Q1a) $E(X \mid X > a) = \dfrac{E(X \{X > a\})}{E(\{X > a\})}$

$$D_X(\varphi) = \int \varphi(t)\, \lambda e^{-\lambda t}\, \mathbb{1}_{[0\_)}(t)\, dt$$

let $\varphi(t) = \mathbb{1}_{(a\_\infty)}(t) \quad x(t)$

$$E(X\{X > a\}) = D_X(\varphi) = \int \varphi(t)\, \lambda e^{-\lambda t}\, \mathbb{1}_{[0\_)}(t)\, dt$$

$$= \int t \lambda e^{-\lambda t}\, \mathbb{1}_{(a\_\infty)}(t)\, \mathbb{1}_{[0\_\infty)}(t)\, dt$$

Since $a > 0$,

$$= \int_a^\infty t \lambda e^{-\lambda t}\, dt$$

let $u = t$, $v' = e^{-\lambda t}$

$$\lambda \int t e^{-\lambda t} dt = \lambda \left(t \int e^{-\lambda t} dt - \int -\frac{1}{\lambda} e^{-\lambda t} dt \right)$$

$$= \lambda \left(t \left(-\frac{1}{\lambda} e^{-\lambda t}\right) + \frac{1}{\lambda} \left(-\frac{1}{\lambda}\right) \left(e^{-\lambda t}\right) \right)$$

$$= t(-1) e^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t} + C$$

$$= e^{-\lambda t} \left(-t - \frac{1}{\lambda}\right)$$

$$\int_a^\infty t \lambda e^{-\lambda t} dt = \left[e^{-\lambda t} \left(-t - \frac{1}{\lambda}\right) \right]_a^\infty$$

$$= 0 - e^{-\lambda a} \left(-t - \frac{1}{\lambda}\right)$$

$$= e^{-\lambda a} \left(a + \frac{1}{\lambda}\right)$$

$$E(\{X > a\}) = \int_a^\infty \lambda e^{-\lambda t} \mathbb{1}_{[0,\infty)}(t)\, dt$$

$$= \int_a^\infty \lambda e^{-\lambda t}\, dt$$

$$= \lambda \int_a^\infty e^{-\lambda t}\, dt$$

$$= \lambda \left[-\frac{1}{\lambda} e^{-\lambda t} \right]_a^\infty$$

$$= \lambda \left[0 + \frac{1}{\lambda} e^{-\lambda a} \right]$$

$$= e^{-\lambda a}$$

$$E(X \mid X > a) = \frac{E(X\{X > a\})}{E(\{X > a\})}$$

$$= e^{-\lambda a}\left(a + \frac{1}{\lambda}\right) \div e^{-\lambda a} = a + \frac{1}{\lambda}$$

Q1b)

$$K_{XZ|Y}(a,c|1) = \frac{K_{XZY}(a,c,1)}{K_Y(1)}$$

$$K_Y(1) = \sum_a \sum_c K_{XYZ}(a,1,c)$$

$$= \sum_{a=-c}^{c} \sum_{c=0}^{2} \frac{2^c}{7} \frac{1}{(2c+1)^2} \mathbb{1}_{[-c \dots c]}(1)$$

$$= \sum_{a=-c}^{c} \sum_{c=1}^{2} \frac{2^c}{7} \frac{1}{(2c+1)^2}$$

$$K_{XZY}(a,c,1) = \frac{2^c}{7} \frac{1}{(2c+1)^2} \mathbb{1}_{[-c \dots c]}(a) \, \mathbb{1}_{[-c \dots c]}(1) \, \mathbb{1}_{[0 \dots 2]}(c)$$

$$K_{XZ|Y}(a,c|1) = \frac{K_{XZY}(a,c,1)}{K_Y(1)}$$

$$= \frac{\frac{2^c}{7} \frac{1}{(2c+1)^2} \mathbb{1}_{[-c \dots c]}(a) \, \mathbb{1}_{[-c \dots c]}(1) \, \mathbb{1}_{[0 \dots 2]}(c)}{\sum_{a=-c}^{c} \sum_{c=1}^{2} \frac{2^c}{7} \frac{1}{(2c+1)^2}}$$

Q1c)

$E\{-4 < V < 0\} = F_V(0) - F_V(-4)$

where $F_V(t)$ is the CDF of V

$=$ Area under graph betwen -4 and 0

$= 4 \times \frac{1}{20} = \frac{1}{5}$

$E\{-4 < W < 0\} = F_W(0) - F_W(-4)$

where $F_W(t)$ is CDF of W

$= \frac{1}{2} \times 1 \times \frac{2}{3} = \frac{1}{3} > \frac{1}{5}$

$E\{-4 < W < 0\} > E\{-4 < V < 0\}$

$$k_V(t) = \frac{1}{8}\mathbb{1}_{(-8,-6)}(t) + \frac{1}{20}\mathbb{1}_{(-5,0)}(t)$$

$$+ \frac{1}{12}\mathbb{1}_{(1,4)}(t) + \frac{1}{16}\mathbb{1}_{(5,9)}(t)$$

let $\varphi(v) = |v|$

$$E(|V|) = D_V(\varphi) = \int \varphi(t)\, k_V(t)\, dt$$

$$= \int |t| \Big(\frac{1}{8}\mathbb{1}_{(-8,-6)}(t) + \frac{1}{20}\mathbb{1}_{(-5,0)}(t)$$

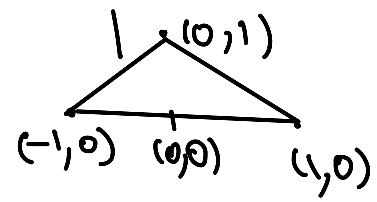
$$+ \frac{1}{12}\mathbb{1}_{(1,4)}(t) + \frac{1}{16}\mathbb{1}_{(5,9)}(t) \Big) dt$$

$$= -\frac{1}{8}\int_{-8}^{-6} t\,dt - \frac{1}{20}\int_{-5}^{0} t\,dt + \frac{1}{12}\int_{1}^{4} t\,dt + \frac{1}{16}\int_{5}^{9} t\,dt$$

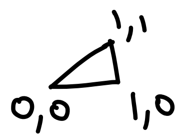
$$\frac{1-0}{0-(-1)} = 1$$

Q1(d)

$y = x+1$



$y = x$



Area of $\Delta = \frac{1}{2}(2)(1) = 1$

Let triangle be T

$$K_{XY}(u,w) = \mathbb{1}_T(u,w)$$

For $-1 \le u \le 0$

$$K_X(u) = \int_{y=0}^{u+1} K_{XY}(u,y)\,dy$$

$$= \int_{y=0}^{u+1} \mathbb{1}_{[-1,0]}(u)\,dy$$

$$= \mathbb{1}_{[-1,0]}(u)\,[y]_0^{u+1}$$

$$= \mathbb{1}_{[-1,0]}(u)\,[u+1]$$

For $0 \le u \le 1$

$$k_X(u) = \int_{y=0}^{-u+1} k_{XY}(u,y)\, dy$$

$$= \int_{y=0}^{-u+1} \mathbb{1}_{[0-1]}(u)\, dy$$

$$= \mathbb{1}_{[0-1]}(u) \left[y \right]_0^{-u+1}$$

$$= \mathbb{1}_{[0-1]}(u) \left[1-u \right]$$

$$k_{Y|X} = \frac{k_{XY}(u,w)}{k_X(u)}$$

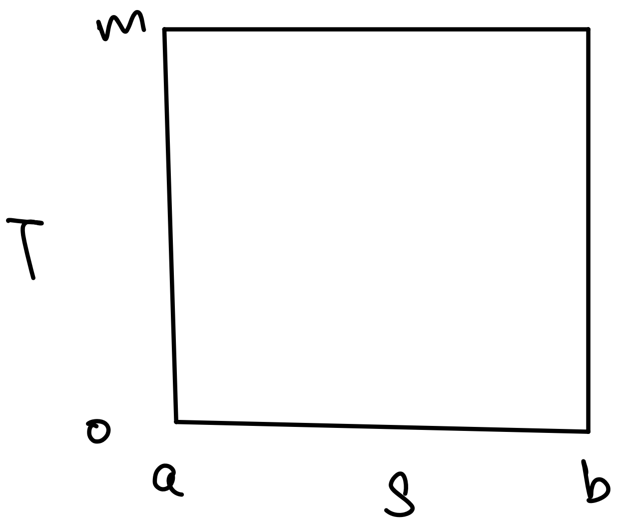
$$= \begin{cases} \dfrac{\mathbb{1}_T(u,w)}{u+1} & -1 \le u < 0 \\[2ex] \dfrac{\mathbb{1}_T(u,w)}{1-u} & 0 \le u \le 1 \\[2ex] 0 & \text{else} \end{cases}$$

$$= \begin{cases} \dfrac{\mathbb{1}_{[0,u+1]}(w)}{u+1} & -1 \leq u < 0 \\ \dfrac{\mathbb{1}_{[0,1-u]}(w)}{1-u} & 0 \leq u \leq 1 \\ 0 & \text{else} \end{cases}$$

Q2a) $\langle S, T \rangle$

S is uniform over $[a,b]$

T is uniform over $[0,m]$



$$E\{\text{we accept}\} = E\{T \le kx(S)\}$$

$$= \frac{\text{Area below } kx}{\text{Total area}}$$

$$= \frac{\int_a^b kx(s)\,ds}{(b-a)(m)}$$

$$k_A(i) = \Pr[\text{accept on } i\text{th}] \Pr[\text{reject } i-1 \text{ times}]$$

$$= \left(\frac{\int_a^b k_X(s)ds}{(b-a)(m)} \right) \left[1 - \frac{\int_a^b k_X(s)ds}{(b-a)(m)} \right]^{i-1}$$

b) $F_Y(s) = E\{Y \leq s\} = \sum_{i=1}^{\infty} E[\{Y \leq s\} | \{A=i\}] E[\{A=i\}]$

$Y = S_i$ if $A = i$

If $A = i$, $Y = S_i$

$$E[\{S_i \leq s\} | \{A=i\}] = E[\{S_i \leq s\} | \{T_i \leq k_X(S_i)\}]$$

$$= \frac{E[\{S_i \leq s\}\{T_i \leq k_X(S_i)\}]}{E[\{T_i \leq k_X(S_i)\}]}$$

since S_i and T_i are independent,

$$\frac{E[\{S_i \leq s\}\{T_i \leq kx(S_i)\}]}{E[\{T_i \leq kx(S_i)\}]} = \frac{E[\{S_i \leq s\}][\{T_i \leq kx(S_i)\}]}{E[\{T_i \leq kx(S_i)\}]}$$

$$= E[\{S_i \leq s\}]$$

$$F_Y(s) = \sum_{i=1}^{\infty} E[\{Y \leq s\} | \{A=i\}] E[\{A=i\}]$$

$$= \sum_{i=1}^{\infty} E[\{S_i \leq s\}] E[\{A=i\}]$$

$$= E[\{S_i \leq s\}] \sum_{i=1}^{\infty} E[\{A=i\}]$$

$$= E[\{S_i \leq s\}]$$

$a \quad s \quad b$

$$= \begin{cases} 1 & \text{if } b \leq s \\ 0 & \text{if } s < a \\ \frac{s-a}{b-a} & \text{if } a \leq s < b \end{cases}$$

c)

```python
import random

def rejection_method(f_X, a, b, m):
    def Y (y):
        i = 1
        while True:
            Si = random.uniform (a,b)
            Ti = random.uniform (0,m)
            if Ti <= f_X (Si):
                return Si
            i += 1
    return Y
```

d)

Q3a) $k_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-\beta_0-\beta_1 x)^2}$

b) Compute log likelihood L

$$L = \sum_{i=1}^{n} \log\left(k_{Y|X}(y_i|x_i)\right)$$

$$= \log\left(\prod_{i=1}^{n} k_{Y|X}(y_i|x_i)\right)$$

$$\log\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-\beta_0-\beta_1 x)^2}\right) = \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}(y_i-\beta_0-\beta_1 x_i)^2$$

$$L = \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}(y_i - \beta_0 - \beta_1 x_i)^2$$

$$= n\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$$

c) L is a function of β_0 and β_1

$$L(\beta_0, \beta_1) = n\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial L}{\partial \beta_0} = 0 - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial \beta_0}(y_i - \beta_0 - \beta_1 x_i)^2$$

$$= -\frac{1}{2}\sum_{i=1}^{n} 2(y_i - \beta_0 - \beta_1 x_i)(-1)$$

$$= \sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)$$

$$\frac{\partial L}{\partial \beta_1} = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial}{\partial \beta_1}(y_i - \beta_0 - \beta_1 x_i)^2$$

$$= -\frac{1}{2}\sum_{i=1}^{n} 2(y_i - \beta_0 - \beta_1 x_i)(-x_i)$$

$$= \sum_{i=1}^{n} x_i(y_i - \beta_0 - \beta_1 x_i)$$

The values of β_0 and β_1 that maximize L satisfy

$$\sum_{i=1}^{n} x_i(y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\sum_{i=1}^{n} y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} x_i = 0$$

To maximize the loglikelihood we must fulfill

$$\sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$$

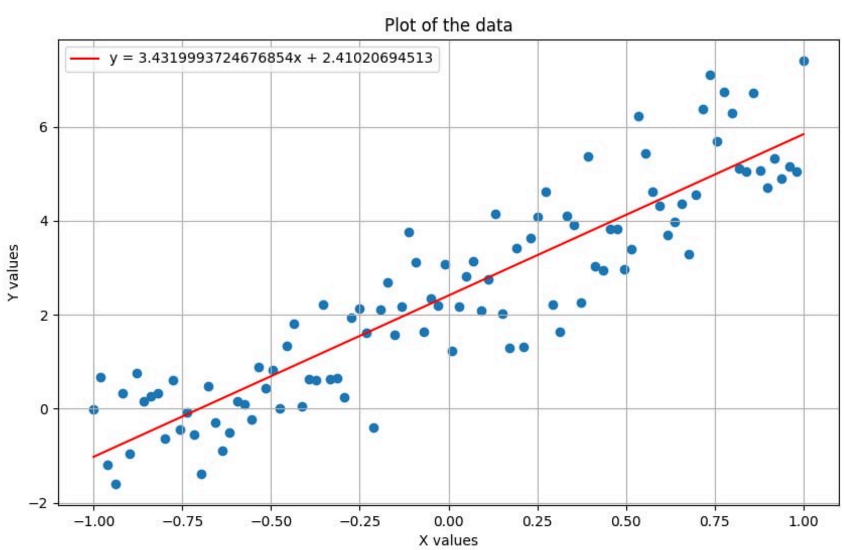
and $$\sum_{i=1}^{n} y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} x_i = 0$$

d)

```python
def loglikelihood (B0, B1 ,data):
    n = len (data)
    # for summation term
    S = 0
    for yi, xi in data:
        S += (yi - B0 - B1 * xi) ** 2
    S *= -1/2
    return S + n * np.log (1/((2 * math.pi)** 0.5))

# solves the linear equations using the dataset to find the values of B0 and B1
# that maximize log likelihood
def findMaxB0B1 (data):
    LHS = []
    RHS = []
    sumxiyi = 0
    sumx = 0
    sumxsquare = 0
    sumy = 0
    for yi, xi in data:
        sumxiyi += yi * xi
        sumx += xi
        sumxsquare += xi ** 2
        sumy += yi

    LHS.append ([sumx, sumxsquare])
    LHS.append ([len(data), sumx])
    RHS.append (sumxiyi)
    RHS.append (sumy)

    LHS = np.asarray (LHS)
    RHS = np.asarray (RHS)
    x = np.linalg.solve(LHS, RHS)
    return x[0], x[1]

print (findMaxB0B1(p3data))
```

```
3.9.exe -i "g:/My Drive/Junior/36218/Week7/hw8.py"
(2.41020694513, 3.4319993724676854)
>>>
```

$\hat{\beta}_0 = 2.41 \qquad \hat{\beta}_1 = 3.43$



Q4a) $E(F \mid R_0 = 5) = 0$ $\quad E(F \mid R_0 = 6) = 1$

b) let $E(F \mid R_0 = r)$ be E_r

Given that $D_{F|R_0}(\varphi|r) = D_{F|R_0}(\langle x \rangle \mapsto D_{F|R_1 R_0}(\varphi(x,r)|r)$

$$E_1 = \tfrac{1}{2}E_4 + \tfrac{1}{2}E_2$$

$$E_2 = \tfrac{1}{2}E_6 + \tfrac{1}{2}E_3 = \tfrac{1}{2} + \tfrac{1}{2}E_3$$

$$E_3 = \tfrac{1}{4}E_1 + \tfrac{1}{4}E_4 + \tfrac{1}{4}E_2 + \tfrac{1}{4}E_5 = \tfrac{1}{4}E_1 + \tfrac{1}{4}E_4 + \tfrac{1}{4}E_2$$

$$E_4 = \tfrac{1}{2}E_3 + \tfrac{1}{2}E_6 = \tfrac{1}{2}E_3 + \tfrac{1}{2}$$

c) We use gaussian elimination to solve

$$\left[\begin{array}{cccc|c} -1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & -1 & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right]$$

$$\Big\downarrow\ R_3 = 4R_3 + R_1$$

$$\left[\begin{array}{cccc|c} -1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{2} & -4 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right]$$

$$\Big\downarrow\ R_3 = R_2 - \frac{2}{3}R_3$$

$$\left[\begin{array}{cccc|c} -1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{13}{6} & -1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} \end{array}\right]$$

$$-\frac{1}{2} + \frac{8}{3} = \frac{-3+16}{6} = \frac{13}{6}$$

$\downarrow \quad R_4 = {}_2 R_3 + \frac{13}{3} R_4$

$$\left[\begin{array}{cccc|c} -1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{13}{6} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{10}{3} & \frac{8}{3} \end{array}\right]$$

$\frac{1}{2} + \frac{13}{6} = \frac{16}{6}$

$$\frac{10}{3} E_4 = \frac{8}{3}$$

$$E_4 = \frac{8}{3} \times \frac{3}{10} = \frac{4}{5}$$

$$\frac{13}{6} E_3 - E_4 = \frac{1}{2}$$

$$E_3 = \frac{6}{13}\left(\frac{1}{2} + \frac{4}{5}\right)$$

$$= \frac{3}{5}$$

$$E_2 - \frac{1}{2} E_3 = \frac{1}{2}$$

$$E_2 = \frac{1}{2} + \frac{3}{10} = \frac{4}{5}$$

$-E_1 + \frac{1}{2} E_2 + \frac{1}{2} E_4 = 0 \quad E_1 = \frac{1}{2}\left(\frac{4}{5}\right) + \frac{1}{2}\left(\frac{4}{5}\right) = \frac{4}{5}$

$\therefore \quad E(F \mid R_0 = 1) = \frac{4}{5}$

$$E(F \mid R_0 = 2) = \frac{4}{5}$$

$$E(F \mid R_0 = 3) = \frac{3}{5}$$

$$E(F \mid R_0 = 4) = \frac{4}{5}$$

$$E(F \mid R_0 = 5) = 0$$

$$E(F \mid R_0 = 6) = 1$$