Notes on Deligne's theorem

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Chapter 1

Introduction

> Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any questions you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.
>
> M. Atiyah, 2002

These are the notes I took while learning about Deligne's theorem on tensor categories and its applications to physics, in particular supersymmetry.

None of the material is original, including the order of presentation, which follows [11] *very* closely.

The fact that it's just a collection of notes means that it assumes as background more or less exactly what I knew before I started (although I have made some effort to keep it self-contained). This means that it probably won't be very useful to anyone other than me.

Chapter 2

Algebra

In this chapter we collect some basic definitions and corollaries so that we may refer to them later. The methods of proof are often non-standard, as we prefer to provide proofs whose methods lend themselves to categorification.

2.1 Monoids

Definition 2.1.1 (monoid)**.** A monoid is a set M together with a binary operation $\cdot\colon M \times M \to M$ such that

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, c, and
- There exists an element $1 \in M$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in M$.

Lemma 2.1.1. *The above definition is equivalent to the following.*

Let M be a set, $\cdot$ an associative binary operation on M. Then we say M is a monoid if there is an element $1 \in M$ such that $1 \cdot 1 = 1$ and the maps $M \to M$, $a \mapsto 1 \cdot a$ and $a \mapsto a \cdot 1$ are bijections.

Proof. If $1 \cdot a = a$ for all a, then the maps $a \to 1 \cdot a$ and $a \mapsto a \cdot 1$ are trivially bijections.

Now, if the map $a \mapsto 1 \cdot a$ is a bijection, then every element a of M can be written $a = 1 \cdot a'$ for some a' in M. Then

$$\begin{aligned} 1 \cdot a &= 1 \cdot (1 \cdot a') \\ &= (1 \cdot 1) \cdot a' \\ &= 1 \cdot a' \\ &= a. \end{aligned}$$

The proof for right-multiplication by 1 is identical. □

2.2 Groups

Definition 2.2.1 (group)**.** A group $(G, \cdot)$ is a monoid with inverses, i.e. a set G with a function $\cdot\colon G \times G \to G$ which is

1. Associative: $(ab)c = a(bc)$
2. Unital: There is an element $e \in G$ such that $eg = ge$ for all $g \in G$
3. Invertible: for all $g \in G$, there is an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$

Lemma 2.2.1. *The above definition is equivalent to the following.*

A group is a monoid such that for all $g \in G$, the maps $G \to G$; $h \mapsto gh$ and $h \mapsto hg$ are bijections.

Proof. If all elements of G are invertible, left-multiplication is obviously bijective. Now suppose left-multiplication by g is a bijection. Then for any $h \in G$, there exists an element $h' \in G$ such that $g \cdot h' = h$. But this is certainly true in particular when $h = e$, so all elements of G are invertible.

The right-multiplication case is identical. □

Definition 2.2.2 (abelian group)**.** A group G is abelian if for all a, $b \in G$, $ab = ba$.

Definition 2.2.3 (free abelian group)**.** Let E be a set. The free abelian group generated by E is the group whose elements are formal finite sums of elements of E.

Definition 2.2.4 (direct sum of abelian groups)**.** Let A, B be abelian groups. Their direct sum, denote $A \oplus B$, is the group whose

1. underlying set is $A \times B$, the cartesian product of the underlying sets of A and B, and whose
2. multiplication is given component-wise.

2.3 Rings

Definition 2.3.1. (ring) A ring $(R, +, \cdot)$ is a set R with two binary operations $+$ and $\cdot$ such that

0. R is closed under $+$ and $\cdot$.
1. R is an Abelian group with respect to $+$.
2. $\cdot$ is associative: $(xy)z = x(yz)$.
3. $\cdot$ is distributive from the left and from the right: $x(y + z) = xy + xz$, $(x + y)z = xz + yz$.

Further,

- R is a commutative ring if $a \cdot b = b \cdot a$ for all a, $b \in R$.
- R is a ring with identity if R contains an element 1_R such that $1_R \cdot x = x \cdot 1_R$ for all $x \in R$.

Theorem 2.3.1 (elementary properties of rings)**.** *Let R be a ring. Then*

1. $0a = a0 = 0$ *for all* $a \in R$
2. $(-a)b = a(-b) = -(ab)$ *for all* a, $b \in R$.
3. $(-a)(-b) = ab$ *for all* a, $b \in R$.
4. $(na)b = a(nb) = n(ab)$ *for all* $n \in \mathbb{Z}$, a, $b \in R$.
5. $$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{m} b_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j.$$

Proof.

1. $0a = (0 + 0)a = 0a + 0a \implies 0a = 0$.
2. $0 = 0b = (a - a)b = ab + (-a)b \implies ab + (-a)b = 0 \implies (-a)b = -(ab)$. The other side is similar.
3. By the previous problem, $(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$.

4. Suppose it is true for n. Then $((n+1)a)b = (na)b + ab = a(nb) + ab = a((n+1)b)$, and similarly for the other.

5. Two uses of induction.

□

Definition 2.3.2 (zero-divisor)**.** A nonzero element a in a ring R is said to be a left (right) zero divisor if there exists $b \neq 0$ such that $ab = 0$ ($ba = 0$). A zero divisor is an element of R which is both a left and a right zero divisor.

Definition 2.3.3 (invertible element)**.** An element a in a ring R is said to be left- (right-)invertible if there exists $b \in R$ such that $ab = 1$ ($ba = 1$). An element $a \in \mathbb{R}$ which is both left- and right-invertible is said to be invertible, or a unit.

Definition 2.3.4 (integral domain)**.** A commutative ring R with identity $1_R \neq 0$ and no zero divisors is called a division ring.

Definition 2.3.5 (division ring)**.** A ring D with identity $1_D \neq 0$ in which every nonzero element is a unit is called a division ring.

Definition 2.3.6 (field)**.** A field is a commutative ring.

Example 2.3.1 (important example)**.** Let X be a topological space. The set

$$C^1(X) = \{ f \colon X \to \mathbb{R} \mid f \text{ continuous} \}$$

is a commutative ring with identity, with addition and multiplication defined pointwise. To see this, we need to check the axioms in Definition 2.3.1:

0. The sum of two continuous functions is continuous; the product of two continuous functions is continuous.
1. $C^1(X)$ inherit its abelian group structure from the real numbers; the additive identity is the function which maps all $x \in X$ to $0 \in \mathbb{R}$.
2. Associativity is inherited from the real numbers
3. Distributivity is inherited from the real numbers.

Definition 2.3.7 (ring homomorphism)**.** Let R and S be rings. A function $f \colon R \to S$ is a ring homomorphism if

$$f(a+b) = f(a) + f(b), \qquad \text{and} \qquad f(ab) = f(a)f(b).$$

Definition 2.3.8 (ring isomorphism)**.** A ring isomorphism is a bijective ring homomorphism.

Definition 2.3.9 (ideal)**.** Let R be a ring and I a nonempty subset of R. I is called a left (right) ideal if

1. I is closed under addition:
$$a, b \in I \implies a + b \in I.$$
2. I absorbs elements of R under multiplication:
$$r \in R \quad \text{and} \quad x \in I \implies rx \in I \quad (xr \in I).$$

If I is both a left and a right ideal, it is called an ideal.

Theorem 2.3.2. *Let R be a ring, and let $\{A_i \mid i \in I\}$ be a set of [left] ideals of X. Then*

$$A = \bigcap_{i \in I} A_i$$

is an ideal.

Proof. According to Definition 2.3.9, we need to check two things:

1. Closure under addition: let a, $b \in A$. Then $a + b$ is in each of the A_i, so it must be in their intersection.
2. Absorption: for any $a \in A$ and any $r \in R$, ra must be in A_i for all i; hence it must be in their intersection.

□

Definition 2.3.10 (ideal generated by a subset)**.** Let X be a subset of a ring R. Let $\{A_i \mid i \in I\}$ be the family of all [left] ideals in R which contain X. Then

$$\bigcap_{i \in I} A_i$$

is called the [left] ideal generated by X, and is denoted (X). The elements of X are called the generators of (X).

Theorem 2.3.3. *Let R be a ring. Let $a \in R$ and $X \subseteq R$. We have the following.*

1. *The principal ideal (a) consists of all elements of the form*

$$ra + as + na + \sum_{i=1}^{m} r_i a s_i; \qquad r, s, r_i, s_i \in R, \quad M \in \mathbb{N}^*, \quad n \in \mathbb{Z}.$$

2. *If R has an identity, then*

$$(a) = \left\{ \sum_{i=1}^{n} r_i a s_i \,\middle|\, r_i, s_i \in R,\ n \in \mathbb{N}^* \right\}.$$

3. *If a is the center of R, then*

$$(a) = \{ra + na \mid r \in \mathbb{R};\ n \in \mathbb{N}^*\}.$$

4. *$Ra = \{ra \mid r \in R\}$ is a left ideal in R. If R has identity, then $a \in Ra$.*
5. *If R has an identity and a is in the center of R then $Ra = (a)$.*
6. *If R has an identity and X is in the center of R, then the ideal (X) consists of all finite sums*

$$\sum_i r_i a_i, \qquad n \in N^*, \quad r_i \in R, \quad a_i \in X.$$

Proof.

1. We prove that any ideal which contains a must contain each term individually; since ideals must be closed under addition, the ideal must contain arbitrary sums of all such terms. Next we prove that the prescription given for (a) is in fact an ideal.
   - Since by definition $a \in (a)$ and (a) is closed under left- and right-multiplication by arbitrary elements of r, for any $r \in R$, ra must be in (a).
   - Similarly, as must be in the

□

Definition 2.3.11 (principal ideal)**.** An ideal generated by a single element is called principal.

Definition 2.3.12 (principal ideal ring)**.** A principal ideal ring is a ring in which every ideal is principal.

Definition 2.3.13 (prime ideal)**.** Let A be a ring. An ideal $\mathfrak{p}$ of A is prime if

- for all ideals $\mathfrak{q}$, $\mathfrak{q}' \subseteq A$, $\mathfrak{q}\mathfrak{q}' \subseteq \mathfrak{p} \implies \mathfrak{q} \subseteq \mathfrak{p}$ or $\mathfrak{q}' \subseteq \mathfrak{p}$.
- $\mathfrak{p} \neq A$

2.4 Modules

Definition 2.4.1 (module)**.** Let R be a ring. A (left) R-module is an additive abelian group A together with a function $*\colon R \times A \to A$ such that for all r, $s \in R$ and a, $b \in A$,

1. $r * (a + b) = r * a + r * b$
2. $(r + s) * a = r * a + s + a$
3. $(rs) * a = r * (s * a)$.

If R has an identity element 1_R and $1_R * a = a$ for all $a \in A$, then A is said to be a unitary R-module.

Note 2.4.1. Right R-modules are defined in the obvious way.

Note 2.4.2. We will now stop notationally differentiating between R-multiplication and A-multiplication, using juxtaposition for both.

Definition 2.4.2 (bimodule)**.** A R-S-bimodule is an Abelian group A which is a left R-module and a right S-module, such that

$$(ra)s = r(as) \qquad \text{for all } r \in R, \quad s \in S, \quad a \in A.$$

We say that the left-multiplication and the right multiplication are *consistent.*

Theorem 2.4.1. *If R is a commutative ring and A is a left R-module, then A can be canonically transformed into a right module or a bimodule.*

Proof. With $ar \equiv ra$, A is both a right R-module and an R-bimodule. The axioms in Definition 2.4.1 are immediate. ☐

Definition 2.4.3 (nilpotent)**.** Let R be a ring. An element $r \in R$ is said to be nilpotent if

$$r^n = 0 \qquad \text{for some } n \in \mathbb{N}^+.$$

Definition 2.4.4 (commutator)**.** Let R be a ring, a, $b \in R$. The commutator of a and b is

$$[a, b] \equiv ab - ba.$$

Theorem 2.4.2. *Let R be a ring, a, $b \in R$. The commutator satisfies the following properties.*

- $[a, b] = -[b, a]$
- $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$

Further, if A is a ring and B is an A-algebra (in the sense of Definition 2.6.1), then for any $a \in A$, b, $c \in B$, the following identity holds.

$$a[b, c] = [ab, c] = [b, ac] = [b, ca] = [b, a]c.$$

Proof.

- $[a, b] = ab - ba = -(ba - ab) = -[b, a]$.
- We have

$$\begin{aligned}
[a, [b, c]] &= a(bc - cb) - (bc - cb)a \\
[b, [c, a]] &= b(ca - ac) - (ca - ac)b \\
[c, [a, b]] &= c(ab - ba) - (ab - ba)c,
\end{aligned}$$

so

$$\begin{aligned}
[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = \quad & abc - acb - bca + cba \\
& + bca - bac - cab + acb \\
& + cab - cba - abc + bac.
\end{aligned}$$

Terms cancel in pairs.

- $a[b, c] = a(bc - cb) = (ab)c - c(ab) = [ab, c] = b(ac) - (ac)b = [b, ac] = (bc - cb)a = [b, c]a.$

$\square$

Definition 2.4.5 (free module)**.** Let E be a set, R be a ring. The free left module of E over R, denoted $R^{(E)}$, is the module whose elements are are finite formal R-linear combinations of the elements of E.

Example 2.4.1. The free abelian group over a set E is also a free $\mathbb{Z}$-module over E with the identification

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = na.$$

2.5 Vector spaces

Definition 2.5.1 (graded vector space)**.** Let M be a monoid. An M-graded vector space is a

2.6 Algebras

Definition 2.6.1 (algebra over a ring)**.** Let R be a commutative ring. An R-algebra is a ring A which is also an R-module (with left-multiplication $*\colon R \times A \to A$) such that the multiplication map $\cdot\colon A \times A \to A$ is R-bilinear, i.e.

$$r * (a \cdot b) = (r * a) \cdot b = a \cdot (r * b), \qquad \text{for any } a, b \in A, \quad r \in R.$$

Theorem 2.6.1. *Let A, R be unital rings, R commutative, and let $f\colon R \to A$ be a unital ring homomorphism. Then A naturally has the structure of an R-module. Furthermore, if the image of R is in the center of A, then A naturally has the structure of an R-algebra.*

Proof. Define the action

$$*\colon R \times A \to A; \qquad (r, a) \mapsto f(r) \cdot a.$$

The verification that this makes A into an R-module is trivial; we need only check left and right distributivity, associativity, and that $1_R * a = a$. We omit this.

To see that what we have is even an algebra, we need to check that $*$ is bilinear, i.e.

$$r * (a \cdot b) = f(r) \cdot (a \cdot b) = (f(r) \cdot a) \cdot b = (r * a) \cdot b$$

and

$$r * (a \cdot b) = f(r) \cdot a \cdot b = a \cdot f(r) \cdot b = a \cdot (r * b).$$

$\square$

Often we are interested specifically in algebras over a field. In that case we have the following.

Definition 2.6.2 (algebra over a field)**.** An algebra A over a field k is a k-vector space with a k-bilinear product $A \times A \to A$.

Further, an algebra A is

- associative if for all a, b, $c \in A$, $(a \times b) \times c = a \times (b \times c)$.
- unitary if there exists an element $1 \in A$ such that $1 \times a = a \times 1 = a$ for all $a \in A$.

Note 2.6.1. In general, algebras are taken to be associative and unital by default. A notable exception is Lie algebras.

2.7 Tensor products

In this section we give a brief overview of the construction of the tensor product.

Definition 2.7.1 (middle-linear map)**.** Let R be a ring, M be a right R-module, N a left R-module, and G an abelian group. A map $\varphi\colon M \times N \to G$ is said to be R-middle-linear if for all m, $m' \in M$, n, $n' \in N$, and $r \in R$ we have

1. $\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$
2. $\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$
3. $\varphi(m \cdot r, n) = \varphi(m, r \cdot n)$.

Definition 2.7.2 (tensor product of modules)**.** Let R be a ring, M be a right R-module, N a left R-module. The tensor product of M and N, denoted $M \otimes_R N$, is constructed as follows.

Let F be the free abelian group generated by $M \times N$ (Definition 2.2.3). Let K be the subgroup of F generated by elements of the form

- $(m, n + n') - (m, n) - (m, n')$
- $(m + m', n) - (m, n) - (m'n)$
- $(m \cdot r, n) - (m, r \cdot n)$.

The tensor product of M and N is then

$$M \otimes_R N = F/K.$$

Definition 2.7.3 (tensor algebra over a module)**.** The tensor algebra

Note 2.7.1. If R is commutative, then we can canonically make M and N into bimodules. In this case we don't need to distinguish between left and right multiplication. If R is a field, we recover the notion of the 'standard' tensor product, which is usually given by the following definition.

Definition 2.7.4 (tensor product of vector spaces)**.** Let k be a field, V and W k-vector spaces. The tensor product $V \otimes W$ is the vector space defined as follows.

Denote by $\mathcal{F}(V \times W)$ the free vector space over $V \times W$. Consider the vector subspace K of $\mathcal{F}(V \times W)$ generated by elements of the forms

- $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$
- $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$
- $(\alpha v, w) - \alpha(v, w)$
- $(v, \alpha w) - \alpha(v, w)$

for all v_1, $v_2 \in V$, w_1, $w_2 \in W$, $\alpha \in k$. Then define

$$V \otimes W = \mathcal{F}(V \times W)/K.$$

Pairs $(v, w) \in V \times W$ are called representing tuples.

Definition 2.7.5 (tensor product of linear maps)**.** Let V, W, X, and Y be k-vector spaces, and let $S\colon V \to X$, $T\colon W \to Y$. Then the tensor product of S and T is the map

$$S \otimes T\colon V \otimes W \to X \otimes Y; \qquad (v \otimes w) \mapsto S(v) \otimes T(w).$$

Of course, one must show that this is well-defined by showing that it vanishes on the subspace by which one quotients in the definition of the tensor product. But this is trivial.

Chapter 3

Categories

In this chapter (and to some extent the following chapters), I will do my best to stick to the conventions used at the nlab ([20]).

3.1 Basic definitions

Definition 3.1.1 (category)**.** A category C consists of

- a class $\mathrm{Obj}(\mathsf{C})$ of *objects*, and
- for every two objects A, $B \in \mathrm{Obj}(\mathsf{C})$, a class $\mathrm{Hom}(A, B)$ of *morphisms* with the following properties.
    1. For $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$, there is an associated morphism
    $$g \circ f \in \mathrm{Hom}(A, C),$$
    called the *composition* of f and g.
    2. This composition is associative.
    3. For every $A \in \mathrm{Obj}(\mathsf{C})$, there is at least one morphism 1_A, called the *identity morphism* which functions as both a left and right identity with respect to the composition of morphisms.

Note 3.1.1. There is a reason we say that the objects and morphisms of a category are a *class* rather than a set. It may be that there may be 'too many' objects to be contained in a set. For example, we will see that there is a category of sets, but there is no set of all sets. Categories whose objects and/or morphisms *are* small enough to be contained in a set will play an especially important role.

Notation 3.1.1. Following Aluffi ([10]), we will use the sans-serif font `mathsf` to denote categories. For example C, Set.

If ever it is potentially unclear which category we are talking about, we will add a subscript to Hom, writing for example $\mathrm{Hom}_{\mathsf{C}}(A, B)$ instead of $\mathrm{Hom}(A, B)$.

Example 3.1.1. The prototypical category is Set, the category whose objects are sets and whose morphisms are set functions.

Example 3.1.2 (category with one object)**.** The category $\mathsf{1}$, where $\mathrm{Obj}(\mathsf{1})$ is the singleton $\{*\}$, and the only morphism is the identity morphism $\mathrm{id}_*\colon * \to *$.

Example 3.1.3. Pretty much all standard algebraic constructions naturally live in categories. For example, we have

- the category Grp, whose objects are groups and whose morphisms are group homomorphisms;
- the category Ab, whose objects are abelian groups and whose morphisms are group homomorphisms;
- the category Ring, whose objects are rings and whose morphisms are ring homomorphisms;

- the category R-Mod, whose objects are modules over a ring R and whose morphisms are module homomorphisms;
- the category Vect_k, whose objects are vector spaces over a field k and whose morphisms are linear maps;
- the category $\mathsf{FinVect}_k$, whose objects are finite-dimensional vector spaces over a field k and whose morphisms are linear maps;
- the category Alg_k, whose objects are algebras over a field k and whose morphisms are algebra homomorphisms

Example 3.1.4. In addition to *algebraic* structures between sets, categories help to formulate geometric structures, such as

- the category Top, whose objects are topological spaces and whose morphisms are continuous maps;
- the category Met, whose objects are metric spaces and whose morphisms are metric maps;
- the category Man^p, whose objects are manifolds of class C^p and whose morphisms are p-times differentiable functions;
- the category $\mathsf{SmoothMfd}$, whose objects are C^∞ manifolds and whose morphisms are smooth functions

and many more.

Here are a few ways we can use existing categories to create new ones.

Definition 3.1.2 (opposite category)**.** Let C be a category. The opposite category C^{op} is the category whose objects are the same as the objects $\mathrm{Obj}(\mathsf{C})$ and whose morphisms $f \in \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B)$ are defined to be the morphisms $\mathrm{Hom}_{\mathsf{C}}(B, A)$.

That is to say, the opposite category is the category one gets by formally reversing all the arrows in a category. If $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, i.e. $f\colon A \to B$, then in C^{op}, $f\colon B \to A$.

Definition 3.1.3 (product category)**.** Let C and D be categories. The product category $\mathsf{C} \times \mathsf{D}$ is the category whose

- objects are ordered pairs (C, D), where $C \in \mathrm{Obj}(\mathsf{C})$ and $D \in \mathrm{Obj}(\mathsf{D})$, whose
- morphisms are ordered pairs (f, g), where $f \in \mathrm{Hom}_{\mathsf{C}}(C_1, C_2)$ and $g \in \mathrm{Hom}_{\mathsf{D}}(D_1, D_2)$, in which
- composition is taken componentwise, so that
$$(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ g_1, f_2 \circ g_2),$$
and
- the identity morphisms are given in the obvious way:
$$1_{(C,D)} = (1_C, 1_D).$$

Definition 3.1.4 (subcategory)**.** Let C be a category. A category S is a subcategory of C if

- The objects $\mathrm{Obj}(\mathsf{S})$ of S are a subcollection of the objects of C
- For S, $T \in \mathrm{Obj}(\mathsf{S})$, the morphisms $\mathrm{Hom}_{\mathsf{S}}(S, T)$ are a subcollection of the morphisms $\mathrm{Hom}_{\mathsf{C}}(S, T)$ such that
    - for every $S \in \mathrm{Obj}(S)$, the identity $1_C \in \mathrm{Hom}_{\mathsf{S}}(S, S)$.
    - for all $f \in \mathrm{Hom}_{\mathsf{S}}(S, T)$ and $g \in \mathrm{Hom}_{\mathsf{S}}(T, U)$, the composite $g \circ f \in \mathrm{Hom}_{\mathsf{S}}(S, U)$.

If S is a subcategory of C, we will write $\mathsf{S} \subseteq \mathsf{C}$.

Definition 3.1.5 (full subcategory)**.** Let C be a category, $\mathsf{S} \subseteq \mathsf{C}$ a subcategory. We say that S is full in C if for every S, $T \in \mathrm{Obj}(\mathsf{S})$, $\mathrm{Hom}_{\mathsf{S}}(S,T) = \mathrm{Hom}_{\mathsf{C}}(S,T)$.

Example 3.1.5. Recall that Vect_k is the category of vector spaces over a field k, and $\mathsf{FinVect}_k$ is the category of finite dimensional vector spaces.

It is not difficult to see that $\mathsf{FinVect}_k \subseteq \mathsf{Vect}_k$: all finite dimensional vector spaces are vector spaces, and all linear maps between finite-dimensional vector spaces are maps between vector spaces. In fact, since for V and W finite-dimensional, one does not gain any maps by moving from $\mathrm{Hom}_{\mathsf{FinVect}_k}(V,W)$ to $\mathrm{Hom}_{\mathsf{Vect}_k}(V,W)$, $\mathsf{FinVect}_k$ is even a *full* subcategory of Vect_k.

Category theory has many essences, one of which is as a major generalization of set theory. We would like to upgrade definitions and theorems about functions between sets to definitions and theorems about morphisms between objects in a category. It is here that we run into our first major challenge: in general, the objects of a category are *not* sets, so we cannot talk about their elements. We therefore have to find definitions which we can give purely in terms objects and morphisms between them.

Definition 3.1.6 (isomorphism)**.** Let C be a category, A, $B \in \mathrm{Obj}(\mathsf{C})$. A morphism $f \in \mathrm{Hom}(A,B)$ is said to be an isomorphism if there exists a morphism $g \in \mathrm{Hom}(B,A)$ such that

$$g \circ f = 1_A, \qquad \text{and} \quad f \circ g = 1_B.$$

If we have an isomorphism $f \colon A \to B$, we say that A and B are isomorphic, and write $A \simeq B$.

Definition 3.1.7 (monomorphism)**.** Let C be a category, A, $B \in \mathrm{Obj}(\mathsf{C})$. A morphism $f \colon A \to B$ is said to be a monomorphism if for all $Z \in \mathrm{Obj}(\mathsf{C})$ and all g_1, $g_2 \colon Z \to A$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

$$\begin{tikzcd} Z \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Note 3.1.2. When we wish to notationally distinguish monomorphisms, we will denote them by hooked arrows: if $f \colon A \to B$ is mono, we will write

$$\begin{tikzcd} A \arrow[r, hook, "f"] & B \end{tikzcd} .$$

Theorem 3.1.1. *In* Set*, a morphism is a monomorphism precisely when it is injective.*

Proof. Suppose $f \colon A \to B$ is a monomorphism. Then for any set Z and any maps g_1, $g_2 \colon Z \to A$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. In particular, take $Z = \{*\}$ and suppose $g_1(*) = a_1$ and $g_2(*) = a_2$. Then $(f \circ g_1)(*) = f(a_1)$ and $(f \circ g_2)(*) = f(a_2)$, so

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

But this is exactly the definition of injectivity.

Now suppose that f is injective. Then for any Z and g_1, g_2 as above,

$$(f \circ g_1)(z) = (f \circ g_2)(z) \implies g_1(z) = g_2(z) \qquad \text{for all } z \in Z.$$

But this means that $g_1 = g_2$, so f is mono. $\square$

Example 3.1.6. In Vect_k, monomorphisms are injective linear maps.

Definition 3.1.8 (epimorphism)**.** Let C be a category, A, $B \in \mathrm{Obj}(\mathsf{C})$. A morphism $f \colon A \to B$ is said to be an epimorphism if for all $Z \in \mathrm{Obj}(\mathsf{C})$ and all g_1, $g_2 \colon B \to Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \end{tikzcd} .$$

Notation 3.1.2. We will denote epimorphisms by two-headed right arrows. That is, if $f \colon A \to B$ is epi, we will write

$$\begin{tikzcd} A \arrow[r, two heads, "f"] & B \end{tikzcd}$$

Example 3.1.7. In Set, epimorphisms are surjections.

Example 3.1.8. In Vect_k, epimorphism are surjective linear maps.

Note 3.1.3. In Set, isomorphisms are bijections, bijections are injective surjections, and injective surjections are monic epimorphisms, so a morphism is iso if and only if it is monic and epic. This is *not* true in general: a morphism can be monic and epic without being an isomorphism.

Take, for example, Top, where the morphisms are continuous maps. In order for a morphism f to be monic and epic, it is necessary only that it be injective and surjective; it must have a set-theoretic inverse. However its inverse does not have to be continuous, and therefore may not be a morphism in Top.

Set theory has some foundational annoyances: not every collection of objects is small enough to be a set, for example. Category theory has its own foundational issues, which for the most part we will avoid. However, there are a few important situations in which foundational questions play an unavoidably important role.

Definition 3.1.9 (small, locally small, hom-set)**.** A category C is

- small if $\mathrm{Obj}(\mathsf{C})$ is a set and for all objects A, $B \in \mathrm{Obj}(\mathsf{C})$, $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is a set.
- locally small if for all A, $B \in \mathrm{Obj}(\mathsf{C})$, $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is a set. In this case, we call $\mathrm{Hom}_{\mathsf{C}}(A, B)$ the hom-set. (Actually, terminology is often abused, and $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is called a hom-set even if it is not a set.)

3.2 Functors

Just as morphisms connect objects in the same category, functors allow us to connect different categories. In fact, functors can be viewed as morphisms in a 'category of categories.'

Definition 3.2.1 (functor)**.** Let C and D be categories. A functor $\mathcal{F}$ from C to D is a mapping which associates

- to each object $X \in \mathrm{Obj}(\mathsf{C})$ an object $\mathcal{F}(X) \in \mathrm{Obj}(\mathsf{D})$.
- to each morphism $f \in \mathrm{Hom}(X, Y)$ a morphism $\mathcal{F}(f)$ such that $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ for all X, and one of the two following properties are satisfied.
  - The morphism $f \in \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$ and if $f : X \to Y$ and $g\colon Y \to Z$, then
    $$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$
    In this case we say that $\mathcal{F}$ is covariant.
  - The morphism $f \in \mathrm{Hom}(\mathcal{F}(Y), \mathcal{F}(X))$, and if $f\colon X \to Y$ and $g\colon Y \to Z$, then
    $$\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g).$$
    In this case, we say that $\mathcal{F}$ is contravariant.

Notation 3.2.1. We will typeset functors with calligraphic letters using the font `eucal`, and notate them with squiggly arrows. For example, if C and D are categories and $\mathcal{F}$ is a functor from C to D, then we would write

$$\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}.$$

Note 3.2.1. One can also define a contravariant functor $\mathsf{C} \rightsquigarrow \mathsf{D}$ as a covariant functor $\mathsf{C}^{\mathrm{op}} \rightsquigarrow \mathsf{D}$.

Note 3.2.2. When the adjective is unspecified, a *functor* will mean a *covariant functor.*

Example 3.2.1. Let 1 be the category with one object $*$ (Example 3.1.2). Let C be any category. Then for each $X \in \mathrm{Obj}(\mathsf{C})$, we have the functor

$$\mathcal{F}_X : 1 \rightsquigarrow \mathsf{C}; \qquad \mathcal{F}(*) = X, \quad \mathcal{F}(\mathrm{id}_*) = \mathrm{id}_X.$$

Example 3.2.2. Recall that Grp is the category of groups. Denote by CRing the category of commutative rings.

We can construct the following functors $\mathsf{CRing} \rightsquigarrow \mathsf{Grp}$.

- The functor GL_n, which assigns to each commutative ring the group of all $n \times n$ matrices with nonzero determinant, and to each morphism $f\colon K \to K'$ the homomorphism $\mathrm{GL}_n(K) \to \mathrm{GL}_n(K')$ which maps a matrix with entries in K to a matrix with entries in K' by mapping each entry individually.
- The functor $(\,\cdot\,)^*$, which maps each commutative ring K to its group of units K^*, and each morphism $K \to K'$ to its restriction to K^*.

One of the nice things about functors is that they map commutative diagrams to commutative diagrams. For example, if $\mathcal{F}$ is a contraviariant functor, then one might see the following.

$$\begin{tikzcd}
A \arrow[rr, "h=g\circ f"] \arrow[dr, "f"'] & & B & {} \arrow[r, rightsquigarrow, "\mathcal{F}"] & {} & \mathcal{F}(A) & & \mathcal{F}(B) \arrow[ll, "\mathcal{F}(h)=\mathcal{F}(f)\circ\mathcal{F}(g)"'] \arrow[dl, "\mathcal{F}(g)"] \\
& C \arrow[ur, "g"'] & & & & & \mathcal{F}(C) \arrow[ul, "\mathcal{F}(f)"] &
\end{tikzcd}$$

Definition 3.2.2 (full, faithful)**.** Let C and D be locally small categories (Definition 3.1.9), and let $\mathcal{F} : \mathsf{C} \rightsquigarrow \mathsf{D}$. Then $\mathcal{F}$ induces a family of set-functions

$$\mathcal{F}_{X,Y}\colon \mathrm{Hom}_{\mathsf{C}}(X,Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X),\mathcal{F}(Y)).$$

We say that $\mathcal{F}$ is

- full if $\mathcal{F}_{X,Y}$ is surjective for all X, $Y \in \mathrm{Obj}(\mathsf{C})$
- faithful if $\mathcal{F}_{X,Y}$ is injective for all X, $Y \in \mathrm{Obj}(\mathsf{C})$,
- fully faithful if $\mathcal{F}$ is full and faithful.

Note 3.2.3. Fullness and faithfulness are *not* the functorial analogs of surjectivity and injectivity. A functor between small categories can be full (faithful) without being surjective (injective) on objects. Instead, we have the following result.

Lemma 3.2.1. *A fully faithful functor is injective on objects up to isomorphism. That is, if $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ is a fully faithful functor and $\mathcal{F}(A) \simeq \mathcal{F}(B)$, then $A \simeq B$.*

Proof. Let $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ be a fully faithful functor, and suppose that $\mathcal{F}(A) \simeq \mathcal{F}(B)$. Then there exist $f'\colon \mathcal{F}(A) \to \mathcal{F}(B)$ and $g'\colon \mathcal{F}(B) \to \mathcal{F}(A)$ such that $f' \circ g' = 1_{\mathcal{F}(B)}$ and $g' \circ f' = 1_{\mathcal{F}(A)}$. Because the function $\mathcal{F}_{A,B}$ is bijective it is invertible, so there is a unique morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$ such that $\mathcal{F}(f) = f'$, and similarly there is a unique $g \in \mathrm{Hom}_{\mathsf{C}}(B,A)$ such that $\mathcal{F}(g) = g'$.

Now,

$$1_{\mathcal{F}(A)} = g' \circ f' = \mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f),$$

and since $\mathcal{F}$ is injective, we must have $g \circ f = 1_A$. Identical logic shows that we must also have $f \circ g = 1_B$. Thus $A \simeq B$. ☐

Definition 3.2.3 (essentially surjective)**.** A functor $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ is essentially surjective if for every $A' \in \mathrm{Obj}(\mathsf{D})$, there exists $A \in \mathrm{Obj}(\mathsf{C})$ such that $A' \simeq \mathcal{F}(A)$.

Example 3.2.3 (diagonal functor)**.** Let C be a category, $\mathsf{C} \times \mathsf{C}$ the product category of C with itself. The diagonal functor $\Delta\colon \mathsf{C} \rightsquigarrow \mathsf{C} \times \mathsf{C}$ is the functor which sends

- each object $A \in \mathrm{Obj}(\mathsf{C})$ to the pair $(A, A) \in \mathrm{Obj}(\mathsf{C} \times \mathsf{C})$, and
- each morphism $f\colon A \to B$ to the ordered pair

$$(f, f) \in \mathrm{Hom}_{\mathsf{C}\times\mathsf{C}}(A \times B, A \times B).$$

Definition 3.2.4 (bifunctor)**.** A bifunctor is a functor whose domain is a product category (Definition 3.1.3).

3.3 Natural transformations

Saunders Mac Lane, one of the fathers of category theory, used to say that he invented categories so he could talk about functors, and he invented functors so he could talk about natural transformations. Indeed, the first paper ever published on category theory, published by Eilenberg and Mac Lane in 1945, was titled "General Theory of Natural Equivalences." [23]

Natural transformations provide a notion of 'morphism between functors.' They allow us much greater freedom in talking about the relationship between two functors than equality.

Here is a motivating example. Let A and B be sets. There is an isomorphism from $A \times B$ to $B \times A$ which is given by switching the order of the ordered pairs:

$$\mathrm{swap}_{A,B} \colon (a,b) \mapsto (b,a).$$

Similarly, for sets C and D, there is an isomorphism $C \times D$ to $D \times C$ which is given by switching the order of ordered pairs.

$$\mathrm{swap}_{C,D} \colon (c,d) \mapsto (d,c).$$

In some obvious intuitive sense, these isomorphisms are really the same isomorphism. However, it is not obvious how to formalize this, since the definition of a function contains the information about the image and coimage, so they cannot be equal as functions.

Here is how it is done. Notice that for *any* functions $f \colon A \to C$ and $g \colon B \to D$, the following diagram commutes.

$$\begin{tikzcd}
A \times B \arrow[r, "{(f,g)}"] \arrow[d, "\mathrm{swap}_{A,B}"'] & C \times D \arrow[d, "\mathrm{swap}_{C,D}"] \\
B \times A \arrow[r, "{(g,f)}"] & D \times C
\end{tikzcd}$$

That is to say, if you're trying to go from $A \times B$ to $D \times C$ and all you've got is functions $f \colon A \to B$ and $g \colon C \to D$ and the two swap isomorphisms, it doesn't matter whether you use the swap isomorphism on $A \times B$ and then use (g,f) to get to $D \times C$, or immediately go to $C \times D$ with (f,g), then use the swap isomorphism there: the result is the same. Furthermore, this is true for *any* functions f and g.

As we will see, the cartesian product of sets is a functor from the product category (Definition 3.1.3) $\mathsf{Set} \times \mathsf{Set}$ to Set which maps (A,B) to $A \times B$. There is another functor $\mathsf{Set} \times \mathsf{Set} \rightsquigarrow \mathsf{Set}$ which sends (A,B) to $B \times A$. The above means that there is a natural isomorphism, called swap, between them.

Definition 3.3.1 (natural transformation)**.** let C and D be categories, and let $\mathcal{F}$ and $\mathcal{G}$ be covariant functors from C to D. a natural transformation η between $\mathcal{F}$ and $\mathcal{G}$ consists of

- for each object $A \in \mathrm{Obj}(\mathsf{C})$ a morphism $\eta_A \colon \mathcal{F}(A) \to \mathcal{G}(A)$, such that
- for all A, $B \in \mathrm{Obj}(C)$, for each morphism $f \in \mathrm{Hom}(A,B)$, the diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_A"'] & \mathcal{F}(B) \arrow[d, "\eta_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(B)
\end{tikzcd}$$

  commutes.

If the functors $\mathcal{F}$ and $\mathcal{G}$ are contravariant, the changes needed to make the definition make sense are obvious: basically, some arrows need to be reversed. However, this is one of those times where it's easier, in line with Note 3.2.1, to just pretend that we have a covariant functor from the opposite category.

Definition 3.3.2 (natural isomorphism)**.** A natural isomorphism $\eta \colon \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation such that each η_A is an isomorphism.

Notation 3.3.1. We will use double-shafted arrows to denote natural transformations: if $\mathcal{F}$ and $\mathcal{G}$ are functors and η is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$, we will write

$$\eta \colon \mathcal{F} \Rightarrow \mathcal{G}.$$

Definition 3.3.3 (set of all natural transformations)**.** Let C and D be categories, $\mathcal{F}$ and $\mathcal{G}$ functors $\mathsf{C} \rightsquigarrow \mathsf{D}$. The set of all natural transformations $\mathcal{F} \Rightarrow \mathcal{G}$ is denoted $\mathrm{Nat}(\mathcal{F}, \mathcal{G})$.

Example 3.3.1. Recall the functors GL_n and $(\cdot)^*$ from Example 3.2.2.

Denote by $\det_K M$ the determinant of a matrix M with its entries in a commutative ring K. Then the determinant is a map

$$\det_K \colon \mathrm{GL}_n(K) \to K^*.$$

Because the determinant is defined by the same formula for each K, the action of f commutes with $\det_K$: it doesn't matter whether we map the entries of M with f first and then take the determinant, or take the determinant first and then feed the result to f.

That is to say, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{GL}_n(K) \arrow[r, "\det_K"] \arrow[d, "\mathrm{GL}_n(f)"'] & K^* \arrow[d, "f^*"] \\
\mathrm{GL}_n K' \arrow[r, "\det_{K'}"] & K'^*
\end{tikzcd}$$

But this means that det is a natural transformation $\mathrm{GL}_n \Rightarrow (\,\cdot\,)^*$.

We can compose natural transformations in the obvious way.

Lemma 3.3.1. *Let* C *and* D *be categories,* $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ *be functors, and* Φ *and* Ψ *be natural transformations as follows.*

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left=60, "\mathcal{F}"{name=F}] \arrow[r, squiggly, "\mathcal{G}"{name=G, description}] \arrow[r, squiggly, bend right=60, "\mathcal{H}"'{name=H}] & \mathsf{D}
\arrow[Rightarrow, from=F, to=G, "\Phi"] \arrow[Rightarrow, from=G, to=H, "\Psi"]
\end{tikzcd}$$

This induces a natural transformation $\mathcal{F} \Rightarrow \mathcal{H}$.

Proof. For each object $A \in \mathrm{Obj}(\mathsf{C})$, the composition $\Psi_A \circ \Phi_A$ exists and maps $\mathcal{F}(A) \to \mathcal{H}(A)$. Let's write

$$\Psi_A \circ \Phi_A = (\Psi \circ \Phi)_A.$$

We have to show that these are the components of a natural transformation, i.e. that they make the following diagram commute for all A, $B \in \mathrm{Obj}(\mathsf{C})$, all $f : A \to B$.

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "(\Phi\circ\Psi)_A"'] & \mathcal{F}(\mathcal{B}) \arrow[d, "(\Phi\circ\Psi)_B"] \\
\mathcal{H}(A) \arrow[r, "\mathcal{H}(f)"] & \mathcal{H}(B)
\end{tikzcd}$$

We can do this by adding a middle row.

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\Phi_A"'] & \mathcal{F}(\mathcal{B}) \arrow[d, "\Phi_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\Psi_A"'] & \mathcal{G}(B) \arrow[d, "\Psi_B"] \\
\mathcal{H}(A) \arrow[r, "\mathcal{H}(f)"] & \mathcal{H}(B)
\end{tikzcd}$$

The top and bottom squares are the naturality squares for Φ and Ψ respectively. The outside square is the one we want to commute, and it manifestly does because each of the inside squares does. ☐

Definition 3.3.4 (vertical composition)**.** The above composition $\Psi \circ \Phi$ is called vertical composition.

Definition 3.3.5 (functor category)**.** Let C and D be categories. The functor category D^{C} (or $[\mathsf{C}, \mathsf{D}]$) is the category whose objects are functors $\mathsf{C} \rightsquigarrow \mathsf{D}$, and whose morphisms are natural transformations between them. The composition is given by vertical composition.

We can also compose natural transformations in a not so obvious way.

Lemma 3.3.2. *Consider the following arrangement of categories, functors, and natural transformations.*

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, squiggly, "\mathcal{F}"] \arrow[r, bend right, squiggly, "\mathcal{F}'"'] \arrow[r, phantom, "\Downarrow \Phi"] & \mathsf{D} \arrow[r, bend left, squiggly, "\mathcal{G}"] \arrow[r, bend right, squiggly, "\mathcal{G}'"'] \arrow[r, phantom, "\Downarrow \Psi"] & \mathsf{E}
\end{tikzcd}$$

This induces a natural transformation $\mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$.

Proof. By definition, Φ and Ψ make the diagrams

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\Phi_A"'] & \mathcal{F}(B) \arrow[d, "\Phi_B"] \\
\mathcal{F}'(A) \arrow[r, "\mathcal{F}'(f)"'] & \mathcal{F}'(B)
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\Psi_A"'] & \mathcal{G}(B) \arrow[d, "\Psi_B"] \\
\mathcal{G}'(A) \arrow[r, "\mathcal{G}'(f)"'] & \mathcal{G}'(B)
\end{tikzcd}$$

commute respectively. Since functors take commutative diagrams to commutative diagrams, we can map everything in the first diagram to E with $\mathcal{G}$.

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F})(f)"] \arrow[d, "\mathcal{G}(\Phi_A)"'] & (\mathcal{G} \circ \mathcal{F})(B) \arrow[d, "\mathcal{G}(\Phi_B)"] \\
(\mathcal{G} \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F}')(f)"] & (\mathcal{G} \circ \mathcal{F}')(B)
\end{tikzcd}$$

Also, since $\mathcal{F}'(f) \colon \mathcal{F}'(A) \to \mathcal{F}'(B)$ is a morphism in D and Ψ is a natural transformation, the following diagram commutes.

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F}')(f)"] \arrow[d, "\Psi_{\mathcal{F}'(A)}"'] & (\mathcal{G} \circ \mathcal{F}')(B) \arrow[d, "\Psi_{\mathcal{F}'(B)}"] \\
(\mathcal{G}' \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}'\circ\mathcal{F}')(f)"] & (\mathcal{G}' \circ \mathcal{F}')(B)
\end{tikzcd}$$

Sticking these two diagrams on top of each other gives a new commutative diagram.

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F})(f)"] \arrow[d, "\mathcal{G}(\Phi_A)"'] & (\mathcal{G} \circ \mathcal{F})(B) \arrow[d, "\mathcal{G}(\Phi_B)"] \\
(\mathcal{G} \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F}')(f)"] \arrow[d, "\Psi_{\mathcal{F}'(A)}"'] & (\mathcal{G} \circ \mathcal{F}')(B) \arrow[d, "\Psi_{\mathcal{F}'(B)}"] \\
(\mathcal{G}' \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}'\circ\mathcal{F}')(f)"] & (\mathcal{G}' \circ \mathcal{F}')(B)
\end{tikzcd}$$

The outside rectangle is nothing else but the commuting square for a natural transformation

$$(\Psi * \Phi) \colon \mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$$

with components $(\Psi * \Phi)_A = \Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$. □

Definition 3.3.6 (horizontal composition)**.** The natural transformation $\Psi * \Phi$ defined above is called the horizontal composition of Φ and Ψ.

Note 3.3.1. Really, the above definition of the horizontal composition is lopsided and ugly. It becomes less so if we notice the following. The first step in our construction of $\Psi * \Phi$ was to apply the functor $\mathcal{G}$ to the commutative diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\Phi_A"'] & \mathcal{F}(B) \arrow[d, "\Phi_B"] \\
\mathcal{F}'(A) \arrow[r, "\mathcal{F}'(f)"'] & \mathcal{F}'(B)
\end{tikzcd}.$$

We could instead have applied the functor $\mathcal{G}'$, giving us the following.

$$\begin{tikzcd}
(\mathcal{G}' \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}' \circ \mathcal{F})(f)"] \arrow[d, "\mathcal{G}'(\Phi_A)"'] & (\mathcal{G}' \circ \mathcal{F})(B) \arrow[d, "\mathcal{G}'(\Phi_B)"] \\
(\mathcal{G}' \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}' \circ \mathcal{F}')(f)"] & (\mathcal{G}' \circ \mathcal{F}')(B)
\end{tikzcd}$$

Then we could have glued to it the bottom of the commuting square

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G} \circ \mathcal{F})(f)"] \arrow[d, "\Psi_{\mathcal{F}(A)}"'] & (\mathcal{G} \circ \mathcal{F})(B) \arrow[d, "\Psi_{\mathcal{F}(B)}"] \\
(\mathcal{G}' \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}' \circ \mathcal{F})(f)"] & (\mathcal{G}' \circ \mathcal{F})(B)
\end{tikzcd};$$

if you do this you get *another* natural transformation $\mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$, with components $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$. Why did we use the first definition rather than this one?

It turns out that $\Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$ and $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$ are equal. To see this, note that from the morphism

$$\Phi_A \colon \mathcal{F}(A) \to \mathcal{F}'(A),$$

the natural transformation Ψ gives us a commuting square

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "\mathcal{G}(\Phi_A)"] \arrow[d, "\Psi_{\mathcal{F}(A)}"'] & (\mathcal{G} \circ \mathcal{F}')(A) \arrow[d, "\Psi_{\mathcal{F}'(A)}"] \\
(\mathcal{G}' \circ \mathcal{F})(A) \arrow[r, "\mathcal{G}'(\Phi_A)"] & (\mathcal{G}' \circ \mathcal{F}')(A)
\end{tikzcd};$$

the two ways of going from top left to bottom right are nothing else but $\Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$ and $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$.

Example 3.3.2 (whiskering)**.** Consider the following assemblage of categories, functors, and natural transformations.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left=50, "\mathcal{F}", ""{name=U, below}] \arrow[r, squiggly, bend right=50, "\mathcal{F}'"', ""{name=L}] & \mathsf{D} \arrow[r, squiggly, "\mathcal{G}"] & \mathsf{E}
\arrow[Rightarrow, from=U, to=L, "\Phi"]
\end{tikzcd}$$

The horizontal composition allows us to Φ to a natural transformation $\mathcal{G} \circ \mathcal{F}$ to $\mathcal{G} \circ \mathcal{F}'$ as follows. First, augment the diagram as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left=50, "\mathcal{F}", ""{name=U, below}] \arrow[r, squiggly, bend right=50, "\mathcal{F}'"', ""{name=L}] & \mathsf{D} \arrow[r, squiggly, bend left=50, "\mathcal{G}", ""{name=U2, below}] \arrow[r, squiggly, bend right=50, "\mathcal{G}"', ""{name=L2}] & \mathsf{E}
\arrow[Rightarrow, from=U, to=L, "\Phi"]
\arrow[Rightarrow, from=U2, to=L2, "1_{\mathcal{G}}"]
\end{tikzcd}$$

We can then take the horizontal composition of Φ and $1_{\mathcal{G}}$ to get a natural transformation from $\mathcal{G} \circ \mathcal{F}$ to $\mathcal{G} \circ \mathcal{F}'$ with components

$$(1_{\mathcal{G}} * \Phi)_A = \mathcal{G}(\Phi_A).$$

This natural transformation is called the *right whiskering whiskering* of Φ with $\mathcal{G}$, and is denoted $\mathcal{G}\Phi$. That is to say, $(\mathcal{G}\Phi)_A = \mathcal{G}(\Phi_A)$.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, squiggly, "\mathcal{G}\circ\mathcal{F}"{name=U}] \arrow[r, bend right=50, squiggly, "\mathcal{G}\circ\mathcal{F}'"'{name=D}] & \mathsf{E} \arrow[Rightarrow, from=U, to=D, "\mathcal{G}\Phi"] \end{tikzcd}$$

The reason for the name is clear: we removed a whisker from the RHS of our diagram.

We can also remove a whisker from the LHS. Given this:

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[r, bend left=50, squiggly, "\mathcal{G}"{name=U}] \arrow[r, bend right=50, squiggly, "\mathcal{G}'"'{name=D}] & \mathsf{E} \arrow[Rightarrow, from=U, to=D, "\Psi"] \end{tikzcd}$$

we can build a natural transformation (denoted $\Psi\mathcal{F}$) with components

$$(\Psi\mathcal{F})_A = \Psi_{\mathcal{F}(A)},$$

making this:

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, squiggly, "\mathcal{G}\circ\mathcal{F}"{name=U}] \arrow[r, bend right=50, squiggly, "\mathcal{G}'\circ\mathcal{F}"'{name=D}] & \mathsf{E} \arrow[Rightarrow, from=U, to=D, "\Psi\mathcal{F}"] \end{tikzcd}.$$

This is called the *left whiskering* of Ψ with $\mathcal{F}$

Lemma 3.3.3. *If we have a natural isomorphism $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$, we can construct a natural isomorphism $\eta^{-1}\colon \mathcal{G} \Rightarrow \mathcal{F}$.*

Proof. The natural transformation gives us for any two objects A and B and morphism $f\colon A \to B$ a naturality square

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_A"'] & \mathcal{F}(B) \arrow[d, "\eta_B"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(B) \end{tikzcd}$$

which tells us that

$$\eta_B \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_A.$$

Since η is a natural isomorphism, its components η_A are isomorphisms, so they have inverses η_A^{-1}. Acting on the above equation with η_A^{-1} from the right and η_B^{-1} from the left, we find

$$\mathcal{F}(f) \circ \eta_A^{-1} = \eta_B^{-1} \circ \mathcal{G}(f),$$

i.e. the following diagram commutes.

$$\begin{tikzcd} \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\eta_A^{-1}"'] & \mathcal{G}(B) \arrow[d, "\eta_B^{-1}"] \\ \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(B) \end{tikzcd}$$

But this is just the naturality square for a natural isomorphism η^{-1} with components $(\eta^{-1})_A = \eta_A^{-1}$. $\square$

We would like to be able to express when two categories are the 'same.' The correct notion of sameness is provided by the following definition.

Definition 3.3.7 (categorical equivalence)**.** Let C and D be categories. We say that C and D are equivalent if there is a pair of functors

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[l, shift left, squiggly, "\mathcal{G}"] \end{tikzcd}$$

and natural isomorphisms $R\colon \mathcal{F} \Rightarrow \mathcal{G}$ and $S\colon \mathcal{G} \Rightarrow \mathcal{F}$ such that $S \circ R = 1_{\mathcal{F}}$ and $R \circ S = 1_{\mathcal{G}}$.

Note 3.3.2. The above definition of categorical equivalence is equivalent to the following: C and D are equivalent if there is a functor $\mathcal{F} : \mathsf{C} \rightsquigarrow \mathsf{D}$ which is fully faithful (Definition 3.2.2) and essentially surjective (Definition 3.2.3). That is to say, if the equivalence $\mathcal{F}$ is 'bijective up to isomorphism.'

Definition 3.3.8 (essentially small category)**.** A category C is said to be essentially small if it is equivalent to a small category (Definition 3.1.9).

3.4 Some special categories

3.4.1 Comma categories

Definition 3.4.1 (comma category)**.** Let A, B, C be categories, $\mathcal{S}$ and $\mathcal{T}$ functors as follows.

$$\begin{tikzcd} \mathsf{A} \arrow[r, squiggly, "\mathcal{S}"] & \mathsf{C} & \mathsf{B} \arrow[l, squiggly, "\mathcal{T}"'] \end{tikzcd}$$

The comma category $(\mathcal{S} \downarrow \mathcal{T})$ is the category whose

- objects are triples (α, β, f) where $\alpha \in \mathrm{Obj}(\mathsf{A})$, $\beta \in \mathrm{Obj}(\mathsf{B})$, and $f \in \mathrm{Hom}_{\mathsf{C}}(\mathcal{S}(\alpha), \mathcal{T}(\beta))$, and whose
- morphisms $(\alpha, \beta, f) \to (\alpha', \beta', f')$ are all pairs (g, h), where $g\colon \alpha \to \alpha'$ and $h\colon \beta \to \beta'$, such that the diagram

$$\begin{tikzcd} \mathcal{S}(\alpha) \arrow[r, "\mathcal{S}(g)"] \arrow[d, "f"'] & \mathcal{S}(\alpha') \arrow[d, "f'"] \\ \mathcal{T}(\beta) \arrow[r, "\mathcal{T}(h)"] & \mathcal{T}(\beta') \end{tikzcd}$$

commutes.

Notation 3.4.1. We will often specify the comma category $(\mathcal{S} \downarrow \mathcal{T})$ by simply writing down the diagram

$$\begin{tikzcd} \mathsf{A} \arrow[r, squiggly, "\mathcal{S}"] & \mathsf{C} & \mathsf{B} \arrow[l, squiggly, "\mathcal{T}"'] \end{tikzcd} .$$

Let us check in some detail that a comma category really is a category. To do so, we need to check the three properties listed in Definition 3.1.1

1. We must be able to compose morphisms, i.e. we must have the following diagram.

$$\begin{tikzcd} (\alpha, \beta, f) \arrow[r, "{(g,h)}"] \arrow[rr, bend left, "{(g',h') \circ (g,h)}"] & (\alpha', \beta', f') \arrow[r, "{(g',h')}"] & (\alpha'', \beta'', f'') \end{tikzcd}$$

But we certainly do, since by definition, each square of the following diagram commutes,

$$\begin{tikzcd} \mathcal{S}(\alpha) \arrow[r, "\mathcal{S}(g)"] \arrow[d, "f"'] & \mathcal{S}(\alpha') \arrow[r, "\mathcal{S}(g')"] \arrow[d, "f'"'] & \mathcal{S}(a'') \arrow[d, "f''"] \\ \mathcal{T}(\alpha) \arrow[r, "\mathcal{T}(h)"] & \mathcal{T}(\alpha') \arrow[r, "\mathcal{T}(h')"] & \mathcal{T}(a'') \end{tikzcd}$$

so the square formed by taking the outside rectangle

$$\begin{tikzcd}
\mathcal{S}(\alpha) \arrow[r, "\mathcal{S}(g')\circ\mathcal{S}(g)"] \arrow[d, "f"'] & \mathcal{S}(\alpha'') \arrow[d, "f''"] \\
\mathcal{T}(\alpha) \arrow[r, "\mathcal{T}(h')\circ\mathcal{T}(h)"] & \mathcal{T}(\alpha'')
\end{tikzcd}$$

commutes. But $\mathcal{S}$ and $\mathcal{T}$ are functors, so

$$\mathcal{S}(g') \circ \mathcal{S}(g) = \mathcal{S}(g' \circ g),$$

and similarly for $\mathcal{T}(h' \circ h)$. Thus, the composition of morphisms is given via

$$(g', h') \circ (g, h) = (g' \circ g, h' \circ h).$$

2. We can see from this definition that associativity in $(\mathcal{S} \downarrow \mathcal{T})$ follows from associativity in the underlying categories A and B.
3. The identity morphism is the pair $(1_{\mathcal{S}(\alpha)}, 1_{\mathcal{T}(\beta)})$. It is trivial from the definition of the composition of morphisms that this morphism functions as the identity morphism.

3.4.2 Slice categories

A special case of a comma category, the so-called *slice category*, occurs when $\mathsf{C} = \mathsf{A}$, $\mathcal{S}$ is the identity functor, and $\mathsf{B} = \mathsf{1}$, the category with one object and one morphism (Example 3.1.2).

Definition 3.4.2 (slice category)**.** A slice category is a comma category

$$\begin{tikzcd}
\mathsf{A} \arrow[r, squiggly, "\mathrm{id}_A"] & \mathsf{A} & \mathsf{1} \arrow[l, squiggly, "\mathcal{T}"']
\end{tikzcd} .$$

Let us unpack this prescription. Taking the definition literally, the objects in our category are triples (α, β, f), where $\alpha \in \mathrm{Obj}(\mathsf{A})$, $\beta \in \mathrm{Obj}(\mathsf{1})$, and $f \in \mathrm{Hom}_{\mathsf{A}}(\mathrm{id}_{\mathsf{A}}(\alpha), \mathcal{T}(\beta))$.

There's a lot of extraneous information here, and our definition can be consolidated considerably. Since the functor $\mathcal{T}$ is given and $\mathsf{1}$ has only one object (call it $*$), the object $\mathcal{T}(*)$ (call it X) is singled out in A. We can think of $\mathcal{T}$ as $\mathcal{F}_X$ (Example 3.2.1). Similarly, since the identity morphism doesn't do anything interesting, Therefore, we can collapse the following diagram considerably.

$$\begin{tikzcd}
\mathcal{F}_X(*) \arrow[r, "\mathcal{F}_X(1_*)"] \arrow[d, "f"'] & \mathcal{F}_X(*) \arrow[d, "f'"] \\
\mathrm{id}_A(\alpha) \arrow[r, "\mathrm{id}_A(g)"] & \mathrm{id}_A(\alpha')
\end{tikzcd}$$

The objects of a slice category therefore consist of pairs (α, f), where $\alpha \in \mathrm{Obj}(A)$ and

$$f : \alpha \to X;$$

the morphisms $(\alpha, f) \to (\alpha', f')$ consist of maps $g \colon \alpha \to \alpha'$. This allows us to define a slice category more neatly.

Let A be a category, $X \in \mathrm{Obj}(\mathsf{A})$. The slice category $(\mathsf{A} \downarrow X)$ is the category whose objects are pairs (α, f), where $\alpha \in \mathrm{Obj}(A)$ and $f \colon \alpha \to X$, and whose morphisms $(\alpha, f) \to (\alpha', f')$ are maps $g : \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd}
\alpha \arrow[rr, "g"] \arrow[dr, "f"'] & & \alpha' \arrow[dl, "f'"] \\
& X &
\end{tikzcd}$$

commutes.

One can also define a coslice category, which is what you get when you take a slice category and turn the arrows around: coslice categories are *dual* to slice categories.

Definition 3.4.3 (coslice category)**.** Let A be a category, $X \in \mathrm{Obj}(A)$. The coslice category $(X \downarrow \mathsf{A})$ is the comma category given by the diagram

$$\begin{tikzcd} 1 \arrow[r, rightsquigarrow, "\mathcal{F}_X"] & \mathsf{A} & \mathsf{A} \arrow[l, leftsquigarrow, "\mathrm{id}_A"'] \end{tikzcd} .$$

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \alpha \arrow[rr, "g"'] & & \alpha' \end{tikzcd}$$

commutes.

3.5 Universal properties

Note 3.5.1. It is assumed that the reader is familar with the basic idea of a universal property and has seen (but not necessarily understood) a few examples, such as the universal properties for products and tensor algebras.

In general, the 'nicest' definition of a structure uses only the category-theoretic information about the category in which the structure lives; the definition of a product (of sets, for example) is best given without making use of hand-wavy statements like "ordered pairs of an element here and an element there." The idea is similar to the situation in linear algebra, where it is aesthetically preferrable to avoid introducing an arbitrary basis every time one needs to prove something, instead using properties intrinsic to the vector space itself.

The easiest way to do this in general is by using the idea of a *universal property.*

Definition 3.5.1 (initial objects, final objects, zero objects)**.** Let C be a category. An object $A \in \mathrm{Obj}(\mathsf{C})$ is said to be an

- initial object if $\mathrm{Hom}(A, B)$ has exactly one element for all $B \in \mathrm{Obj}(\mathsf{C})$, i.e. if there is exactly one arrow from A to every object in C.
- final object (or *terminal object*) if $\mathrm{Hom}(B, A)$ has exactly one element for all $B \in \mathrm{Obj}(\mathsf{C})$, i.e. if there is exactly one arrow from every object in C to A.
- zero object if it is both initial and final.

Example 3.5.1. In Set, there is exactly one map from any set S to any one-element set $\{*\}$. Thus $\{*\}$ is a terminal object in Set.

Furthermore, it is conventional that there is exactly one map from the empty set $\emptyset$ to any set B. Thus the $\emptyset$ is initial in Set.

Example 3.5.2. The trivial group is a zero object in Grp.

Theorem 3.5.1. *Let C be a category, let I and I' be two initial objects in C. Then there exists a unique isomorphism between I and I'.*

Proof. Since I is initial, there exists exactly one morphism from I to *any* object, including I itself. By Definition 3.1.1, Part 3, I must have at least one map to itself: the identity morphism 1_I. Thus, the only morphism from I to itself is the identity morphism. Similarly, the only morphism from I' to itself is also the identity morphism.

But since I is initial, there exists a unique morphism from I to I'; call it f. Similarly, there exists a unique morphism from I' to I; call it g. By Definition 3.1.1, Part 1, we can take the composition $g \circ f$ to get a morphism $I \to I$.

But there is only one isomorphism $I \to I$: the identity morphism! Thus

$$g \circ f = 1_I.$$

Similarly,

$$f \circ g = 1_{I'}.$$

This means that, by Definition 3.1.6, f and g are isomorphisms. They are clearly unique because of the uniqueness condition in the definition of an initial object. Thus, between any two initial objects there is a unique isomorphism, and we are done.

Here is a bad picture of this proof.

$$\begin{tikzcd} I \arrow[loop left, "1_I = g\circ f"'] \arrow[r, bend left, "f"] & I' \arrow[l, bend left, "g"] \arrow[loop right, "1_{I'} = f\circ g"] \end{tikzcd}$$

□

Definition 3.5.2 (category of morphisms from an object to a functor)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ be a functor. Further let $X \in \mathrm{Obj}(C)$. The category of morphisms $(X \downarrow \mathcal{U})$ is the following comma category (see Definition 3.4.1):

$$\begin{tikzcd} 1 \arrow[r, squiggly, "\mathcal{F}_X"] & \mathsf{C} & \mathsf{D} \arrow[l, squiggly, "\mathcal{U}"'] \end{tikzcd}.$$

Just as for (co)slice categories (Definitions 3.4.2 and 3.4.3), there is some unpacking to be done. In fact, the unpacking is very similar to that of coslice categories. The LHS of the commutative square diagram collapses because the functor $\mathcal{F}_X$ picks out a single element X; therefore, the objects of $(X \downarrow \mathcal{U})$ are ordered pairs

$$(\alpha, f); \qquad \alpha \in \mathrm{Obj}(\mathsf{D}), \quad f\colon X \to \mathcal{U}(\alpha),$$

and the morphisms $(\alpha, f) \to (\alpha', f')$ are morphisms $g\colon \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \mathcal{U}(\alpha) \arrow[rr, "\mathcal{U}(g)"'] & & \mathcal{U}(\alpha') \end{tikzcd}$$

commutes.

Just as slice categories are dual to coslice categories, we can take the dual of the previous definition.

Definition 3.5.3 (category of morphisms from a functor to an object)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ be a functor. The category of morphisms $(\mathcal{U} \downarrow X)$ is the comma category

$$\begin{tikzcd} \mathsf{D} \arrow[r, squiggly, "\mathcal{U}"] & \mathsf{C} & 1 \arrow[l, squiggly, "\mathcal{F}_X"'] \end{tikzcd}.$$

The objects in this category are pairs (α, f), where $\alpha \in \mathrm{Obj}(\mathsf{D})$ and $f\colon \mathcal{U}(\alpha) \to X$. The morphisms $(\alpha, f) \to (\alpha', f')$ are morphisms $g\colon \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd} \mathcal{U}(\alpha) \arrow[rr, "\mathcal{U}(g)"] \arrow[dr, "f"'] & & \mathcal{U}(\alpha') \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

commutes.

Definition 3.5.4 (initial morphism)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ be a functor, and let $X \in \mathrm{Obj}(\mathsf{C})$. An initial morphism (called a *universal arrow* in [17]) is an initial object in the category $(X \downarrow \mathcal{U})$, i.e. the comma category which has the diagram

$$\begin{tikzcd} 1 \arrow[r, squiggly, "\mathcal{F}_X"] & \mathsf{C} & \mathsf{D} \arrow[l, squiggly, "\mathcal{U}"'] \end{tikzcd}$$

This is not by any stretch of the imagination a transparent definition, but decoding it will be good practice.

The definition tells us that an initial morphism is an object in $(X \downarrow \mathcal{U})$, i.e. a pair (I, φ) for $I \in \mathrm{Obj}(\mathsf{D})$ and $\varphi \colon X \to \mathcal{U}(\alpha)$. But it is not just any object: it is an initial object. This means that for any other object (α, f), there exists a unique morphism $(I, \varphi) \to (\alpha, f)$. But such morphisms are simply maps $g \colon I \to \alpha$ such that the diagram

$$\begin{tikzcd}
& X \arrow[dl, "\varphi"'] \arrow[dr, "f"] & \\
\mathcal{U}(I) \arrow[rr, "\mathcal{U}(g)"'] & & \mathcal{U}(\alpha)
\end{tikzcd}$$

commutes.

We can express this schematically via the following diagram (which is essentially the above diagram, rotated to agree with the literature).

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathcal{U}(I) \arrow[d, "\mathcal{U}(g)"] & & I \arrow[d, "\exists! g"] \\
& \mathcal{U}(\alpha) & & \alpha
\end{tikzcd}$$

As always, there is a dual notion.

Definition 3.5.5 (terminal morphism)**.** Let C, D be categories, let $\mathcal{U} \colon \mathsf{D} \rightsquigarrow \mathsf{C}$ be a functor, and let $X \in \mathrm{Obj}(\mathsf{C})$. A terminal morphism is a terminal object in the category $(\mathcal{U} \downarrow X)$.

This time "terminal object" means a pair (I, φ) such that for any other object (α, f) there is a unique morphism $g \colon \alpha \to I$ such that the diagram

$$\begin{tikzcd}
\mathcal{U}(\alpha) \arrow[rr, "\mathcal{U}(g)"] \arrow[dr, "f"'] & & \mathcal{U}(I) \arrow[dl, "\varphi"] \\
& X &
\end{tikzcd}$$

commutes.

Again, with the diagram helpfully rotated, we have the following.

$$\begin{tikzcd}
\mathcal{U}(\alpha) \arrow[d, "\mathcal{U}(g)"'] \arrow[dr, "f"] & & & \alpha \arrow[d, "\exists! g"] \\
\mathcal{U}(I) \arrow[r, "\varphi"'] & X & & I
\end{tikzcd}$$

Definition 3.5.6 (universal property)**.** There is no hard and fast definition of a universal property. In general, an object with a universal property is just an object which is initial or terminal in some category. However, enough interesting universal properties are given in terms of initial and terminal morphisms that it will pay to study them in some detail.

Note 3.5.2. One often hand-wavily says that an object I satisfies a universal property if (I, φ) is an initial or terminal morphism. This is actually rather annoying; one has to remember that when one states a universal property, one is defining not only an object I but also a morphism φ, which is often left implicit.

Example 3.5.3 (tensor algebra)**.** One often sees some variation of the following universal characterization of the tensor algebra, which was taken (almost) verbatim from Wikipedia. We will try to stretch it to fit our definition, following the logic through in some detail.

> Let V be a vector space over a field k, and let A be an algebra over k. The tensor algebra $T(V)$ satisfies the following universal property.
>
> Any linear transformation $f\colon V \to A$ from V to A can be uniquely extended to an algebra homomorphism $T(V) \to A$ as indicated by the following commutative diagram.
>
> $$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & T(V) \arrow[d, "\tilde{f}"] \\ & A \end{tikzcd}$$

As it turns out, it will take rather a lot of stretching.

Let $\mathcal{U}\colon k\text{-}\mathsf{Alg} \rightsquigarrow \mathsf{Vect}_k$ be the forgetful functor which assigns to each algebra over a field k its underlying vector space. Pick some k-vector space V. We consider the category $(V \downarrow \mathcal{U})$, which is given by the following diagram.

$$\begin{tikzcd} \mathsf{1} \arrow[r, squiggly, "\mathcal{F}_V"] & \mathsf{Vect}_k & k\text{-}\mathsf{Alg} \arrow[l, squiggly, "\mathcal{U}"'] \end{tikzcd}$$

By Definition 3.5.4, the objects of $(V \downarrow \mathcal{U})$ are pairs (A, L), where A is a k-algebra and L is a linear map $V \to \mathcal{U}(A)$. The morphisms are algebra homomorphisms $\rho\colon A \to A'$ such that the diagram

$$\begin{tikzcd} & V \arrow[dl, "L"'] \arrow[dr, "L'"] & \\ \mathcal{U}(A) \arrow[rr, "\mathcal{U}(\rho)"'] & & \mathcal{U}(A') \end{tikzcd}$$

commutes. An object $(T(V), i)$ is initial if for any object (A, f) there exists a unique morphism $g\colon T(V) \to A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "L"'] & \mathcal{U}(T(V)) \arrow[d, "\mathcal{U}(g)"] & & T(V) \arrow[d, "\exists! g"] \\ & \mathcal{U}(A) & & A \end{tikzcd}$$

commutes.

Thus, the pair $(i, T(V))$ is the initial object in the category $(V \downarrow \mathcal{U})$. We called $T(V)$ the *tensor algebra* over V.

But what is i? Notice that in the Wikipedia definition above, the map i is from V to $T(V)$, but in the diagram above, it is from V to $\mathcal{U}(T(V))$. What gives?

The answer that the diagram in Wikipedia's definition does not take place in a specific category. Instead, it implicitly treats $T(V)$ only as a vector space. But this is exactly what the functor $\mathcal{U}$ does.

Example 3.5.4 (tensor product)**.** According to the excellent book [3], the tensor product satisfies the following universal property.

> Let V_1 and V_2 be vector spaces. Then we say that a vector space V_3 together with a bilinear map $\iota\colon V_1 \times V_2 \to V_3$ has the *universal property* provided that for any bilinear map $B\colon V_1 \times V_2 \to W$, where W is also a vector space, there exists a unique linear map $L\colon V_3 \to W$ such that $B = L\iota$. Here is a diagram describing this 'factorization' of B through ι:
>
> $$\begin{tikzcd} V_1 \times V_2 \arrow[r, "\iota"] \arrow[dr, "B"'] & V_3 \arrow[d, "L"] \\ & W \end{tikzcd}$$

It turns out that the tensor product defined in this way is neither an initial or final morphism. This is because in the category Vect_k, there is no way of making sense of a bilinear map.

We will discover the correct way of dealing with tensor products a bit later, after we have defined a *symmetric monoidal category.*

Example 3.5.5 (categorical product)**.** Here is the universal property for a product, taken verbatim from Wikipedia ([21]).

> Let C be a category with some objects X_1 and X_2. A product of X_1 and X_2 is an object X (often denoted $X_1 \times X_2$) together with a pair of morphisms $\pi_1 \colon X \to X_1$ and $\pi_2 \colon X \to X_2$ such that for every object Y and pair of morphisms $f_1 \colon Y \to X_1$, $f_2 \colon Y \to X_2$, there exists a unique morphism $f \colon Y \to X_1 \times X_2$ such that the following diagram commutes.
>
> $$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2 \end{tikzcd}$$

Consider the comma category $(\Delta \downarrow (X_1, X_2))$ (where Δ is the diagonal functor, see Example 3.2.3) given by the following diagram.

$$\begin{tikzcd} \mathsf{C} \arrow[r, rightsquigarrow, "\Delta"] & \mathsf{C} \times \mathsf{C} & 1 \arrow[l, rightsquigarrow, "\mathcal{F}_{(X_1, X_2)}"'] \end{tikzcd}$$

The objects of this category are pairs $(A, (s, t))$, where $A \in \mathrm{Obj}(\mathsf{C})$ and

$$(s, t) \colon \Delta(A) = (A, A) \to (X_1, X_2).$$

The morphisms $(A, (s, t)) \to (B, (u, v))$ are morphisms $r \colon A \to B$ such that the diagram

$$\begin{tikzcd} (A, A) \arrow[rr, "{(r,r)}"] \arrow[dr, "{(s,t)}"'] & & (B, B) \arrow[dl, "{(u,v)}"] \\ & (X_1, X_2) & \end{tikzcd}$$

commutes.

An object $(X_1 \times X_2, (\pi_1, \pi_2))$ is final if for any other object $(Y, (f_1, f_2))$, there exists a unique morphism $f \colon Y \to X_1 \times X_2$ such that the diagram

$$\begin{tikzcd} (Y, Y) \arrow[d, "{(f,f)}"'] \arrow[dr, "{(f_1,f_2)}"] & & Y \arrow[d, "\exists ! f"] \\ (X_1 \times X_2, X_1 \times X_2) \arrow[r, "{(\pi_1,\pi_2)}"'] & (X_1, X_2) & X_1 \times X_2 \end{tikzcd}$$

commutes. If we re-arrange our diagram a bit, it is not too hard to see that it is equivalent to the one given above. Thus, we can say: a product of two sets X_1 and X_2 is a final object in the category $(\Delta \downarrow (X_1, X_2))$

3.6 Cartesian closed categories

This section loosely follows [24].

Cartesian closed categories are the prototype for many of the structures possessed by monoidal categories. They are interesting in their own right, but will not be essential in what follows.

3.6.1 Products

We saw in Example 3.5.5 the definition for a categorical product. In some categories, we can naturally take the product of any two objects; one generally says that such a category *has products.* We formalize that in the following.

Definition 3.6.1 (category with products)**.** Let C be a category such that for every two objects A, $B \in \mathrm{Obj}(\mathsf{C})$, there exists an object $A \times B$ which satisfies the universal product (Example 3.5.5). Then we say that C has products.

Note 3.6.1. Sometimes people call a category with products a *Cartesian category*, but others use this terminology to mean a category with all finite limits.[1] We will avoid it altogether.

Theorem 3.6.1. *Let C be a category with products. Then the product can be extended to a bifunctor (Definition 3.2.4) $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$.*

Proof. Let X, $Y \in \mathrm{Obj}(C)$. We need to check that the assignment $(X, Y) \mapsto \times(X, Y) \equiv X \times Y$ is functorial, i.e. that $\times$ assigns

- to each pair $(X, Y) \in \mathrm{Obj}(\mathsf{C} \times \mathsf{C})$ an object $X \times Y \in \mathrm{Obj}(\mathsf{C})$ and
- to each pair of morphisms $(f, g)\colon (X, Y) \to (X', Y')$ a morphism

  $$\times(f, g) = f \times g\colon X \times Y \to X' \times Y'$$

  such that

  - $1_X \times 1_Y = 1_{X\times Y}$, and
  - $\times$ respects composition as follows.

$$\begin{tikzcd}
(X,Y) \arrow[r, "{(f,g)}"] \arrow[rr, bend left, "{(f',g')\circ(f,g)=(f'\circ f,g'\circ g)}"] & (X',Y') \arrow[r, "{(f',g')}"] \arrow[d, squiggly, "\times"] & (X'',Y'') \\
X \times Y \arrow[r, "f\times g"] \arrow[rr, bend right, "f'\times g'\circ f\times g=(f'\circ f)\times(g'\circ g)"'] & X' \times Y' \arrow[r, "f'\times g'"] & X'' \times Y''
\end{tikzcd}$$

□

We know how $\times$ assigns objects in $\mathsf{C} \times \mathsf{C}$ to objects in C. We need to study how $\times$ assigns to a morphism (f, g) a morphism $f \times g$. We do this by diagram chasing.

Suppose we are given two maps $f\colon X_1 \to X_2$ and $g\colon Y_1 \to Y_2$. This induces a morphism (f, g) in $\mathsf{C} \times \mathsf{C}$.

Recall the universal property for products: a product $X_1 \times Y_1$ is a final object in the category $(\Delta \downarrow (X_1, Y_1))$. Objects in this category can be thought of as diagrams in $\mathsf{C} \times \mathsf{C}$.

$$(X_1 \times Y_1, X_1 \times Y_1) \xrightarrow{\pi_1,\pi_2} (X_1, Y_1)\ .$$

By assumption, we can take the product of both X_1 and Y_1, and X_2 and Y_2. This gives us two diagrams living in $\mathsf{C} \times \mathsf{C}$, which we can put next to each other.

$$(X_1 \times Y_1, X_1 \times Y_1) \xrightarrow{(\pi_1,\pi_2)} (X_1, Y_1)$$

$$(X_2 \times Y_2, X_2 \times Y_2) \xrightarrow{(\rho_1,\rho_2)} (X_2, Y_2)$$

---

[1] We will see later that any category with both products and equalizers has all finite limits.

We can draw in our morphism (f, g), and take its composition with (π_1, π_2).

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[r, "(\pi_1,\pi_2)"] \arrow[dr, "(f\circ\pi_1, g\circ\pi_2)"'] & (X_1, Y_1) \arrow[d, "(f,g)"] \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[r, "(\rho_1,\rho_2)"'] & (X_2, Y_2)
\end{tikzcd}$$

Now forget about the top right of the diagram. I'll erase it to make this easier.

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[dr, "(f\circ\pi_1, g\circ\pi_2)"] & \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[r, "(\rho_1,\rho_2)"'] & (X_2, Y_2)
\end{tikzcd}$$

The universal property for products says that there exists a unique map $h\colon X_1 \times Y_1 \to X_2 \times Y_2$ such that the diagram below commutes.

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[d, "(h,h)"'] \arrow[dr, "(f\circ\pi_1, g\circ\pi_2)"] & & X_1 \times Y_1 \arrow[d, "\exists! h"] \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[r, "(\rho_1,\rho_2)"'] & (X_1, X_2) & X_2 \times Y_2
\end{tikzcd}$$

And h is what we will use for the product $f \times g$.

Of course, we must also check that h behaves appropriately. Draw two copies of the diagram for the terminal object in $(\Delta \downarrow (X, Y))$ and identity arrows between them.

$$\begin{tikzcd}
(X \times Y, X \times Y) \arrow[r, "(\pi_1,\pi_2)"] \arrow[d, "(1_{X\times Y}, 1_{X\times Y})"'] & (X, Y) \arrow[d, "(1_X, 1_Y)"] \\
(X \times Y, X \times Y) \arrow[r, "(\pi_1,\pi_2)"'] & (X, Y)
\end{tikzcd}$$

Just as before, we compose the morphisms to and from the top right to draw a diagonal arrow, and then erase the top right object and the arrows to and from it.

$$\begin{tikzcd}
(X \times Y, X \times Y) \arrow[d, "(1_{X\times Y},\, 1_{X\times Y}) = (1_X \times 1_Y,\, 1_X \times 1_Y)"'] \arrow[dr, "(1_X\circ\pi_1,\, 1_Y\circ\pi_2)"] & & X \times Y \arrow[d, "1_X \times 1_Y = 1_X \times 1_Y"] \\
(X \times Y, X \times Y) \arrow[r, "(\pi_1,\pi_2)"'] & (X, Y) & X \times Y
\end{tikzcd}$$

By definition, the arrow on the right is $1_X \times 1_Y$. It is also $1_{X\times Y}$, and by the universal property it is unique. Therefore $1_{X\times Y} = 1_X \times 1_Y$.

Next, put three of these objects together. The proof of the last part is immediate.

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[d, "(f\times g, f\times g)"] \arrow[dd, bend right=60, "(f'\times g')\circ(f\times g)=(g'\circ g)\times(f'\circ f)"'] & (X_1, Y_1) \arrow[d, "(f,g)"'] \arrow[dd, bend left=60, "(g'\circ g, f'\circ f)"] \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[d, "(f'\times g', f'\times g')"] & (X_2, Y_2) \arrow[d, "(f',g')"'] \\
(X_3 \times Y_3, X_3 \times Y_3) & (X_3, Y_3)
\end{tikzcd}$$

The meaning of this theorem is that in any category where you can take products of the objects, you can also take products of the morphisms.

Example 3.6.1. The category Vect_k of vector spaces over a field k has the direct sum $\oplus$ as a product.

The product is, in an appropriate way, commutative.

Lemma 3.6.1. *There is a natural isomorphism between the following functors* $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$

$$\times\colon (A, B) \to A \times B \qquad \text{and} \qquad \tilde{\times}\colon (A, B) \to B \times A.$$

Proof. We define a natural transformation Φ whose components are

$$\Phi_{A,B}\colon A \times B \to B \times A$$

as follows. Denote the canonical projections for the product $A \times B$ by π_A and π_B. Then (π_B, π_A) is a map $A \times B \to (B, A)$, and the universal property for products gives us a map $A \times B \to B \times A$. We can pull the same trick to go from $B \times A$ to $A \times B$, using the pair (π_A, π_B) and the universal property. Furthermore, these maps are inverse to each other, so $\Phi_{A,B}$ is an isomorphism.

We need only check naturality, i.e. that for $f\colon A \to A'$, $g\colon B \to B'$, the following square commutes.

$$\begin{tikzcd}
A \times B \arrow[r] \arrow[d] & A' \times B' \arrow[d] \\
B \times A \arrow[r] & B' \times A'
\end{tikzcd}$$

□

Note 3.6.2. It is an important fact that the product is associative. We shall see this in Theorem 3.7.2, after we have developed the machinery to do it cleanly.

3.6.2 Coproducts

Definition 3.6.2. Let C be a category, X_1 and $X_2 \in \mathrm{Obj}(\mathsf{C})$. The coproduct of X_1 and X_2, denoted $X_1 \amalg X_2$, is the initial object in the category $((X_1, X_2) \downarrow \Delta)$. In everyday language, we have the following.

> An object $X_1 \amalg X_2$ is called the coproduct of X_1 and X_2 if
>
> 1. there exist morphisms $i_1\colon X_1 \to X_1 \amalg X_2$ and $i_2\colon X_2 \to X_1 \amalg X_2$ called *canonical injections* such that
> 2. for any object Y and morphisms $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ there exists a unique morphism $f\colon X_1 \amalg X_2 \to Y$ such that $f_1 = f \circ i_1$ and $f = f_2 = f \circ i_2$, i.e. the following diagram commutes.
>
> $$\begin{tikzcd}
& Y & \\
X_1 \arrow[ur, "f_1"] \arrow[r, "i_1"'] & X_1 \amalg X_2 \arrow[u, "f"] & X_2 \arrow[ul, "f_2"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Definition 3.6.3 (category with coproducts)**.** We say that a category C has coproducts if for all A, $B \in \mathrm{Obj}(C)$, the coproduct $A \amalg B$ is in $\mathrm{Obj}(C)$.

We have the following analog of Theorem 3.6.1.

Theorem 3.6.2. *Let C be a category with coproducts. Then we have a functor* $\amalg\colon C \times C \rightsquigarrow C$.

Proof. We verify that $\amalg$ allows us to define canonically a coproduct of morphisms. Let X_1, X_2, Y_1, $Y_2 \in \mathrm{Obj}(C)$, and let $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$. We have the following diagram.

$$\begin{tikzcd}
X_1 \arrow[r, "i_{1,X}"] \arrow[d, "f"'] \arrow[dr, "i_{i,Y} \circ f"] & X_1 \amalg X_2 & X_2 \arrow[l, "i_{2,X}"'] \arrow[d, "g"] \arrow[dl, "i_{2,Y} \circ g"'] \\
Y_1 \arrow[r, "i_{1,Y}"] & Y_1 \amalg Y_2 & Y_2 \arrow[l, "i_{2,Y}"']
\end{tikzcd}$$

But by the universal property of coproducts, the diagonal morphisms induce a map $X_1 \amalg X_2 \to Y_1 \amalg Y_2$, which we define to be $f \amalg g$.

$$\begin{tikzcd}
X_1 \arrow[r, "i_{1,X}"] \arrow[dr, "i_{i,Y}\circ f"'] & X_1 \amalg X_2 \arrow[d, "f\amalg g"] & X_2 \arrow[l, "i_{2,X}"'] \arrow[dl, "i_{2,Y}\circ g"] \\
& Y_1 \amalg Y_2 &
\end{tikzcd}$$

The rest of the verification that $\amalg$ really is a functor is identical to that in Theorem 3.6.1. ☐

Example 3.6.2. In Set, the coproduct is the disjoint union.

Example 3.6.3. In Vect_k, the direct sum $\oplus$ is the coproduct.

3.6.3 Biproducts

Definition 3.6.4 (category with biproducts)**.** We say that a category C has biproducts if

1. there is a zero object (Definitionh 3.5.1) $0 \in \mathrm{Obj}(\mathsf{C})$,
2. C has products $\times$,
3. C has coproducts $\amalg$, and
4. there is a natural isomorphsim (Definition 3.3.2) $\Phi\colon \amalg \Rightarrow \times$.

We can cast this in the form of a diagram.

$$\begin{tikzcd}
\mathsf{C}\times\mathsf{C} \arrow[r, bend left=50, squiggly, "\amalg", ""{name=U, below}] \arrow[r, bend right=50, squiggly, "\times"', ""{name=D}] & \mathsf{C}
\arrow[Rightarrow, from=U, to=D, "\Phi"]
\end{tikzcd}$$

Example 3.6.4. In the category Vect_k of vector spaces over a field k, the direct sum $\oplus$ is a categorical biproduct.

Example 3.6.5. In the category Ab of abelian groups, the direct sum $\oplus$ is both a product and a coproduct. Hence Ab has biproducts.

3.6.4 Exponentials

Astonishingly, we can talk about functional evaluation purely in terms of products and universal properties.

Definition 3.6.5 (exponential)**.** Let C be a category, A, $B \in \mathrm{Obj}(\mathsf{C})$. The exponential of A and B is an object $B^A \in \mathrm{Hom}(\mathsf{C})$ together with a morphism $\varepsilon\colon B^A \times A \to B$, called the *evaluation morphism*, which satisfies the following universal property.

> For any object $X \in \mathrm{Obj}(\mathsf{C})$ and morphism $f\colon X \times A \to B$, there exists a unique morphism $\bar{f}\colon X \to B^A$ which makes the following diagram commute.
>
> $$\begin{tikzcd}
B^A & B^A \times A \arrow[r, "\varepsilon"] & B \\
X \arrow[u, "\exists!\bar{f}"] & X \times A \arrow[u, "\bar{f}\times 1_A"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Example 3.6.6. In Set, the exponential object B^A is the set of functions $A \to B$, and the evaluation morphism ε is the map which assigns $(f, a) \mapsto f(a)$. Let us check that these objects indeed satisfy the universal property given.

Suppose we are given a function $f \colon X \times A \to B$. We want to construct from this a function $\bar{f} \colon X \to B^A$. i.e. a function which takes an element $x \in X$ and returns a function $\bar{f}(x) \colon A \to B$. There is a natural way to do this: fill the first slot of f with x! That is to say, define $\bar{f}(x) = f(x, -)$. It is easy to see that this makes the diagram commute:

$$\begin{tikzcd}
(f(x,-),a) \arrow[r, mapsto, "\varepsilon"] & f(x,a) \\
(x,a) \arrow[u, mapsto, "\bar{f}\times 1_A"] \arrow[ur, mapsto, "f"'] &
\end{tikzcd}$$

Furthermore, $\bar{f}$ is unique since if it sent x to any function other than f the diagram would not commute.

Example 3.6.7. One might suspect that the above generalizes to all sorts of categories. For example, one might hope that in Grp, the exponential H^G of two groups G and H would somehow capture all homomorphisms $G \to H$. This turns out to be impossible; we will see why in REF.

3.6.5 Cartesian closed categories

Definition 3.6.6 (cartesian closed category)**.** A category C is cartesian closed if it has

1. products: for all A, $B \in \mathrm{Obj}(\mathsf{C})$, $A \times B \in \mathrm{Obj}(\mathsf{C})$;
2. exponentials: for all A, $B \in \mathrm{Obj}(\mathsf{C})$, $B^A \in \mathrm{Obj}(\mathsf{C})$;
3. a terminal element (Definition 3.5.1) $1 \in \mathsf{C}$.

3.7 Hom functors and the Yoneda lemma

3.7.1 The Hom functor

Given any locally small category (Definition 3.1.9) C and two objects A, $B \in \mathrm{Obj}(\mathsf{C})$, we have thus far notated the set of all morphisms $A \to B$ by $\mathrm{Hom}_{\mathsf{C}}(A, B)$. This may have seemed a slightly odd notation, but there was a good reason for it: $\mathrm{Hom}_{\mathsf{C}}$ is really a functor.

To be more explicit, we can view $\mathrm{Hom}_{\mathsf{C}}$ as a functor $\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \rightsquigarrow \mathsf{Set}$, which sends each pair $(A', A) \in \mathrm{Obj}(\mathsf{C}^{\mathrm{op}} \times \mathsf{C})$ to the set $\mathrm{Hom}_{\mathsf{C}}(A', A)$ of morphisms $A' \to A$.

This is fine, and probably the definition you were expecting, but we're not done: as a functor it also has to send the morphisms in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ to Set-arrows. How does *that* work?

A morphism in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ between two objects (A', A) and (B', B) is an ordered pair (f', f) of two morphisms:

$$f' \colon B' \to A' \qquad \text{and} \qquad f \colon A \to B,$$

where A, $B \in \mathrm{Obj}(\mathsf{C})$ and A', $B' \in \mathrm{Obj}(\mathsf{C}^{\mathrm{op}})$. We can draw this as follows.

$$\begin{tikzcd}
A' & B' \arrow[l, "f'"'] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

We have to send this to a set-function $\mathrm{Hom}_{\mathsf{C}}(A', A) \to \mathrm{Hom}_{\mathsf{C}}(B', B)$. The way to do this is as follows: send each $m \in \mathrm{Hom}(A', A)$ to the element

$$f \circ m \circ f' \in \mathrm{Hom}_{\mathsf{C}}(B', B).$$

As the function $m \mapsto f \circ m \circ f'$ is the action of the functor $\mathrm{Hom}_{\mathsf{C}}$ on the morphism (f', f), it is natural to call it $\mathrm{Hom}_{\mathsf{C}}(f', f)$.

Of course, we should check that this *really is* a functor, i.e. that it treats identities and compositions correctly. First, we need to show that $\mathrm{Hom}_{\mathsf{C}}(1_{A'}, 1_A)$ is the identity function $1_{\mathrm{Hom}_{\mathsf{C}}(A',A)}$. It is, since if $m \in \mathrm{Hom}_{\mathsf{C}}(A', A)$,

$$\mathrm{Hom}_{\mathsf{C}}(1_{A'}, 1_A)\colon m \mapsto 1_A \circ m \circ 1_{A'} = m.$$

Next, we have to check that compositions work the way they're supposed to. Suppose we have objects and morphisms like this:

$$\begin{tikzcd} A' & B' \arrow[l, "f'"'] & C' \arrow[l, "g'"'] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

where the primed stuff is in C^{op} and the unprimed stuff is in C. This can be viewed as three objects (A', A), etc. in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ and two morphisms (f', f) and (g', g) between them.

Okay, so we can compose (f', f) and (g', g) to get a morphism

$$(g', g) \circ (f', f) = (f' \circ g', g \circ f)\colon (A', A) \to (C', C).$$

Note that the order of the composition in the first argument has been turned around: this is expected since the first argument lives in C^{op}. To check that $\mathrm{Hom}_{\mathsf{C}}$ handles compositions correctly, we need to verify that

$$\mathrm{Hom}_{\mathsf{C}}(g, g') \circ \mathrm{Hom}_{\mathsf{C}}(f, f') = \mathrm{Hom}_{\mathsf{C}}(f' \circ g', g \circ f).$$

Let $m \in \mathrm{Hom}_{\mathsf{C}}(A', A)$. We victoriously compute

$$\begin{aligned} [\mathrm{Hom}_{\mathsf{C}}(g, g') \circ \mathrm{Hom}_{\mathsf{C}}(f, f')](m) &= \mathrm{Hom}_{\mathsf{C}}(g, g')(f \circ m \circ f') \\ &= g \circ f \circ m \circ f' \circ g' \\ &= \mathrm{Hom}_{\mathsf{C}}(f' \circ g', g \circ f). \end{aligned}$$

Let's formalize this in a definition.

Definition 3.7.1 (hom functor)**.** Let C be a locally small category. The hom functor $\mathrm{Hom}_{\mathsf{C}}$ is the functor

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \rightsquigarrow \mathsf{Set}$$

which sends the object (A', A) to the set $\mathrm{Hom}_{\mathsf{C}}(A', A)$ of morphisms $A' \to A$, and the morphism $(f', f)\colon (A', A) \to (B', B)$ to the function

$$\mathrm{Hom}_{\mathsf{C}}(f', f)\colon \mathrm{Hom}_{\mathsf{C}}(A', A) \to \mathrm{Hom}_{\mathsf{C}}(B', B); \qquad m \mapsto f \circ m \circ f'.$$

Now recall from computer science the concept of currying. Suppose we are given a function of two arguments, say $f(x, y)$. The idea of currying is this: if we like, we can view f as a family of functions of only one variable y, indexed by x:

$$h_x(y) = f(x, y).$$

We can even view h as a function which takes one argument x, and which returns a *function* h_x of one variable y.

This sets up a correspondence between functions $f\colon A \times B \to C$ and functions $h\colon A \to C^B$, where C^B is the set of functions $B \to C$. The map which replaces f by h is called *currying.* We can also go the other way (i.e. $h \mapsto f$), which is called *uncurrying.*

We have been intentionally vague about the nature of our function f, and what sort of arguments it might take; everything we have said also holds for, say, bifunctors.

In particular, it gives us two more ways to view our bifunctor $\mathrm{Hom}_{\mathsf{C}}$. We can fix any $A \in \mathrm{Obj}(\mathsf{C})$ and curry either argument. This gives us the following.

Definition 3.7.2 (curried hom functor)**.** Let C be a locally small category, $A \in \mathrm{Obj}(\mathsf{C})$. We can construct from the hom functor $\mathrm{Hom}_{\mathsf{C}}$

- a functor $Y^A\colon \mathsf{C} \rightsquigarrow \mathsf{Set}$ which maps

  - an object $B \in \mathrm{Obj}(\mathsf{C})$ to the set $\mathrm{Hom}_{\mathsf{C}}(A, B)$, and
  - a morphism $f\colon B \to B'$ to a Set-function

$$Y^A(f)\colon \mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{C}}(A, B'); \qquad m \mapsto f \circ m.$$

- a functor $Y_A\colon \mathsf{C}^{\mathrm{op}} \rightsquigarrow \mathsf{Set}$ which maps
  - an object $B \in \mathrm{Obj}(\mathsf{C})$ to the set $\mathrm{Hom}_{\mathsf{C}}(B, A)$, and
  - a morphism $f\colon B \to B'$ to a Set-function

$$Y_A(f)\colon \mathrm{Hom}_{\mathsf{C}}(B', A) \to \mathrm{Hom}_{\mathsf{C}}(B, A); \qquad m \mapsto m \circ f.$$

Example 3.7.1. Hom functors give us a new way of looking at the universal property for products, coproducts, and exponentials.

Let C be a locally small category with products, and let X, A, $B \in \mathrm{Obj}(\mathsf{C})$. Recall that the universal property for the product $A \times B$ allows us to exchange two morphisms

$$f_1\colon X \to A \qquad \text{and} \qquad f_2\colon X \to B$$

for a morphism

$$f\colon X \to A \times B.$$

We can also compose a morphism $g\colon X \to A \times B$ with the canonical projections

$$\pi_A\colon A \times B \to A \qquad \text{and} \qquad \pi_B\colon A \times B \to B$$

to get two morphisms

$$\pi_A \circ f\colon X \to A \qquad \text{and} \qquad \pi_B \circ f\colon X \to B.$$

This means that there is a bijection

$$\mathrm{Hom}_{\mathsf{C}}(X, A \times B) \simeq \mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B).$$

In fact this bijection is natural in X. That is to say, there is a natural bijection between the following functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$:

$$Y_{A\times B}\colon X \to \mathrm{Hom}_{\mathsf{C}}(X, A \times B) \qquad \text{and} \qquad Y_A \times Y_B\colon X \to \mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B).$$

Let's prove naturality. Let

$$\Phi_X\colon \mathrm{Hom}(X, A \times B) \to \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B); \qquad f \mapsto (\pi_A \circ f, \pi_B \circ f)$$

be the components of the above transformation, and let $g\colon X' \to X$. Naturality follows from the fact that the diagram below commutes.

$$\begin{tikzcd}
\mathrm{Hom}(X, A \times B) \arrow[r, "\mathrm{Hom}(g, 1_{A\times B})"] \arrow[d, "\Phi_X"'] & \mathrm{Hom}(X', A \times B) \arrow[d, "\Phi_{X'}"] \\
\mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B) \arrow[r, "\mathrm{Hom}(g, 1_A) \times \mathrm{Hom}(g, 1_B)"] & \mathrm{Hom}(X', A) \times \mathrm{Hom}(X', B)
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & f \circ g \arrow[d, mapsto] \\
(\pi_A \circ f, \pi_B \circ g) \arrow[r, mapsto] & (\pi_A \circ f \circ g, \pi_B \circ f \circ g)
\end{tikzcd}$$

The coproduct does a similar thing: it allows us to trade two morphisms

$$A \to X \qquad \text{and} \qquad B \to X$$

for a morphism

$$A \amalg B \to X,$$

and vice versa. Similar reasoning yields a natural bijection between the following functors $\mathsf{C} \rightsquigarrow \mathsf{Set}$:

$$Y^{A \amalg B} \Rightarrow Y^A \times Y^B.$$

The exponential is a little bit more complicated: it allows us to trade a morphism

$$X \times A \to B$$

for a morphism

$$X \to B^A,$$

and vice versa. That is to say, we have a bijection between two functors $\mathsf{C}^{\mathrm{op}} \rightsquigarrow \mathsf{Set}$

$$X \mapsto \mathrm{Hom}_{\mathsf{C}}(X \times A, B) \qquad \text{and} \qquad X \mapsto \mathrm{Hom}_{\mathsf{C}}(X, B^A).$$

The fact that the functors are more complicated does not hurt us: the above bijection is still natural.

3.7.2 Representable functors

Definition 3.7.3 (representable functor)**.** Let C be a category. A functor $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{Set}$ is representable if there is an object $A \in \mathrm{Obj}(\mathsf{C})$ and a natural isomorphism (Definition 3.3.2)

$$\eta\colon \mathcal{F} \Rightarrow \begin{cases} Y^A, & \text{if } \mathcal{F} \text{ is covariant} \\ Y_A, & \text{if } \mathcal{F} \text{ is contravariant.} \end{cases}$$

Note 3.7.1. Since natural isomorphisms are invertible, we could equivalently define a representable functor with the natural isomorphism going the other way.

Example 3.7.2. Consider the forgetful functor $\mathcal{U}\colon \mathsf{Grp} \rightsquigarrow \mathsf{Set}$ which sends a group to its underlying set, and a group homomorphism to its underlying function. This functor is represented by the group $\mathbb{Z}$.

To see this, we have to check that there is a natural isomorphism η between $\mathcal{U}$ and $Y^{\mathbb{Z}} = \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, -)$. That is to say, for each group G, there a Set-isomorphism (i.e. a bijection)

$$\eta_G\colon \mathcal{U}(G) \to \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G)$$

satisfying the naturality conditions in Definition 3.3.1.

First, let's show that there's a bijection by providing an injection in both directions. Pick some $g \in G$. Then there is a unique group homomorphism which sends $1 \mapsto g$, so we have an injection $\mathcal{U}(G) \hookrightarrow \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G)$.

Now suppose we are given a group homomorphism $\mathbb{Z} \to G$. This sends 1 to some element $g \in G$, and this completely determines the rest of the homomorphism. Thus, we have an injection $\mathrm{Hom}_{\mathsf{Grp}(\mathbb{Z},G)} \hookrightarrow \mathcal{U}(G)$.

All that is left is to show that η satisfies the naturality condition. Let F and H be groups, and $f\colon G \to H$ a homomorphism. We need to show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{U}(G) \arrow[r, "\mathcal{U}(f)"] \arrow[d, "\eta_G"'] & \mathcal{U}(H) \arrow[d, "\eta_H"] \\
\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G) \arrow[r, "Y^{\mathbb{Z}}(f)"] & \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, H)
\end{tikzcd}$$

The upper path from top left to bottom right assigns to each $g \in G$ the function $\mathbb{Z} \to G$ which maps $1 \mapsto f(g)$. Walking down η_G from $\mathcal{U}(G)$ to $\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G)$, g is mapped to the function $\mathbb{Z} \to G$ which maps $1 \mapsto g$. Walking right, we compose this function with f to get a new function $\mathbb{Z} \to H$ which sends $1 \mapsto f(g)$. This function is really the image of a homomorphism and is therefore unique, so the diagram commutes.

3.7.3 The Yoneda embedding

The Yoneda embedding is a very powerful tool which allows us to prove things about any locally small category by embedding that category into Set, and using the structure which Set has.

In the thread [19] on the sci.math google group, John Baez explains the importance of the Yoneda lemma in the following way.

> Category theory can be viewed in many ways, but one is as a massive generalization of set theory. The category Set has sets as objects and functions between them as morphisms. In practice, other categories are often built by taking as objects "sets with extra structure" and as morphisms "functions preserving the extra structure." For example, Vect is the category whose objects are vector spaces and whose morphisms are linear functions. The extra structure in this case is the linear structure. One can easily list hundreds more such examples.
>
> However, an abstract category needn't have as its objects "sets with structure"—its objects are simply abstract thingamabobs! (I would have said "simply abstract objects", but that would sound circular here, since it is, so I resorted to a more technical term.) At this point the Yoneda lemma leaps to our rescue by saying that all (small enough) categories can be embedded in categories in which the objects are sets with structure and the morphisms are structure-preserving functions between these.
>
> Here's the basic idea of how it works, in watered-down form. To each object x, we associate the set $S(x)$ of all morphisms *to* x from all other objects. This set has a certain structure which one can work out, and a morphism $f\colon x \to y$ gives rise to a structure-preserving function $S(f)$ from $S(x)$ to $S(y)$ in the obvious way: given a morphism from something to x, just compose it with f to get a morphism to y.
>
> So we have taken our original category and embedded in a category of "sets with structure."

Definition 3.7.4 (Yoneda embedding)**.** Let C, D be locally small categories. The Yoneda embedding is the functor

$$\mathcal{Y}\colon \mathsf{C} \rightsquigarrow [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}], \qquad A \mapsto Y_A = \mathrm{Hom}_{\mathsf{C}}(-, A).$$

where the notation $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$, defined in Definition 3.3.5, is the category of functors $\mathsf{C}^{\mathrm{op}} \rightsquigarrow \mathsf{Set}$.

Of course, we also need to say how $\mathcal{Y}$ behaves on morphisms. Let $f\colon A \to A'$. Then $\mathcal{Y}(f)$ will be a morphism from Y_A to $Y_{A'}$, i.e. a natural transformation $Y_A \Rightarrow Y_{A'}$. We can specify how it behaves by specifying its components $(\mathcal{Y}(f))_B$.

Plugging in definitions, we find that $(\mathcal{Y}(f))_B$ is a map $\mathrm{Hom}_{\mathsf{C}}(B, A) \to \mathrm{Hom}_{\mathsf{C}}(B, A')$. But we know how to get such a map—we can compose all of the maps $B \to A$ with f to get maps $B \to A'$. So $\mathcal{Y}(f)$, as a natural transformation $\mathrm{Hom}_{\mathsf{C}}(-, A) \to \mathrm{Hom}_{\mathsf{C}}(-, A')$, simply composes everything in sight with f.

Theorem 3.7.1 (Yoneda lemma)**.** *Let $\mathcal{F}$ be a functor from C^{op} to Set. Let $A \in \mathrm{Obj}(\mathsf{C})$. Then there is a set-isomorphism (i.e. a bijection) η between the set $\mathrm{Nat}(Y_A, \mathcal{F})$ of natural transformations $Y_A \Rightarrow \mathcal{F}$ and the set $\mathcal{F}(A)$. Furthermore, this isomorphism is natural in A.*

Proof. A natural transformation $\Phi\colon Y_A \Rightarrow \mathcal{F}$ consists of a collection of Set-morphisms (that is to say, functions) $\Phi_B\colon \mathrm{Hom}_{\mathsf{C}}(B, A) \to \mathcal{F}(A)$, one for each $B \in \mathrm{Obj}(\mathsf{C})$. We need to show that to each element of $\mathcal{F}(A)$ there corresponds exactly one Φ. We will do this by showing that any Φ is completely determined by where Φ_A sends the identity morphism $1_A \in \mathrm{Hom}_{\mathsf{C}}(A, A)$, so there is exactly one natural transformation for each place Φ can send 1_A.

The proof that this is the case can be illustrated by the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[rrr, "Y_A(f)"] \arrow[ddd, "\Phi_A"'] & & & \mathrm{Hom}(B,A) \arrow[ddd, "\Phi_B"] \\
& 1_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& a \arrow[r, mapsto] & (\mathcal{F}f)(a) \overset{!}{=} \Phi_B(f) & \\
\mathcal{F}(A) \arrow[rrr, "\mathcal{F}(f)"'] & & & \mathcal{F}(B)
\end{tikzcd}$$

The natural transformation $\Phi\colon Y_A \Rightarrow \mathcal{F}$ has a component $\Phi_A\colon \mathrm{Hom}_{\mathsf{C}}(A,A) \to \mathcal{F}(A)$, and Φ_A has to send the identity transformation 1_A *somewhere*. It can send it to any element of $\mathcal{F}(A)$; let's call $\Phi_A(1_A) = a$.

Now the naturality conditions force our hand. For any $B \in \mathrm{Obj}(\mathsf{C})$ and any $f\colon B \to A$, the naturality square above means that $\Phi_B(f)$ *has to* be equal to $(\mathcal{F}f)(a)$. We get no choice in the matter.

But this completely determines Φ! So we have shown that there is exactly one natural transformation for every element of $\mathcal{F}(A)$. We are done!

Well, almost. We still have to show that the bijection we constructed above is natural. To that end, let $f\colon B \to A$. We need to show that the following square commutes.

$$\begin{tikzcd}
\mathrm{Nat}(Y_A, \mathcal{F}) \arrow[r, "\eta_A"] \arrow[d, "\mathrm{Nat}(\mathcal{Y}(f),\mathcal{F})"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(f)"] \\
\mathrm{Nat}(Y_B, \mathcal{F}) \arrow[r, "\eta_B"] & \mathcal{F}(B)
\end{tikzcd}$$

This deserves a lot of explanation. The natural transformation $\mathrm{Nat}(\mathcal{Y}(f), \mathcal{F})$ looks complicated, but it's really not. It takes every natural transformation $Y_A \Rightarrow \mathcal{F}$ and pre-composes it with $\mathcal{Y}(f)$. The natural transformation η_A takes a natural transformation $\Phi\colon Y_A \Rightarrow \mathcal{F}$ and sends it to the element $\Phi_A(1_A) \in \mathcal{F}(A)$.

So, starting at the top left with a natural transformation $\Phi\colon Y_A \Rightarrow \mathcal{F}$, we can go to the bottom right in two ways.

1. We can head down to $\mathrm{Nat}(Y_B, \mathcal{F})$, mapping
$$\Phi \mapsto \Phi \circ \mathcal{Y}(f),$$
then use η_B to map this to $\mathcal{F}(B)$:
$$\Phi \circ \mathcal{Y}(f) \mapsto (\Phi \circ \mathcal{Y}(f))_B(1_B).$$
We can simplify this right away. The component of the composition of natural transformations is the composition of the components, i.e.
$$(\Phi \circ \mathcal{Y}(f))(1_B) = (\Phi_B \circ \mathcal{Y}(f)_B)(1_B).$$
We also know how $\mathcal{Y}(f)$ behaves: it composes everything in sight with f.
$$\mathcal{Y}(f)(1_B) = f.$$
Thus, we have
$$(\Phi \circ \mathcal{Y}(f))(1_B) = \Phi_B(f).$$

2. We can first head to the right using η_A. This sends Φ to
$$\Phi_A(1_A) \in \mathcal{F}(A).$$
We can then map this to $\mathcal{F}(B)$ with f, getting
$$\mathcal{F}(f)(\Phi_A(1_A)).$$

The naturality condition is thus

$$(\mathcal{F}(f))(\Phi_A(1_A)) = \Phi_B(f),$$

which we saw above was true for any natural transformation Φ. ☐

Lemma 3.7.1. *The Yoneda embedding is fully faithful (Definition 3.2.2).*

Proof. We have to show that for all A, $B \in \mathrm{Obj}(\mathsf{C})$, the map

$$\mathcal{Y}_{A,B} \colon \mathrm{Hom}_{\mathsf{C}}(A,B) \to \mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(Y_A, Y_B)$$

is a bijection.

Fix $B \in \mathrm{Obj}(\mathsf{C})$, and consider the functor Y_B. By the Yoneda lemma, there is a bijection between $\mathrm{Nat}(Y_A, Y_B)$ and $Y_B(A)$. By definition,

$$Y_B(A) = \mathrm{Hom}_{\mathsf{C}}(A,B).$$

But $\mathrm{Nat}(Y_A, Y_B)$ is nothing else but $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(Y_A, Y_B)$, so we are done. ☐

Note 3.7.2. We defined the Yoneda embedding to be the functor $A \mapsto Y_A$; this is sometimes called the *contravariant Yoneda embedding.* We could also have studied to map A to Y^A, called the *covariant Yoneda embedding*, in which case a slight modification of the proof of the Yoneda lemma would have told us that the map

$$\mathrm{Hom}_{\mathsf{C}}(B,A) \to \mathrm{Hom}_{[\mathsf{C},\mathsf{Set}]}(Y^A, Y^B)$$

is a natural bijection. This is often called the *covariant Yoneda lemma.*

Here is a situation in which the Yoneda lemma is commonly used.

Corollary 3.7.1. *Let* C *be a locally small category. Suppose for all* $A \in \mathrm{Obj}(\mathsf{C})$ *there is a bijection*

$$\mathrm{Hom}_{\mathsf{C}}(A,B) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(A,B')$$

which is natural in A. *Then* $B \simeq B'$

Proof. We have a natural isomorphism $Y_B \Rightarrow Y_{B'}$. Since the Yoneda embedding is fully faithful, it is injective on objects up to isomorphism (Lemma 3.2.1). Thus, since Y_B and $Y_{B'}$ are isomorphic, so must be $B \simeq B'$. ☐

3.7.4 Applications

We have now built up enough machinery to make the proofs of a great variety of things trivial, as long as we accept a few trivial assertions.

It is possible, for example, to prove that the product is associative in *any* locally small category, just from the fact that it is in Set.

Lemma 3.7.2. *There is a natural isomorphism* α *between* $(A \times B) \times C$ *and* $A \times (B \times C)$ *for any sets* A, B, *and* C.

Proof. The isomorphism is given by $\alpha_{A,B,C} \colon ((a,b),c) \mapsto (a,(b,c))$. This is natural because for functions like this,

$$A \xrightarrow{f} A'$$
$$B \xrightarrow{g} B'$$
$$C \xrightarrow{h} C'$$

the following diagram commutes.

$$\begin{tikzcd}
((a,b),c) \arrow[r, mapsto, "((f,g),h)"] \arrow[d, mapsto, "\alpha_{A,B,C}"'] & ((f(a),g(b)),h(c)) \arrow[d, mapsto, "\alpha_{A',B',C'}"] \\
(a,(b,c)) \arrow[r, mapsto, "(f,(g,h))"'] & (f(a),(g(b),h(c)))
\end{tikzcd}$$

☐

Theorem 3.7.2. *In any locally small category* C *with products, there is a natural isomorphism* $(A \times B) \times C \simeq A \times (B \times C)$.

Proof. We have the following string of natural isomorphisms for any X, A, B, $C \in \mathrm{Obj}(\mathsf{C})$.

$$\begin{aligned}\mathrm{Hom}_{\mathsf{C}}(X, (A \times B) \times C) &\simeq \mathrm{Hom}_{\mathsf{C}}(X, A \times B) \times \mathrm{Hom}_{\mathsf{C}}(X, C)\\ &\simeq (\mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B)) \times \mathrm{Hom}_{\mathsf{C}}(X, C)\\ &\simeq \mathrm{Hom}_{\mathsf{C}}(X, A) \times (\mathrm{Hom}_{\mathsf{C}}(X, B) \times \mathrm{Hom}_{\mathsf{C}}(X, C))\\ &\simeq \mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B \times C)\\ &\simeq \mathrm{Hom}_{\mathsf{C}}(X, A \times (B \times C)).\end{aligned}$$

Thus, by Corollary 3.7.1, $(A \times B) \times C \simeq A \times (B \times C)$. □

Note 3.7.3. By induction, all finite products are associative.

Theorem 3.7.3. *In any category with products* $\times$, *coproducts* $+$, *initial object* 0, *final object* 1, *and exponentials, we have the following natural isomorphisms.*

1. $A \times (B + C) \simeq (A \times B) + (A \times C)$
2. $C^{A+B} \simeq C^A \times C^B$.
3. $(C^A)^B \simeq C^{A \times B}$
4. $(A \times B)^C \simeq A^C \times B^C$
5. $C^0 \simeq 1$
6. $C^1 \simeq C$

Proof.

1. We have the following list of natural isomorphisms.

$$\begin{aligned}\mathrm{Hom}_{\mathsf{C}}(A \times (B + C), X) &\simeq \mathrm{Hom}_{\mathsf{C}}(B + C, X^A)\\ &\simeq \mathrm{Hom}_{\mathsf{C}}(B, X^A) \times \mathrm{Hom}_{\mathsf{C}}(C, X^A)\\ &\simeq \mathrm{Hom}_{\mathsf{C}}(B \times A, X) \times \mathrm{Hom}_{\mathsf{C}}(C \times A, X)\\ &\simeq \mathrm{Hom}_{\mathsf{C}}((B \times A) + (C \times A), X).\end{aligned}$$

2. We have the following list of natural isomorphisms.

$$\begin{aligned}\mathrm{Hom}(X, C^{A+B}) &\simeq \mathrm{Hom}(X \times (A + B), C)\\ &\simeq \mathrm{Hom}_{\mathsf{C}}((X \times A) + (X \times B), C)\\ &\simeq \mathrm{Hom}_{\mathsf{C}}(X \times A, C) \times \mathrm{Hom}_{\mathsf{C}}(X \times B, C)\\ &\simeq \mathrm{Hom}_{\mathsf{C}}(X, C^A) \times \mathrm{Hom}_{\mathsf{C}}(X, C^B)\\ &\simeq \mathrm{Hom}_{\mathsf{C}}(X, C^A \times C^B).\end{aligned}$$

Etc. □

3.8 Limits

3.8.1 Limits and colimits

As we have seen, one often gets categorical concepts by 'categorifying' concepts from set theory. One example of this is the notion of a *diagram*, which is the categorical generalization of an indexed family.

Definition 3.8.1 (indexed family)**.** Let J and X be sets. A family of elements in X indexed by J is a function

$$x\colon J \to X; \qquad j \mapsto x_j.$$

To categorify this, one considers a functor from one category J, called the *index category*, to another category C. Using a functor from a category instead of a function from a set allows us to index the morphisms as well as the objects.

Definition 3.8.2 (diagram)**.** Let J and C be categories. A diagram of type J in C is a covariant functor

$$\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}.$$

One thinks of the functor $\mathcal{D}$ embedding the index category J into C.

Definition 3.8.3 (cone)**.** Let C be a category, J an index category, and $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$ be a diagram. Let 1 be the category with one object and one morphism (Example 3.1.2) and $\mathcal{F}_X$ the functor $1 \rightsquigarrow \mathsf{C}$ which picks out $X \in \mathrm{Obj}(\mathsf{C})$ (see Example 3.2.1). Let $\mathcal{K}$ be the unique functor $\mathsf{J} \rightsquigarrow 1$.

$$\begin{tikzcd}
1 \arrow[dr, squiggly, "\mathcal{F}_X"] & \\
\mathsf{J} \arrow[u, squiggly, "\mathcal{K}"] \arrow[r, squiggly, "\mathcal{D}"] & \mathsf{C}
\end{tikzcd}$$

A cone of shape J from X is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with a natural transformation

$$\varepsilon\colon \mathcal{F}_X \circ \mathcal{K} \Rightarrow \mathcal{D}.$$

That is to say, a cone to J is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with a family of morphisms $\Phi_A\colon X \to \mathcal{F}(A)$ (one for each $A \in \mathrm{Obj}(\mathsf{J})$) such that for all A, $B \in \mathrm{Obj}(\mathsf{J})$ and all $f\colon A \to B$ the following diagram commutes.

$$\begin{tikzcd}
& X \arrow[dl, "\Phi_A"'] \arrow[dr, "\Phi_B"] & \\
\mathcal{F}(A) \arrow[rr, "\mathcal{F}(f)"] & & \mathcal{F}(B)
\end{tikzcd}$$

Note 3.8.1. Here is an alternate definition. Let Δ be the functor $\mathsf{C} \rightsquigarrow [\mathsf{J}, \mathsf{C}]$ (the category of functors $\mathsf{J} \rightsquigarrow \mathsf{C}$, see TODO) which assigns to each object $X \in \mathrm{Obj}(\mathsf{C})$ the constant functor $\Delta_X : \mathsf{J} \rightsquigarrow \mathsf{C}$, i.e. the functor which maps every object of J to X and every morphism to 1_X. A cone over $\mathcal{D}$ is then an object in the comma category (Definition 3.4.1) $(\Delta \downarrow \mathcal{D})$ given by the diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, "\Delta"] & {[\mathsf{J}, \mathsf{C}]} & 1 \arrow[l, squiggly, "\mathcal{F}_D"']
\end{tikzcd} .$$

The objects of this category are pairs (X, f), where $X \in \mathrm{Obj}(\mathsf{C})$ and $f\colon \Delta(X) \to \mathcal{D}$; that is to say, f is a natural transformation $\Delta_X \Rightarrow \mathcal{D}$.

This allows us to make the following definition.

Definition 3.8.4 (category of cones over a diagram)**.** Let C be a category, J an index category, and $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ a diagram. The category of cones over $\mathcal{D}$ is the category $(\Delta \downarrow \mathcal{D})$.

Note 3.8.2. The alternate definition given in Note 3.8.1 not only reiterates what cones look like, but even prescribes what morphisms between cones look like. Let (X, Φ) be a cone over a diagram $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$, i.e.

- an object in the category $(\Delta \downarrow \mathcal{D})$, i.e.
- a pair (X, Φ), where $X \in \mathrm{Obj}(\mathsf{C})$ and $\Phi\colon \Delta_X \Rightarrow \mathcal{D}$ is a natural transformation, i.e.
- for each $J \in \mathrm{Obj}(\mathsf{J})$ a morphism $\Phi_J\colon X \to \mathcal{D}(J)$ such that for any other object $J' \in \mathrm{Obj}(\mathsf{J})$ and any morphism $f\colon J \to J'$ the following diagram commutes.

$$\begin{tikzcd}
& X \arrow[dl, "\Phi_J"'] \arrow[dr, "\Phi_{J'}"] & \\
\mathcal{D}(J) \arrow[rr, "\mathcal{D}(f)"] & & \mathcal{D}(J')
\end{tikzcd}$$

This agrees with our previous definition of a cone.

Let (Y, Γ) be another cone over $\mathcal{D}$. Then a morphism $\Xi\colon (X, \Phi) \to (Y, \Gamma)$ is

- a morphism $\Xi \in \mathrm{Hom}_{(\Delta \downarrow \mathcal{D})}((X, \Phi), (Y, \Gamma))$, i.e.
- a natural transformation $\Xi\colon \Delta_X \Rightarrow \Delta_Y$ (i.e. a morphism $\Delta_X \to \Delta_Y$ in the category $[\mathsf{J}, \mathsf{C}]$) such that the diagram

$$\begin{tikzcd}
\Delta_X \arrow[rr, Rightarrow, "\Xi"] \arrow[dr, Rightarrow, "\Phi"] & & \Delta_Y \arrow[dl, Rightarrow, "\Gamma"'] \\
& \mathcal{D} &
\end{tikzcd}$$

  commutes, i.e.
- a morphism $\xi\colon X \to Y$ such that for each $J \in \mathrm{Obj}(\mathsf{J})$, the diagram

$$\begin{tikzcd}
X \arrow[rr, "\xi"] \arrow[dr, "\Phi_J"] & & Y \arrow[dl, "\Gamma_J"'] \\
& \mathcal{D}(J) &
\end{tikzcd}$$

  commutes.

Cocones are the dual notion to cones. We make the following definition.

Definition 3.8.5 (cocone)**.** A cocone over a diagram $\mathcal{D}$ is an object in the comma category $(\mathcal{D} \downarrow \Delta)$.

Definition 3.8.6 (category of cocones)**.** The category of cocones over a diagram $\mathcal{D}$ is the category $(\mathcal{D} \downarrow \Delta)$.

The categorical definitions of cones and cocones allow us to define limits and colimits succinctly.

Definition 3.8.7 (limits, colimits)**.** A limit of a diagram $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$ is a final object in the category $(\Delta \downarrow \mathcal{D})$. A colimit is an initial object in the category $(\mathcal{D} \downarrow \Delta)$.

Note 3.8.3. The above definition of a limit unwraps as follows. The limit of a diagram $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$ is a cone (X, Φ) over $\mathcal{D}$ such that for any other cone (Y, Γ) over $\mathcal{D}$, there is a unique map $\xi\colon Y \to X$ such that for each $J \in \mathrm{Obj}(\mathsf{J})$, the following diagram commutes.

$$\begin{tikzcd}
Y \arrow[rr, "\xi"] \arrow[dr, "\Gamma_J"'] & & X \arrow[dl, "\Phi_J"] \\
& \mathcal{D}(J) &
\end{tikzcd}$$

Example 3.8.1. Here is a definition of the product $A \times B$ equivalent to that given in Example 3.5.5: it is the limit of the following somewhat trivial diagram.

$$\begin{tikzcd}
A \arrow[loop left, "1_A"] & B \arrow[loop right, "1_B"]
\end{tikzcd}$$

Let us unwrap this definition. We are saying that the product $A \times B$ is a cone over A and B

$$\begin{tikzcd}
& A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
A & & B
\end{tikzcd}$$

But not just any cone: a cone which is universal in the sense that any *other* cone factors through it uniquely.

$$\begin{tikzcd}
& X \arrow[dl, "f_1"'] \arrow[d, "\exists! f"] \arrow[dr, "f_2"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

In fact, this allows us to generalize the product: the product of n objects $\prod_{i=1}^{n} A_i$ is the limit over diagram consisting of all the A_i with no morphisms between them.

Definition 3.8.8 (equalizer)**.** Let J be the category with objects and morphisms as follows. (The necessary identity arrows are omitted.)

$$\begin{tikzcd} J \arrow[r, shift left, "1"] \arrow[r, shift right, "2"'] & J' \end{tikzcd}$$

A diagram $\mathcal{D}$ of shape J in some category C looks like the following.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

The equalizer of f and g is the limit of the diagram $\mathcal{D}$; that is to say, it is an object $\mathrm{eq} \in \mathrm{Obj}(\mathsf{C})$ and a morphism $e\colon \mathrm{eq} \to A$

$$\begin{tikzcd} \mathrm{eq} \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

such that for any *other* object Z and morphism $i\colon Z \to A$ such that $f \circ i = g \circ i$, there is a unique morphism $e\colon Z \to \mathrm{eq}$ making the following diagram commute.

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists! u"'] \arrow[dr, "i"] & & \\ \mathrm{eq} \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Example 3.8.2. What does it mean for the above diagram to commute in, say, Set? We need $(f \circ e)(x) = (g \circ e)(x)$ for all $x \in \mathrm{eq}$. That is, once we have been mapped by e into A, we need to be taken to the same place by f and g. The range of e must lie entirely within the set

3.8.2 Pullbacks and kernels

In what follows, C will be a category and A, B, etc. objects in $\mathrm{Obj}(\mathsf{C})$.

Definition 3.8.9 (pullback)**.** Let f, g be morphisms as follows.

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

A pullback of f along g (also called a pullback of g over f, sometimes notated $A \times_C B$) is a commuting square

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

such that for any other commuting square

$$\begin{tikzcd} V \arrow[r, "t"] \arrow[d, "s"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

there is a unique morphism $h\colon V \to U$ such that the diagram

$$\begin{tikzcd} V \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

Note 3.8.4. Here is another definition: the pullback $A \times_C B$ is the limit of the diagram

$$\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

This might at first seem odd; after all, don't we also need an arrow $A \times_C B \to C$? But this arrow is completely determined by the commutativity conditions, so it is superfluous.

Example 3.8.3. In Set, U is given (up to unique isomorphism) by

$$U = \left\{ (a, b) \in A \times B \,\middle|\, f(a) = g(b) \right\}.$$

The morphisms p and q are given by the projections $p(a, b) = a$, $q(a, b) = b$.

To see that this really does satisfy the universal property, consider any other set V and functions $s\colon V \to A$ and $t\colon V \to B$ making the above diagram commute. Then for all $v \in V$, $f(s(v)) = t(g(v))$.

Now consider the map $V \to U$ sending v to $(s(v), t(v))$. This certainly makes the above diagram commute; furthermore, any other map from V to U would not make the diagram commute. Thus U and h together satisfy the universal property.

Lemma 3.8.1. *Let $f\colon X \to Y$ be a monomorphism. Then any pullback of f is a monomorphism.*

Proof. Suppose we have the following pullback square.

$$\begin{tikzcd}
X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

Our aim is to show that p_2 is a monomorphism.

Suppose we are given an object Z and two morphisms α, $\beta\colon Z \to X'$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "\alpha"] \arrow[dr, shift right, "\beta"'] & & \\
& X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

We can compose α and β with p_2. Suppose that these agree, i.e.

$$p_2 \circ \alpha = p_2 \circ \beta.$$

We will be done if we can show that this implies that $p_1 = p_2$.

We can compose with g to find that

$$g \circ p_2 \circ \alpha = g \circ p_2 \circ \beta.$$

but since the pullback square commutes, we can replace $g \circ p_2$ by $f \circ p_1$.

$$f \circ p_1 \circ \alpha = f \circ p_1 \circ \beta.$$

since f is a monomorphism, this implies that

$$p_1 \circ \alpha = p_1 \circ \beta.$$

Now forget α and β for a minute. we have constructed a commuting square as follows.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "p_1\circ\alpha=p_1\circ\beta"] \arrow[ddr, bend right, "p_2\circ\alpha=p_2\circ\beta"'] & & \\
& X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

the universal property for pullbacks tells us that there is a unique morphism $Z \to X'$ making the diagram commute. But either α or β will do! So $\alpha = \beta$. $\square$

Definition 3.8.10 (kernel of a morphism)**.** Let C be a category with an initial object (Definition 3.5.1) 0 and pullbacks. The kernel $\ker(f)$ of a morphism $f\colon A \to B$ is the pullback along f of the unique morphism $0 \to B$.

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

That is to say, the kernel of f is a pair $(\ker(f), \iota)$, where $\ker(f) \in \mathrm{Obj}(\mathsf{C})$ and $\iota\colon \ker(f) \to A$ which satisfies the above universal property.

Note 3.8.5. Although the kernel of a morphism f is a pair $(\ker(f), \iota)$ as described above, we will sometimes sloppily say that the object $\ker(f)$ is the kernel of f, especially when the the morphism ι is obvious or understood. Such abuses of terminology are common; one occasionally even sees the morphism ι being called the kernel of f.

Example 3.8.4. In Vect_k, the initial object is the zero vector space $\{0\}$. For any vector spaces V and W and any linear map $f\colon V \to W$, the kernel of f is the pair $(\ker(f), \iota)$ where $\ker(f)$ is the vector space

$$\ker(f) = \left\{ v \in V \,\middle|\, f(v) = 0 \right\}$$

and ι is the obvious injection $\ker(f) \to V$.

Lemma 3.8.2. *Let $f\colon A \to B$, and let $(\iota, \ker(f))$ be the kernel of f. Then ι is a monomorphism (Definition 3.1.7).*

Proof. Suppose we have an object $Z \in \mathrm{Obj}(\mathsf{C})$ and two morphisms g_1, $g_2\colon Z \to \ker(f)$. We have the following diagram.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "g_1"] \arrow[dr, shift right, "g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Further suppose that $\iota \circ g_1 = \iota \circ g_2$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "g_1"] \arrow[dr, shift right, "g_2"'] \arrow[drr, bend left] \arrow[ddr, bend right, "\iota\circ g_1=\iota\circ g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Now pretend that we don't know about g_1 and g_2.

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[ddr, bend right, "\iota\circ g_1=\iota\circ g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

The universal property for kernels tells us that there is a unique map $Z \to \ker(f)$ making the above diagram commute. But since g_1 and g_2 both make the diagram commute, g_1 and g_2 must be the same map, i.e. $g_1 = g_2$. $\square$

3.8.3 Pushouts and cokernels

Pushouts are the dual notion to pullbacks.

Definition 3.8.11 (pushouts)**.** Let f, g be morphisms as follows.

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \\ A & \end{tikzcd}$$

The <u>pushout of f along g</u> (or g along f) is a commuting square

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \\ A \arrow[r, "p"'] & U \end{tikzcd}$$

such that for any other commuting square

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "t"] \\ A \arrow[r, "s"'] & V \end{tikzcd}$$

there exists a unique morphism $h\colon U \to V$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\ A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\ & & V \end{tikzcd}$$

commutes.

Note 3.8.6. As with pullbacks, we can also define a pushout as the colimit of the following diagram.

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \\ A & \end{tikzcd}$$

Example 3.8.5. Let us construct the pushout in Set. If we ignore the object C and the morphisms f and g, we discover that U must satisfy the universal property of the coproduct of A and B.

$$\begin{tikzcd} & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\ A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\ & & V \end{tikzcd}$$

Let us therefore make the ansatz that $U = A \amalg B = A \sqcup B$ and see what happens when we add C, f, and g back in.

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\ A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\ & & V \end{tikzcd}$$

In doing so, we find that the square A-C-B-U must also commute, i.e. we must have that $(q \circ g)(c) = (p \circ f)(c)$ for all $c \in C$. Since p and q are just inclusions, we see that

$$U = A \amalg B / \sim,$$

where $\sim$ is the equivalence relation generated by the relations $f(c) \sim g(c)$ for all $c \in C$.

Definition 3.8.12 (cokernel of a morphism)**.** Let C be a category with terminal object 1. The cokernel of a morphism $f\colon A \to B$ is the pushout of f along the unique morphism $A \to 1$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "\pi"] \\ 1 \arrow[r] & \operatorname{coker}(f) \end{tikzcd}$$

Example 3.8.6. In Vect_k, the terminal object is the vector space $\{0\}$. If V and W are k-vector spaces and f is a linear map $V \to W$, then $\operatorname{coker}(f)$ is

$$W / \sim,$$

where $\sim$ is the relation generated by $f(v) \sim 0$ for all $v \in V$. But this relation is exactly the one which mods out by $\operatorname{im}(f)$, so $\operatorname{coker}(f) = W/\operatorname{im}(f)$.

Lemma 3.8.3. *For any morphism $f\colon A \to B$, the canonical projection $\pi\colon B \to \operatorname{coker}(f)$ is an epimorphism.*

Proof. The proof is dual to the proof that the canonical injection ι is mono (Lemma 3.8.2). □

Definition 3.8.13 (normal monomorphism)**.** A monomorphism (Definition 3.1.7) $f\colon A \to B$ is normal if it the kernel of some morphism. To put it more plainly, f is normal if there exists an object C and a morphism $g\colon B \to C$ such that (A, f) is the kernel of g.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

Example 3.8.7. In Vect_k, monomorphisms are injective linear maps (Example 3.1.6). If f is injective then sequence

$$\begin{tikzcd} \{0\} \arrow[r] & V \arrow[r, "f"] & W \arrow[r, "\pi"] & W/\operatorname{im}(f), \arrow[r] & \{0\} \end{tikzcd}$$

is exact, and we always have that $\operatorname{im}(f) = \ker(\pi)$. Thus in Vect_k, every monomorphism is normal.

Definition 3.8.14 (conormal epimorphism)**.** An epimorphism $f\colon A \to B$ is conormal if it is the cokernel of some morphism. That is to say, if there exists an object C and a morphism $g\colon C \to A$ such that (B, f) is the cokernel of g.

Example 3.8.8. In Vect_k, epimorphisms are surjective linear maps. If $f\colon V \to W$ is a surjective linear map, then the sequence

$$\begin{tikzcd} \{0\} \arrow[r] & \ker(f) \arrow[r, "\iota"] & V \arrow[r, "f"] & W \arrow[r] & \{0\} \end{tikzcd}$$

is exact. But then $\operatorname{im}(\iota) = \ker(f)$, so f is conormal. Thus in Vect_k, every epimorphism is conormal.

Note 3.8.7. To show that in our proofs that in Vect_k monomorphisms were normal and epimorphisms were conormal, we showed that monomorphisms were the kernels of their cokernels, and epimorphisms were the cokernels of their kernels. This will be a general feature of Abelian categories.

Definition 3.8.15 (binormal category)**.** A category is binormal if all monomorphisms are normal and all epimorphisms are conormal.

Example 3.8.9. As we have seen, Vect_k is binormal.

3.8.4 A necessary and sufficient condition for the existence of finite limits

In this section we prove a simple criterion to check that a category has all finite limits (i.e. limits over diagrams with a finite number of objects and a finite number of morphisms). The idea is as follows.

In general, adding more objects to a diagram makes the limit over it larger, and adding more morphisms makes the limit smaller. In the extreme case in which the only morphisms are the identity morphisms, the limit is simply the categorical product. As we add morphisms, roughly speaking, we have to get rid of the parts of the product which are preventing the necessary triangles from commuting. Thus, to make the universal cone over a diagram, we can start with the product, and cut out the bare minimum we need to make everything commute: the way to do that is with an equalizer.

The following proof was adapted from [25].

Theorem 3.8.1. *Let C be a category. The following are equivalent.*

1. *C has all finite limits.*
2. *C has all finite products and equalizers.*
3. *C has all pullbacks and a terminal object.*

Proof. Since finite products, equalizers, pullbacks, and terminal objects are all instances of finite limits (the terminal object is the limit of the empty diagram), 1 clearly implies 2 and 3.

Next we show that $2 \implies 1$. Assume 2, i.e. suppose that C has all finite products and equalizers.

Let $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$ be a finite diagram. We want to prove that $\mathcal{D}$ has a limit; we will do this by constructing a universal cone over it, i.e.

- an object $L \in \mathrm{Obj}(\mathsf{C})$, and
- for each $j \in \mathrm{Obj}(\mathsf{J})$, a morphism $P_j\colon L \to \mathcal{D}(j)$

such that

1. for any i, $j \in \mathrm{Obj}(\mathsf{J})$ and any $\alpha\colon i \to j$ the following diagram commutes,

$$\begin{tikzcd} & L \arrow[dl, "P_i"'] \arrow[dr, "P_j"] & \\ \mathcal{D}(i) \arrow[rr, "\mathcal{D}(\alpha)"] & & \mathcal{D}(j) \end{tikzcd}$$

   and

2. for any other object $L' \in \mathrm{Obj}(\mathsf{C})$ and family of morphisms $Q_j\colon L' \to \mathcal{D}(j)$ which make the diagrams

$$\begin{tikzcd} & L' \arrow[dl, "Q_i"'] \arrow[dr, "Q_j"] & \\ \mathcal{D}(i) \arrow[rr, "\mathcal{D}(\alpha)"] & & \mathcal{D}(j) \end{tikzcd}$$

   commute for all i, j, and α, there is a unique morphism $f\colon L' \to L$ such that $Q_j = P_j \circ f$ for all $j \in \mathrm{Obj}(\mathsf{J})$.

Denote by $\mathrm{Mor}(\mathsf{J})$ the set of all morphisms in J. For any $\alpha \in \mathrm{Hom}_{\mathsf{J}}(i,j) \subseteq \mathrm{Mor}(\mathsf{J})$, let $\mathrm{dom}(\alpha) = i$ and $\mathrm{cod}(\alpha) = j$.

Consider the following finite products:

$$A = \prod_{j \in \mathrm{Obj}(\mathsf{J})} \mathcal{D}(j) \qquad \text{and} \qquad B = \prod_{\alpha \in \mathrm{Mor}(\mathsf{J})} \mathcal{D}(\mathrm{cod}(\alpha)).$$

From the universal property for products, we know that we can construct a morphism $f\colon A \to B$ by specifying a family of morphisms $f_\alpha\colon A \to \mathcal{D}(\mathrm{cod}(\alpha))$, one for each $\alpha \in \mathrm{Mor}(\mathsf{J})$. We will define two morphisms R, $S\colon A \to B$ in this way:

$$R_\alpha = \pi_{\mathcal{D}(\mathrm{cod}(\alpha))}; \qquad S_\alpha = \mathcal{D}(\alpha) \circ \pi_{\mathcal{D}(\mathrm{dom}(\alpha))}.$$

Now let $e\colon L \to A$ be the equalizer of R and S (we are guaranteed the existence of this equalizer by assumption). Further, define $P_j\colon L \to \mathcal{D}(j)$ by

$$P_j = \pi_{\mathcal{D}(j)} \circ e$$

for all $j \in \mathrm{Obj}(\mathsf{J})$.

The claim is that L together with the P_j is the limit of $\mathcal{D}$. We need to verify conditions 1 and 2 on L and P_j listed above.

1. We need to show that for all i, $j \in \mathrm{Obj}(\mathsf{J})$ and all $\alpha\colon i \to j$, we have the equality $\mathcal{D}(\alpha) \circ P_i = P_j$. Now, for every $\alpha\colon i \to j$ we have

$$\begin{aligned} \mathcal{D}(\alpha) \circ P_i &= \mathcal{D}(\alpha) \circ \pi_{\mathcal{D}(i)} \circ e \\ &= S_\alpha \circ e \\ &= R_\alpha \circ e \\ &= \pi_{\mathcal{D}(j)} \circ e \\ &= P_j. \end{aligned}$$

2. We need to show that for any other $L' \in \mathrm{Obj}(\mathsf{C})$ and any other family of morphisms $Q_j\colon L' \to \mathcal{D}(j)$ such that for all $\alpha\colon i \to j$, $Q_j = \mathcal{D}(\alpha) \circ Q_i$, there is a unique morphism $h\colon L' \to L$ such that $Q_j = P_j \circ h$ for all $j \in \mathrm{Obj}(\mathsf{J})$. Suppose we are given such an L' and Q_j.

   The universal property for products allows us to construct from the family of morphisms Q_j a morphism $Q\colon L' \to A$ such that $Q_j = \pi_{\mathcal{D}(j)} \circ Q$. Now, for any $\alpha\colon i \to j$,

$$\begin{aligned} R_\alpha \circ Q &= \pi_{\mathcal{D}(j)} \circ Q \\ &= Q_j \\ &= \mathcal{D}(\alpha) \circ Q_i \\ &= \mathcal{D}(\alpha) \circ \pi_i \circ Q \\ &= S_\alpha \circ Q. \end{aligned}$$

   Thus, $Q\colon L' \to A$ equalizes R and S. But the universal property for equalizers guarantees us a unique morphism $h\colon L' \to L$ such that $Q = P \circ h$. We can compose both sides of this equation on the left with $\pi_{\mathcal{D}(j)}$ to find

$$\pi_{\mathcal{D}(j)} \circ Q = \pi_{\mathcal{D}(j)} \circ P \circ h,$$

   i.e.

$$Q_j = P_j \circ h$$

   as required.

It remains only to show that 3 $\implies$ 1, i.e. that if C has pullbacks and a terminal object 1, then it has all finite limits. We do this by showing that □

Note 3.8.8. The meat of this theorem is the fact that finite products and equalizers yield all finite limits.

3.9 Adjunctions

Consider the following functors:

- $\mathcal{U}\colon \mathsf{Grp} \rightsquigarrow \mathsf{Set}$, which sends a group to its underlying set, and

- $\mathcal{F}\colon \mathsf{Set} \rightsquigarrow \mathsf{Grp}$, which sends a set to the free group on it.

The functors $\mathcal{U}$ and $\mathcal{F}$ are dual in the following sense: $\mathcal{U}$ is the most efficient way of moving from Grp to Set since all groups are in particular sets; $\mathcal{F}$ might be thought of as providing the most efficient way of moving from Set to Grp. But how would one go about formalizing this?

Well, these functors have the following property. Let S be a set, G be a group, and let $f\colon S \to \mathcal{U}(G)$, $s \mapsto f(s)$ be a set-function. Then there is an associated group homomorphism $\tilde{f}\colon \mathcal{F}(S) \to G$, which sends $s_1 s_2 \ldots s_n \mapsto f(s_1 s_2 \ldots s_n) = f(s_1) \cdots f(s_n)$. In fact, $\tilde{f}$ is the unique homomorphism $\mathcal{F}(S) \to G$ such that $f(s) = \tilde{f}(s)$ for all $s \in S$.

Similarly, for every group homomorphism $g\colon \mathcal{F}(S) \to G$, there is an associated function $S \to \mathcal{U}(G)$ given by restricting g to S. In fact, this is the unique function $\mathcal{F}(S) \to G$ such that $f(s) = \tilde{f}(s)$ for all $s \in S$.

Thus for each $f \in \mathrm{Hom}_{\mathsf{Grp}}(S, \mathcal{U}(G))$ we can construct an $\tilde{f} \in \mathrm{Hom}_{\mathsf{Set}}(\mathcal{F}(S), G)$, and vice versa.

Let us add some mathematical scaffolding to the ideas explored above. We build two functors $\mathsf{Set}^{\mathrm{op}} \times \mathsf{Grp} \rightsquigarrow \mathsf{Set}$ as follows.

1. Our first functor maps the object $(S, G) \in \mathrm{Obj}(\mathsf{Set}^{\mathrm{op}} \times \mathsf{Grp})$ to the hom-set $\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G)$, and a morphism $(\alpha, \beta)\colon (S, G) \to (S', G')$ to a function
$$\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S'), G'); \qquad m \mapsto \mathcal{F}(\alpha) \circ m \circ \beta$$
2. Our second functor maps (S, G) to $\mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G))$, and (α, β) to
$$m \mapsto \alpha \circ m \circ \mathcal{U}(\beta).$$

We can define a natural isomorphism Φ between these functors with components
$$\Phi_{S,G}\colon \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G)); \qquad f \to \tilde{f}.$$

This mathematical structure turns out to be a recurring theme in the study of categories, called an *adjuction.* We have encountered it before: we saw in Example 3.7.1 that there was a natural bijection between
$$\mathrm{Hom}_{\mathsf{C}}(X \times (-), B) \qquad \text{and} \qquad \mathrm{Hom}_{\mathsf{C}}(X, B^{(-)}).$$

Definition 3.9.1 (hom-set adjunction)**.** Let C, D be categories and $\mathcal{F}$, $\mathcal{G}$ functors as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, squiggly, "\mathcal{G}"]
\end{tikzcd}$$

We say that $\mathcal{F}$ is left-adjoint to $\mathcal{G}$ (or equivalently $\mathcal{G}$ is right-adjoint to $\mathcal{F}$) and write $\mathcal{F} \dashv \mathcal{G}$ if there is a natural isomorphism
$$\Phi\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-), -) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(-, \mathcal{G}(-)),$$
which fits between $\mathcal{F}$ and $\mathcal{G}$ like this.

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \times \mathsf{D} \arrow[r, bend left=50, squiggly, "\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-){,}-)", ""{name=U, below}] \arrow[r, bend right=50, squiggly, "\mathrm{Hom}_{\mathsf{D}}(-{,}\mathcal{G}(-))"', ""{name=L}] & \mathsf{Set} \arrow[from=U, to=L, Rightarrow, "\Phi"]
\end{tikzcd}$$

The natural isomorphsim amounts to a family of bijections
$$\Phi_{A,B}\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \to \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$$
which satisfies the coherence conditions for a natural transformation.

Here are two equivalent definitions which are often used.

Definition 3.9.2 (unit-counit adjunction)**.** We say that two functors $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ form a unit-counit adjunction if there are two natural transformations

$$\eta\colon 1_{\mathsf{C}} \Rightarrow \mathcal{G}\circ\mathcal{F}, \qquad \text{and} \qquad \varepsilon\colon \mathcal{F}\circ\mathcal{G} \Rightarrow 1_{\mathsf{D}},$$

called the *unit* and *counit* respectively, which make the following so-called *triangle diagrams*

$$\begin{tikzcd} \mathcal{F} \arrow[r, Rightarrow, "\mathcal{F}\eta"] \arrow[dr, Rightarrow, "1_{\mathcal{F}}"'] & \mathcal{F}\mathcal{G}\mathcal{F} \arrow[d, Rightarrow, "\varepsilon\mathcal{F}"] \\ & \mathcal{F} \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{G} \arrow[r, Rightarrow, "\eta\mathcal{G}"] \arrow[dr, Rightarrow, "1_{\mathcal{G}}"'] & \mathcal{G}\mathcal{F}\mathcal{G} \arrow[d, Rightarrow, "\mathcal{G}\varepsilon"] \\ & \mathcal{G} \end{tikzcd}$$

commute.

The triangle diagrams take quite some explanation. The unit η is a natural transformation $1_{\mathsf{C}} \to \mathcal{G}\circ\mathcal{F}$. We can draw it like this.

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left=50, "1_{\mathsf{C}}"{name=U}] \arrow[r, squiggly, bend right=50, "\mathcal{G}\circ\mathcal{F}"'{name=L}] & \mathsf{C} \arrow[from=U, to=L, Rightarrow, "\eta"] \end{tikzcd} .$$

Analogously, we can draw ε like this.

$$\begin{tikzcd} \mathsf{D} \arrow[r, squiggly, bend left=50, "\mathcal{F}\circ\mathcal{G}"{name=U}] \arrow[r, squiggly, bend right=50, "1_{\mathsf{D}}"'{name=L}] & \mathsf{D} \arrow[from=U, to=L, Rightarrow, "\varepsilon"] \end{tikzcd} .$$

We can arrange these artfully like so. Notice that we haven't actually *done* anything; this diagram is just the diagrams for the unit and counit, plus some extraneous information.

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] \arrow[rr, squiggly, bend left=70, "1_{\mathsf{C}}"{name=A}] \arrow[rr, squiggly, bend left=30, "\mathcal{G}\circ\mathcal{F}"'{name=B}] & \mathsf{D} \arrow[r, squiggly, "\mathcal{G}"] \arrow[rr, squiggly, bend right=30, "\mathcal{F}\circ\mathcal{G}"{name=E}] \arrow[rr, squiggly, bend right=70, "1_{\mathsf{D}}"'{name=F}] & \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[from=A, to=B, Rightarrow, "\eta"'] \arrow[from=E, to=F, Rightarrow, "\varepsilon"] \end{tikzcd}$$

We can whisker the η on top from the right, and the ε below from the left, to get the following diagram,

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] \arrow[rrr, squiggly, bend left=70, "\mathcal{F}"{name=A}] \arrow[rrr, squiggly, bend left=30, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"'{name=B}] \arrow[rrr, squiggly, bend right=30, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"{name=E}] \arrow[rrr, squiggly, bend right=70, "\mathcal{F}"'{name=F}] & \mathsf{D} \arrow[r, squiggly, "\mathcal{G}"] & \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[from=A, to=B, Rightarrow, "\mathcal{F}\eta"'] \arrow[from=E, to=F, Rightarrow, "\varepsilon\mathcal{F}"] \end{tikzcd}$$

then consolidate to get this:

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left=70, "\mathcal{F}"{name=A}] \arrow[r, squiggly, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"{name=M}] \arrow[r, squiggly, bend right=70, "\mathcal{F}"'{name=F}] & \mathsf{D} \arrow[from=A, to=M, Rightarrow, "\mathcal{F}\eta"] \arrow[from=M, to=F, Rightarrow, "\varepsilon\mathcal{F}"] \end{tikzcd}$$

We can then take the composition $\varepsilon\mathcal{F}\circ\mathcal{F}\eta$ to get a natural transformation $\mathcal{F}\Rightarrow\mathcal{F}$

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left=50, "\mathcal{F}"{name=U}] \arrow[r, squiggly, bend right=50, "\mathcal{F}"'{name=L}] & \mathsf{D} \arrow[from=U, to=L, Rightarrow, "\varepsilon\mathcal{F}\circ\mathcal{F}\eta"] \end{tikzcd};$$

the first triangle diagram says that this must be the same as the identity natural transformation $1_{\mathcal{F}}$.

The second triangle diagram is analogous.

Lemma 3.9.1. *The functors $\mathcal{F}$ and $\mathcal{G}$ form a unit-counit adjunction if and only if they form a hom-set adjunction.*

Proof. Suppose $\mathcal{F}$ and $\mathcal{G}$ form a hom-set adjunction with natural isomorphism Φ. Then for any $A \in \mathrm{Obj}(\mathsf{C})$, we have $\mathcal{F}(A) \in \mathrm{Obj}(\mathsf{D})$, so Φ give us a bijection

$$\Phi_{A,\mathcal{F}(A)} \colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \to \mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$

We don't know much in general about $\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A))$, but the category axioms tell us that it always contains $1_{\mathcal{F}(A)}$. We can use $\Phi_{A,\mathcal{F}(A)}$ to map this to

$$\Phi_{A,\mathcal{F}(A)}(1_{\mathcal{F}(A)}) \in \mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$

Let's call $\Phi_{A,\mathcal{F}(A)}(1_{\mathcal{F}(A)}) = \eta_A$.

Similarly, if $B \in \mathrm{Obj}(\mathsf{D})$, then $\mathcal{G}(B) \in \mathrm{Obj}(\mathsf{C})$, so Φ gives us a bijection

$$\Phi_{\mathcal{G}(B),B} \colon \mathrm{Hom}_{\mathsf{D}}((\mathcal{F} \circ \mathcal{G})(B), B) \to \mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(B), \mathcal{G}(B)).$$

Since $\Phi_{\mathcal{G}(B),B}$ is a bijection, it is invertible, and we can evaluate the inverse on $1_{\mathcal{G}(B)}$. Let's call

$$\Phi^{-1}_{\mathcal{G}(B),B}(1_{\mathcal{G}(B)}) = \varepsilon_B.$$

Clearly, η_A and ε_B are completely determined by Φ and Φ^{-1} respectively. It turns out that the converse is also true; in a manner reminiscent of the proof of the Yoneda lemma, we can express $\Phi_{A,B}$ in terms of η, and $\Phi^{-1}_{A,B}$ in terms of ε, for *any* A and B. Here's how this is done.

We use the naturality of Φ. We know that for any $A \in \mathrm{Obj}(\mathsf{C})$, $B \in \mathrm{Obj}(\mathsf{D})$, and $g \colon \mathcal{F}(A) \to B$, the following diagram has to commute.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \arrow[r, "g\circ(-)"] \arrow[d, "\Phi_{A,\mathcal{F}(A)}"'] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \arrow[d, "\Phi_{A,B}"] \\
\mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)) \arrow[r, "\mathcal{G}(g)\circ(-)"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).
\end{tikzcd}$$

Let's start at the top left with $1_{\mathcal{F}(A)}$ and see what happens. Taking the top road to the bottom right, we have $\Phi_{A,B}(g)$, and from the bottom road we have $\mathcal{G}(g) \circ \eta_A$. The diagram commutes, so we have

$$\Phi_{A,B}(g) = \mathcal{G}(g) \circ \eta_A.$$

Similarly, the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}((\mathcal{F} \circ \mathcal{G})(B), B) \arrow[r, "(-)\circ\mathcal{F}(f)"] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \\
\mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(B), \mathcal{G}(B)) \arrow[r, "(-)\circ f"] \arrow[u, "\Phi^{-1}_{\mathcal{G}(B),B}"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)) \arrow[u, "\Phi^{-1}_{A,B}"']
\end{tikzcd}$$

means that, for any $f \colon A \to \mathcal{G}(B)$,

$$\Phi^{-1}_{A,B}(f) = \varepsilon_B \circ \mathcal{F}(f)$$

To show that η and ε as defined here satisfy the triangle identities, we need to show that for all $A \in \mathrm{Obj}(\mathsf{C})$ and all $B \in \mathrm{Obj}(\mathsf{D})$,

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = (1_{\mathcal{F}})_A \qquad \text{and} \qquad (\mathcal{G}\varepsilon)_B \circ (\eta\mathcal{G})_B = (1_{\mathcal{G}})_B.$$

We have

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = \varepsilon_{\mathcal{F}(A)} \circ \mathcal{F}(\eta_A) = \Phi^{-1}_{A,\mathcal{F}(A)}(\eta_A) = 1_A = (1_{\mathcal{F}})_A$$

and

$$(\mathcal{G}_\varepsilon)_B \circ (\eta\mathcal{G})_B = \mathcal{G}(\varepsilon_B) \circ \eta_{\mathcal{G}(B)} = \Phi_{\mathcal{G}(B),B}(\varepsilon_B) = 1_B = (1_{\mathcal{G}})_B.$$

□

Definition 3.9.3 (adjunct)**.** Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction as follows.

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, squiggly, bend left, "\mathcal{G}"] \end{tikzcd}$$

Then for each $A \in \mathrm{Obj}(\mathsf{C})$ $B \in \mathrm{Obj}(\mathsf{D})$, we have a natural isomorphism (i.e. a bijection)

$$\Phi_{A,B} : \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \to \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).$$

Thus, for each $f \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B)$ there is a corresponding element $\tilde{f} \in \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$, and vice versa. The morphism $\tilde{f}$ is called the adjunct of f, and f is called the adjunct of $\tilde{f}$.

Lemma 3.9.2. *Let* C *and* D *be categories,* $\mathcal{F} \colon \mathsf{C} \rightsquigarrow \mathsf{D}$ *and* $\mathcal{G}$*,* $\mathcal{G}' \colon \mathsf{D} \rightsquigarrow \mathsf{C}$ *functors,*

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[l, squiggly, bend right=60, "\mathcal{G}"'] \arrow[l, squiggly, bend left=60, "\mathcal{G}'"] \end{tikzcd}$$

and suppose that $\mathcal{G}$ *and* $\mathcal{G}'$ *are both right-adjoint to* $\mathcal{F}$*. Then there is a natural isomorphism* $\mathcal{G} \Rightarrow \mathcal{G}'$*.*

Proof. Since any adjunction is a hom-set adjunction, we have two isomorphisms

$$\Phi_{C,D} \colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \qquad \text{and} \qquad \Psi_{C,D} \colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D))$$

which are natural in both C and D. By Lemma 3.3.3, we can construct the inverse natural isomorphism

$$\Phi^{-1}_{C,D} \colon \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D),$$

and compose it with Ψ to get a natural isomorphsim

$$(\Psi \circ \Phi^{-1})_{C,D} \colon \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)).$$

Thus for any morphism $f \colon D \to E$, the following diagram commutes.

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \arrow[r, "\mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(f))"] \arrow[d, "(\Psi\circ\Phi^{-1})_{C,D}"'] & \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(E)) \arrow[d, "(\Psi\circ\Phi^{-1})_{C,E}"] \\ \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)) \arrow[r, "\mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(f))"] & \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(E)) \end{tikzcd}$$

But the fully faithfulness of the Yoneda embedding tells us that there exist isomorphisms μ_D and μ_E making the following diagram commute,

$$\begin{tikzcd} \mathcal{G}(D) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\mu_D"'] & \mathcal{G}(E) \arrow[d, "\mu_E"] \\ \mathcal{G}'(D) \arrow[r, "\mathcal{G}'(f)"] & \mathcal{G}'(E) \end{tikzcd}$$

and taking the collection of all such $\mu_{(-)}$ gives us a natural isomorphism $\mu \colon \mathcal{G} \Rightarrow \mathcal{G}'$. ☐

Note 3.9.1. We do not give very many examples of adjunctions now because of the frequency with which category theory graces us with them. Howver, it is worth mentioning a specific class of adjunctions: the so-called *free-forgetful adjunctions*. There are many *free* objects in mathematics: free groups, free modules, free vector spaces, free categories, etc. These are all unified by the following property: the functors defining them are all left adjoints.

Let us take a specific example: the free vector space over a set. This takes a set S and constructs a vector space which has as a basis the elements of S.

There is a forgetful functor $\mathcal{U} \colon \mathsf{Vect}_k \rightsquigarrow \mathsf{Set}$ which takes any set and returns the set underlying it. There is a functor $\mathcal{F} \colon \mathsf{Set} \rightsquigarrow \mathsf{Vect}_k$, which takes a set and returns the free vector space on it. It turns out that there is an adjunction $\mathcal{F} \dashv \mathcal{U}$.

And this is true of any free object! (In fact by definition.) In each case, the functor giving the free object is left adjoint to a forgetful functor.

3.10 Monoidal categories

Recall the definition of a monoid (Definition 2.1.1). subsectionMonoidal categories. Basically, a monoid is a group without inverses.

Monoidal categories are the first ingredient in the categorification and generalization of the tensor product. Roughly speaking, the tensor product allows us to multiply two vector spaces to produce a new vector space. There are natural isomorphisms making this multiplication associative, and an 'identity vector space' given by the ground field regarded as a one-dimensional vector space over itself. That is to say the tensor product gives the set of all vector spaces the structure of a monoid.

A monoidal category will be a category in which the objects have the structure of a monoid, i.e. there is a suitably defined 'multiplication' which is unital and associative. The prototypical example of categorical multiplication is the product (see Example 3.5.5), although it is far from the only one.

Definition 3.10.1 (monoidal category)**.** A monoidal category is a category C equipped with a monoidal structure. A monoidal structure is the following:

- A bifunctor (Definition 3.2.4) $\otimes\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}$ called the *tensor product*,
- An object I called the *unit object*, and
- Three natural isomorphisms (Definition 3.3.2) subject to coherence conditions expressing the fact that the tensor product
  - is associative: there is a natural isomorphism α

$$\begin{tikzcd}
\mathsf{C}\times\mathsf{C}\times\mathsf{C} \arrow[r, squiggly, bend left=50, "(\_\otimes\_)\otimes\_"{name=U}] \arrow[r, squiggly, bend right=50, "\_\otimes(\_\otimes\_)"'{name=D}] & \mathsf{C} \arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

    called the *associator*, with components

$$\alpha_{A,B,C}\colon (A\otimes B)\otimes C \xrightarrow{\sim} A\otimes(B\otimes C)$$

  - has left and right identity: there are two natural isomorphisms λ

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left=50, "I\otimes\_"{name=U}] \arrow[r, squiggly, bend right=50, "1_{\mathsf{C}}"'{name=D}] & \mathsf{C} \arrow[from=U, to=D, Rightarrow, "\lambda"]
\end{tikzcd}$$

    and ρ

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left=50, "\_\otimes I"{name=U}] \arrow[r, squiggly, bend right=50, "1_{\mathsf{C}}"'{name=D}] & \mathsf{C} \arrow[from=U, to=D, Rightarrow, "\rho"]
\end{tikzcd}$$

    respectively called the *left unitor* and *right unitor* with components

$$\lambda_A\colon I\otimes A\xrightarrow{\sim} A$$

    and

$$\rho_A\colon A\otimes I\xrightarrow{\sim} A.$$

The coherence conditions are that the following diagrams commute for all A, B, C, and $D\in\mathrm{Obj}(C)$.

- The *triangle diagram*

$$\begin{tikzcd}
(A\otimes I)\otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A\otimes 1_B"'] & & A\otimes(I\otimes B) \arrow[dl, "1_A\otimes\lambda_B"] \\
& A\otimes B &
\end{tikzcd}$$

- The *home plate diagram.*[2] (usually the *pentagon diagram*)

$$\begin{tikzcd}
 & (A\otimes(B\otimes C))\otimes D \arrow[dr, "\alpha_{A,B\otimes C,D}"] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ur, "\alpha_{A,B,C}\otimes 1_D"] \arrow[d, "\alpha_{A\otimes B,C,D}"'] & & A\otimes((B\otimes C)\otimes D) \arrow[d, "1_A\otimes\alpha_{B,C,D}"] \\
(A\otimes B)\otimes(C\otimes D) \arrow[rr, "\alpha_{A,B,C\otimes D}"'] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}$$

More succinctly, a monoidal structure on a category C is a quintuple $(\otimes, 1, \alpha, \lambda, \rho)$.

Notation 3.10.1. The notation $(\otimes, 1, \alpha, \lambda, \rho)$ is prone to change. If the associator and unitors are not important, or understood from context, they are often left out. We will often say "Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category."

Example 3.10.1. The simplest (though not the prototypical) example of a monoidal category is Set with the Cartesian product. We have already studied this structure in some detail in Section 3.6. We check that it satisfies the axioms in Definition 3.10.1.

- The Cartesian product on Set *is* a set-theoretic product, and can be naturally viewed, thanks to Theorem 3.6.1, as the bifunctor.
- Any set with one element $I = \{*\}$ functions as the unit object.
- For all sets A, B, C
  - The universal property of products guarantees us an isomorphism $\alpha_{A,B,C}\colon (A\times B)\times C \to A\times(B\times C)$ which sends $((a,b),c)\mapsto(a,(b,c))$.
  - Since $\{*\}$ is terminal, we get an isomorphism $\lambda_A\colon \{*\}\times A\to A$ which sends $(*,a)\mapsto a$.
  - Similarly, we get a map $\rho_A\colon A\times\{*\}\to A$ which sends $(a,*)\mapsto a$.

The pentagon and triangle diagram commute vacuously since the cartesian product is associative.

Lemma 3.10.1. *The tensor product $\otimes$ of vector spaces is a bifunctor* $\mathsf{Vect}_k\times\mathsf{Vect}_k \rightsquigarrow \mathsf{Vect}_k$.

Proof. It only remains to check that the standard definition Definition 2.7.5 respects composition. ☐

Example 3.10.2. The category Vect_k is a monoidal category with

1. The bifunctor $\otimes$ is given by the tensor product.
2. The unit 1 is given by the field k regarded as a 1-dimensional vector space over itself.
3. The associator is the map which sends $(v_1\otimes v_2)\otimes v_3$ to $v_1\otimes(v_2\otimes v_3)$. It is not *a priori* obvious that this is well-defined, but it is also not difficult to check.
4. The left unitor is the map which sends $(x,v)\in k\times V$ to $xv\in V$.
5. The right unitor is the map which sends

$$(v,x)\mapsto xv.$$

---

[2] Unfortunately, the home plate, as drawn, is actually upside down.

Definition 3.10.2 (monoidal subcategory)**.** Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category. A monoidal subcategory of C is a subcategory $\mathsf{S} \subseteq \mathsf{C}$ which is closed under the tensor product, and which contains 1.

Example 3.10.3. Recall that $\mathsf{FinVect}_k$ is the category of finite dimensional vector spaces over a field k. We saw in Example 3.1.5 that $\mathsf{FinVect}_k$ was a full subcategory of Vect_k.

We have just seen that Vect_k is a monoidal category with unit object $1 = k$. Since k is a one-dimensional vector space over itself, $k \in \mathrm{Obj}(\mathsf{FinVect}_k)$, and since the dimension of the tensor product of two finite-dimensional vector spaces is the product of their dimensions, the tensor product is closed in $\mathsf{FinVect}_k$. Hence $\mathsf{FinVect}_k$ is a monoidal subcategory of Vect_k.

Lemma 3.10.2. *Let C be a category with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$. Then for all X, $Y \in \mathrm{Obj}(\mathsf{C})$, the diagram*

$$\begin{tikzcd}
(1 \otimes X) \otimes Y \arrow[d, "\alpha_{1,X,Y}"'] \arrow[dr, "\lambda_X \otimes 1_Y"] & \\
1 \otimes (X \otimes Y) \arrow[r, "\lambda_{X \otimes Y}"'] & X \otimes Y
\end{tikzcd}$$

commutes.

Proof. □

Note 3.10.1. This insight is due to [14].

The triangle and pentagon diagram are the first in a long list of hulking coherence diagrams designed to pacify categorical structures into behaving like their algebraic counterparts. For a monoid, multiplication is associative by fiat, and the extension of this to n-fold products follows by a trivial application of induction. In monoidal categories, associators are isomorphisms rather than equalities, and we have no a priori guarantee that different ways of composing associators give the same isomorphism.

Having said that, these diagrams are not as unnatural as they appear. The pentagon diagram is exactly what you get when you write down all possible ways of bracketing the tensor product of four objects and draw arrows between the ones which differ by exactly one associator. Since the associators are isomorphsims, we are allowed to

[3]

Theorem 3.10.1 (Mac Lane's coherence theorem)**.** *Any two ways of freely composing unitors and associators to go from one expression to another coincide.*

Sketch of proof. The proof involves showing that any two such natural transformation can be put in a canonical form in which they agree. □

Definition 3.10.3 (invertible object)**.** Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category. An invertible object (sometimes called a *line object*) is an object $L \in \mathrm{Obj}(\mathsf{C})$ such that both of the functors $\mathsf{C} \rightsquigarrow \mathsf{C}$

- $\ell_L \colon A \mapsto L \otimes A$
- $r_L \colon A \mapsto A \otimes L$

are categorical equivalences.

The following lemma is the categorification of Lemma 2.1.1.

Lemma 3.10.3. *If $(\mathsf{C}, \otimes, 1)$ is a monoidal category and L is an invertible object, then there is an object $L^{-1} \in \mathrm{Obj}(\mathsf{C})$, unique up to isomorphism, such that $L \otimes L^{-1} \simeq L^{-1} \otimes L \simeq 1$. Furthermore, L is invertible only if there exists such an L^{-1}.*

---

[3] The number of ways to parenthesize a product involving n objects is the $(n-1)$st Catalan number

$$C_{n-1} = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

Proof. Suppose L is invertible. Then the functor $\ell_L\colon A \mapsto L \otimes A$ is bijective up to isomorphism, i.e. for any $A \in \mathrm{Obj}(\mathsf{C})$, there is an object $A' \in \mathrm{Obj}(\mathsf{C})$, unique up to isomorphism, such that

$$\ell_L(A') = L \otimes A' \simeq A.$$

But if this is true for *any* A, it must also be true for 1, so there exists an object L^{-1}, unique up to isomorphism, such that

$$\ell_L(L^{-1}) = L \otimes L^{-1} \simeq 1.$$

The same logic tells us that there exists some other element L'^{-1}, such that

$$\reflectbox{ℓ}_L(L'^{-1}) = L'^{-1} \otimes L \simeq 1.$$

Now

$$1 \simeq L'^{-1} \otimes L \simeq L'^{-1} \otimes (L \otimes L^{-1}) \otimes L \simeq (L'^{-1} \otimes L) \otimes (L^{-1} \otimes L) \simeq (L'^{-1} \otimes L) \otimes 1 \simeq L'^{-1} \otimes L,$$

so L'^{-1} is also a left inverse for L. But since ℓ_L is an equivalence of categories, L only has one left inverse up to isomorphism, so L^{-1} and L'^{-1} must be isomorphic. □

Definition 3.10.4. Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category. Then the full subcategory (Definition 3.1.5) $(\mathrm{Line}(\mathsf{C}), \otimes, 1) \subseteq (\mathsf{C}, \otimes, 1)$ whose objects are the line objects in C is called the line subcategory of $(\mathsf{C}, \otimes, 1)$. Since invertibility is closed under the tensor product and 1 is invertible ($1^{-1} \simeq 1$), $\mathrm{Line}(\mathsf{C})$ is a monoidal subcategory of C.

The following definition was taken *mutatis mutandis* from [13].

Definition 3.10.5 (monoidal functor)**.** Let C and C' be monoidal categories. A functor $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{C}'$ is lax monoidal if it is equipped with

- a natural transformation $\Phi_{X,Y}\colon \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$, and
- a morphism $\varphi\colon 1_{C'} \to \mathcal{F}(1_C)$ such that
- the following diagrams commute for any X, Y, $Z \in \mathrm{Obj}(\mathsf{C})$.

$$\begin{tikzcd}
(\mathcal{F}(X) \otimes \mathcal{F}(Y)) \otimes \mathcal{F}(Z) \arrow[r, "\Phi_{X,Y} \otimes 1_{\mathcal{F}(Z)}"] \arrow[d, "\alpha_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)}"'] & \mathcal{F}(X \otimes Y) \otimes \mathcal{F}(Z) \arrow[r, "\Phi_{X \otimes Y, Z}"] & \mathcal{F}((X \otimes Y) \otimes Z) \arrow[d, "\mathcal{F}(\alpha_{X,Y,Z})"] \\
\mathcal{F}(X) \otimes (\mathcal{F}(Y) \otimes \mathcal{F}(Z)) \arrow[r, "1_{\mathcal{F}(X)} \otimes \Phi_{Y,Z}"'] & \mathcal{F}(X) \otimes \mathcal{F}(Y \otimes Z) \arrow[r, "\varphi_{X, Y \otimes Z}"'] & \mathcal{F}(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes \mathcal{F}(X) \arrow[r, "\lambda_{\mathcal{F}(X)}"] \arrow[d, "\varphi \otimes 1_{\mathcal{F}(X)}"'] & \mathcal{F}(X) \\
\mathcal{F}(1) \otimes \mathcal{F}(X) \arrow[r, "\Phi_{1,X}"] & \mathcal{F}(1 \otimes X) \arrow[u, "\mathcal{F}(\lambda_X)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(X) \otimes 1 \arrow[r, "\rho_{\mathcal{F}(X)}"] \arrow[d, "1_{\mathcal{F}(X) \otimes \varphi}"'] & \mathcal{F}(X) \\
\mathcal{F}(X) \otimes \mathcal{F}(1) \arrow[r, "\Phi_{X,1}"] & \mathcal{F}(X \otimes 1) \arrow[u, "F(\lambda_X)"']
\end{tikzcd}$$

If Φ is a natural isomorphism and φ is an isomorphism, then $\mathcal{F}$ is called a strong monoidal functor.

We will denote the above monoidal functor by $(\mathcal{F}, \Phi, \varphi)$.

Note 3.10.2. The above diagrams above are exactly those necessary to ensure that the monoidal structure is preserved. They do this by demanding that the associator and the unitors be $\mathcal{F}$-equivariant.

Note 3.10.3. In much of the literature, a strong monoidal functor is simply called a monoidal functor.

Lemma 3.10.4. *The composition of lax (strong) monoidal functors is lax (strong) monoidal.*

Proof. □

Definition 3.10.6 (monoidal natural transformation)**.** Let (F, Φ, φ) and (G, Γ, γ) be monoidal functors. A natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ is monoidal if the following diagrams commute.

$$\begin{tikzcd}
\mathcal{F}(X) \otimes \mathcal{F}(Y) \arrow[r, "\eta_X \otimes \eta_Y"] \arrow[d, "\Phi_{X,Y}"'] & \mathcal{G}(X) \otimes \mathcal{G}(Y) \arrow[d, "\Gamma_{X,Y}"] \\
\mathcal{F}(X \otimes Y) \arrow[r, "\eta_{X \otimes Y}"] & \mathcal{G}(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[d, "\varphi"'] \arrow[dr, "\gamma"] & \\
\mathcal{F}(1) \arrow[r, "\eta_1"] & \mathcal{G}(1)
\end{tikzcd}$$

3.10.1 Braided monoidal categories

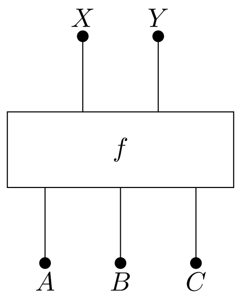
Any insightful remarks in this section are due to [12].

Braided monoidal categories capture the idea that we should think of morphisms between tensor products spatially, as diagrams embedded in 3-space. This sounds odd, but it turns out to be the correct way of looking at a wide class of problems.

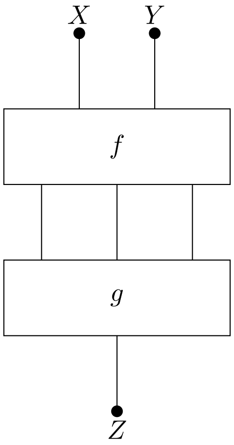
To be slightly more precise, we can think of the objects X, Y, etc. in any monoidal category as little dots.

We can express the tensor product $X \otimes Y$ by putting the dots representing X and Y next to each other.

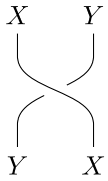
A morphism f between, say, $X \otimes Y$ and $A \otimes B \otimes C$ can be drawn as a diagram consisting of some lines and boxes.



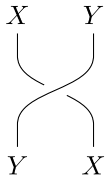
We can compose morphisms by concatenating their diagrams.



In a braided monoidal category, we require that for any two objects X and Y we have an isomorphism $\gamma_{XY} \colon X \otimes Y \to Y \otimes X$, which we draw like this.



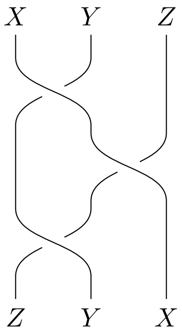
Since γ_{AB} is an isomorphism, it has an inverse γ_{AB}^{-1} (not necessarily equal to γ_{BA}!) which we draw like this.



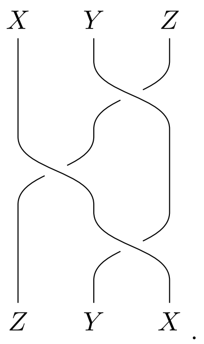
The idea of a braided monoidal category is that we want to take these pictures seriously: we want two expressions involving repeated applications of the $\gamma_{\cdot\cdot}$ and their inverses to be equivalent if and only if the braid diagrams representing them are homotopic. Thus we want, for example,

$$\gamma_{XY} \circ \gamma_{YZ} \circ \gamma_{XY} = \gamma_{YZ} \circ \gamma_{XY} \circ \gamma_{YZ}$$

since



is homotopic to



A digression into the theory of braid groups would take us too far afield. The punchline is that to guarantee that all such compositions involving the γ are identified in the correct way, we must define braided monoidal categories as follows.

Definition 3.10.7 (braided monoidal category)**.** A catgory C with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$ is braided if for every two objects A and $B \in \mathrm{Obj}(\mathsf{C})$, there is an isomorphism $\gamma_{A,B} \colon A \otimes B \to B \otimes A$ such that the following *hexagon diagrams* commute.

$$\begin{tikzcd}
 & A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{BCA}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{ABC}"] \arrow[dr, "\gamma_{AB}\otimes 1"'] & & & B \otimes (C \otimes A) \\
 & (B \otimes A) \otimes C \arrow[r, "\alpha_{BAC}"'] & (B \otimes A) \otimes C \arrow[ur, "1\otimes\gamma"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (A \otimes B) \otimes C \arrow[r, "\gamma_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}_{CAB}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}_{ABC}"] \arrow[dr, "1\otimes\gamma_{BC}"'] & & & (C \otimes A) \otimes B \\
 & A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{ACB}"'] & (A \otimes C) \otimes B \arrow[ur, "\gamma_{AC}\otimes 1"'] &
\end{tikzcd}$$

The collection of such γ form a natural isomorphism betweem the bifunctors

$$(A, B) \mapsto A \otimes B \qquad \text{and} \qquad (A, B) \mapsto B \otimes A,$$

and is called a braiding.

Definition 3.10.8 (braided monoidal functor)**.** A lax monoidal functor $(\mathcal{F}, \Phi, \phi)$ (Definition 3.10.5) is braided monoidal if it makes the following diagram commute.

$$\begin{tikzcd}
\mathcal{F}(x) \otimes \mathcal{F}(y) \arrow[r, "\gamma_{\mathcal{F}(x),\mathcal{F}(y)}"] \arrow[d, "\Phi_{x,y}"'] & \mathcal{F}(y) \otimes \mathcal{F}(x) \arrow[d, "\Phi_{x,y}"] \\
\mathcal{F}(x \otimes y) \arrow[r, "\mathcal{F}(\gamma_{x,y})"] & \mathcal{F}(y \otimes x)
\end{tikzcd}$$

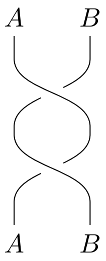
Note 3.10.4. There are no extra conditions imposed on a monoidal natural transformation to turn it into a braided natural transformation.

3.10.2 Symmetric monoidal categories

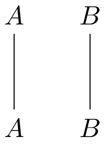
Until now, we have been calling the bifunctor $\otimes$ in Definition 3.10.1 a tensor product. This has been an abuse of terminology: in general, one defines tensor products not to be those bifunctors which come from any monoidal category, but only those which come from *symmetric* monoidal categories. We will define these shortly.

Conceptually, passing from the definition of a braided monoidal category to that of a symmetric monoidal category is rather simple. One only requires that for any two objects A and B, $\gamma_{BA} = \gamma_{AB}^{-1}$, i.e. $\gamma_{BA} \circ \gamma_{AB} = 1_{A \otimes B}$.

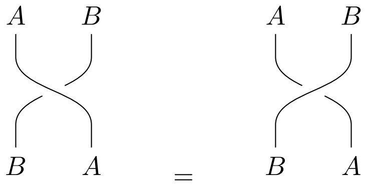
We can interpret this nicely in terms of our braid diagrams. We can draw $\gamma_{BA} \circ \gamma_{AB}$ like this.



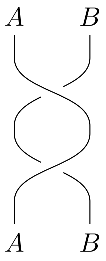
The requirement that this must be homotopic to the identity transformation



can be expressed by making the following rule: in a *symmetric* monoidal category, we don't care about the difference between undercrossings and overcrossings:



Then we can exchange the diagram representing $\gamma_{BA} \circ \gamma_{AB}$ for



which is clearly homotopic to the identity transformation on $A \otimes B$.

Definition 3.10.9 (symmetric monoidal category)**.** Let C be a braided monoidal category with braiding γ. We say that C is a symmetric monoidal category if for all A, $B \in \mathrm{Obj}(\mathsf{C})$, $\gamma_{BA} \circ \gamma_{AB} = 1_{A \otimes B}$. A braiding γ which satisfies such a condition is called symmetric.

Note 3.10.5. There are no extra conditions imposed on a monoidal natural transformation to turn it into a symmetric natural transformation.

3.11 Internal hom functors

We can now generalize the notion of an exponential object (Definiton 3.6.5) to any monoidal category.

3.11.1 The internal hom functor

Recall the definition of the hom functor on a locally small category C (Definition 3.7.1): it is the functor which maps two objects to the set of morphisms between them, so it is a functor

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \rightsquigarrow \mathsf{Set}.$$

If we take $\mathsf{C} = \mathsf{Set}$, then our hom functor never really leaves Set; it is *internal* to Set. This is our first example of an *internal hom functor*. In fact, it is the prototypical internal hom functor, and we can learn a lot by studying its properties.

Let X and Y be sets. Denote the set of all functions $X \to Y$ by $[X, Y]$.

Let S be any other set, and consider a function $f \colon S \to [X, Y]$. For each element $s \in S$, f picks out a function $h_s \colon X \to Y$. But this is just a curried version of a function $S \times X \to Y$! So as we saw in Section 3.7.1, we have a bijection between the sets $[S, [X, Y]]$ and $[S \times X, Y]$. In fact, this is even a *natural* bijection, i.e. a natural transformation between the functors

$$[-, [-, -]] \qquad \text{and} \quad [- \times -, -] \colon \mathsf{Set}^{\mathrm{op}} \times \mathsf{Set}^{\mathrm{op}} \times \mathsf{Set} \rightsquigarrow \mathsf{Set}.$$

Let's check this. First, we need to figure out how our functors act on functions. The first one is the most complicated, so let's get it out of the way. Suppose we have sets and functions like so.

$$\begin{tikzcd} A'' & B'' \arrow[l, "f''"'] \\ A' & B' \arrow[l, "f'"'] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Our functor maps

$$(A'', A', A) \mapsto [A'', [A', A]] = \mathrm{Hom}_{\mathsf{Set}}(A'', \mathrm{Hom}_{\mathsf{Set}}(A', A)),$$

so it should map (f'', f', f) to a function

$$\mathrm{Hom}_{\mathsf{Set}}(A'', \mathrm{Hom}_{\mathsf{Set}}(A', A)) \to \mathrm{Hom}_{\mathsf{Set}}(B'', \mathrm{Hom}_{\mathsf{Set}}(B', B)).$$

The notation $\mathrm{Hom}_{\mathsf{Set}}(-, -)$ is getting a little lengthy, so I'll stick to $[-, -]$ for now. In this notation, we want to map (f'', f', f) to a function

$$[f'', [f', f] \colon [A'', [A', A]] \to [B'', [B', B]].$$

The way to do that is by sending $m \in [A'', [A', A]]$ to

$$[f', f] \circ m \circ f''.$$

You can check that this works as advertised.

The other one's not so tough. Our functor maps an object (A'', A', A) to $[A'' \times A', A]$. We need to map (f'', f', f) to a function

$$[f'' \times f', f] \colon [A'' \times A', A] \to [B'' \times B', B].$$

We do that by sending $m \in [A'' \times A', A]$ to

$$f \circ m \circ (f'', f') \in [B'' \times B', B].$$

Checking that $[-, [-, -]]$ and $[- \times -, -]$ really *are* functorial would be a bit much; each is just a few applications of the Yoneda embedding. We will however check that there is a natural isomorphism between them, which amounts to checking that the following diagram commutes.

$$\begin{tikzcd}
{[A''\times A',A]} \arrow[rrr, "{[f''\times f',f]}"] \arrow[ddd, "\Phi_{[A''\times A',A]}"'] & & & {[B''\times B',B]} \arrow[ddd, "\Phi_{[B''\times B',B]}"] \\
& m \arrow[r, mapsto] \arrow[d, mapsto] & f\circ m\circ (f',f'') \arrow[d, mapsto] & \\
& \Phi(m) \arrow[r, mapsto] & {[f',f]\circ\Phi(m)\circ f'' \overset{!}{=} \Phi(f\circ m\circ (f',f''))} & \\
{[A'',[A',A]]} \arrow[rrr, "{[f'',[f',f]]}"'] & & & {[B'',[B',B]]}
\end{tikzcd}$$

In other words, we have to show that

$$\Phi_{[A''\times A',A]}(f\circ m\circ (f',f'')) = [f',f]\circ \Phi_{[B''\times B',B]}(m)\circ f''.$$

So what is each of these? Well, $f\circ m\circ (f',f'')$ is a map $B''\times B'\to B$, which maps (say) $(b'',b')\mapsto b$.

The natural transformation Φ tells us to curry this, i.e. turn it into a map $B''\to [B',B]$. Not just any map, though: a map which when evaluated on b'' turns into a map which, when evaluated on b', yields b.

We know that $f\circ m\circ (f',f'')\colon (b'',b')\mapsto b$, i.e.

$$f(m(f''(b''),f'(b'))) = b.$$

If we can show that this is *also* what $[f',f]\circ\Phi_{[B''\times B',B]}(m)\circ f''$ is equal to when evaluated on b', we are done.

Well, let's go through what this definition means. First, we take b'' and feed it to f''. Next, we let $\Phi_{[B''\times B',B]}(m)$ act on the result, i.e. we fill the first argument of m with $f''(b'')$. What we get is the following:

$$m(f''(b''),-).$$

Then we are to precompose this with f' and stick the result into f:

$$f(m(f''(b''),f'(-))).$$

Indeed, when we evaluate this on b', we get b, so the diagram commutes.

In other words, Φ is a natural bijection between the hom-sets $[-,[-,-]]$ and $[-\times -,-]$.

We picked this example because the collection $[A,B]$ of all functions between two sets A and B is itself a set. Therefore it makes sense to think of the hom-sets $\mathrm{Hom}_{\mathsf{C}}(A,B)$ as living within the same category as A and B. We saw in Section 3.6 that in a category with products, we could sometimes view hom-sets as exponential objects. However, we now have the technology to be even more general.

Definition 3.11.1 (internal hom functor)**.** Let $(\mathsf{C},\otimes)$ be a monoidal category. An internal hom functor is a functor

$$[-,-]_{\mathsf{C}}\colon \mathsf{C}^{\mathrm{op}}\times\mathsf{C}\rightsquigarrow\mathsf{C}$$

such that for every $X\in\mathrm{Obj}(\mathsf{C})$ we have a pair of adjoint functors

$$(-)\otimes X\dashv [X,-]_{\mathsf{C}}.$$

The objects $[A,B]_{\mathsf{C}}$ are called internal hom objects.

Notation 3.11.1. The convention at the nLab is to denote the internal hom by square braces $[A,B]$, and this is for the most part what we will do. Unfortunately, we have already used this notation for the *regular* hom functor. To remedy this, we will add a subscript if the category to which the hom functor belongs is not clear: $[-,-]_{\mathsf{C}}$ for a hom functor internal to C, $[-,-]_{\mathsf{Set}}$ for the standard hom functor (or the hom functor internal to Set, which amounts to the same).

There is no universally accepted notation for the internal hom functor. One often sees it denoted by a lower-case hom: $\mathrm{hom}_{\mathsf{C}}(A,B)$. Many sources (for example DMOS [16]) distinguish the internal hom with an underline: $\underline{\mathrm{Hom}}_{\mathsf{C}}(A,B)$. Deligne typesets it with a script H: $\mathscr{H}om_{\mathsf{C}}(A,B)$

Definition 3.11.2 (closed monoidal category)**.** A monoidal category equipped with an internal hom functor is called a closed monoidal category.

Note 3.11.1. Here is another definition of $[X, Y]_{\mathsf{C}}$: it is the object representing (Definition 3.7.3) the functor

$$T \mapsto \mathrm{Hom}_{\mathsf{C}}(T \otimes X, Y).$$

Example 3.11.1. In many locally small categories whose objects can be thought of as "sets with extra structure," it is possible to pile structure on top of the hom sets until they themselves can be viewed as bona fide objects in their categories. It often happens that these beefed-up hom sets coincide (up to isomorphism) with the internal hom objects.

Take for example Vect_k. For any vector spaces V and W, we can turn $\mathrm{Hom}_{\mathsf{Vect}_k}(V, W)$ into a vector space by defining addition and scalar multiplication pointwise; we can then view $\mathrm{Hom}_{\mathsf{Vect}_k}(V, W)$ as belonging to $\mathrm{Obj}(\mathsf{Vect}_k)$. It turns out that this is precisely (up to isomorphism) the internal hom object $[V, W]_{\mathsf{Vect}_k}$.

To see this, we need to show that there is a natural bijection

$$\mathrm{Hom}_{\mathsf{Vect}_k}(A, \mathrm{Hom}_{\mathsf{Vect}_k}(B, C)) \simeq \mathrm{Hom}_{\mathsf{Vect}_k}(A \otimes B, C).$$

Suppose we are given a linear map $f\colon A \to \mathrm{Hom}_{\mathsf{Vect}_k}(B, C)$. If we act with this on an element of A, we get a linear map $B \to C$. If we evaluate this on an element of B, we get an element of C. Thus, we can view f as a bilinear map $A \times B \to C$, hence as a linear map $A \otimes B \to C$.

Now suppose we are given a linear map $g\colon A \otimes B \to C$. We can view this as a bilinear map $A \times B \to C$, and by currying this we get a linear map $A \to \mathrm{Hom}_{\mathsf{Vect}_k}(B, C)$.

For the remainder of this chapter, let $(\mathsf{C}, \otimes, 1)$ be a closed monoidal category with internal hom functor $[-, -]_{\mathsf{C}}$.

In a closed monoidal category, the adjunction between the internal hom and the tensor product even holds internally.

Lemma 3.11.1. *For any X, Y, $Z \in \mathrm{Obj}(\mathsf{C})$ there is a natural isomorphism*

$$[X \otimes Y, Z]_{\mathsf{C}} \xrightarrow{\sim} [X, [Y, Z]_{\mathsf{C}}]_{\mathsf{C}}.$$

Proof. Let $A \in \mathrm{Obj}(\mathsf{C})$. We have the following string of natural isomorphisms.

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(A, [X \otimes Y, Z]_{\mathsf{C}}) &\simeq \mathrm{Hom}_{\mathsf{C}}(A \otimes (X \otimes Y), Z) \\ &\simeq \mathrm{Hom}_{\mathsf{C}}((A \otimes X) \otimes Y, Z) \\ &\simeq \mathrm{Hom}_{\mathsf{C}}(A \otimes X, [Y, Z]_{\mathsf{C}}) \\ &\simeq \mathrm{Hom}_{\mathsf{C}}(A, [X, [Y, Z]_{\mathsf{C}}]_{\mathsf{C}}). \end{aligned}$$

Since this is true for each A we have, by Corollary 3.7.1,

$$[X \otimes Y, Z]_{\mathsf{C}} \xrightarrow{\sim} [X, [Y, Z]_{\mathsf{C}}]_{\mathsf{C}}.$$

□

3.11.2 The evaluation map

The internal hom functor gives us a way to talk about evaluating morphisms $f\colon X \to Y$ without mentioning elements of X.

Definition 3.11.3 (evaluation map)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. We have seen that the adjunction

$$(-) \otimes X \dashv [X, -]_{\mathsf{C}}$$

gives us, for any A, X, $Y \in \mathrm{Obj}(\mathsf{C})$, a natural bijection

$$\mathrm{Hom}_{\mathsf{C}}(A \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(A, [X, Y]_{\mathsf{C}}).$$

In particular, with $A = [X,Y]_{\mathsf{C}}$, we have a bijection

$$\mathrm{Hom}_{\mathsf{C}}([X,Y]_{\mathsf{C}} \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}([X,Y]_{\mathsf{C}}, [X,Y]_{\mathsf{C}}).$$

The adjunct (Definition 3.9.3) of $1_{[X,Y]_{\mathsf{C}}} \in \mathrm{Hom}_{\mathsf{C}}([X,Y]_{\mathsf{C}}, [X,Y]_{\mathsf{C}})$ is an object in $\mathrm{Hom}_{\mathsf{C}}\,([X,Y]_{\mathsf{C}} \otimes X, Y)$, denoted

$$\mathrm{eval}_{X,Y} \colon [X,Y]_{\mathsf{C}} \otimes X \to Y,$$

and called the evaluation map.

Example 3.11.2. As we saw in Example 3.10.1, the category Set is a monoidal category with a bifunctor given by the cartesian product. The internal hom is simply the regular hom functor

$$\mathrm{Hom}_{\mathsf{Set}}(-,-) = [-,-].$$

Let us explore the evaluation map on Set. It is the adjunct of the identity map $1_{[X,Y]}$ under the adjunction

$$[[X,Y] \times X, Y] \dashv [[X,Y],[X,Y]].$$

Thus, it is a function

$$\mathrm{eval}_{X,Y} \colon [X,Y] \times X \to Y; \qquad (f,x) \mapsto \mathrm{eval}_{X,Y}(f,x).$$

So far, we don't know what $\mathrm{eval}_{X,Y}$ sends (f,x) to; we just know that we'd *like it* if it sent it to $f(x)$.

The above adjunction is given by currying: we start on the LHS with a map $\mathrm{eval}_{X,Y}$ with two arguments, and we turn it into a map which fills in only the first argument. Thus the map on the RHS adjunct to $\mathrm{eval}_{X,Y}$ is given by

$$f \mapsto \mathrm{eval}_{X,Y}(f,-).$$

If we want the map $f \mapsto \mathrm{eval}_{X,Y}(f,-)$ to be the identity map, f and $\mathrm{eval}_{X,Y}(f,-)$ must agree on all elements x, i.e.

$$f(x) = \mathrm{eval}_{X,Y}(f,x) \qquad \text{for all } x \in X.$$

Thus, the evaluation map is the map which sends $(f,x) \mapsto f(x)$.

3.11.3 The composition morphism

The evaluation map allows us to define composition of morphisms without talking about internal hom objects as if they have elements.

Definition 3.11.4 (composition morphism)**.** For X, Y, $Z \in \mathrm{Obj}(\mathsf{C})$, the composition morphism

$$\circ_{X,Y,Z} \colon [Y,Z]_{\mathsf{C}} \otimes [X,Y]_{\mathsf{C}} \to [X,Z]_{\mathsf{C}}$$

is the $(-) \otimes X \vdash [X,-]_{\mathsf{C}}$-adjunct of the composition

$$\begin{tikzcd} {[Y,Z]_{\mathsf{C}} \otimes [X,Y]_{\mathsf{C}} \otimes X} \arrow[r, "{\left(1_{[Y,Z]_{\mathsf{C}}}, \mathrm{eval}_{X,Y}\right)}"] & {[Y,Z]_{\mathsf{C}} \otimes Y} \arrow[r, "\mathrm{eval}_{Y,Z}"] & Z \end{tikzcd} .$$

Example 3.11.3. In Set, the composition morphism $\circ_{X,Y,Z}$ lives up to its name. Let $f \colon X \to Y$, $g \colon Y \to Z$, and $x \in X$. The above composition goes as follows.

1. The map $\left(1_{[Y,Z]}, \mathrm{eval}_{X,Y}\right)$ turns the triple (g,f,x) into the pair $(g, f(x))$.
2. The map $\mathrm{eval}_{Y,Z}$ turns $(g, f(x))$ into $g(f(x)) = (g \circ f)(x)$.

The evaluation morphism $\circ_{X,Y,Z}$ is the currying of this, i.e. it sends

$$(f,g) \mapsto (f \circ g)(-).$$

3.11.4 Dual objects

Recall that for any k-vector space V, there is a dual vector space

$$V^* = \{L\colon V \to k\}.$$

This definition generalizes to any closed monoidal category.

Definition 3.11.5 (dual object)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. The dual object to X, denoted X^*, is defined to be the object

$$[X, 1]_{\mathsf{C}}.$$

That is to say, X^* is the internal hom object modelling the hom set of morphisms from X to the identity object 1.

Notation 3.11.2. The evaluation morphism has a component

$$\mathrm{eval}_{X^*,X}\colon X^* \otimes X \to 1.$$

To clean things up a bit, we will write eval_X instead of $\mathrm{eval}_{X^*,X}$.

Notation 3.11.3. In many sources, e.g. DMOS ([16]), the dual object to X is denoted $X^\vee$ instead of X^*.

Lemma 3.11.2. *There is a natural isomorphism between the functors*

$$\mathrm{Hom}_{\mathsf{C}}(-, X^*) \qquad \textit{and} \qquad \mathrm{Hom}_{\mathsf{C}}((-) \otimes X, 1).$$

Proof. For any X, $T \in \mathrm{Obj}(\mathsf{C})$, the definition of the internal hom $[-,-]_{\mathsf{C}}$ gives us a natural isomorphism

$$\mathrm{Hom}_{\mathsf{C}}(T \otimes X, 1) \simeq \mathrm{Hom}_{\mathsf{C}}(T, [X, 1]_{\mathsf{C}}) = \mathrm{Hom}_{\mathsf{C}}(T, X^*).$$

□

Theorem 3.11.1. *The map $X \mapsto X^*$ can be extended to a contravariant functor.*

Proof. We need to figure out how our functor should act on morphisms. We define this by analogy with the familiar setting of vector spaces. Recall that for a linear map $L\colon V \to W$, the dual map $L^t\colon W^* \to V^*$ is defined by

$$(L^t(w))(v) = w(L(v)).$$

By analogy, for $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, we should define the dual morphism $f^t \in \mathrm{Hom}_{\mathsf{C}}(B^*, A^*)$ by demanding that the following diagram commutes.

$$\begin{tikzcd}
Y^* \otimes X \arrow[r, "f^t \otimes 1_X"] \arrow[d, "1_Y \otimes f"'] & X^* \otimes X \arrow[d, "\mathrm{eval}_X"] \\
Y^* \otimes Y \arrow[r, "\mathrm{eval}_Y"] & 1
\end{tikzcd}$$

To check that this is functorial, we must check that it respects compositions, i.e. that the following diagram commutes.

$$\begin{tikzcd}
Z^* \otimes X \arrow[r, "(f^t \circ g^t) \otimes 1_X"] \arrow[d, "1_Z \otimes (g \circ f)"'] & X^* \otimes X \arrow[d, "\mathrm{eval}_X"] \\
Z^* \otimes Z \arrow[r, "\mathrm{eval}_Z"] & 1
\end{tikzcd}$$

Let's add in some more objects and morphisms.

$$\begin{tikzcd}
Z^* \otimes X \arrow[r, "g^t \otimes 1_X"] \arrow[d, "1_{Z^*} \otimes f"'] & Y^* \otimes X \arrow[r, "f^t \otimes 1_X"] \arrow[d, "1_{Y^*} \otimes f"'] & X^* \otimes X \arrow[dd, "\mathrm{eval}_X"] \\
Z^* \otimes Y \arrow[r, "g^t \otimes 1_Y"] \arrow[d, "1_{Z^*} \otimes g"'] & Y^* \otimes Y \arrow[dr, "\mathrm{eval}_Y"] & \\
Z^* \otimes Z \arrow[rr, "\mathrm{eval}_Z"'] & & 1
\end{tikzcd}$$

We want to show that the outer square commutes. But it clearly does: that the top left square commutes is trivial, and the right and bottom 'squares' are the commutativity conditions defining f^t and g^t. □

The above is one, but not the only, way to define dual objects. We can be more general.

Definition 3.11.6 (right duality)**.** Let C be a category with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$. Right duality of two objects A and $A^* \in \mathrm{Obj}(\mathsf{C})$ consists of

1. A morphism of the form
$$\mathrm{eval}_A \colon A^* \otimes A \to 1,$$
called the *evaluation map* (or *counit* if you're into Hopf algebras)
2. A morphism of the form
$$i_A \colon 1 \to A \otimes A^*,$$
called the *coevaluation* map (or *unit*)

such that the compositions

$$X \xrightarrow{i_A \otimes 1_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{1_X \otimes \mathrm{eval}_X} X$$

$$X^* \xrightarrow{1_{X^*} \otimes \mathrm{eval}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{eval}_X \otimes 1_{X^*}} X^*$$

are the identity morphism.

Definition 3.11.7 (rigid monoidal category)**.** A monoidal category $(\mathsf{C}, \otimes, 1)$ is rigid if every object has a left and right dual.

Theorem 3.11.2. *Every rigid monoidal category is a closed monoidal category (Definition 3.11.2) with internal hom object*

$$[A, B]_{\mathsf{C}} \simeq B \otimes A^*.$$

Proof. We can prove the existence of this isomorphism by showing, thanks to Corollary 3.7.1, that for any $X \in \mathrm{Obj}(\mathsf{C})$ there is an isomorphsm

$$\mathrm{Hom}_{\mathsf{C}}(X, [A, B]_{\mathsf{C}}) \simeq \mathrm{Hom}_{\mathsf{C}}(X, B \otimes A^*).$$

The defining adjunction of the internal hom gives us

$$\mathrm{Hom}_{\mathsf{C}}(X, [A, B]_{\mathsf{C}}) \simeq \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B).$$

Now we can map any $f \in \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B)$ to

$$(f \otimes 1_A) \circ (1_X \otimes i_A) \in \mathrm{Hom}_{\mathsf{C}}(X, B \otimes A^*).$$

We will be done if we can show that the assignment

$$f \mapsto (f \otimes 1_A) \circ (1_X \otimes i_A)$$

is an isomorphism. We'll do this by exhibiting an inverse:

$$\mathrm{Hom}_{\mathsf{C}}(X, B \otimes A^*) \ni g \mapsto (1_W \otimes \mathrm{eval}_V) \circ (g \otimes 1_V) \in \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B).$$

Of course, first we should show that $(f \otimes 1_A) \circ (1_X \otimes i_A)$ really does map $X \to B \otimes A^*$. But it does; it does this by first acting on X with i_A:

$$X \to X \otimes A \otimes A^*$$

and then acting on the $X \otimes A$ with f and letting the A^* hang around:

$$X \otimes A \otimes A^* \to B \otimes A^*.$$

To show that

$$g \mapsto (1_B \otimes \mathrm{eval}_A) \circ (g \otimes 1_A)$$

really is an inverse, we can shove the assignment

$$f \mapsto (f \otimes 1_A) \circ (1_X \otimes i_A)$$

into it and show that we get f right back out. That is to say, we need to show that

$$(1_B \otimes \mathrm{eval}_A) \circ ([(f \otimes 1_A) \circ (1_X \otimes i_A)] \otimes 1_A) = f.$$

This is easy to see but hard to type. Write it out. You'll need to use first of the two composition identities.

To show that the other composition yields g, you have to use the other. □

3.12 Abelian categories

This section draws heavily from [22].

3.12.1 Additive categories

Recall that Ab is the category of abelian groups (Definition 2.2.2).

Definition 3.12.1 (Ab-enriched category)**.** A category C is Ab-enriched if

1. for all objects A, $B \in \mathrm{Obj}(\mathsf{C})$, the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of an abelian group (i.e. one can add morphisms), such that
2. the composition
$$\circ \colon \mathrm{Hom}_{\mathsf{C}}(B, C) \times \mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{C}(A, C)$$
is additive in each slot: for any f_1, $f_2 \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, we must have
$$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g,$$
and similarly in the second slot.

Already in any Ab-enriched category we can make the following definition.

Definition 3.12.2 (endomorphism ring)**.** Let C be an Ab-enriched category, and let $A \in \mathrm{Obj}(\mathsf{C})$. The endomorphism ring of A, denoted $\mathrm{End}(A)$, is $\mathrm{Hom}_{\mathsf{C}}(A, A)$, with addition given by the abelian structure and multiplication given by composition.

Definition 3.12.3 (additive category)**.** A category C is additive if it has (Definition 3.6.4) and is Ab-enriched.

Lemma 3.12.1. *In an* Ab*-enriched category* C*, a finite product is also a coproduct, and vice versa. In particular, initial objects and terminal objects coincide.*

Proof. First, we show that initial and terminal objects coincide. Let $\emptyset$ be initial and $*$ terminal in C. Since $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ has only one element, it must be both the identity morphism on $\emptyset$ *and* the identity element of $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ as an abelian group. Similarly, 1_* must be the identity element of $\mathrm{Hom}_{\mathsf{C}}(*, *)$. □

Example 3.12.1. The category Ab of Abelian groups is an additive category. We have already seen that it has the direct sum $\oplus$ as biproduct. Given any two abelian groups A and B and morphisms f, $g\colon A \to B$, we can define the sum $f + g$ via

$$(f+g)(a) = f(a) + g(a) \qquad \text{for all } a \in A.$$

Then for another abelian group C and a morphism $h\colon B \to C$, we have

$$[h \circ (f+g)]\,(a) = h(f(a) + g(a)) = h(f(a)) + h(g(a)) = [h \circ f + h \circ g]\,(a),$$

so

$$h \circ (f+g) = h \circ f + h \circ g,$$

and similarly in the other slot.

Example 3.12.2. The category Vect_k is additive. Since vector spaces are in particular abelian groups under addition, it is naturally Ab-enriched,

Definition 3.12.4 (additive functor)**.** Let $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ be a functor between additive categories. We say that $\mathcal{F}$ is additive if for each X, $Y \in \mathrm{Obj}(\mathsf{C})$ the map

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a homomorphism of abelian groups.

Lemma 3.12.2. *For any additive functor* $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$*, there exists a natural isomorphism*

$$\Phi\colon \mathcal{F}(-) \oplus \mathcal{F}(-) \Rightarrow \mathcal{F}(- \oplus -).$$

Proof. The commutativity of the following diagram is immediate.

$$\begin{tikzcd}
\mathcal{F}(X \oplus Y) \arrow[r, "\mathcal{F}(f\oplus g)"] \arrow[d, "\Phi_{X,Y}"'] & \mathcal{F}(X' \oplus Y') \arrow[d, "\Phi_{X',Y'}"] \\
\mathcal{F}(X) \oplus \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)\oplus\mathcal{F}(g)"] & \mathcal{F}(X') \oplus \mathcal{F}(Y')
\end{tikzcd}$$

The (X, Y)-component $\Phi_{X,Y}$ is an isomorphism because □

Definition 3.12.5 (k-linear category)**.** An additive category C is k-linear if for all A, $B \in \mathrm{Obj}(\mathsf{C})$ the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of a k-vector space whose additive structure is the abelian structure, and for which the composition of morphisms is k-linear.

Example 3.12.3. The category Vect_k is k-linear.

Definition 3.12.6 (k-linear functor)**.** let C and D be two k-linear categories, and $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ a functor. Suppose that for all objects C, $D \in \mathrm{Obj}(\mathsf{C})$ all morphisms f, $g\colon C \to D$, and all α, $\beta \in k$, we have

$$\mathcal{F}(\alpha f + \beta g) = \alpha \mathcal{F}(f) + \beta \mathcal{G}(g).$$

Then we say that $\mathcal{F}$ is k-linear.

3.12.2 Pre-abelian categories

Definition 3.12.7 (pre-abelian category)**.** A category C is pre-abelian if it is additive and every morphism has a kernel (Definition 3.8.10) and a cokernel (Definition 3.8.12).

Lemma 3.12.3. *Pre-abelian categories have equalizers (Definition 3.8.8).*

Proof. It suffices to show that the kernel of $f-g$ satisfies the universal property for the equalizer of f and g.

Here is the diagram for the universal property of the kernel of $f-g$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!\bar{i}"] \arrow[drr, bend left] \arrow[ddr, bend right, "i"'] & & \\
& \ker(f-g) \arrow[r] \arrow[d, "\iota_{f-g}"'] & 0 \arrow[d] \\
& A \arrow[r, "f-g"] & B
\end{tikzcd}$$

The universal property tells us that for any object $Z \in \mathrm{Obj}(\mathsf{C})$ and any morphism $i\colon Z \to A$ with $i \circ (f-g) = 0$ (i.e. $i \circ f = i \circ g$), there exists a unique morphism $\bar{i}\colon Z \to \ker(f-g)$ such that $i = \bar{i} \circ \iota_{f-g}$. ☐

Corollary 3.12.1. *Every pre-abelian category has all finite limits.*

Proof. By Theorem 3.8.1, a category has finite limits if and only if it has finite products and equalizers. Pre-abelian categories have finite products by definition, and equalizers by Lemma 3.12.3. ☐

Definition 3.12.8 (zero morphism)**.** Let C be a category with zero object 0. For any two objects A, $B \in \mathrm{Obj}(\mathsf{C})$, the zero morphism $0_{A,B}$ is the unique morphism $A \to B$ which factors through 0.

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left, "0_{A,B}"] & 0 \arrow[r] & B
\end{tikzcd}$$

Notation 3.12.1. It will often be clear what the source and destination of the zero morphism are; in this case we will drop the subscripts, writing 0 instead of 0_{AB}.

It is easy to see that the left- or right-composition of the zero morphism with any other morphism results in the zero morphism: $f \circ 0 = 0$ and $0 \circ g = 0$.

Lemma 3.12.4. *Every morphism $f\colon A \to B$ in a pre-abelian catgory has a canonical decomposition*

$$\begin{tikzcd}
A \arrow[r, "p"] & \mathrm{coker}(\ker(f)) \arrow[r, "\bar{f}"] & \ker(\mathrm{coker}(f)) \arrow[r, "i"] & B
\end{tikzcd},$$

where p is an epimorphism (Definition 3.1.8) and i is a monomorphism (Definition 3.1.7).

Proof. We start with a map

$$\begin{tikzcd}
A \arrow[r, "f"] & B
\end{tikzcd}.$$

Since we are in a pre-abelian category, we are guaranteed that f has a kernel $(\ker(f), \iota)$ and a cokernel $(\mathrm{coker}(f), \pi)$. From the universality squares it is immediate that $f \circ \iota = 0$ and $\pi \circ f = 0$ This tells us that the composition $\pi \circ f \circ \iota = 0$, so the following commutes.

$$\begin{tikzcd}
& A \arrow[r, "f"] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & & & \mathrm{coker}(f) \\
& 0 \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

We know that π has a kernel $(\ker(\pi), i)$ and ι has a cokernel $(\operatorname{coker}(\iota), p)$, so we can add their commutativity squares as well.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) & \ker(\pi) \arrow[u, "i"'] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

If we squint hard enough, we can see the following diagram.

$$\begin{tikzcd}
A \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\
 & \ker(\pi) \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\
 & B \arrow[r, "\pi"] & \operatorname{coker}(f)
\end{tikzcd}$$

The outer square commutes because $\pi \circ f = 0$, so the universal property for $\ker(\pi)$ gives us a unique morphism $A \to \ker(\pi)$. Let's add this to our diagram, along with a morphism $0 \to \ker(\pi)$ which trivially keeps everything commutative.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) & \ker(\pi) \arrow[u, "i"'] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[r] \arrow[ur] & 0 \arrow[ur] &
\end{tikzcd}$$

Again, buried in the bowels of our new diagram, we find the following.

$$\begin{tikzcd}
\ker(f) \arrow[r, "\iota"] \arrow[d] & A \arrow[d, "\pi"] \arrow[ddr, bend left] & \\
0 \arrow[r] \arrow[drr, bend right] & \operatorname{coker}(\iota) & \\
 & & \ker(\pi)
\end{tikzcd}$$

And again, the universal property of cokernels gives us a unique morphism $\bar{f}\colon \operatorname{coker}(\iota) \to \ker(\pi)$.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"'] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[r] \arrow[ur] & 0 \arrow[ur] &
\end{tikzcd}$$

The fruit of our laborious construction is the following commuting square.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "i"] \\
\operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi)
\end{tikzcd}$$

We have seen (Lemma 3.8.2) that i is mono, and (Lemma 3.8.3) that p is epi.

Now we abuse terminology by calling $\iota = \ker(f)$ and $\pi = \operatorname{coker}(f)$. Then we have the required decomposition. □

Note 3.12.1. The abuse of notation above is ubiquitous in the literature.

3.12.3 Abelian categories

This section is under very heavy construction. Don't trust anything you read here.

Definition 3.12.9 (abelian category)**.** A pre-abelian category C is abelian if for each morphism f, the canonical morphism guaranteed by Lemma 3.12.4

$$\bar{f} \colon \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$$

is an isomorphism.

Note 3.12.2. The above piecemeal definition is equivalent to the following.

A category C is *abelian* if

1. it is Ab-enriched Definition 3.12.3, i.e. each hom-set has the structure of an abelian group and composition is bilinear;
2. it admits finite coproducts, hence (by Lemma 3.12.1) biproducts and zero objects;
3. every morphism has a kernel and a cokernel;
4. for every morphism, f, the canonical morphism $\bar{f} \colon \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$ is an isomorphism.

For the remainder of the section, let C be an abelian category.

Lemma 3.12.5. *In an abelian category, every morphism decomposes into the composition of an epimorphism and a monomorphism.*

Proof. For any morphism f, bracketing the decomposition $f = i \circ \bar{f} \circ p$ as

$$i \circ (\bar{f} \circ p).$$

gives such a composition. □

Note 3.12.3. The above decomposition is unique up to unique isomorphism.

Definition 3.12.10 (image of a morphism)**.** Let $f \colon A \to B$ be a morphism. The object $\ker(\operatorname{coker}(f))$ is called the image of f, and is denoted $\operatorname{im}(f)$.

Lemma 3.12.6.

1. *A morphism $f \colon A \to B$ is mono iff for all $Z \in \operatorname{Obj}(\mathsf{C})$ and for all $g \colon Z \to A$, $f \circ g = 0$ implies $g = 0$.*
2. *A morphism $f \colon A \to B$ is epi iff for all $Z \in \operatorname{Obj}(\mathsf{C})$ and for all $g \colon B \to Z$, $g \circ f = 0$ implies $g = 0$.*

Proof.

1. First, suppose f is mono. Consider the following diagram.

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

   If the above diagram commutes, i.e. if $f \circ g = 0$, then $g = 0$, so $1 \implies 2$.

   Now suppose that for all $Z \in \operatorname{Obj}(\mathsf{C})$ and all $g \colon Z \to A$, $f \circ g = 0$ implies $g = 0$.

   Let $g, g' \colon Z \to A$, and suppose that $f \circ g = f \circ g'$. Then $f \circ (g - g') = 0$. But that means that $g - g' = 0$, i.e. $g = g'$. Thus, $2 \implies 1$.

2. Dual to the proof above.

□

Lemma 3.12.7. *Let* $f\colon A \to B$*. We have the following.*

1. *The morphism* f *is mono iff* $\ker(f) = 0$
2. *The morphism* f *is epi iff* $\operatorname{coker}(f) = 0$.

Proof.

1. We first show that if $\ker(f) = 0$, then f is mono. Suppose $\ker(f) = 0$. By the universal property of kernels, we know that for any $Z \in \mathrm{Obj}(\mathsf{C})$ and any $g\colon Z \to A$ with $f \circ g = 0$ there exists a unique map $\bar{g}\colon Z \to \ker(f)$ such that $g = \iota \circ \bar{g}$.

   $$\begin{tikzcd}
   Z \arrow[dr, dashed, "\exists! g"] \arrow[ddr, bend right, "g"'] & & \\
   & \ker(f) = 0 \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
   & A \arrow[r, "f"] & B
   \end{tikzcd}$$

   But then g factors through the zero object, so we must have $g = 0$. This shows that $f \circ g = 0 \implies g = 0$, and by Lemma 3.12.6 f must be mono.

   Next, we show that if f is mono, then $\ker(f) = 0$. To do this, it suffices to show that $\ker(f)$ is final, i.e. that there exists a unique morphism from every object to $\ker(f)$.

   Since $\mathrm{Hom}_{\mathsf{C}}(Z, \ker(f))$ has the structure of an abelian group, it must contain at least one element. Suppose it contains two morphisms h_1 and h_2.

   $$\begin{tikzcd}
   Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] & & \\
   & \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
   & A \arrow[r, "f"] & B
   \end{tikzcd}$$

   Our aim is to show that $h_1 = h_2$. To this end, compose each with ι.

   $$\begin{tikzcd}
   Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] \arrow[ddr, bend right, shift left, "\iota\circ h_1"] \arrow[ddr, bend right, shift right, "\iota\circ h_2"'] & & \\
   & \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
   & A \arrow[r, "f"] & B
   \end{tikzcd}$$

   Since $f \circ \iota = 0$, we have $f \circ (\iota \circ h_1) = 0$ and $f \circ (\iota \circ h_2) = 0$. But since f is mono, by Lemma 3.12.6, we must have $\iota \circ h_1 = 0 = \iota \circ h_2$. But by Lemma 3.8.2, ι is mono, so again we have

   $$h_1 = h_2 = 0,$$

   and we are done.

2. Dual to the proof above.

□

Lemma 3.12.8. *We have the following.*

1. *The kernel of the zero morphism* $0 : A \to B$ *is the pair* $(A, 1_A)$.

2. *The cokernel of the zero morphism* $0 : A \to B$ *is the pair* $(B, 1_B)$.

Proof.

1. We need only verify that the universal property is satisfied. That is, for any object $Z \in \mathrm{Obj}(\mathsf{C})$ and any morphism $h\colon Z \to A$ such that $0 \circ g = 0$, there exists a unique morphism $\bar{g}\colon Z \to \ker(0)$ such that the following diagram commutes.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\bar{g}=g"] \arrow[ddr, bend right, "g"'] & & \\
& \ker(0) = A \arrow[r, "0"] \arrow[d, "\iota=1_A"'] & 0 \arrow[d] \\
& A \arrow[r, "0"] & B
\end{tikzcd}$$

But this is pretty trivial: $\bar{g} = g$.

2. Dual to above.

□

Theorem 3.12.1. *All abelian categories are binormal (Definition 3.8.15). That is to say:*

1. *all monomorphisms are kernels*

2. *all epimorphisms are cokernels.*

Proof.

1. Consider the following diagram taken Lemma 3.12.4, which shows the canonical factorization of any morphism f.

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \mathrm{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"] \arrow[d] & \mathrm{coker}(f) \\
& 0 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

By definition of a pre-abelian category, we know that $\bar{f}$ is an isomorphism.

□

Note 3.12.4. The above theorem is actually an equivalent definition of an abelian category, but the proof of equivalence is far from trivial.

Definition 3.12.11 (subobject, quotient object, subquotient object)**.** Let $Y \in \mathrm{Obj}(\mathsf{C})$.

1. A subobject of Y is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with a monomorphism $i\colon X \hookrightarrow Y$. If X is a subobject of Y we will write $X \subseteq Y$.

2. A quotient object of Y is an object Z together with an epimorphism $p\colon Y \twoheadrightarrow Z$.

3. A subquotient object of Y is a quotient object of a subobject of Y.

Lemma 3.12.9. *Let O be an object, Z be a subquotient of O, and Z' a subquotient of Z. Then Z' is a subquotient of O.*

Proof. A subquotient Z of O is a quotient Z of a subobject X of O.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, two heads] & O \\ Z & \end{tikzcd}$$

A subquotient Z' of Z looks like this.

$$\begin{tikzcd} & X \arrow[r, hook] \arrow[d, two heads] & O \\ X' \arrow[r, hook] \arrow[d, two heads] & Z & \\ Z' & & \end{tikzcd}$$

Take pullback yadda yadda. Will finish later. □

Definition 3.12.12 (quotient)**.** Let $X \subseteq Y$, i.e. let there exist a monomorphism $f\colon X \hookrightarrow Y$. The quotient Y/X is the cokernel $(\mathrm{coker}(f), \pi_f)$.

Example 3.12.4. Let V be a vector space, $W \subseteq V$ a subspace. Then we have the canonical inclusion map $\iota\colon W \hookrightarrow V$, so W is a subobject of V in the sense of Definition 3.12.11.

According to Definition 3.12.12, the quotient V/W is the cokernel $(\mathrm{coker}(\iota), \pi_\iota)$ of ι. We saw in Example 3.8.6 that the cokernel of ι was $V/\mathrm{im}(\iota)$. However, $\mathrm{im}(\iota)$ is exactly W! So the categorical notion of the quotient V/W agrees with the linear algebra notion.

3.12.4 Exact sequences

Definition 3.12.13 (exact sequence)**.** A sequence of morphisms

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is called exact in degree i if the image (Definition 3.12.10) of f_{i-1} is equal to the kernel (Definition 3.8.10) of f_i. A sequence is exact if it is exact in every degree.

Lemma 3.12.10. *If a sequence*

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is exact in degree i, then $f_i \circ f_{i-1} = 0$.

Definition 3.12.14 (short exact sequence)**.** A short exact sequence is an exact sequence of the following form.

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

This definition has some immediate trivial consequences.

Lemma 3.12.11. *Let*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence. Then

1. *f is mono*
2. *g is epi*
3. *$Z \simeq Y/X$.*

Proof.

1. An easy consequence of Lemma 3.12.7, part 1.
2. An easy consequence of Lemma 3.12.7, part 2.
3. We need to show that

□

Definition 3.12.15 (exact functor)**.** Let C, D be abelian categories, $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ a functor. We say that $\mathcal{F}$ is

- left exact if it preserves biproducts and kernels
- right exact if it preserves biproducts and cokernels
- exact if it is both left exact and right exact.

Theorem 3.12.2. *Let $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ be a functor between abelian categories. Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence in C. Then

- *if $\mathcal{F}$ is left exact, then*
  $$0 \longrightarrow \mathcal{F}(A) \longrightarrow \mathcal{F}(B) \longrightarrow \mathcal{F}(C)$$
  is an exact sequence in D.
- *if $\mathcal{F}$ is right exact, then*
  $$\mathcal{F}(A) \longrightarrow \mathcal{F}(B) \longrightarrow \mathcal{F}(C) \longrightarrow 0$$
  is an exact sequence in D.
- *if $\mathcal{F}$ is exact, then*
  $$0 \longrightarrow \mathcal{F}(A) \longrightarrow \mathcal{F}(B) \longrightarrow \mathcal{F}(C) \longrightarrow 0$$
  is an exact sequence in D.

Proof. □

3.12.5 Length of objects

Definition 3.12.16 (simple object)**.** A nonzero object $X \in \mathrm{Obj}(\mathsf{C})$ is called simple if 0 and X are its only subobjects.

Example 3.12.5. In Vect_k, the only simple object (up to isomorphism) is k, taken as a one-dimensional vector space over itself.

Definition 3.12.17 (semisimple object)**.** An object $Y \in \mathrm{Obj}(\mathsf{C})$ is semisimple if it is isomorphic to a dirct sum of simple objects.

Example 3.12.6. In Vect_k, all finite-dimensional vector spaces are simple.

Definition 3.12.18 (semisimple category)**.** An abelian category C is semisimple if every object of C is semisimple.

Example 3.12.7. The category $\mathsf{FinVect}_k$ is semisimple.

Definition 3.12.19 (Jordan-Hölder series)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. A filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

of X such that X_i/X_{i-1} is simple for all i is called a Jordan Hölder series for X. The integer n is called the length of the series X_i.

The importance of Jordan-Hölder series is the following.

Theorem 3.12.3 (Jordan-Hölder)**.** *Let X_i and Y_i be two Jordan-Hölder series for some object $X \in \mathrm{Obj}(\mathsf{C})$. Then the length of X_i is equal to the length of Y_i, and the objects Y_i/Y_{i-1} are a reordering of X_i/X_{i-1}.*

Proof. □

Definition 3.12.20 (length)**.** The length of an object X is defined to be the length of its Jordan-Hölder series.

Definition 3.12.21 (finite tensor category)**.** A k-tensor category A is called finite (over k) if

1. There are only finitely many simple objects in A, and each of them admits a projective presentation.[4]
2. Each object A of A is of finite length.
3. For any two objects A, B of A, the hom-object (i.e. k-vector space) $\mathrm{Hom}_{\mathsf{A}}(A,B)$ is finite-dimensional.

Example 3.12.8. The category $\mathsf{FinVect}_k$ is finite.

1. The only simple object is k taken as a one-dimensional vector space over itself.
2. The length of a finite-dimensional vector space is simply its dimension.
3. The vector space $\mathrm{Hom}_{\mathsf{FinVect}}(V,W)$ has dimension $\dim(V) \times \dim(W)$.

Definition 3.12.22 (finitely $\otimes$-generated)**.** A k-tensor category A is called finitely $\otimes$-generated if there exists an object $E \in \mathrm{Obj}(\mathsf{A})$ such that every other object $X \in A$ is a subquotient (Definition 3.12.11) of a finite direct sum of tensor products of E; that is to say, if there exists a finite collection of integers n_i such that X is a subquotient of $\bigoplus_i E^{\otimes^{n_i}}$.

$$\begin{tikzcd}
 & \bigoplus_i E^{\otimes^{n_i}} \arrow[d, two heads, "\pi"] \\
X \arrow[r, hook, "\iota"] & \left(\bigoplus_i E^{\otimes^{n_i}}\right)/Q
\end{tikzcd}$$

Example 3.12.9. The category $\mathsf{FinVect}_k$ is finitely generated since any finite-dimensional vector space is isomorphic to $k^n = k \oplus \cdots \oplus k$ for some n.

Definition 3.12.23 (subexponential growth)**.** A tensor category A has subexponential growth if, for each object X there exists a natural number N_X such that

$$\mathrm{len}(X^{\otimes_n}) \leq (N_X)^n.$$

Example 3.12.10. The monoidal category $\mathsf{FinVect}_k$ has subexponential growth. For any finite-dimensional vector space V, we always have

$$\dim(V^{\otimes^n}) = (\dim(V))^n,$$

so we can take $N_V = \dim(V)$.

3.13 Tensor Categories

The following definition is taken almost verbatim from [11].

Definition 3.13.1 (tensor category)**.** Let k be a field. A k-tensor category A (as considered by Deligne in [30]) is an

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[4] This is satisfied trivially by pretty much every 'obviously finite' category we'll look at.

1. essentially small (Definition 3.3.8)
2. k-linear (Definition 3.12.5)
3. rigid (Definition 3.11.7)
4. symmetric (Definition 3.10.9)
5. monoidal category (Definition 3.10.1)

such that

1. the tensor product functor $\otimes\colon \mathsf{A} \times \mathsf{A} \rightsquigarrow \mathsf{A}$ is, in both arguments separately,
    (a) k-linear (Definition 3.12.6)
    (b) exact (Definition 3.12.15)
2. $\mathrm{End}(1) \simeq k$, where End denotes the endomorphism ring (Definition 3.12.2).

3.14 Higher categories

This section follows [18] closely.

One of the main benefits of categories is that they let us distinguish between different notions of sameness. In a set, two elements are either the same or different, and that is that. In a category, however, objects can be the same in certain ways without being identical: there are many n-dimensional real vector spaces, but they are all isomorphic. However this notion of 'sameness without identity' is not omnipresent in category theory; we still have to resort to the limited, set-theoretic sense of equivalence when talking about morphisms.

In higher category theory, this is remedied by introducing 2-morphisms, which are 'morphisms between morphisms.' We can then say that two 1-morphisms are isomorphic if there is a pair of mutually inverse 2-morphisms between them. If you take a category and add 2-morphisms, what you get is called a 2-category.

$$\begin{tikzcd}
A \arrow[r, bend left=50, "f", ""{name=U, below}] \arrow[r, bend right=50, "g"', ""{name=D}] & B
\arrow[Rightarrow, from=U, to=D, shift right]
\arrow[Rightarrow, from=D, to=U, shift right]
\end{tikzcd}$$

Of course, we now have no notion of isomorphism for 2-categories, so we must add 3-morphisms, etc. The result of the nth iteration of this procedure is called an n-category. If you keep going forever (or more rigorously, formally consider the object you'd get if you did), you get an ∞-category.

You may have noticed that we have been using the same notation ($\Rightarrow$) for 2-morphisms and natural transformations. The reason for this, as you may have guessed, is that natural transformations are the morphisms in the 2-category Cat of (small) categories.

Chapter 4

A few concepts from algebraic geometry

4.1 Elementary notions

The elementary notions of algebraic geometry are assumed, but a few definitions are given here for concreteness.

Definition 4.1.1 (Zariski topology)**.** Let k be an algebraically closed field of characteristic 0. We define a topology on k^n, called the Zariski topology, as follows. Let $A = k[x_1, \ldots, x_n]$ be the ring of polynomials on k^n. For any subset $S \subseteq A$, define

$$Z(S) = \left\{ (x_1, \ldots, x_n) \in k^n \,\middle|\, f(x_1, \ldots, x_n) = 0 \text{ for all } f \in S \right\}.$$

Clearly, if $(S) = \mathfrak{a}$ is the ideal generated by S, then $Z(S) = Z(\mathfrak{a})$.

The Zariski topology on k^n is then the topology whose closed sets are given by $Z(\mathfrak{a})$ for ideals $\mathfrak{a}$ of A.

Definition 4.1.2 (affine space)**.** Denote by $\mathbb{A}^n$ the space k^n, together with the Zariski topology.

Note that if this were a real textbook on algebraic geometry, we would be careful about the definition of $\mathbb{A}^n$, defining it as a torsor over the the vector space k^n.

Definition 4.1.3 (algebraic set)**.** A subset $S \subseteq \mathbb{A}^n$ is called an algebraic set if it is closed in the Zariski topology.

Definition 4.1.4 (affine variety)**.** An affine variety is an irreducible algebraic set.

Lemma 4.1.1. *The maximal ideals of $A = k[x_1, \ldots, x_n]$ correspond to the points of $\mathbb{A}^n$.*

4.2 Sheaves

4.2.1 Presheaves

Definition 4.2.1 (category of opens)**.** Let (X, τ) be a topological space. The category of opens of X, $\mathsf{Open}(X)$ is the category whose objects are

$$\mathrm{Obj}(\mathsf{Open(X)}) = \tau$$

and whose morphisms are, for U, $V \in \tau$,

$$\mathrm{Hom}(U, V) = \begin{cases} r(V, U), & U \subseteq V \\ \emptyset, & \text{otherwise.} \end{cases}$$

To put it another way, if U and V are open sets in X, then there is exactly one morphism from U to V if $U \subseteq V$, and none otherwise.

Of course, we must check that $\mathsf{Open}(X)$ really is a category.

1. If $U \subseteq V$ and $V \subseteq W$, then $U \subseteq W$, so we are forced to define the composition
$$r(W,V) \circ r(V,U) = r(W,U).$$
2. The composition $\circ$ is associative since set inclusion is.
3. The identity morphism $r(U,U)$ is the identity with respect to composition: restricting a set to itself is the same thing as not restricting.

Definition 4.2.2 (presheaf)**.** A presheaf on a topological space (X, τ) is a contravariant functor from $\mathsf{Open}(X)$ to some category C. Usually, C will be the category Ring of rings.

Example 4.2.1 (important example)**.** The prototypical example of a presheaf on a topological space X is the presheaf $\mathcal{C}_X$ of real continuous functions on X. Since by Definition 4.2.2 $\mathcal{C}_X$ must be a contravariant functor, we need to specify what $\mathcal{C}_X$ does to the objects $\mathrm{Obj}(\mathsf{Open}(X))$ and the morphisms $\mathrm{res}_{V,U}$.

For every open set $U \in \mathrm{Obj}(\mathsf{Open}(X))$, define
$$\mathcal{C}_X(U) = \left\{ f \colon U \to \mathbb{R} \,\middle|\, f \text{ continuous} \right\},$$
the ring of all real continuous functions $U \to \mathbb{R}$.

Recall that for two open sets U and $V \in \mathrm{Obj}(\mathsf{Open}(X))$, there is a unique morphism $r(V,U)$ from U to V if and only if $V \subseteq U$. Thus, we need to assign to each $r(V,U)$ a ring homomorphism from $\mathcal{C}_X(V)$ to $\mathcal{C}_X(U)$. We use the restriction homomorphism, which maps $f \in \mathcal{C}_X(V)$ to $f|_U$, its restriction to U. We denote this by
$$\mathrm{res}_{V,U}(f) = f|_U.$$
In other words,
$$\mathcal{C}_X(r(V,U)) = \mathrm{res}_{V,U}.$$

Example 4.2.2. Let M be a C^∞ manifold, and let $\mathsf{Open}(M)$ be its category of opens. Then for each $U \in \mathrm{Obj}(\mathsf{Open}(M))$, define a functor $C^\infty \colon \mathsf{Open}(M)^{\mathrm{op}} \rightsquigarrow \mathsf{Ring}$ on objects by
$$C^\infty(U) = \left\{ f \colon U \to \mathbb{R} \,\middle|\, f \text{ smooth} \right\},$$
and on morphisms by restriction. Then C^∞ is a presheaf.

4.2.2 Sheaves

Definition 4.2.3 (sheaf)**.** Let X be a topological space. A presheaf $\mathcal{F}$ on X is called a sheaf if it satisfies the following:

1. **Identity:** Let $U \subseteq X$ be an open set, and let $\{U_i\}$ be an open cover of U. If f_1, $f_2 \in \mathcal{F}(U)$ such that
$$f_1|_{U_i} = f_2|_{U_i}$$
for all i, then $f_1 = f_2$. That is to say, if two sections of $\mathcal{F}$ over U agree on every element of an open cover of U, then they must agree on U.
2. **Glueability:** Suppose $\{U_i\}$ is an open cover of U, and $f_i \in \mathcal{F}(U_i)$ is a collection of sections of $\mathcal{F}$ such that if $U_i \cap U_j \neq \emptyset$ then
$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$
Then there exists some $f \in \mathcal{F}(U)$ such that
$$f|_{U_i} = f_i$$
for all i. In other words, If we have sections of an open cover of U which agree on overlaps, then we can glue them together to get a section on all of U.

Example 4.2.3 (important example continued)**.** The presheaf $\mathcal{C}_X$ from Example 4.2.1 is a sheaf, as is the presheaf from Example 4.2.2.

Example 4.2.4 (a presheaf which is not a sheaf)**.** Let $X = \mathbb{R}$, and define a presheaf $\mathcal{F}$ on X via

$$\mathcal{F}(U) = \left\{ f\colon U \to \mathbb{R} \,\middle|\, f \text{ bounded} \right\}.$$

This is a presheaf since the restriction of a bounded map is bounded.

However, we do not have condition 2 of Definition 4.2.3: glueability. To see this, consider the following open cover

$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} U_n, \qquad U_n = (n-1, n+1).$$

Clearly, the identity function 1_{U_n} on U_n is bounded for all n. However, the function $f : \mathbb{R} \to \mathbb{R}$ which agrees with 1_{U_n} on all the U_n is the identity function $1_{\mathbb{R}}$, which is unbounded.

Definition 4.2.4 (stalk)**.** Let X be a topological space, $x \in X$, and let $\mathcal{F}$ be a presheaf on X. The stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \left\{ (f, U) \,\middle|\, x \in U,\ f \in \mathcal{F}(U) \right\} / \sim,$$

where

$$(f, U) \sim (g, V)$$

if there exists an open set $W \subseteq U \cap V$ with $x \in W$ such that $f|_W = g|_W$.

Lemma 4.2.1. *If $\mathcal{F}$ is a sheaf of objects with some algebraic structure, like rings, then the stalk $\mathcal{F}_x$ inherits this algebraic structure.*

Proof. For the sake of concreteness, consider the case in which $\mathcal{F}$ is a presheaf of rings. Let $[(f, U)]$, $[(g, V)] \in \mathcal{F}_x$. We need to define

$$[(f, U)] + [(g, V)] \qquad \text{and} \qquad [(f, U)] \cdot [(g, V)].$$

In each case, we can simply use representatives of the equivalence classes, letting

$$[(f, U)] + [(g, V)] = [(f + g, U \cap V)], \qquad \text{and} \qquad [(f, U)] \cdot [(g, V)] = [(f \cdot g, U \cap V)].$$

We need of course to prove well-definition: that if $(f, U) \sim (f', U')$ and $(g, V) \sim (g', V')$, then

$$[f + g] = [f' + g'],$$

and similarly for multiplication.

But if $(f, U) \sim (f', U')$, then there exists an open set $U'' \in U \cap U'$ with $x \in U''$ such that $f|_{U''} = f'|_{U''}$, and similarly V''. But since $x \in U''$ and $x \in V''$, we must also have that $x \in U'' \cap V''$. Since f and f' (and g and g') agree on $U'' \cap V''$, so must $f + g$ and $f' + g'$. Thus $[(f + g, U \cap V)]$ is well-defined.

The proof of the well-definition of multiplication is exactly analogous. □

We now drop the open set in the notation of an element of a stalk, writing $[f]$ instead of $[(f, U)]$.

Example 4.2.5 (important example continued)**.** What do the stalks of $\mathcal{C}_X$ (Example 4.2.1) look like? Let $x \in X$, and define $\varphi\colon (\mathcal{C}_X)_x \to \mathbb{R}$ via

$$\varphi\colon [f] \mapsto f(x).$$

This is a ring homomorphism since

$$\varphi([f][g]) = \varphi([fg]) = (fg)(x) = f(x)g(x) = \varphi([f])\varphi([g]),$$

and similarly for addition. It is also surjective; to see this, we need only find a single function which maps to any $r \in \mathbb{R}$. (The constant function will do.)

Denote $\ker(\varphi) = \mathfrak{m}_x$. By the first isomorphism theorem,

$$(\mathcal{C}_X)_x/\mathfrak{m}_x \simeq \mathbb{R}.$$

But since $\mathbb{R}$ is a field, $\mathfrak{m}_x$ must be maximal.

If $\mathfrak{M} \neq \mathfrak{m}_x$ is another maximal ideal, then there must exist $[g] \in \mathfrak{M} \setminus \mathfrak{m}_x$ (since if $\mathfrak{M} \subseteq \mathfrak{m}_x$, $\mathfrak{M}$ isn't maximal). Since $[g] \notin \ker(\varphi)$, $g(x) \neq 0$. But since g is continuous, there exists a neighborhood V of x such that $g \neq 0$ on V.

But then g is invertible on V, and hence is invertible in the stalk $(\mathcal{C}_X)_x$; thus, $\mathfrak{M}$ contains an invertible element, hence contains the identity 1_R, hence is the whole ring; and we have a contradiction.

Thus each stalk of $\mathcal{C}_X$ has a unique maximal ideal.

4.2.3 Ringed spaces

Definition 4.2.5 (ringed space)**.** A ringed space is a double $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

Definition 4.2.6 (local ring)**.** A ring R is local if it has a unique maximal ideal.

Definition 4.2.7 (locally ringed space)**.** A locally ringed space is a ringed space whose stalks are local rings.

Example 4.2.6 (important example continued)**.** By the toil of Example 4.2.5, each stalk of $\mathcal{C}_X$ is a local ring, hence $\mathcal{C}_X$ is a locally ringed space.

Note 4.2.1. It is *not* necessary that each local section of a locally ringed space be a local ring; only the stalks must be local.

There are many ways to build sheaves. Here are two.

Definition 4.2.8 (restriction sheaf)**.** Let X be a topological space, and $\mathcal{F}$ a sheaf on X. Let U be an open subset of X. The restriction sheaf $\mathcal{F}|_U$ is the sheaf which maps open subsets $V \subseteq U$ to their images under $\mathcal{F}$:

$$\mathcal{F}_U(V) = \mathcal{F}(V), \qquad \text{for } V \subseteq U.$$

For open sets V, $W \subseteq U$, the restriction $r(W, V)$ is mapped to

$$\mathcal{F}|_U(r(W, V)) \equiv \mathcal{F}(r(W, V)) = \mathrm{res}_{W,V},$$

the same restriction as in $\mathcal{F}$.

Definition 4.2.9 (pushforward sheaf)**.** Let X and Y be topological spaces, $\pi \colon X \to Y$ a continuous function. Let $\mathcal{F}$ be a sheaf on X. Define the pushforward of $\mathcal{F}$ by π, denoted $\pi_*\mathcal{F}$, by

$$(\pi_*\mathcal{F})(V) = \mathcal{F}(\pi^{-1}(V)), \qquad V \subseteq Y \text{ open},$$

and

$$(\pi_*\mathcal{F})(r(V, U)) = \mathrm{res}_{\pi^{-1}(V), \pi^{-1}(U)}.$$

We can view sheaves as objects in a category. We can then talk about their morphisms, and find the correct definition of a sheaf isomorphism.

First, we need to define a homomorphism of presheaves. Since we defined a presheaf on a topological space X as a functor $\mathsf{Open}(X) \rightsquigarrow \mathsf{C}$ for some category C (Definition 4.2.2), we can define a sheaf homomorphism as a morphism in the functor category $\mathsf{C}^{\mathsf{Open}(X)}$, which is to say

Definition 4.2.10 (presheaf homomorphism)**.** Let $\mathcal{C}$, $\mathcal{D}$ be C-presheaves on a topological space X. A presheaf homomorphism $\mathcal{C} \to \mathcal{D}$ is a morphism in the category $\mathsf{C}^{\mathsf{Open}(X)}$, i.e. a natural transformation between $\mathcal{C}$ and $\mathcal{D}$.

Definition 4.2.11 (sheaf homomorphism)**.** Since sheaves are in particular presheaves, a sheaf homomorphism is simply a presheaf homomorphism between sheaves.

Definition 4.2.12 (sheaf isomorphism)**.** A sheaf isomorphism is defined in the obvious way.

Definition 4.2.13 (locally isomorphic)**.** Let X be a topological space, $\mathcal{F}$, and let $\mathcal{G}$ be C-sheaves on X. We say that $\mathcal{F}$ is locally isomorphic to $\mathcal{G}$ if for every $x \in X$, there exists a neighborhood U of x such that the restriction sheaf (Definition 4.2.8) $\mathcal{F}|_U$ is sheaf-isomorphic to $\mathcal{G}$.

4.3 Schemes

We follow loosely the treatment in [2].

Definition 4.3.1 (spectrum of a ring)**.** Let A be a ring. The set of prime ideals (Definition 2.3.13) is called the spectrum of A, and denoted $\operatorname{Spec}(A)$.

Definition 4.3.2 (distinguished open set)**.** Let A be a ring, $f \in A$. Define the distinguished open set

$$\mathcal{D}(f) = \left\{ \mathfrak{p} \in \operatorname{Spec}(A) \,\middle|\, f \notin \mathfrak{p} \right\} .$$

Example 4.3.1. Let $A = \mathbb{R}[x]$, the ring of polynomials in one variable.

Chapter 5

Spinors

5.1 A few more algebraic concepts

Definition 5.1.1 (graded algebra)**.** Let A be an algebra, G be a monoid. We say that A is G-graded if A decomposes additively as

$$A = \bigoplus_{i \in G} A_i$$

with the property that $A_i \cdot A_j \subseteq A_{i+j}$, where the addition $i + j$ is understood to be taking place in G.

Definition 5.1.2 (filtered algebra)**.** Let A be an algebra. A filtration on A is an increasing sequence

$$\{0\} \subset F_0 \subset F_2 \subset \cdots \subset A$$

of sub vector spaces of A such that

$$A = \bigcup_{i \geq 0} F_i,$$

and which are compatible with the multiplication law in the sense that

$$F_r \cdot F_s \subseteq F_{r+s}.$$

A filtered algebra is an algebra A together with a filtration $\{F_i\}_{i=0}^{\infty}$ of A.

Definition 5.1.3 (associated graded algebra)**.** Let A be an algebra with a filtration $\{F_i\}_{i=1}^{\infty}$. The associated graded algebra can be defined as follows.

Define vector spaces

$$G_i = F_i/F_{i-1} \quad \text{for } i \geq 1, \qquad G_0 = F_0.$$

Then for all $i \geq 1$, $F_i = G_i \oplus F_{i-1}$. Iterating this, we have

$$F_i = G_i \oplus G_{i-1} \oplus \cdots \oplus G_0,$$

and by induction or colimits or something

$$A = \bigoplus_{i=0}^{\infty} G_i,$$

where the equality is to be taken as equality of vector spaces.

In order to turn $\bigoplus_{i=0}^{\infty} G_i$ into a graded algebra we need to define multiplication on it in such a way that it is compatible with the grading, i.e. for $\alpha \in G_i$ and $\beta \in G_j$ we should have $\alpha \cdot \beta \in G_{i+j}$. Let $\alpha = \alpha_i + F_{i-1}$ be the equivalence class of α, and similarly $\beta = \beta_j + F_{j-1}$.

This can be accomplished by defining

$$\alpha\beta = (\alpha_i + F_{i-1})(\beta_j + F_{j-1}) = \alpha_i\beta_j + F_{i+j-1} \in G_{i+j}.$$

Note 5.1.1. Any graded algebra

$$A = \bigoplus_n A_n$$

is also a filtered algebra with

$$F_n = \bigoplus_{i=0}^{n} A_i.$$

Thus from any graded algebra one can, via the canonical filtration, arrive at an associated graded algebra. This is isomorphic to A as a vector space, but not in general as an algebra.

Definition 5.1.4 (tensor product of algebras)**.** Let R be a ring, and A and B be R-algebras. Since in particular every algebra is an R-module if you forget about the multiplication, we can take the tensor product of A and B as $R-$modules:

$$A \otimes_R B.$$

We can then define a multiplication on $A \otimes_R B$ by defining

$$(a \otimes b) \cdot (a' \otimes b') = (a \cdot a') \otimes (b \cdot b')$$

and extending via linearity.

The module $A \otimes_R B$ *together with* the multiplication rule described above is called the tensor product of A and B.

Note 5.1.2. This is a perfectly good definition of a tensor product for general algebras, and is even a graded algebra with the grading

$$(A \otimes B)_n = \bigoplus_{i+j=n} a_i \otimes b_j.$$

However, as we will see in the next chapter, for graded algebras there is exactly one other choice. In the language of category theory, there are two choices of braiding that make the category of G-graded vector spaces into a symmetric monoidal category. It turns out that this other choice is the more interesting one.

Definition 5.1.5 (graded tensor product of algebras)**.** Let A and B be $\mathbb{Z}_r$-graded algebras. Then we have a decomposition

$$A = \bigoplus_{i+j=n}^{r} A_i$$

and similarly for B. The graded tensor product, denoted $\hat{\otimes}$, of A and B is defined by the grading

$$(A \hat{\otimes} B)_n = \bigoplus_{i+j=n}^{r} A_i \otimes B_j$$

and the multiplication law

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b) \cdot \deg(c)} (ac) \otimes (bd),$$

where a-d are of pure degree.

Definition 5.1.6 (Koszul sign rule)**.** The multiplication law

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\deg(b) \cdot \deg(c)} (ac) \otimes (bd),$$

is called the Koszul sign rule.

Definition 5.1.7 ($\mathbb{Z}_2$-graded tensor product)**.** In the case we are interested in, $d = 2$. In this case we denote the tensor product of two $\mathbb{Z}_2$ graded algebras A and B by $A \hat{\otimes} B$, whose grading is

$$(A \otimes B)_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1), \qquad (A \otimes B)_1 = (A_0 \otimes B_1) \oplus (A_1 \otimes B_0).$$

5.2 Clifford algebras

Clifford algebras can be defined as universal objects in an appropriate category. We first define this category.

Definition 5.2.1 (quadratic vector space)**.** A quadratic vector space is a pair (V, q) where V is a vector space over a field k and q is a (possibly degenerate) quadratic form on V.

Note 5.2.1. Quadratic forms and symmetric bilinear forms can be freely exchanged via a polarization; we will use the same notation for both, differentiating them only be the number of arguments they take. For example,

$$q(v) = q(v, v).$$

Definition 5.2.2 (category of quadratic vector spaces)**.** Let QVec be the category whose objects are quadratic vector spaces and whose morphisms $(V, q_V) \to (W, q_W)$ are linear maps $f\colon V \to W$ such that $f^* q_W = q_V$.

Definition 5.2.3 (Clifford map)**.** Let (V, q) be a quadratic vector space and A be an associative k-algebra with unit 1_A. We say that a k-linear map $\varphi\colon V \to A$ is Clifford if for all $x \in V$,

$$\varphi(x)^2 = -q(x)1_A.$$

Definition 5.2.4 (category $\mathsf{Cliff}(V, q)$)**.** $\mathsf{Cliff}(V, q)$ is the category whose objects are Clifford maps $f\colon V \to A$ to some associative unital k-algebra A and whose morphisms are algebra homomorphisms $\rho\colon A \to A'$ such that the diagram

$$\begin{tikzcd}
 & V \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
A \arrow[rr, "\rho"] & & A'
\end{tikzcd}$$

commutes.

Definition 5.2.5 (Clifford algebra)**.** Let (V, Q) be a quadratic vector space. The Clifford algebra $\mathrm{C}\ell(V, q)$ is the algebra from the initial object $(\iota, \mathrm{C}\ell(V, q))$ in the category $\mathsf{Cliff}(V, q)$, if it exists.

Theorem 5.2.1. *The Clifford algebra always exists. That is to say for any quadratic vector space (V, q) there is a pair $(\iota, \mathrm{C}\ell(V, q))$ where ι is a Clifford map and $\mathrm{C}\ell(V, q)$ is an associative unital algebra such that for any Clifford map $f\colon V \to A$, there is a unique algebra homomorphism $\mathrm{C}\ell(f)\colon \mathrm{C}\ell(V) \to A$ such that the following diagram commutes.*

$$\begin{tikzcd}
 & V \arrow[dl, "\iota"'] \arrow[dr, "f"] & \\
\mathrm{C}\ell(V, q) \arrow[rr, "\mathrm{C}\ell(f)"] & & A
\end{tikzcd}$$

Proof. Denote by $T(V)$ the tensor algebra over V and by ϕ the canonical injection $V \hookrightarrow T(V)$.

The universal property of the tensor algebra tells us that any linear map $f\colon V \to A$ to an associative, unital algebra A can be extended to a unique algebra homomorphism $\bar{f}\colon T(V) \to A$ making the following diagram commute.

$$\begin{tikzcd}
 & V \arrow[dl, "\phi"'] \arrow[dr, "f"] & \\
T(V) \arrow[rr, "\bar{f}"] & & A
\end{tikzcd}$$

Now consider the case in which V is taken to be a quadratic vector space (V, q) and f is Clifford. Then the algebra homomorphism $\bar{f}$ has the property that for any $v \in V^{\otimes 1}$, $\bar{f}(v \otimes v + q(v)1) = 0$. Thus, all elements of $T(V)$ which can be written in the form

$$v \otimes v + q(v)1, \qquad v \in V^{\otimes 1}$$

are in the kernel of $\bar{f}$, and so must be the ideal generated by such elements, which we will call $\mathscr{I}_q$. We can take the quotient $T(V)/\mathscr{I}_q$, calling the associated canonical projection $\pi_q\colon T(V) \to T(V)/\mathscr{I}_q$.

Since $\mathscr{I}_q$ is in the kernel of f, we can replace f by a map $\tilde{f}\colon T(V)/\mathscr{I}_q \to A$. This allows us to fill in the diagram below.

$$\begin{tikzcd}
T(V) \arrow[dd, "\pi_q"'] \arrow[rrd, bend left, "\bar{f}"] & & \\
& V \arrow[lu, "\phi"'] \arrow[ld, "\pi_q\circ\phi"] \arrow[r, "f"] & A \\
T(V)/\mathscr{I}_q \arrow[rru, bend right, "\tilde{f}"'] & &
\end{tikzcd}$$

Notice that $\pi_q \circ \phi$ is Clifford.

The above discussion means that for any Clifford map $f\colon V \to A$, we have a unique algebra homomorphism $\tilde{f}\colon T(V)/\mathscr{I}_q \to A$ such that the diagram above commutes. But that means that the pair $(\pi_q \circ \phi, T(V)/\mathscr{I}_q)$ satisfies the universal property for Clifford algebras. Thus we must have that, up to a unique isomorphism, $\mathrm{C}\ell(V,q) = T(V)/\mathscr{I}_q$, and $\iota = \pi_q \circ \phi$. ☐

Theorem 5.2.2. *The Clifford algebra defines a functor* $\mathrm{C}\ell$ *from* QVec *to the category of associative unital algebras.*

Proof. We need two pieces of information to determine that $\mathrm{C}\ell$ is a functor: how it acts on the objects (V,q) of QVec, and how it acts on the morphisms $f\colon V \to V'$. We have studied in some detail how it acts on objects by sending (V,q) to $\mathrm{C}\ell(V,q)$. Now we must see how it acts on morphisms.

By definition, the morphisms $(V,q) \to (V',q')$ in QVec are linear maps $L\colon V \to V'$ which preserve the quadratic form. We can write down the following commutative diagram.

$$\begin{tikzcd}
(V,q) \arrow[r, "L"] \arrow[d, "\iota_V"'] \arrow[rd, "\iota_{V'}\circ L"] & (V',q') \arrow[d, "\iota_{V'}"] \\
\mathrm{C}\ell(V,q) & \mathrm{C}\ell(V',q')
\end{tikzcd}$$

Notice that the map L preserves the quadratic form q and $\iota_{V'}$ is Clifford. Thus, we have

$$[(\iota_{V'} \circ L)(v)]^2 = [\iota_{V'}(L(v))]^2 = \iota_{V'}(L(v)^2) = \iota_{V'}(-q'(L(v))) = \iota_{V'}(-q(v)1) = -q(v)1,$$

so the map $\iota_{V'} \circ L$ is Clifford. But the universal property of Clifford algebras guarantees us that any Clifford map from (V,q) to an associative, unital algebra (of which $\mathrm{C}\ell(V',q')$ is an example) must factor through $\mathrm{C}\ell(V,q)$! This gives us, for any such L, a unique map $\mathrm{C}\ell(V,q) \to \mathrm{C}\ell(V',q')$ which we will call $\mathrm{C}\ell(L)$. This fills in the diagram as follows.

$$\begin{tikzcd}
(V,q) \arrow[r, "L"] \arrow[d, "\iota_V"'] \arrow[rd, "\iota_{V'}\circ L"] & (V',q') \arrow[d, "\iota_{V'}"] \\
\mathrm{C}\ell(V,q) \arrow[r, "\exists!\mathrm{C}\ell(L)"'] & \mathrm{C}\ell(V',q')
\end{tikzcd}$$

Checking that the assignment $L \mapsto \mathrm{C}\ell(L)$ really is functorial amounts to checking that it is covariant and that it is associative, both of which follow from juxtaposition of diagrams. ☐

Note 5.2.2. The automorphism group of any quadratic vector space (V,q) is the group of linear transformations which preserve the quadratic form, and is called the *orthogonal group*, denoted

$$\mathrm{O}(V) = \left\{ M \in \mathrm{GL}(V) \,\middle|\, M^*q = q \right\}.$$

The functoriality of $\mathrm{C}\ell$ means that any $M \in \mathrm{O}(V)$ induces an algebra automorphsim of $\mathrm{C}\ell(M)$, the unique map which makes the following diagram commute.

$$\begin{tikzcd}
V \arrow[r, "M"] \arrow[d, "\iota"'] & V \arrow[d, "\iota"] \\
\mathrm{C}\ell(V,q) \arrow[r, "\mathrm{C}\ell(M)"] & \mathrm{C}\ell(V,q)
\end{tikzcd}$$

To put it another way, Theorem 5.2.2 ensures us of an embedding $\mathrm{O}(V,q) \hookrightarrow \mathrm{Aut}(\mathrm{C}\ell(V,q))$.

Theorem 5.2.3. *There is a $\mathbb{Z}_2$ grading on $\mathrm{C}\ell(V,q)$.*

Proof. Take $P \in \mathrm{O}(V)$ to be the map which sends $v \mapsto -v$ for all $v \in V$. By the functoriality of $\mathrm{C}\ell$, this extends to an algebra automorphism $\mathrm{C}\ell(P) = \alpha$ such that $\alpha^2 = 1_{\mathrm{C}\ell(V,q)}$. This means that $\mathrm{C}\ell(V,q)$ splits, as a vector space, into two parts

$$\mathrm{C}\ell(V,q) = \mathrm{C}\ell^0(V,q) \oplus \mathrm{C}\ell^1(V,q).$$

It is not hard to check that the multiplication has the grading in Definition 5.1.7. ☐

Lemma 5.2.1. *The Clifford algebra $\mathrm{C}\ell(V,q)$ inherits a filtration from the tensor algebra $T(V)$.*

Proof. The tensor algebra has a natural filtration given by

$$F_r = \bigoplus_{n=0}^{r} V^{\otimes n};$$

we have

$$T(V) = \bigcup_{i=0}^{\infty} F_i$$

and

$$F_r \otimes F_s \subseteq F_{r+s}.$$

This makes $T(V)$ into a filtered algebra.

Now consider the image of each F_r under the canonical projection $\pi_q \colon T(V) \to T(V)/\mathscr{I}_q$, say

$$\mathscr{F}_r = \pi_q(F_r).$$

Certainly, this makes $\mathrm{C}\ell(V,q)$ into a filtered vector space. We must check that the filtration respects the algebraic structure. We have, for $\alpha \in F_r$ and $\beta \in F_s$, $\alpha \otimes \beta \in F_{r+s}$, so

$$\pi_q(\alpha \otimes \beta) \in \mathscr{F}_{r+s}.$$

However, π_q is an algebra homomorphism, so

$$\pi_q(\alpha \otimes \beta) = \pi_q(\alpha) \cdot \pi_q(\beta) \in \mathscr{F}_r \cdot \mathscr{F}_s.$$

☐

Theorem 5.2.4. *For any quadratic form q, the associated graded algebra to $\mathrm{C}\ell(V,q)$ is isomorphic (as a vector space) to the exterior algebra $\Lambda(V)$.*

Proof. Very easy to see by writing everything down in a basis. I'd really like a categorical proof of this, however. ☐

Theorem 5.2.5. *Let $V = V_1 \oplus V_2$ be a q-orthogonal decomposition of the vector space V. Then there is an isomorphism of Clifford algebras*

$$\mathrm{C}\ell(V,q) \to \mathrm{C}\ell(V_1,q_1) \hat{\otimes} \mathrm{C}\ell(V_2,q_2).$$

Proof. Consider the map

$$f\colon V \to \mathrm{C}\ell(V_1, q_1)\hat{\otimes}\mathrm{C}\ell(V_2, q_2); \qquad v \mapsto v_1 \otimes 1 + 1 \otimes v_2$$

where $v = v_1 + v_2$ is the q-orthogonal decomposition of v.

$$\begin{aligned} f(v) \cdot f(v) &= (v_1 \otimes 1 + 1 \otimes v_2) \cdot (v_1 \otimes 1 + 1 \otimes v_2) \\ &= (v_1 \cdot v_1) \otimes 1 + v_1 \otimes v_2 - v_1 \otimes v_2 + 1 \otimes (v_2 \cdot v_2) \\ &= (v_1^2 + v_2^2)1 \\ &= -(q(v_1) + q(v_2))1 \\ &= -q(v)1, \end{aligned}$$

where the minus sign on the second line comes from the fact that the image of V under the canonical inclusion is in $\mathrm{C}\ell^1(V, q)$ and the Koszul sign rule. Thus the map f is Clifford, so it extends to an algebra homomorphism $\tilde{f}\colon \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V_1, q_1)\hat{\otimes}\mathrm{C}\ell(V_2, q_2)$.

This is injective since ☐

Definition 5.2.6 (transpose)**.** On any tensor power $V^{\otimes n}$ of a vector space V there is an action of the symmetric group S_n which sends

$$v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \qquad \sigma \in S_n.$$

In particular, for any n we have the involution

$$v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes \cdots \otimes v_1.$$

This extends to an algebra involution on

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n},$$

called the <u>transpose</u> and denoted

$$(v_1 \otimes \cdots \otimes v_n)^t = (v_n \otimes \cdots \otimes v_1).$$

Any element I of the ideal $\mathscr{I}_q$ can be written

$$I = \sum_i a_i \otimes (v_i \otimes v_i + q(v_i)1) \otimes b_i,$$

where a_i and b_i can be assumed to be of pure degree. Under the action of the transpose we have

$$I^t = \sum_i (b_i)^t \otimes (v_i \otimes v_i + q(v)1) \otimes (a_i)^t,$$

which is still in $\mathscr{I}_q$. Thus $\mathscr{I}_q$ is invariant under the transpose and the transpose extends to an involution on $\mathrm{C}\ell(V, q) = T(V)/\mathscr{I}_q$, which we will also denote by $(\cdot)^t$.

Note 5.2.3. The transpose is an antiautomorphism, i.e. $(\psi\varphi)^t = \varphi^t\psi^t$.

5.3 Pin and Spin

Definition 5.3.1 (group of units of $\mathrm{C}\ell(v, q)$)**.** Denote by <u>$\mathrm{C}\ell^\times(V, q)$</u> the group of units of $\mathrm{C}\ell(V, q)$.

Lemma 5.3.1 (properties of $\mathrm{C}\ell^\times(V, q)$)**.**

1. *For all* $v \in V$ *with* $q(v) \neq 0$, $v \in \mathrm{C}\ell^\times(V, q)$.
2. *When* $\dim(V) = n < \infty$ *and* $k = \mathbb{R}$ *(or* $\mathbb{C}$*),* $\mathrm{C}\ell^\times(V, q)$ *is a Lie group of dimension* 2^n.

3. The universal enveloping algebra of the Lie algebra $T_e(\mathrm{C}\ell^\times(V,q))$ is isomorphic to $\mathrm{C}\ell(V,q)$.

Proof.

1. Since $v \otimes v = -q(v)1$, $v \otimes (-v/q(v)) = 1$.
2. We specialize to the case $k = \mathbb{R}$; $k = \mathbb{C}$ is the same.

   As a vector space, $\mathrm{C}\ell(V,q) \simeq \Lambda(V) \simeq \mathbb{R}^{2n}$. Thus $\mathrm{C}\ell(V,q)$ can canonically be viewed both as topological manifold with a single chart and a topological vector space. On a finite-dimensional topological vector space any (multi)linear map is continuous, so addition and multiplication are continuous. Thus the group of units is a C^0-submanifold of $\mathrm{C}\ell(V,q)$, and multiplication restricts to a continuous map on it.

   But by a theorem of Gleason, Montgomery, and Zippin ([15]) any such topological group automatically has a compatible smooth atlas, and is therefore a Lie group.
3. This (or at least something equivalent to it) is just stated in [1]; I would have no idea how one would go about proving it.

☐

Definition 5.3.2 (adjoint representation of $\mathrm{C}\ell^\times(V,q)$)**.** The adjoint representation of $\mathrm{C}\ell^\times(V,q)$ is the map

$$\mathrm{Ad}\colon \mathrm{C}\ell^\times(V,q) \to \mathrm{Aut}(\mathrm{C}\ell(V,q)); \qquad \varphi \mapsto \mathrm{Ad}_\varphi = \varphi(\,\cdot\,)\varphi^{-1}$$

i.e. $\mathrm{Ad}_\varphi(x) = \varphi x \varphi^{-1}$.

Lemma 5.3.2. *For any v such that $q(v) \neq 0$, i.e. when v belongs to the set*

$$\tilde{V} = \left\{ v \in V \,\middle|\, q(v) \neq 0 \right\},$$

then Ad_v is an endomorphism on V. Furthermore, for any $w \in V$, $\mathrm{Ad}_v w$ has the explicit formula

$$-\mathrm{Ad}_v(w) = w - 2\frac{q(v,w)}{q(v)}v.$$

Proof. For any $v \in \tilde{V}$, $v^{-1} = -v/q(v)$. Thus for any $w \in V$,

$$-q(v)\mathrm{Ad}_v(w) = -q(v)vwv^{-1} = vwv = -v^2w - 2q(v,w)v = q(v)w - 2q(v,w)v.$$

Clearly, the assignment

$$w \mapsto w - 2\frac{q(v,w)}{q(v)}v$$

is a linear map $V \to V$, hence an endomorphism. ☐

Lemma 5.3.3. *For every $v \in \tilde{V}$, $\mathrm{Ad}_v \in \mathrm{O}(V)$.*

Proof. Let $w \in V$.

$$\begin{aligned} q(\mathrm{Ad}_v(w)) &= q\left(w - 2\frac{q(v,w)}{q(v)}v,\, w - 2\frac{q(v,w)}{q(v)}v \right) \\ &= q(w) - 4\frac{q(v,w)}{q(v)}q(v,w) + 4\frac{q(v,w)^2}{q(v)^2}q(v) \\ &= q(w), \end{aligned}$$

so $\mathrm{Ad}_v^*(q) = q$. ☐

Definition 5.3.3 (pin group, spin group)**.** Let $\mathrm{P}(V,q)$ denote the subgroup of $\mathrm{C}\ell^\times(V,q)$ generated by $\tilde{V}$. This group is not in and of itself terribly important, but it has two important subgroups:

- Pin(V, q), the subgroup of $\mathrm{P}(V, q)$ generated by those $v \in V$ with $q(v) = \pm 1$. That is to say, each element of Pin(V, q) can be written
$$v_1 v_2 \cdots v_r,$$
where each belongs to V, and $q(v_i) = \pm 1$ for all i.
- Spin(V, q), defined by
$$\mathrm{Spin}(V, q) = \mathrm{Pin}(V, q) \cap \mathrm{C}\ell^0(V, q).$$

Notation 5.3.1. For $v \in V$, denote by R_v the reflection across the hyperplane
$$v^\perp = \left\{ w \in V \,\middle|\, q(v, w) = 0 \right\}.$$

That is to say, R_v fixes $v^\perp$ and maps $v \to -v$. Explicitly,
$$R_v w = w - 2\frac{q(v, w)}{q(v)} v.$$

Theorem 5.3.1 (Cartan-Dieudonné). *Any orthogonal transformation can be expressed as a finite number of reflections. That is, for any $A \in \mathrm{O}(V, q)$, we have the decomposition*
$$A = R_{v_1} R_{v_2} \cdots R_{v_n}.$$
for some $n \in \mathbb{N}$ and some $\{v_i\}_{i=1}^n \subseteq V$.

From Lemma 5.3.2, we have the following expression for $\mathrm{Ad}_v(w)$ when $w \in \tilde{V}$:
$$-\mathrm{Ad}_v(w) = w - 2\frac{q(v, w)}{q(v)} v.$$

The RHS is simply the reflection of w over the hyperplane $v^\perp$, i.e.
$$-\mathrm{Ad}_v = R_v.$$

However, there is a pesky minus sign in the LHS of the above. We can rectify this by modifying our definition of the adjoint slightly.

Definition 5.3.4 (twisted adjoint representation). For $\varphi \in \mathrm{C}\ell^\times(V, q)$, define the twisted adjoint representation
$$\widetilde{\mathrm{Ad}}_\varphi \colon \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q); \qquad w \mapsto \alpha(\varphi) w \varphi^{-1}.$$

The word representation is justified: we can view $\widetilde{\mathrm{Ad}}$ as a map $\mathrm{C}\ell(V, q) \to \mathrm{GL}(\mathrm{C}\ell(V, q))$, and this map is a homomorphism since for all $w \in \mathrm{C}\ell^\times(V, q)$
$$\widetilde{\mathrm{Ad}}_{\varphi_1\varphi_2}(w) = \alpha(\varphi_1\varphi_2) \cdot w \cdot (\varphi_1\varphi_2)^{-1} = \alpha(\varphi_1) \left(\alpha(\varphi_2) \cdot w \cdot \varphi_2^{-1} \right) \varphi_1^{-1} = (\widetilde{\mathrm{Ad}}_{\varphi_1} \circ \widetilde{\mathrm{Ad}}_{\varphi_2})(w).$$

Note 5.3.1. The twisted adjoint representation fixes the problem it was meant to solve, i.e. for $v \in \tilde{V}$, $w \in V$,
$$\widetilde{\mathrm{Ad}}_v w = w - 2\frac{q(v, w)}{q(v)} v.$$

This is easy to see: $v \in \tilde{V} \subseteq V^{\otimes 1}$, so $\alpha(v) = -v$ and
$$\widetilde{\mathrm{Ad}}_v w = \alpha(v) w v^{-1} = -v w v^{-1} = -\mathrm{Ad}_v w.$$

Definition 5.3.5 ($\tilde{\mathrm{P}}(V, q)$). Let
$$\tilde{\mathrm{P}}(V, q) = \left\{ \varphi \in \mathrm{C}\ell^\times(V, q) \,\middle|\, \widetilde{\mathrm{Ad}}_\varphi V = V \right\},$$

the set of all elements of $\mathrm{C}\ell^\times(V, q)$ which fix V under the twisted adjoint action.

Lemma 5.3.4. $\mathrm{P}(V, q) \subseteq \tilde{\mathrm{P}}(V, q)$.

Proof. Let $\varphi \in \mathrm{P}(V, q)$. Then $\varphi = \varphi_1 \cdots \varphi_r$ for $\varphi_i \in \tilde{V}$. Since for any $v \in V$,

$$\widetilde{\mathrm{Ad}}_{\varphi_i} v = v - 2\frac{q(\varphi, v)}{q(\varphi)}\varphi \in V$$

and

$$\widetilde{\mathrm{Ad}_\varphi}(v) = (\widetilde{\mathrm{Ad}_{\varphi_1}} \circ \cdots \circ \widetilde{\mathrm{Ad}_{\varphi_r}})(v),$$

the result follows by induction. □

Theorem 5.3.2. *Suppose V is finite-dimensional and q is nondegenerate. Then the kernel of the homomorphism*

$$\widetilde{\mathrm{Ad}} \colon \tilde{\mathrm{P}}(V, q) \to \mathrm{GL}(V)$$

is exactly the group $k^\times$ of nonzero multiples of 1.

Proof. Choose an orthonormal basis for V, i.e. a basis $\{v_i\}_{i=1}^n$ for V such that $q(v_i, v_j) = \delta_{ij}$.

Suppose $\varphi \in \ker(\widetilde{\mathrm{Ad}})$, i.e.

$$\widetilde{\mathrm{Ad}}_\varphi(v) = v \qquad \text{for all } v \in V.$$

Then

$$\alpha(\varphi)v = v\varphi.$$

One can continue by showing that φ must be independent of each v_i, and thus equal to a nonzero constant, but this is not very elegant. I think there is probably a better way. □

Note 5.3.2. The image $\widetilde{\mathrm{Ad}}(\mathrm{P}(V, q))$ is exactly the subgroup of $\mathrm{O}(V, q)$ generated by reflections. But by Theorem 5.3.1 (Cartan-Dieudonné), any element $A \in \mathrm{O}(V, q)$ can be written as a product of reflections. Therefore the restriction $\widetilde{\mathrm{Ad}} \colon \mathrm{P}(V, q) \to \mathrm{O}(V, q)$ is surjective.

Definition 5.3.6 (norm map)**.** The norm map is the function

$$N \colon \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q); \qquad N \colon \varphi \mapsto \varphi \cdot \alpha(\varphi^t).$$

Lemma 5.3.5. *If V is finite-dimension and q is nondegenerate, then the restriction of N to $\tilde{\mathrm{P}}(V, q)$ is a homomorphism*

$$N \colon \tilde{\mathrm{P}}(V, q) \to k^\times.$$

Proof. By definition, for any $\varphi \in \tilde{\mathrm{P}}(V, q)$ we have

$$\alpha(\varphi) \cdot v \cdot \varphi^{-1} \in V.$$

The transpose $(\cdot)^t$ is the identity on V. Thus

$$\begin{aligned} \alpha(\varphi) \cdot v \cdot \varphi^{-1} &= (\alpha(\varphi) \cdot v \cdot \varphi^{-1})^t \\ &= (\varphi^{-1})^t \cdot v \cdot \alpha(\varphi^t). \end{aligned}$$

Thus

$$\begin{aligned} v &= \varphi^t \cdot \alpha(\varphi) \cdot v \cdot \varphi^{-1} \cdot \alpha(\varphi^t)^{-1} \\ &= [\varphi^t \alpha(\varphi)] v [\alpha(\varphi^t)\varphi]^{-1} \\ &= \alpha[\alpha(\varphi^t)\varphi] v [\alpha(\varphi^t)\varphi]^{-1} \\ &= \widetilde{\mathrm{Ad}}_{\alpha(\varphi^t)\varphi}(v), \end{aligned}$$

so $\alpha(\varphi^t)\varphi \in k^\times$. Applying α we find that

$$N(\varphi) = \varphi^t \alpha(\varphi) \in k^\times$$

as desired.

To see that N is indeed a homomorphism and not just a function, it is enough to see that for any φ, $\psi \in \tilde{\mathrm{P}}(V, q)$,

$$\begin{aligned} N(\varphi\psi) &= (\varphi\psi) \cdot \alpha(\varphi\psi)^t \\ &= \varphi\left(\psi \cdot \alpha(\psi)^t\right)\alpha(\varphi)^t \\ &= \varphi N(\psi)\alpha(\varphi)^t \\ &= \varphi \cdot \alpha(\varphi)^t N(\psi) \\ &= N(\varphi) \cdot N(\psi). \end{aligned}$$

□

Theorem 5.3.3. *For each $\varphi \in \tilde{\mathrm{P}}(V, q)$, the map $\widetilde{\mathrm{Ad}}_\varphi \colon V \to V$ preserves the quadratic form q, and thus belongs to $\mathrm{O}(V, q)$. This gives us a representation*

$$\widetilde{\mathrm{Ad}} \colon \tilde{\mathrm{P}}(V, q) \to \mathrm{O}(V, q).$$

Proof. First, note that $N(\alpha(\varphi)) = N(\varphi)$ since

$$N(\alpha(\varphi)) = \alpha(\varphi)\alpha(\alpha(\varphi)^t) = \alpha(\varphi\alpha(\varphi)^t) = \alpha(N(\varphi)) = N(\varphi).$$

Also, for all $v \in V$,

$$N(v) = v\alpha(v^t) = v \cdot (-v) = -v^2 = q(v).$$

Hence, for any $v \in \tilde{V}$ and any $\varphi \in \tilde{\mathrm{P}}(V, q)$, we have

$$N(\widetilde{\mathrm{Ad}}_\varphi(v)) = N(\varphi \cdot v \cdot \alpha(\varphi)^{-1}) = N(\varphi) \cdot N(v) \cdot N(\varphi^{-1}) = N(v) = q(v).$$

But by definition, $\widetilde{\mathrm{Ad}}_\varphi(v) \in V$, so

$$N(\widetilde{\mathrm{Ad}}_\varphi(v)) = q(\widetilde{\mathrm{Ad}}_\varphi(v)).$$

Thus, $q(\widetilde{\mathrm{Ad}}_\varphi(v)) = q(v)$, so $\widetilde{\mathrm{Ad}}_\varphi^*(q) = q$. □

Definition 5.3.7 ($\mathrm{SP}(V, q)$)**.** Let

$$\mathrm{SP}(V, q) = \mathrm{P}(V, q) \cap \mathrm{C}\ell^0(V, q),$$

i.e. the subset of $\mathrm{P}(V, q)$ generated by products of the form

$$v_1 \cdot v_2 \cdots v_n, \qquad n \text{ even.}$$

Lemma 5.3.6. *The restriction*

$$\widetilde{\mathrm{Ad}} \colon \mathrm{SP}(V, q) \to \mathrm{SO}(V, q)$$

is surjective.

Proof. Since for any $v \in \tilde{V}$ the determinant $\det(R_v) = -1$, any element R of $\mathrm{SO}(V, q)$ must be those expressible as an even number of reflections

$$R = R_{v_1} \circ R_{v_2} \cdots \circ R_{v_n} = R_{v_1 \cdot v_2 \cdots v_n}.$$

But since n is even $v_1 \cdot v_2 \cdots v_n \in \mathrm{SP}(V, q)$ by definition, so R is in the image $\widetilde{\mathrm{Ad}}(\mathrm{SP}(V, q))$. □

Note 5.3.3. We know that the maps

$$\widetilde{\mathrm{Ad}} \colon \mathrm{P}(V, q) \to \mathrm{O}(V, q) \qquad \text{and} \qquad \widetilde{\mathrm{Ad}} \colon \mathrm{SP}(V, q) \to \mathrm{SO}(V, q)$$

are surjective.

It is reasonable to wonder if the restrictions

$$\widetilde{\mathrm{Ad}} \colon \mathrm{Pin}(V, q) \to \mathrm{O}(V, q) \qquad \text{and} \quad \widetilde{\mathrm{Ad}} \colon \mathrm{Spin}(V, q) \to \mathrm{SO}(V, q)$$

are also surjective. This seems likely because R_v is invariant under the scaling of v, i.e. $R_{tv} = R_v$ for $t \neq 0$, so one could try to normalize any v so that $q(v) = \pm 1$.

This sort of works. Since q is quadratic, for any $v \in V$, $t \in k$ we have $q(tv) = t^2 q(v)$. The above restrictions will be surjective only if for any $a \neq 0$, we can solve at least one of the equations $t^2 = \pm a$. This leads us to the following definition.

Definition 5.3.8 (spin field)**.** A field k is called spin if for all $a \in k$ at least one of the equations

$$t^2 = a, \qquad t^2 = -a$$

can be solved.

Example 5.3.1. The fields $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_p$ where p is a prime congruent to 3 (mod 4) are spin. The field $\mathbb{Q}$ of rationals is not.

Lemma 5.3.7. *If V is a k-vector space and k is spin, then the restrictions*

$$\widetilde{\mathrm{Ad}}\colon \mathrm{Pin}(V,q) \to \mathrm{O}(V,q) \qquad \textit{and} \quad \widetilde{\mathrm{Ad}}\colon \mathrm{Spin} \to \mathrm{SO}(V,q)$$

are surjections.

Proof. We already sort of know how to prove this.

Suppose $O \in \mathrm{O}(V,q)$. Then we can express O as a finite product of reflections

$$O = R_{v_1} \circ R_{v_2} \circ \cdots \circ R_{v_n}, \qquad n \in \mathbb{N}.$$

But since k is by assumption spin we can find for each $i \in \{1,\dots,n\}$ a number t_i such that $q(t_i v_i) = \pm 1$. Since $R_{t_i v_i} = R_{v_i}$, we have

$$O = R_{t_1 v_1} \circ R_{t_2 v_2} \circ \cdots \circ R_{t_n v_n} = R_{t_1 v_1 \cdot t_2 v_2 \cdots t_n v_n},$$

where $t_1 v_1 \cdot t_2 v_2 \cdots t_n v_n \in \mathrm{Pin}(V,q)$.

The proof that the second restriction is a surjection is the same except that n must be even. □

Theorem 5.3.4. *Let V be a finite-dimensional vector space over a spin field k, and suppose q is a non-degenerate quadratic form on V. Then there are short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] & \mathrm{Spin}(V,q) \arrow[r, "\widetilde{\mathrm{Ad}}"] & \mathrm{SO}(V,q) \arrow[r] & 1 \\
0 \arrow[r] & F \arrow[r] & \mathrm{Pin}(V,q) \arrow[r, "\widetilde{\mathrm{Ad}}"] & \mathrm{O}(V,q) \arrow[r] & 1
\end{tikzcd}$$

where

$$F = \left\{\varphi \in k \,\middle|\, \varphi^2 = \pm 1\right\} = \begin{cases} \mathbb{Z}_2 = \{1,-1\} & \textit{if } \sqrt{-1} \notin k \\ \mathbb{Z}_4 = \left\{\pm 1, \pm\sqrt{-1}\right\} & \textit{otherwise.} \end{cases}$$

Proof. We already know that both $\mathrm{Pin}(V,q)$ and $\mathrm{Spin}(V,q)$ contain F so the maps from F are injective. We have just shown that $\widetilde{\mathrm{Ad}}$ is surjective. Therefore, all we have to show is that in both cases,

$$\ker(\widetilde{\mathrm{Ad}}) = F.$$

To this end, let $\varphi \in \mathrm{Pin}(V,q) = v_1 \cdots v_n$ be in $\ker(\widetilde{\mathrm{Ad}})$. Then

$$\varphi^2 = N(\varphi) = N(v_1)\cdots N(v_n) = v_1^2 \cdots v_n^2 = (\pm 1)\cdots(\pm 1) = \pm 1,$$

so $\varphi \in K$. In fact, this logic also holds for $\varphi \in \mathrm{Spin}(V,q)$ if we imagine that n is even. But we already know that $K \subseteq \ker(\widetilde{\mathrm{Ad}})$, so $K = \ker(\widetilde{\mathrm{Ad}})$ and we are done. □

Chapter 6

Some basic superalgebra

In this chapter we give some basic definitions of various super-* structures in a haphazard, rushed way. The clarifying categorical aspects will be left until the following chapter.

6.1 Super rings

Definition 6.1.1 (super ring)**.** A super ring (or $\mathbb{Z}/2\mathbb{Z}$ graded ring) A is, additively, an abelian group A with a direct sum decomposition

$$A = A_0 \oplus A_1.$$

Elements $a \in A_0$ are said to be of degree zero, denoted $\tilde{a} = 0$; elements $b \in A_1$ are said to be of degree one, denoted $\tilde{b} = 1$.

Additionally, there is a multiplicative structure which is associative and distributive from the left and the right; the only change from the normal definition of a ring is that the multiplicative structure must obey the axiom

$$\widetilde{ab} = \tilde{a} + \tilde{b},$$

where the addition is taken modulo 2.

Note 6.1.1. Of course, not all elements $c \in A$ are in A_0 or A_1; formulae which talk about the grading must be understood to be extended via additivity. The axiom concerning the multiplicative grading, for example, can be taken to say that if $a = a_0 + a_1$ and $b = b_0 + b_1$, then

$$ab = a_0 b_0 + a_1 b_0 + a_0 b_1 + a_1 b_1,$$

and the axiom applies to each monomial.

Example 6.1.1 (exterior algebra)**.** Consider the exterior algebra

$$\bigwedge \mathbb{R}^n = \bigoplus_{i=0}^{n} \bigwedge^i \mathbb{R}^n.$$

With multiplication given by the wedge product, $\bigwedge \mathbb{R}^n$ is a super ring with grading given by

$$\left(\bigoplus_{i \text{ even}} \bigwedge^i \mathbb{R}^n \right) \oplus \left(\bigoplus_{i \text{ odd}} \bigwedge^i \mathbb{R}^n \right).$$

Example 6.1.2. Let V be a vector space over a field k of characteristic not 2. The so-called *parity transformation* on V is the map

$$P \colon V \to V; \qquad v \mapsto -v.$$

This preserves any quadratic form, and hence extends to an algebra homomorphism $\mathrm{C}\ell(P) \equiv \alpha$ of any quadratic vector space $\mathrm{C}\ell(V, q)$. Since

$$\alpha^2 = \mathrm{C}\ell(P) \circ \mathrm{C}\ell(P) = \mathrm{C}\ell(P \circ P) = \mathrm{C}\ell(\mathrm{id}_{(V,q)}) = \mathrm{id}_{\mathrm{C}\ell(V,q)},$$

α is invertible, hence an isomorphism. Indeed, from the above equation we can say more. For the moment ignoring the multiplicative structure of $\mathrm{C}\ell(V,q)$ and considering it only as a vector space, α can be thought of as a linear bijection whose square is the identity. This means α has two eigenvalues $+1$ and -1, and additively $\mathrm{C}\ell(V,q)$ decomposes into a direct sum

$$\mathrm{C}\ell(V,q) = \mathrm{C}\ell^0(V,q) \oplus \mathrm{C}\ell^1(V,q),$$

where

$$\mathrm{C}\ell^0(V,q) = \left\{\varphi \in \mathrm{C}\ell(V,q) \,\middle|\, \alpha(\varphi) = \varphi\right\}, \qquad \text{and} \qquad \mathrm{C}\ell^0(V,q) = \left\{\varphi \in \mathrm{C}\ell(V,q) \,\middle|\, \alpha(\varphi) = -\varphi\right\}.$$

For $\varphi_1 \in \mathrm{C}\ell^i(V,q)$ and $\varphi_2 \in \mathrm{C}\ell^j(V,q)$,

$$\alpha(\varphi_1\varphi_2) = \alpha(\varphi_1)\alpha(\varphi_2);$$

this gives us the following multiplication table.

$\times$	$\mathrm{C}\ell^0(V,q)$	$\mathrm{C}\ell^1(V,q)$
$\mathrm{C}\ell^0(V,q)$	$\mathrm{C}\ell^0(V,q)$	$\mathrm{C}\ell^1(V,q)$
$\mathrm{C}\ell^1(V,q)$	$\mathrm{C}\ell^1(V,q)$	$\mathrm{C}\ell^0(V,q)$

Thus, $\mathrm{C}\ell(V,q)$ is a superalgebra.

Definition 6.1.2 (supercommutator)**.** Let A be a commutative ring. The supercommutator is the map

$$[\cdot,\cdot]\colon A \times A \to A; \qquad (a,b) \to [a,b] = ab - (-1)^{\tilde{a}\cdot\tilde{b}}ba.$$

Definition 6.1.3 (supercommutation)**.** We will say that a supercommutes with b if $[a,b] = 0$; this means in particular that two even elements commute if $ab = ba$, and two odd elements supercommute if $ab = -ba$.

Note 6.1.2. Supercommutativity should not be confused with regular commutativity. Here is an example of a super ring which is commutative but not supercommutative.

Example 6.1.3. Consider the ring $C^0(\mathbb{R})$ of continuous functions $\mathbb{R} \to \mathbb{R}$. This is a super ring with the direct sum decomposition is given by

$$C^0(\mathbb{R}) = \{\text{even functions}\} \oplus \{\text{odd functions}\};$$

this can be seen since any $f \in C^0(\mathbb{R})$ can be written

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}.$$

The multiplication defined pointwise inherits its commutativity from the real numbers; but *all* elements are commutative, even the odd elements. Hence $C^0(\mathbb{R})$ is commutative but not supercommutative.

Definition 6.1.4 (super ring homomorphism)**.** Let A, B be super rings. A super ring homomorphism $f\colon A \to B$ is a ring homomorphism which preserve the grading.

One might wonder if, in fact, *all* ring homomorphisms between super rings are super ring homomorphisms. This is not the case.

Counterexample 6.1.1 (Not all super ring homomorphisms preserve the grading)**.** Consider the set

$$R = \left\{f\colon \mathbb{R} \to \bigwedge \mathbb{R}^2\right\}.$$

A general element $f \in R$ is of the form

$$f(x) = f_{00}(x) + f_{10}(x)\hat{\boldsymbol{x}} + f_{01}(x)\hat{\boldsymbol{y}} + f_{11}(x)\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}},$$

where $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are basis vectors for $\mathbb{R}^2$. Define multiplication pointwise:

$$\begin{aligned}(f\cdot g)(x) = f(x)g(x) =& f_{00}(x)g_{00}(x)\\ &+ (f_{00}(x)g_{10}(x) + f_{10}(x)g_{00}(x))\hat{\boldsymbol{x}}\\ &+ (f_{00}(x)g_{01}(x) + f_{01}(x)g_{00}(x))\hat{\boldsymbol{y}}\\ &+ (f_{11}(x)g_{00}(x) + f_{00}(x)g_{11}(x) + f_{10}(x)g_{01}(x) - f_{01}(x)g_{10}(x))\hat{\boldsymbol{x}}\wedge\hat{\boldsymbol{y}}.\end{aligned}$$

Then R becomes a ring. In fact it is more; it is easy (if tedious) to check from the above multiplication law that R can be seen as a super ring with grading given by

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{R_0} + \underbrace{\frac{f(x)-f(-x)}{2}}_{R_1}.$$

Now consider the evaluation homomorphism

$$\iota\colon R \to \bigwedge\mathbb{R}^2; \qquad f\mapsto f(0).$$

This is a ring homomorphism which does not preserve grading. To see this, consider the constant function

$$\varphi(x) = \hat{\boldsymbol{x}} \qquad \text{for all } x.$$

Non-zero constant functions such as φ are even, but $\hat{\boldsymbol{x}}$ is odd in $\bigwedge\mathbb{R}^2$.

Definition 6.1.5 (supercenter)**.** The supercenter of A is the set

$$Z(A) = \left\{a\in A \,\middle|\, \text{for all } b\in A,\ [a,b]=0\right\}.$$

In the theory of unexceptional (=not super) algebra, we have Definition 2.6.1 of an algebra over a ring. This definition generalizes naturally to the case where A and R are super rings:

Definition 6.1.6 (super algebra over a ring)**.** Let R be a supercommutative super ring. An R-super-algebra is a super ring A which is also an R-module (with left multiplication $*\colon R\times A\to A$) such that the multiplication map $\cdot\colon A\times A\to A$ is R-bilinear, i.e.

$$r*(a\cdot b) = (r*a)\cdot b = (-1)^{\tilde{a}\cdot\tilde{r}}a\cdot(r*b).$$

Theorem 2.6.1 also generalizes nicely.

Theorem 6.1.1. *Let A, R be unital super rings, R supercommutative, and let $f\colon R\to A$ be a unital super ring homomorphism. Then A naturally has the structure of an R-module. Furthermore, if the image $f(R)$ is in the supercenter of A, then A naturally has the structure of an R-super algebra.*

Proof. The module axioms (Definition 2.4.1) don't involve commutivity, so their verification is trivial as before. We must only check that if $f(R)$ is in the supercenter of A then $*$ is R-bilinear, i.e. that

$$r*(a\cdot b) = f(r)\cdot a\cdot b = (f(r)\cdot a)\cdot b = (r*a)\cdot b,$$

and

$$r*(a\cdot b) = f(r)\cdot a\cdot b = (-1)^{\tilde{a}\cdot\widetilde{f(r)}}a\cdot f(r)\cdot b = (-1)^{\tilde{a}\cdot\tilde{r}}a\cdot f(r)\cdot b = (-1)^{\tilde{a}\cdot\tilde{r}}(r*b).$$

□

Theorem 6.1.2. *Let R be a supercommutative super ring, $r\in R_1$. Then r is nilpotent (Definition 2.4.3).*

Proof.

$$r^2 = \frac{1}{2}[r,r] = 0.$$

□

We have the following analog to Theorem 2.4.2.

Lemma 6.1.1. *Let B be an associative superring. The supercommutator satisfies the following identities.*

- $[a,b] = -(-1)^{\tilde{a}\cdot\tilde{b}}[b,a]$.
- $[a,[b,c]] + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}[b,[c,a]] + (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}[c,[a,b]] = 0$

Furthermore, if B is an A-algebra, then the following holds for all $a \in A$, $b, c \in B$.

- $a[b,c] \overset{1}{=} [ab,c] \overset{2}{=} (-1)^{\tilde{a}\cdot\tilde{b}}[ba,c] \overset{3}{=} (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}[b,ca] \overset{4}{=} (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}[b,c]a$

Proof.

- $[a,b] = ab - (-1)^{\tilde{a}\cdot\tilde{b}}ba = ba(-1)^{\tilde{a}\cdot\tilde{b}} - \left((-1)^{\tilde{a}\cdot\tilde{b}}\right)^2 ab = (-1)^{\tilde{a}\cdot\tilde{b}}\left(ba - (-1)^{\tilde{a}\cdot\tilde{b}}ab\right) = (-1)^{\tilde{a}\cdot\tilde{b}}[b,a]$
- Nightmarish.

  $$\begin{aligned} [a,[b,c]] &= a(bc - (-1)^{\tilde{b}\cdot\tilde{c}}cb) - (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}(bc - (-1)^{\tilde{b}\cdot\tilde{c}}cb)a \\ [b,[c,a]] &= b(ca - (-1)^{\tilde{c}\cdot\tilde{a}}ac) - (-1)^{\tilde{b}(\tilde{c}+\tilde{a})}(ca - (-1)^{\tilde{c}\cdot\tilde{a}}ac)b \\ [c,[a,b]] &= c(ab - (-1)^{\tilde{a}\cdot\tilde{b}}ba) - (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}(ab - (-1)^{\tilde{a}\cdot\tilde{b}}ba)c \end{aligned}$$

  Truly nightmarish.
- My apologies for the formatting.

Equality 1:

$$\begin{aligned} a[b,c] = a(bc - (-1)^{\tilde{b}\cdot\tilde{c}}cb) = abc - (-1)^{\tilde{b}\cdot\tilde{c}}acb = abc - (-1)^{\tilde{b}\cdot\tilde{c}}(-1)^{\tilde{a}\cdot\tilde{c}}cab \\ = (ab)c - (-1)^{\tilde{ab}\cdot\tilde{c}}c(ab) = [ab,c] \end{aligned}$$

Equality 2:

$$[ab,c] = (ab)c - (-1)^{\widetilde{ab}\cdot\tilde{c}}c(ab) = (-1)^{\tilde{a}\cdot\tilde{b}}\left((ba)c - (-1)^{\widetilde{ab}\cdot\tilde{c}}c(ba)\right) = (-1)^{\tilde{a}\cdot\tilde{b}}[ba,c]$$

Equality 3:

$$\begin{aligned} (-1)^{\tilde{a}\cdot\tilde{b}}[ba,c] &= (-1)^{\tilde{a}\cdot\tilde{b}}\left((ba)c - (-1)^{\widetilde{ab}\cdot\tilde{c}}c(ba)\right) \\ &= (-1)^{\tilde{a}\cdot\tilde{b}}\left(b(ac) - (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}(-1)^{\tilde{a}\cdot\tilde{b}}(-1)^{\tilde{a}\cdot\tilde{c}}(ac)b\right) = (-1)^{\tilde{a}\cdot\tilde{b}}\left(b(ac) - (-1)^{\tilde{b}(\tilde{a}+\tilde{c})}(ac)b\right) \\ &= (-1)^{\tilde{a}\cdot\tilde{b}}[b,ac] \end{aligned}$$

Equality 4: Similar.

□

6.2 Supermodules

The definition of a supermodule is very similar to that of a regular module (Definition 2.4.1).

Definition 6.2.1 (supermodule)**.** Let A be an abelian group with a $\mathbb{Z}/2\mathbb{Z}$ grading, i.e.

$$A = A_0 \oplus A_1.$$

Further, let R be a superring with identity. A (left) supermodule is a triple $(A, R, *)$, with A and R as above and $*$ a function $R \times A \to A$ such that

1. if $r \in R_i$ and $a \in A_j$ then $r * a \in A_{i+j \mod 2}$.
2. $*$ satisfies all the module axioms listed in Definition 2.4.1.

If R is supercommutative and A is a left supermodule, we can make A into a right supermodule or a super bimodule as in Theorem 2.4.1.

Definition 6.2.2 (super bimodule)**.** An abelian group A which is both a left and a right R module is a super bimodule if left and right multiplication obey the compatibility condition

$$ra = (-1)^{\tilde{r}\cdot\tilde{a}} ar \qquad \text{for all } a \in A, \quad r \in R.$$

Lemma 6.2.1. *Let R be a supercommutative superring, A be a left R-module. Then A can naturally be given the structure of a right R-module or a bimodule.*

Proof. Define a right action $ra \equiv (-1)^{\tilde{r}\cdot\tilde{a}} ar$. This makes A into a right R-module, and □

Definition 6.2.3 (superlinear map)**.** Let R be a super ring, A and B be R-supermodules. A map $f \colon A \to B$ is called

6.3 Super vector spaces

Definition 6.3.1 ($\mathbb{Z}_2$-graded vector space)**.** A $\mathbb{Z}_2$-graded vector space over a field k (for simplicity of characteristic zero) is a $\mathbb{Z}_2$-graded vector space

$$V = V_0 \oplus V_1.$$

As before elements of V_0 are *even*, elements of V_1 are *odd*, and elements of $V_1 \cup V_2 \setminus \{0\}$ are *homogeneous.*

Definition 6.3.2 (dimension of a $\mathbb{Z}_2$-graded vector space)**.** Let

$$V = V_0 \oplus V_1.$$

be a $\mathbb{Z}_2$-graded vector space. The dimension of V is

$$(\dim V_0 | \dim V_1) \in (\mathbb{N} \cup \{\infty\})^2.$$

Definition 6.3.3 (finite-dimensional $\mathbb{Z}_2$-graded vector space)**.** We say that a $\mathbb{Z}_2$-graded vector space V is finite dimensional if V_1 and V_2 are.

Definition 6.3.4 ($\mathbb{Z}_2$-graded vector space morphism)**.** Let V and W be $\mathbb{Z}_2$-graded vector spaces. A morphism from V to W (also called a $\mathbb{Z}_2$-graded linear map) is a linear map $V \to W$ which preserves the grading, i.e. which maps $V_0 \to W_0$ and $V_1 \to W_1$.

Definition 6.3.5 (direct sum of $\mathbb{Z}_2$-graded vector spaces)**.** Let V and W be $\mathbb{Z}_2$-graded vector spaces. Their direct sum, denoted $V \oplus W$, is the $\mathbb{Z}_2$-graded vector space with grading

$$(V \oplus W)_i = V_i \oplus W_i, \qquad i = 1, 2.$$

Definition 6.3.6 (tensor product of $\mathbb{Z}_2$-graded vector spaces)**.** Let V and W be $\mathbb{Z}_2$-graded vector spaces. Their tensor product denoted $V \otimes W$, is the tensor product of the underlying ungraded vector spaces, with the grading

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes V_1), \qquad (V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes V_0).$$

6.4 Superalgebras

Definition 6.4.1 (superalgebra)**.** A superalgebra A is a $\mathbb{Z}/2$-graded vector space $A = A_0 \oplus A_1$ together with a bilinear map

$$\cdot \colon A \times A \to A,$$

which respects the grading in the sense that $A_i \cdot A_j \subseteq A_{i+j \mod(2)}$.

Definition 6.4.2 (supercommutative superalgebra)**.** A supercommutative superalgebra is a superalgebra which obeys the Koszul sign rule, i.e. $\tilde{a \cdot b} = \tilde{a} \cdot \tilde{b}$ for elements a, b of pure degree.

Chapter 7

Superalgebra and supergeometry in categories

Until now, we have been defining super-* strucutures haphazardly, imposing the Koszul sign rule by hand as we went. Seemingly miraculously, this worked in the sense that if one added the appropriate sign changes to the definitions, they would turn up in the right places in the theorems. In the first part of this chapter, we will see that the appearance of the Koszul sign rule is much more natural: it is a manifestation of the fact that the supercommutativity is really just a different sort of commutativity, one which shows up naturally in the study of $\mathbb{Z}_2$-graded vector spaces.

In the second section, we will apply the machinery of algebraic geometry to our new understanding of supercommutative algebra, and arrive at a natural definition of a supermanifold.

7.1 Supercommutativity

The goal of this section is to explain the ubiquity of supercommutativity (i.e. the Koszul sign rule) in the study of $\mathbb{Z}_2$-graded spaces. Stated roughly, the reason is this: it is the only possible multiplication law on a $\mathbb{Z}_2$-graded algebra which knows about the grading. That is to say, the only other possible multiplication law (up to an appropriate sort of isomorphism) on a $\mathbb{Z}_2$-graded algebra is the trivial one, which is equivalent to treating the vector space as ungraded.

Of course, this doesn't explain why we should care about $\mathbb{Z}_2$-graded vector spaces in the first place. The reason for their importance to theoretical physics is Deligne's theorem, which is the topic of these notes.

Definition 7.1.1 (category of $\mathbb{Z}_2$-graded vector spaces)**.** The category of $\mathbb{Z}_2$-graded vector spaces, i.e. the category whose objects are $\mathbb{Z}_2$-graded vector spaces (Definition 6.3.1) and whose morphisms are $\mathbb{Z}_2$-graded vector space morphisms, is notated $\mathsf{Vect}_k^{\mathbb{Z}_2}$.

Note 7.1.1. We call this category the category of $\mathbb{Z}_2$-graded vector spaces rather than the category of super vector spaces because we will reserve the latter name for the category with the appropriate symmetric monoidal structure, i.e. that which yields the Koszul sign rule. We will explore this structure now.

Lemma 7.1.1. *The $\mathbb{Z}_2$-graded tensor product (Definition 5.1.7) can be extended to a bifunctor*

$$\otimes \colon \mathsf{Vect}_k^{\mathbb{Z}_2} \times \mathsf{Vect}_k^{\mathbb{Z}_2} \rightsquigarrow \mathsf{Vect}_k^{\mathbb{Z}_2}.$$

Proof. The behavior of the $\mathbb{Z}_2$ tensor product on objects and morphisms is inherited from the ungraded tensor product. It is not hard to check that the tensor product of two $\mathbb{Z}_2$-graded linear maps defined in this way is $\mathbb{Z}_2$-graded. □

Theorem 7.1.1. *There are, up to braided monoidal equivalence of categories (that is to say, a categorical equivalence Definition 3.3.7 whose functors are braided monoidal (Definition 3.10.8)), only two choices for a symmetric braiding γ (Definition 3.10.9) on $\mathsf{Vect}_k^{\mathbb{Z}_2}$, with components*

$$\gamma_{V,W} \colon V \otimes W \to W \otimes V :$$

1. *The* trivial braiding, *which acts on representing tuples (Definition 2.7.4) by*

$$\gamma_{V,W}\colon (v,w)\mapsto (w,v)$$

2. *The* super braiding, *which acts on representing tuples of pure degree by*

$$\gamma_{V,W}\colon (v,w)\mapsto (-1)^{\tilde{v}\cdot\tilde{w}}(w,v).$$

Proof. Consider the category Line($\mathsf{Vect}_k^{\mathbb{Z}_2}$) □

Definition 7.1.2 (category of super vector spaces)**.** The category of super vector spaces, denoted SVect_k, is the category $\mathsf{Vect}_k^{\mathbb{Z}_2}$ together with the super braiding γ.

Lemma 7.1.2. *The evident forgetful functor*

$$\mathcal{U}\colon \mathsf{SVect}_k \rightsquigarrow \mathsf{Vect}_k$$

is strong monoidal (Definition 3.10.5).

Proof. We need to find a natural isomorphism Φ with components

$$\Phi_{X,Y}\colon \mathcal{U}(X)\otimes\mathcal{U}(Y)\to\mathcal{U}(X\otimes Y)$$

and an isomorphism

$$\varphi\colon 1_{\mathsf{Vect}_k}\to\mathcal{U}(\mathsf{SVect}_k)$$

which make some diagrams commute.

But the tensor product on SVect_k is inherited from that on Vect_k, so we can take $\Phi_{X,Y}$ to be the identity transformation for all X and Y, and since the field k is the identity object in both categories, we can take $\varphi = 1_k$.

The necessary diagrams in Definition 3.10.5 commute trivially with these definitions. □

Definition 7.1.3 (internal monoid)**.** Let C be a monoidal category (Definition 3.10.1) with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$. A monoid internal to C is

1. an object $A\in \mathrm{Obj}(\mathsf{C})$,
2. a morphism $e\colon 1\to A$, called the unit, and
3. a morphism $\mu\colon A\otimes A\to A$, called the product,

such that the following diagrams commute.

1. Associativity

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "\alpha"] \arrow[d, "\mu"'] & & A\otimes(A\otimes A) \arrow[d, "\mu"] \\
A\otimes A \arrow[dr, "\mu"'] & & A\otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

2. Unitality

$$\begin{tikzcd}
1\otimes A \arrow[r, "e"] \arrow[dr, "\lambda"'] & A\otimes A \arrow[d, "\mu"] & A\otimes 1 \arrow[l, "e"'] \arrow[dl, "\rho"] \\
& A &
\end{tikzcd}$$

Moreover, if C is a symmetric monoidal category with symmetric braiding γ, then the monoid (A, e, μ) is called commutative if the following diagram commutes.

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\gamma_{A,A}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}$$

A homomorphism of monoids $(A_1, \mu_1, e_1) \to (A_2, e_2, \mu_2)$ is a morphism $f\colon A_1 \to A_2$ such that the following diagrams commute.

$$\begin{tikzcd}
A_1 \otimes A_1 \arrow[r, "f\otimes f"] \arrow[d, "\mu_1"'] & A_2 \otimes A_2 \arrow[d, "\mu_2"] \\
A_1 \arrow[r, "f"] & A_2
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "e_1"] \arrow[dr, "e_2"'] & A_1 \arrow[d, "f"] \\
& A_2
\end{tikzcd}$$

Definition 7.1.4 (category of monoids)**.** Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category. Denote by $\mathsf{Mon}(\mathsf{C}, \otimes, 1)$ the category of monoids in C, i.e. the category whose objects are monoids of C and whose morphism are homomorphisms.

Denote by $\mathsf{CMon}(\mathsf{C}, \otimes, 1)$ the full subcategory of $\mathsf{Mon}(\mathsf{C}, \otimes, 1)$ whose objects are commutative monoids.

Example 7.1.1. A monoid internal to Vect_k is is nothing else but an associative k-algebra with unity. Let us see that this agrees with Definition 2.6.2.

Let (A, e, μ) be a monoid internal to Vect. We need to find a product $\times\colon A \times A \to A$ and a unit element $1 \in A$; show that the product is bilinear and associative; and that the unit behaves like a unit.

We have a linear map $\mu\colon V \otimes V \to V$, which is a bilinear map $V \times V \to V$ if we pre-compose it with the tensor product:

$$(v, v') \mapsto \mu(v \otimes v').$$

It's associative since the associativity diagram says that

$$\mu((v_1 \otimes v_2) \otimes v_3) = \mu(v_1 \otimes (v_2 \otimes v_3)).$$

The unit element is even easier: we take the image of the unit in $1 = k$ under the map e; call it $1_V = e(1_k)$. To see that this behaves like a unit, we let it act on $v \in V$:

$$v \mapsto \mu(1 \otimes v).$$

However, the unitality diagram says that this has to equal $\lambda\colon 1 \otimes V \to V$, and we know how that behaves from Example 3.10.2: it sends $r \otimes v \mapsto rv$. With $r = 1$, it just sends $1 \otimes v \mapsto v$ as we'd like.

The story for multiplication on the right is identical, but with ρ instead of λ.

Now, let A be a unital, associative k-algebra. We need to show that A is a monoid internal to Vect_k. The map $A \otimes A \to A$ is exactly that guaranteed by the universal property of the tensor product; the map e sends $r \in k$ to $r_{1_A} \in A$. It is easy to show that the appropriate diagrams commute.

Example 7.1.2. A commutative monoid internal to Vect_k is exactly a commutative, associative k-algebra with unity.

Theorem 7.1.2. *A supercommutative superalgebra (Definition 6.4.2) is nothing else but a commutative monoid in the symmetric monoidal category* SVect_k*.*

Proof. Let $V = V_0 \oplus V_1$ be a super vector space. We define, as in Example 7.1.1, a bilinear map $V \times V \to V$ by pre-composing μ with the tensor product on SVect_k:

$$\cdot\colon (v, v') \mapsto \mu(v \otimes v').$$

The product as defined here inherits the correct grading from the graded tensor product. The unit element is the image of the unit in k under e, and the verification that this behaves correctly is identical to that in Example 7.1.1.

All that remains is the verification of the Koszul sign rule: we must have the multiplication law

$$v_1 \cdot v_2 = (-1)^{\tilde{v_1} \cdot \tilde{v_2}} v_2 \cdot v_1.$$

But we do by the following:

$$\begin{aligned} v_1 \cdot v_2 &= \mu(v_1 \otimes v_2) \\ &= \mu(\gamma_{V,V}(v_1 \otimes v_2)) && \text{(by the commutativity diagram)} \\ &= \mu((-1)^{\tilde{v_1} \cdot \tilde{v_2}} v_2 \cdot v_1) \\ &= (-1)^{\tilde{v_1} \cdot \tilde{v_2}} \mu(v_2 \otimes v_1) \\ &= (-1)^{\tilde{v_1} \cdot \tilde{v_2}} v_2 \cdot v_1. \end{aligned}$$

□

Theorem 7.1.3. *There is a full subcategory inclusion*

$$\begin{tikzcd} \mathsf{CAlg}_k \arrow[r, hook] \arrow[d, equal] & \mathsf{SCAlg}_k \arrow[d, equal] \\ \mathsf{CMon}(\mathsf{Vect}_k) \arrow[r, hook] & \mathsf{CMon}(\mathsf{SVect}_k) \end{tikzcd}$$

7.2 Supergeometry

Definition 7.2.1. Let M be a C^∞ manifold (that is, recalling Example 3.1.4, an object in the category $\mathsf{SmoothMfd}$.) The functor

$$C^\infty \colon \mathsf{SmoothMfd} \rightsquigarrow \mathsf{Alg}_{\mathbb{R}}$$

is the defined by sending M to the $\mathbb{R}$-algebra of smooth functions $M \to \mathbb{R}$, and morphisms $\varphi \colon M \to N$ to homomorphisms of $\mathbb{R}$-algebras $C^\infty(N) \to C^\infty(M)$ via the pullback.

Theorem 7.2.1. *The functor $C^\infty \colon \mathsf{SmoothMfd} \rightsquigarrow \mathsf{Alg}_{\mathbb{R}}$ is fully faithful (Definition 3.2.2).*

Proof. See [31], lemma 35.8, corollaries 35.9 and 35.10. □

As we saw in Lemma 3.2.1, the meaning of this theorem is that C^∞ is injective on objects up to isomorphism. Thus, we can identify any smooth manifold with (the formal dual of) its algebra of functions, and if we are given any $\mathbb{R}$-algebra in the range of the functor C^∞, we can always find a smooth manifold which gives rise to it which is unique up to isomorphism.

We might take this idea even further: given *any* associative algebra A, we can think of its formal dual as some sort of generalized manifold.

Definition 7.2.2 (affine scheme of an algebra)**.** Let C be a symmetric monoidal category. Write

$$\mathsf{Aff}(\mathsf{C}) = \mathsf{CMon}(\mathsf{C})^{\mathrm{op}}.$$

For any $A \in \mathsf{CMon}(\mathsf{C})$, denote by $\mathrm{Spec}(A)$ the same object in the opposite category $\mathsf{Aff}(\mathsf{C})$. Call this the affine scheme of A.

Conversely, for any $X \in \mathsf{Aff}(\mathsf{C})$, write

$$\mathscr{O}(X) \in \mathrm{Obj}(\mathsf{CMon}(\mathsf{C})).$$

for the same object viewed in the opposite category.

Definition 7.2.3 (affine super scheme)**.** When $\mathsf{C} = \mathsf{SVect}_k$, we call the objects in $\mathsf{Aff}(\mathsf{SVect}_k)$ affine super schemes over k.

7.3 Modules in tensor categories; super vector bundles

7.4 Super groups as super-commutative Hopf algebras

Chapter 8

Representations

8.1 Linear super-representations as Comodules

8.2 Super fiber functors and their automorphism groups

8.3 Super-exterior powers and Schur functors

Chapter 9

Deligne's theorem on tensor categories

9.1 Statement

9.2 Proof

Chapter 10

Relevance to physics

10.1 Wigner's classification of fundamental particle species

This section follows [26] closely. Some stuff was taken from [27].

The axioms of quantum mechanics tell us that the states of a quantum system correspond to vectors ψ belonging to some Hilbert space $\mathscr{H}$. However, this is not the end of the story. The observable quantities built from the states, i.e. the conditional probabilities

$$P(\varphi|\psi) = \frac{|\langle\varphi|\psi\rangle|^2}{\|\varphi\|^2 \cdot \|\psi\|^2}$$

are invariant under a rescaling of either φ or ψ by any nonzero complex number: that is,

$$\psi \sim \psi' \qquad \text{if} \qquad \psi' = \lambda\psi \qquad \text{for some } \lambda \in \mathbb{C} \setminus \{0\},$$

where $\sim$ should be read as 'is physically indistinguishable from.'

One often 'fixes' this problem by working only with normalized state vectors, but this does not completely solve the problem. Instead we should think about two states which differ only by a nonzero complex factor as being physically equivalent. That is, the true space of states of our quantum system is not $\mathscr{H}$, but $\mathscr{H}/\sim$, *projective Hilbert space*. Physically inequivalent quantum states are in one-to-one correspondence to the equivalence classes in $\mathscr{H}/\sim$.

Now suppose we want our quantum theory to have Poincaré symmetry. For us, the Poincaré group will mean the proper orthochromous Poincaré group. Its elements are ordered pairs (a, Λ), with $a \in \mathbb{R}^4$ and $\Lambda \in \mathcal{L}_+^\uparrow$.

Pick any ray $[\psi_0] \in \mathscr{H}/\sim$. For any Poincaré transformation $g \in \mathcal{P}$, there should be another state (i.e. ray) $[\psi]_g$, which corresponds to the result of acting on the system with g. Furthermore, we want the result of acting on $[\psi_0]$ with a sequence of Poincaré transformations to be the same no matter how we imagine them to be bracketed. Mathematically, this means that if we are to consider a quantum theory which is Poincaré-invariant, we want a Hilbert space $\mathscr{H}$ which admits an injective homomorphism

$$\rho\colon \mathbb{R}^4 \rtimes \mathcal{L}_+^\uparrow \hookrightarrow \operatorname{Aut}(\mathscr{H}/\sim).$$

Of course, having such a ρ is nice, but quantum mechanics does not deal with projective spaces and ray transformations. If we are to use these results as-is, we need to translate them to results about $\mathscr{H}$ itself.

Pick one specific element $g \in \mathcal{P}$. To this element there is associated a bijective ray map on $\mathscr{H}$. Obviously, we can mimic this bijective ray map with a function $\bar{U}\colon \mathscr{H} \to \mathscr{H}$; we need only ensure that the vectors making up individual rays are mapped to the correct target rays. Equally obviously however, this does not specify $\bar{U}(g)$ uniquely; there are many possible ways of mixing the vectors composing the rays. We have some freedom to choose this mixing in as nice as possible a way.

By a theorem of Wigner, we can get away with demanding that $\bar{U}(g)$ be additive, length-preserving, and either

- linear and unitary or
- antilinear and antiunitary.

In doing so, we specify $\bar{U}(g)$ up to an arbitrary phase.

But we have many $\bar{U}(g)$—one for each Poincaré transformation. The arbitrarity in the phases translates to a change in representation law for multiplication of the $\bar{U}(g)$; we now have

$$\bar{U}(g)\bar{U}(g') = \eta(g, g')\bar{U}(gg')$$

where the phases η are arbitrary.

Recall that we have restricted our attention to the component of $\mathcal{P}$ connected to the identity, the *proper orthochronous Poincaré group* $\mathcal{P}^{\uparrow}_{+}$. In this setting, we can use the fact that on a connected Lie group, every element has a square root, so

$$\bar{U}(g) = \eta(g', g')(\bar{U}(g'))^2 \qquad \text{for some } g' \in \mathcal{P}.$$

But even if $\bar{U}(g')$ is antilinear and antiunitary, its square will be linear and unitary. Since this is true for any g, we must have that for all $g \in \mathcal{P}^{\uparrow}_{+}$, $\bar{U}(g)$ is linear and unitary.

In fact, we can do even more. By another theorem of Wigner, we can use the phase factor η to replace our unitary ray representation of $\mathcal{P}^{\uparrow}_{+}$ by a bona fide unitary representation of the universal covering group $\overline{\mathcal{P}^{\uparrow}_{+}}$. To save on symbols, we'll call this $\widetilde{\mathcal{P}}$.

The moral of the story so far: the Hilbert space of any quantum mechanical theory which is invariant under the proper orthochronous Poincaré group must admit a representation of the covering group of the proper orthochronous Poincaré group, $\widetilde{\mathcal{P}}$.

Denote the representation of (a, α) by $U(a, \alpha)$.

Consider some vector in $\mathbb{R}^4$, say

$$\hat{e}_0 = (1, 0, 0, 0).$$

The one-dimensional subspace of $\mathbb{R}^4$ generated by $\hat{e}_0$ is a one-dimensional Lie subgroup of $\widetilde{\mathcal{P}}$, so by Stone's theorem, we can write its action on $\mathscr{H}$ as follows:

$$U(\hat{e}_0 t, I) = e^{iP_0 t}, \qquad P_0 = -i\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} U(\hat{e}_0 t, I).$$

For each of the other canonical basis vectors $\hat{e}_i$ we get similar P_i. Since $\mathbb{R}^4$ is abelian, these commute, so by the BCH formula,

$$U(a^\mu \hat{e}_\mu, I) = e^{iP_\mu a^\mu}.$$

Each of the P_μ are Hermitian, and they commute. Thus, they have a common eigenbasis. A priori, all we know is that their spectrum must inhabit some subspace $S \subseteq \mathbb{R}^4$. For each spectral component $p_\mu \in S$, we have an associated degeneracy space $\mathscr{H}_p \subseteq \mathscr{H}$, and if we wave our hands a bit, we can decompose $\mathscr{H}$ into these subspaces.

$$\mathscr{H} = \bigoplus_{p \in S} \mathscr{H}_p$$

That is to say, for any $\psi \in \mathscr{H}_p$, we have $P_\mu \psi = p_\mu \psi$.

Under the action of $\alpha \in \mathrm{SL}(2, \mathbb{C})$, we have

$$\begin{aligned} P_\mu \mapsto P'_\mu = U^{-1}(\alpha, 0) P_\mu U(\alpha, 0) &= U^{-1}(\alpha, 0)\left[-i\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} U(I, \hat{e}_\mu t)\right] U(\alpha, 0) \\ &= -i\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} U^{-1}(\alpha, 0) U(I, \hat{e}_\mu t) U(\alpha, 0) = -i\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} U\left(I, \Lambda^{-1}(\alpha)\hat{e}_\mu t\right) = \Lambda^{-1}(\alpha)^{\nu}_{\mu} P_\nu. \end{aligned}$$

Thus, if p_μ is in S, so must be the entire orbit of p_μ under the Lorentz group, i.e. the invariant hyperboloid on which p_μ lives. Furthermore, these orbits are never mixed by the action of $\widetilde{\mathcal{P}}$.

Perhaps now is a good time to make the following remark. Henceforth, everything we have said applies to *any* quantum system which admits a Poincaré symmetry. As is well know, if we wish to view two quantum systems with Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ as a single composite system, the Hilbert space we consider is $\mathscr{H}_1 \otimes \mathscr{H}_2$. That is to say, composite systems are the tensor product of their constituents. And the composite system's representation of $\widetilde{\mathcal{P}}$ is given by the tensor product of the representations of its constituents.

This would seem to imply that given a Poincaré-invariant quantum system and its associated representation of $\widetilde{\mathcal{P}}$, we could break it and its representation down into smaller and smaller subsystems, until we hit 'atomic' subsystems whose representations were irreducible, i.e. could not be broken down any further. The resulting atomic representations would be the building blocks of our original system.

But we can pull this trick with *any* system! This means that cataloging the irreducible representations of $\widetilde{\mathcal{P}}$ will give us a list of the building blocks of any Poincaré-invariant system—the fundamental particles.

Now let us concentrate on our previous analysis again. Recall: the orbit of any point p_μ in the spectrum of P_μ is the invariant hyperboloid on which p_μ lives. Different hyperboloids are not mixed by $U(a, \alpha)$. Thus, the spectrum of an *irreducible representation* must be concentrated on one hyperboloid, or else the Hilbert space could be decomposed into invariant subspaces. This gives us our first breakdown of the irreducible representations of $\widetilde{\mathcal{P}}$. The following table was taken, mutatis mutandis, from [26].

class	orbit			
m_+	Hyperboloid in forward cone;	$p^2 = m^2$	and	$p_0 > 0$.
0_+	Surface of forward cone;	$p^2 = 0$	and	$p_0 \geq 0$.
0_0	The single point $p_\mu = 0$.			
κ	Space-like hyperboloid;	$p^2 = -\kappa^2$	(κ real).	
m_-	Hyperboloid in backward cone;	$p^2 = m^2$	and	$p_0 < 0$.
0_-	Surface of backward cone;	$p^2 = 0$	and	$p_0 \leq 0$.

Only the first two of these are physically interesting.

Case 1: positive mass

Let us focus first on the case m_+.

If we can understand how U acts on each of the $\mathscr{H}_p$, and we understand how it maps the $\mathscr{H}_p$ amongst each other, we understand everything about it. Pick some $\bar{p}$, and for every other p on its orbit, find some $\beta(\alpha) \in \mathrm{SL}(2, \mathbb{C})$ such that $\Lambda(\beta(\alpha))\bar{p} = p$. If we pick a basis in $\mathscr{H}_{\bar{p}}$, we can use $U(\beta(\alpha))$ to map it to a basis for $\mathscr{H}_p$; in this way we arrive at a basis for our entire Hilbert space

$$\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathscr{H}_{\bar{p}}.$$

Now here is the claim: we can understand the action of *any* $\alpha \in \bar{\mathcal{L}}$ if we understand its action on $\mathscr{H}_{\bar{p}}$. For any $\alpha \in \bar{\mathcal{L}}$, we can make the decomposition

$$\alpha = \beta(\Lambda(\alpha)p)\gamma(\alpha, p)\beta^{-1}(p) :$$

we map $\mathscr{H}_p$ to our proto-little-Hilbert-space $\mathscr{H}_{\bar{p}}$, act on it with some $\gamma \in \bar{\mathcal{L}}$ which fixes $\mathscr{H}_{\bar{p}}$, then map it back to where it needs to be. Since we can pull this trick for any α and p, to get an irreducible representation on $\mathscr{H}$, we only need an irreducible representation of the stabilizer of $\bar{p}$ in $\mathscr{H}_{\bar{p}}$.

There is a particularly easy choice: we can choose

$$\bar{p}_\mu = (m, 0, 0, 0).$$

With this choice, it becomes manifest that the subgroup of $\mathcal{P}$ which leaves $\bar{p}_\mu$ invariant is exactly the covering group of SO(3), i.e. Spin(3) = SU(2).

Thus, to obtain an irreducible representation of $\widetilde{\mathcal{P}}$ on $\mathscr{H}$, we need only find an irreducible representation of SU(2) on $\mathscr{H}_{\bar{p}_\mu}$. These are classified by half-integers.

Case 2: zero mass

Next, let us focus on the class 0_+. In this case, we do not have the 'canonical' choice of $\bar{p}_\mu = (m, 0, 0, 0)$. We can make a different choice, however:

$$\bar{p}_\mu = (1/2, 0, 0, 1/2).$$

As far as the action of $\widetilde{\mathcal{P}}$ is concerned, this vector is really the matrix

$$\hat{p} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

via the correspondence between 2×2 matrices $\hat{V}$ and vectors $V^\mu \in \mathbb{R}^4$

$$\hat{V} = V^\mu \sigma_\mu; \qquad V^\mu = \frac{1}{2}\mathrm{tr}(\hat{V}\sigma_\mu).$$

The action of $\mathrm{SL}(2, \mathbb{C})$ is via conjugation:

$$\hat{V} \mapsto \alpha \hat{V} \alpha^\dagger$$

The subgroup of $\mathrm{SL}(2, \mathbb{C})$ which fixes $\hat{p}$ is generated by the following subgroups of $\mathrm{SL}(2, \mathbb{C})$:

$$\gamma_\varphi = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \qquad \text{and} \qquad \gamma_\eta = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix},$$

where $\varphi \in [0, 2\pi)$ and $\eta \in \mathbb{C}$.

The non-trivial representations of the group γ_η are infinite-dimensional, and appear not to be realized in experiments. The irreducible representations of γ_φ are all one-dimensional, and are classified by integers:

$$U(\gamma_\varphi) = e^{in\varphi}, \qquad n \in \mathbb{Z}.$$

Since

$$\Lambda(\gamma_\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & \sin(2\varphi) & \cos(2\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the matrix representing rotation by 2φ in the x^1-x^2 plane, it makes sense to interpret $n/2$ as angular momentum along the axis of $\bar{p}_\mu$, i.e. the *helicity.*

10.2 Feynman diagrams: representations and intertwiners

10.3 Why SUSY? The standard explanation

The standard argument for the existence of supersymmetry relies on the combination of the Coleman-Mandula and the Haag-Łopuszański-Sohnius theorem. Roughly speaking, the Coleman-Mandula theorem says that the only possible connected Lie groups which give the symmetries of the S-matrix of a four-dimensional relativistic quantum field theory are a direct product of the Poincaré group and an internal symmetry group.

The Haag-Łopuszański-Sohnius theorem generalizes the assumptions of this theorem by expanding the definition of 'symmetry' to include super Lie groups. The Coleman-Mandula theorem still applies to the even part; however, one finds that the odd-graded elements are allowed to mix internal and spacetime symmetries in a non-trivial way. In particular, one has the following schematic multiplication laws:

$$\{\text{odd}, \text{odd}\} = \text{even}; \qquad [\text{even}, \text{even}] = \text{even}; \qquad [\text{odd}, \text{even}] = \text{odd},$$

where the even part is the direct product guaranteed by the Coleman-Mandula theorem, and the odd part is the super- component of the super-Poincaré algebra.

We will prove neither the Coleman-Mandula nor the Haag-Łopuszański-Sohnius theorem. We will, however, give their statements.

The statement of the Coleman-Mandula theorem, taken from [29] is as follows.

Theorem 10.3.1 (Coleman-Mandula). *Let G be a connected symmetry group of the S-matrix, i.e. a group whose generators commute with the S-matrix, and make the following five assumptions.*

1. Lorentz-invariance: *G contains a subgroup which is locally isomorphic to the Poincaré group.*
2. Particle finiteness: *All particles types correspond to positive-energy representations of the Poincaré group. For any finite mass M, there is only a finite number of particles with mass less than M.*
3. Weak elastic analyticity: *Elastic scattering amplitudes are analytic functions of center-of-mass energy squared s and invariant momentum transfer squared t in some neighbourhood of the physical region, except at normal thresholds.*
4. Occurence of scattering: *Let $|p\rangle$ and $|p'\rangle$ be any two one-particle momentum eigenstates, and let $|p,p'\rangle$ be the two-particle state constructed from these. Then*
$$T\,|p,p'\rangle \neq 0$$
where T is the T-matrix defined by
$$S = \mathbf{1} = i(2\pi)^4\delta^4(p_\mu - p'_\mu)T,$$
except, perhaps, for certain values of S. In simpler terms this assumption means: Two plane waves scatter at almost any energy.
5. Technical assumption: *The generators of G, considered as integral operators in momentum space, have distributions for their kernels.*

Then the group G is locally isomorphic to the direct product of a compact symmetry group and the Poincaré group.

The Haag-Łopuszański-Sohnius theorem extends the above theorem by allowing G to be a super group. In the end, one finds that the Poincaré algebra can be extended to the more general *super-Poincaré algebra*.

If one is to use Haag-Łopuszański-Sohnius theorem to motivate supersymmetry, one must navigate a minefield of caveats and grains of salt. In order to use either theorem at all, one must conform to the assumptions in the statement of the Coleman-Mandula theorem, which are very restrictive. To view it as a motivation for supersymmetry, one must believe the following.

1. Spacetime is 4-dimensional, and its symmetry group is the Poincaré group.
2. Spacetime and internal symmetries ought to be unified.
3. There are no more loopholes that no one has thought of yet.

10.4 What does Deligne's theorem do for us?

Chapter 11

Questions

1. Let M be a C^∞ manifold. How much is known about the topology of

$$M^n/\Sigma_n,$$

where Σ_n is the nth symmetric group? Specifically, how many inequivalent $\mathbb{C}$-line bundles does it admit?

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