Viterbi Algorithm for Hidden Markov Models

1 Viterbi Algorithm

The Viterbi algorithm is used to compute the most likely sequence of hidden states, such as weather conditions, given a sequence of observations (e.g., umbrella usage). For the observations $[\text{true}, \text{true}, \text{false}, \text{true}, \text{true}]$, the goal is to infer the most probable sequence of weather conditions (rain or no rain).

1.1 Initialization

At time $t = 0$:

$$P(\text{Rain}_0 = \text{true}) = 0.5$$

1.2 Recursion (Forward Pass)

We compute the probabilities step by step for each time t from 1 to 5 using the following transition probabilities:

- $P(\text{Rain}_t = \text{true} \mid \text{Rain}_{t-1} = \text{true}) = 0.7$
- $P(\text{Rain}_t = \text{false} \mid \text{Rain}_{t-1} = \text{true}) = 0.3$
- $P(\text{Rain}_t = \text{true} \mid \text{Rain}_{t-1} = \text{false}) = 0.3$
- $P(\text{Rain}_t = \text{false} \mid \text{Rain}_{t-1} = \text{false}) = 0.7$

The observation probabilities are:

- $P(U_t = \text{true} \mid \text{Rain}_t = \text{true}) = 0.9$
- $P(U_t = \text{false} \mid \text{Rain}_t = \text{false}) = 0.9$
- $P(U_t = \text{true} \mid \text{Rain}_t = \text{false}) = 0.1$
- $P(U_t = \text{false} \mid \text{Rain}_t = \text{true}) = 0.1$

For time $t = 1$, we calculate:

$$P(\text{Rain}_1 = \text{true}) = \max(0.5 \times 0.7, 0.5 \times 0.3) \times 0.9 = \max(0.35, 0.15) \times 0.9 = 0.315$$

$$P(\text{Rain}_1 = \text{false}) = \max(0.5 \times 0.3, 0.5 \times 0.7) \times 0.1 = \max(0.15, 0.35) \times 0.1 = 0.035$$

For time $t = 2$:

$$P(\text{Rain}_2 = \text{true}) = \max(0.315{\times}0.7, 0.035{\times}0.3){\times}0.9 = \max(0.2205, 0.0105){\times}0.9 = 0.198$$

$$P(\text{Rain}_2 = \text{false}) = \max(0.315{\times}0.3, 0.035{\times}0.7){\times}0.9 = \max(0.0945, 0.0245){\times}0.9 = 0.00945$$

For time $t = 3$:

$$P(\text{Rain}_3 = \text{true}) = \max(0.198{\times}0.7, 0.00945{\times}0.3){\times}0.1 = \max(0.1386, 0.002835){\times}0.1 = 0.0139$$

$$P(\text{Rain}_3 = \text{false}) = \max(0.198{\times}0.3, 0.00945{\times}0.7){\times}0.9 = \max(0.0594, 0.006615){\times}0.9 = 0.0535$$

1.3 Backtrack

After computing the probabilities at each step, we backtrack to identify the most likely sequence of weather conditions.

Starting from $t = 5$:

- At $t = 5$, $P(\text{Rain}_5 = \text{true})$
- At $t = 4$, $\text{Rain}_4 = \text{true}$
- At $t = 3$, $\text{Rain}_3 = \text{false}$
- At $t = 2$, $\text{Rain}_2 = \text{true}$
- At $t = 1$, $\text{Rain}_1 = \text{true}$

The most likely sequence is:

$$\text{true}, \text{true}, \text{false}, \text{true}, \text{true}$$