# Backward Probability and Smoothing

## 1 Backward Probability

The backward probability $P(\text{Rain}_t = \text{true} \mid U_t, U_{t+1} = \text{true})$ represents the probability of observing evidence from time $t+1$ to the final observation, given the hidden state at time $t$.

We begin with the backward probability calculation:

$$P(U_{t+1} = \text{true} \mid \text{Rain}_t = \text{true}) = \sum_{\text{Rain}_{t+1}} P(U_{t+1} = \text{true} \mid \text{Rain}_{t+1}) \cdot P(* \mid \text{Rain}_{t+1}) \cdot P(\text{Rain}_{t+1} \mid \text{Rain}_t = \text{true}$$

Where $P(* \mid \text{Rain}_{t+1})$ is the transition probability from Rain at time $t+1$ to the evidence at time $t+2$.

Example Calculation

Let's compute:

$$P(U_{t+1} = \text{true} \mid \text{Rain}_t = \text{true}) = 0.9 \times 1.0 \times 0.7 + 0.2 \times 1.0 \times 0.3$$

Simplifying:

$$P(U_{t+1} = \text{true} \mid \text{Rain}_t = \text{true}) = 0.63 + 0.06 = 0.69$$

## 2 Smoothing Probability

The smoothing probability is given by:

$$P(\text{Rain}_t = \text{true} \mid U_t = \text{true}, U_{t+1} = \text{true}) \propto P(R_t = \text{true} \mid U_t = \text{true}) \cdot P(U_{t+1} = \text{true} \mid R_t = \text{true})$$

Substituting values:

$$P(\text{Rain}_t = \text{true} \mid U_t = \text{true}, U_{t+1} = \text{true}) \propto \alpha \cdot 0.818 \cdot 0.69 = \alpha \cdot 0.56$$

Normalizing

After normalizing the smoothing probability:

$$P(\text{Rain}_t = \text{true} \mid U_t = \text{true}, U_{t+1} = \text{true}) \approx 0.883$$

# 3 Conclusion

The forward-backward algorithm provides an efficient way to compute the smoothing probabilities. These calculations involve summing over possible future hidden states and weighting them by their corresponding transition and evidence probabilities.