is taken modulo 2.

Note 5.1.1. Of course, not all elements $c \in A$ are in A_0 or A_1 ; formulae which talk about the grading must be understood to be extended via additivity. The axiom concerning the multiplicative grading, for example, can be taken to say that if $a = a_0 + a_1$ and $b = b_0 + b_1$, then

$$ab = a_0b_0 + a_1b_0 + a_0b_1 + a_1b_1,$$

and the axiom applies to each monomial.

Example 5.1.1 (exterior algebra). Consider the exterior algebra

$$\bigwedge \mathbb{R}^n = \bigoplus_{i=0}^n \bigwedge^i \mathbb{R}^n.$$

With multiplication given by the wedge product, $\bigwedge \mathbb{R}^n$ is a super ring with grading given by

$$\left(\bigoplus_{i \text{ even}} \bigwedge^i \mathbb{R}^n \right) \oplus \left(\bigoplus_{i \text{ odd}} \bigwedge^i \mathbb{R}^n \right).$$

Example 5.1.2. Let V be a vector space over a field k of characteristic not 2. The so-called *parity transformation* on V is the map

$$P: V \rightarrow V; \quad v \mapsto -v.$$

This preserves any quadratic form, and hence extends to an algebra homomorphism $\text{Cl}(P) \equiv \alpha$ of any quadratic vector space $\text{Cl}(V, q)$. Since

$$\alpha^2 = \text{Cl}(P) \circ \text{Cl}(P) = \text{Cl}(P \circ P) = \text{Cl}(\text{id}_{(V, q)}) = \text{id}_{\text{Cl}(V, q)},$$

α is invertible, hence an isomorphism. Indeed, from the above equation we can say more. For the moment ignoring the multiplicative structure of $\mathcal{C}\ell(V, q)$ and considering it only as a vector space, α can be thought of as a linear bijection whose square is the identity. This means α has two eigenvalues $+1$ and -1 , and additively $\mathcal{C}\ell(V, q)$ decomposes into a direct sum

$$\mathcal{C}\ell(V, q) = \mathcal{C}\ell^0(V, q) \oplus \mathcal{C}\ell^1(V, q),$$

where

$$\mathcal{C}\ell^0(V, q) = \{\varphi \in \mathcal{C}\ell(V, q) \mid \alpha(\varphi) = \varphi\}, \quad \text{and} \quad \mathcal{C}\ell^1(V, q) = \{\varphi \in \mathcal{C}\ell(V, q) \mid \alpha(\varphi) = -\varphi\}.$$

For $\varphi_1 \in \mathcal{C}\ell^i(V, q)$ and $\varphi_2 \in \mathcal{C}\ell^j(V, q)$,

$$\alpha(\varphi_1 \varphi_2) = \alpha(\varphi_1) \alpha(\varphi_2);$$

this gives us the following multiplication table.

\times	$\mathcal{C}\ell^0(V, q)$	$\mathcal{C}\ell^1(V, q)$
$\mathcal{C}\ell^0(V, q)$	$\mathcal{C}\ell^0(V, q)$	$\mathcal{C}\ell^1(V, q)$
$\mathcal{C}\ell^1(V, q)$	$\mathcal{C}\ell^1(V, q)$	$\mathcal{C}\ell^0(V, q)$

Thus, $\mathcal{C}\ell(V, q)$ is a superalgebra.

Definition 5.1.2 (supercommutator). Let A be a commutative ring. The supercommutator is the map

$$[\cdot, \cdot]: A \times A \rightarrow A; \quad (a, b) \rightarrow [a, b] = ab - (-1)^{\tilde{a}\tilde{b}}ba.$$

Definition 5.1.3 (supercommutation). We will say that a supercommutes with b if $[a, b] = 0$; this means in particular that two even elements commute if $ab = ba$, and two odd elements supercommute if $ab = -ba$.

Note 5.1.2. Supercommutativity should not be confused with regular commutativity. Here is an example of a super ring which is commutative but not supercommutative.

Example 5.1.3. Consider the ring $C^0(\mathbb{R})$ of continuous functions $\mathbb{R} \rightarrow \mathbb{R}$. This is a super ring with the direct sum decomposition is given by

$$C^0(\mathbb{R}) = \{\text{even functions}\} \oplus \{\text{odd functions}\};$$

this can be seen since any $f \in C^0(\mathbb{R})$ can be written

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd}}.$$

The multiplication defined pointwise inherits its commutativity from the real numbers; but *all* elements are commutative, even the odd elements. Hence $C^0(\mathbb{R})$ is commutative but not supercommutative.

Definition 5.1.4 (super ring homomorphism). Let A, B be super rings. A super ring homomorphism $f: A \rightarrow B$ is a ring homomorphism which preserve the grading.

One might wonder if, in fact, *all* ring homomorphisms between super rings are super ring homomorphisms. This is not the case.

Counterexample 5.1.1 (Not all super ring homomorphisms preserve the grading). Consider the set

$$R = \left\{ f: \mathbb{R} \rightarrow \bigwedge \mathbb{R}^2 \right\}.$$

A general element $f \in R$ is of the form

$$f(x) = f_{00}(x) + f_{10}(x)\hat{\mathbf{x}} + f_{01}(x)\hat{\mathbf{y}} + f_{11}(x)\hat{\mathbf{x}} \wedge \hat{\mathbf{y}},$$

where \hat{x} and \hat{y} are basis vectors for \mathbb{R}^2 . Define multiplication pointwise:

$$\begin{aligned} (f \cdot g)(x) &= f(x)g(x) = f_{00}(x)g_{00}(x) \\ &\quad + (f_{00}(x)g_{10}(x) + f_{10}(x)g_{00}(x))\hat{x} \\ &\quad + (f_{00}(x)g_{01}(x) + f_{01}(x)g_{00}(x))\hat{y} \\ &\quad + (f_{11}(x)g_{00}(x) + f_{00}(x)g_{11}(x) + f_{10}(x)g_{01}(x) - f_{01}(x)g_{10}(x))\hat{x} \wedge \hat{y}. \end{aligned}$$

Then R becomes a ring. In fact it is more; it is easy (if tedious) to check from the above multiplication law that R can be seen as a super ring with grading given by

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{R_0} + \underbrace{\frac{f(x) - f(-x)}{2}}_{R_1}.$$

Now consider the evaluation homomorphism

$$\iota: R \rightarrow \bigwedge \mathbb{R}^2; \quad f \mapsto f(0).$$

This is a ring homomorphism which does not preserve grading. To see this, consider the constant function

$$\varphi(x) = \hat{x} \quad \text{for all } x.$$

Non-zero constant functions such as φ are even, but \hat{x} is odd in $\bigwedge \mathbb{R}^2$.

Definition 5.1.5 (supercenter). The supercenter of A is the set

$$Z(A) = \{a \in A \mid \text{for all } b \in A, [a, b] = 0\}.$$

In the theory of unexceptional (=not super) algebra, we have Definition 1.6.1 of an algebra over a ring. This definition generalizes naturally to the case where A and R are super rings:

Definition 5.1.6 (super algebra over a ring). Let R be a supercommutative super ring. An R -super-algebra is a super ring A which is also an R -module (with left multiplication $*$: $R \times A \rightarrow A$) such that the multiplication map \cdot : $A \times A \rightarrow A$ is R -bilinear, i.e.

$$r * (a \cdot b) = (r * a) \cdot b = (-1)^{\tilde{a} \cdot \tilde{r}} a \cdot (r * b).$$

Theorem 1.6.1 also generalizes nicely.

Theorem 5.1.1. Let A, R be unital super rings, R supercommutative, and let $f: R \rightarrow A$ be a unital super ring homomorphism. Then A naturally has the structure of an R -module. Furthermore, if the image $f(R)$ is in the supercenter of A , then A naturally has the structure of an R -super algebra.

Proof. The module axioms (Definition 1.4.1) don't involve commutivity, so their verification is trivial as before. We must only check that if $f(R)$ is in the supercenter of A then $*$ is R -bilinear, i.e. that

$$r * (a \cdot b) = f(r) \cdot a \cdot b = (f(r) \cdot a) \cdot b = (r * a) \cdot b,$$

and

$$r * (a \cdot b) = f(r) \cdot a \cdot b = (-1)^{\tilde{a} \cdot \widetilde{f(r)}} a \cdot f(r) \cdot b = (-1)^{\tilde{a} \cdot \tilde{r}} a \cdot f(r) \cdot b = (-1)^{\tilde{a} \cdot \tilde{r}} (r * b).$$

□

Theorem 5.1.2. Let R be a supercommutative super ring, $r \in R_1$. Then r is nilpotent (Definition 1.4.3).

Proof.

$$r^2 = \frac{1}{2}[r, r] = 0.$$

□

We have the following analog to Theorem 1.4.2.

Lemma 5.1.1. Let B be an associative superring. The supercommutator satisfies the following identities.

- $[a, b] = -(-1)^{\tilde{a} \cdot \tilde{b}}[b, a]$.
- $[a, [b, c]] + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}[b, [c, a]] + (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}[c, [a, b]] = 0$

Furthermore, if B is an A -algebra, then the following holds for all $a \in A$, $b, c \in B$.

- $a[b, c] \stackrel{1}{=} [ab, c] \stackrel{2}{=} (-1)^{\tilde{a} \cdot \tilde{b}}[ba, c] \stackrel{3}{=} (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}[b, ca] \stackrel{4}{=} (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}[b, c]a$

Proof.

- $[a, b] = ab - (-1)^{\tilde{a} \cdot \tilde{b}}ba = ba(-1)^{\tilde{a} \cdot \tilde{b}} - \left((-1)^{\tilde{a} \cdot \tilde{b}}\right)^2 ab = (-1)^{\tilde{a} \cdot \tilde{b}}(ba - (-1)^{\tilde{a} \cdot \tilde{b}}ab) = (-1)^{\tilde{a} \cdot \tilde{b}}[b, a]$
- Nightmarish.

$$\begin{aligned} [a, [b, c]] &= a(bc - (-1)^{\tilde{b} \cdot \tilde{c}}cb) - (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}(bc - (-1)^{\tilde{b} \cdot \tilde{c}}cb)a \\ [b, [c, a]] &= b(ca - (-1)^{\tilde{c} \cdot \tilde{a}}ac) - (-1)^{\tilde{b}(\tilde{c}+\tilde{a})}(ca - (-1)^{\tilde{c} \cdot \tilde{a}}ac)b \\ [c, [a, b]] &= c(ab - (-1)^{\tilde{a} \cdot \tilde{b}}ba) - (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}(ab - (-1)^{\tilde{a} \cdot \tilde{b}}ba)c \end{aligned}$$

Truly nightmarish.

- My apologies for the formatting.

Equality 1:

$$\begin{aligned} a[b, c] &= a(bc - (-1)^{\tilde{b} \cdot \tilde{c}}cb) = abc - (-1)^{\tilde{b} \cdot \tilde{c}}acb = abc - (-1)^{\tilde{b} \cdot \tilde{c}}(-1)^{\tilde{a} \cdot \tilde{c}}cab \\ &= (ab)c - (-1)^{\tilde{a} \cdot \tilde{b}}c(ab) = [ab, c] \end{aligned}$$

Equality 2:

$$[ab, c] = (ab)c - (-1)^{\tilde{a} \cdot \tilde{b}}c(ab) = (-1)^{\tilde{a} \cdot \tilde{b}}((ba)c - (-1)^{\tilde{a} \cdot \tilde{b}}c(ba)) = (-1)^{\tilde{a} \cdot \tilde{b}}[ba, c]$$

Equality 3:

$$\begin{aligned} (-1)^{\tilde{a} \cdot \tilde{b}}[ba, c] &= (-1)^{\tilde{a} \cdot \tilde{b}}((ba)c - (-1)^{\tilde{a} \cdot \tilde{b}}c(ba)) \\ &= (-1)^{\tilde{a} \cdot \tilde{b}}(b(ac) - (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}(-1)^{\tilde{a} \cdot \tilde{b}}(-1)^{\tilde{a} \cdot \tilde{c}}(ac)b) = (-1)^{\tilde{a} \cdot \tilde{b}}(b(ac) - (-1)^{\tilde{b}(\tilde{a}+\tilde{c})}(ac)b) \\ &= (-1)^{\tilde{a} \cdot \tilde{b}}[b, ac] \end{aligned}$$

Equality 4: Similar.

□

5.2 Supermodules

The definition of a supermodule is very similar to that of a regular module (Definition 1.4.1).

Definition 5.2.1 (supermodule). Let A be an abelian group with a $\mathbb{Z}/2\mathbb{Z}$ grading, i.e.

$$A = A_0 \oplus A_1.$$

Further, let R be a superring with identity. A (left) supermodule is a triple $(A, R, *)$, with A and R as above and $*$ a function $R \times A \rightarrow A$ such that

1. if $r \in R_i$ and $a \in A_j$ then $r * a \in A_{i+j \bmod 2}$.
2. $*$ satisfies all the module axioms listed in Definition 1.4.1.

If R is supercommutative and A is a left supermodule, we can make A into a right supermodule or a super bimodule as in Theorem 1.4.1.

Definition 5.2.2 (super bimodule). An abelian group A which is both a left and a right R module is a super bimodule if left and right multiplication obey the compatibility condition

$$ra = (-1)^{\tilde{r} \cdot \tilde{a}} ar \quad \text{for all } a \in A, \quad r \in R.$$

Lemma 5.2.1. Let R be a supercommutative superring, A be a left R -module. Then A can naturally be given the structure of a right R -module or a bimodule.

Proof. Define a right action $ra \equiv (-1)^{\tilde{r} \cdot \tilde{a}} ar$. This makes A into a right R -module, and \square

Definition 5.2.3 (superlinear map). Let R be a super ring, A and B be R -supermodules. A map $f: A \rightarrow B$ is called

5.3 Super vector spaces

Definition 5.3.1 (\mathbb{Z}_2 -graded vector space). A \mathbb{Z}_2 -graded vector space over a field k (for simplicity of characteristic zero) is a \mathbb{Z}_2 -graded vector space

$$V = V_0 \oplus V_1.$$

As before elements of V_0 are *even*, elements of V_1 are *odd*, and elements of $V_1 \cup V_2 \setminus \{0\}$ are *homogeneous*.

Definition 5.3.2 (\mathbb{Z}_2 -graded vector space morphism). Let V and W be \mathbb{Z}_2 -graded vector spaces. A morphism from V to W (also called a \mathbb{Z}_2 -graded linear map) is a linear map $V \rightarrow W$ which preserves the grading, i.e. which maps $V_0 \rightarrow W_0$ and $V_1 \rightarrow W_1$.

Definition 5.3.3 (direct sum of \mathbb{Z}_2 -graded vector spaces). Let V and W be \mathbb{Z}_2 -graded vector spaces. Their direct sum, denoted $V \oplus W$, is the \mathbb{Z}_2 -graded vector space with grading

$$(V \oplus W)_i = V_i \oplus W_i, \quad i = 1, 2.$$

Definition 5.3.4 (tensor product of \mathbb{Z}_2 -graded vector spaces). Let V and W be \mathbb{Z}_2 -graded vector spaces. Their tensor product denoted $V \otimes W$, is the \mathbb{Z}_2 -graded vector space with grading

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes V_1), \quad (V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes V_0).$$

Lemma 5.3.1. The \mathbb{Z}_2 -graded tensor product can be extended to a bifunctor

Chapter 6

Superalgebra in categories

6.1 Supercommutativity in symmetric monoidal categories

The goal of this section is to explain the ubiquity of supercommutativity (i.e. the Koszul sign rule) in the study of \mathbb{Z}_2 -graded spaces. Stated roughly, the reason is this: it is the only possible multiplication law on a \mathbb{Z}_2 -graded algebra which knows about the grading. That is to say, the only other possible multiplication law (up to an appropriate sort of isomorphism) on a \mathbb{Z}_2 -graded algebra is the trivial one, which is equivalent to treating the vector space as ungraded.

Of course, this doesn't explain why we should care about \mathbb{Z}_2 -graded vector spaces in the first place. The reason for their importance to theoretical physics is Deligne's theorem, which is the topic of these notes.

Definition 6.1.1 (category of \mathbb{Z}_2 -graded vector spaces). The category of \mathbb{Z}_2 -graded vector spaces, i.e. the category whose objects are \mathbb{Z}_2 -graded vector spaces (Definition 5.3.1) and whose morphisms are \mathbb{Z}_2 -graded vector space morphisms, is notated $\mathbf{Vect}_k^{\mathbb{Z}_2}$.

Note 6.1.1. We call this category the category of \mathbb{Z}_2 -graded vector spaces rather than the category of super vector spaces because we will reserve the latter name for the category with the appropriate symmetric monoidal structure, i.e. that which yields the Koszul sign rule. We will explore this structure now.

To make $\mathbf{Vect}_k^{\mathbb{Z}_2}$ into a symmetric monoidal category, we will have to

Lemma 6.1.1. The \mathbb{Z}_2 -graded tensor product can be extended to a bifunctor d

Theorem 6.1.1. There are, up to braided monoidal equivalence of categories (that is to say, a categorical equivalence Definition 2.1.20 whose functors are braided monoidal (Definition 2.2.17)), only two choices for a symmetric braiding γ (Definition 2.2.18) on $\mathbf{Vect}_k^{\mathbb{Z}_2}$, with components

$$\gamma_{V,W}: V \otimes W \rightarrow W \otimes V :$$

1. The *trivial braiding*, which acts on representing tuples (Definition 1.7.4) by

$$\gamma_{V,W}: (v, w) \mapsto (w, v)$$

2. The *super braiding*, which acts on representing tuples of pure degree by

$$\gamma_{V,W}: (v, w) \mapsto (-1)^{\tilde{v} \cdot \tilde{w}}(w, v).$$

Proof. Consider the category $\mathbf{Line}(\mathbf{Vect}_k^{\mathbb{Z}_2})$

□

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