Generalization of the Chain Rule

For three events A, B, and C, the chain rule is given by:

$$P(A \wedge B \wedge C) = P(A \mid B, C) \cdot P(B \mid C) \cdot P(C)$$

This can be generalized for n events:

$$P(A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_n) = P(A_1 \mid A_2, A_3, \ldots, A_n) \cdot P(A_2 \mid A_3, \ldots, A_n) \cdot \cdots \cdot P(A_{n-1} \mid A_n) \cdot P(A_n)$$

Conditional Independence

Unconditional Independence

Two events A and B are independent if:

$$P(A \mid B) = P(A)$$
$$P(A \wedge B) = P(A) \cdot P(B)$$

Conditional Independence

Two events A and B are conditionally independent given C if:

$$P(A \wedge B \mid C) = P(A \mid C) \cdot P(B \mid C)$$

Examples of Independence

Unconditional Independence

Let:

- A = It's raining
- B = The stock market goes up

The events A and B are unconditionally independent if:

$$P(\text{Stock market goes up} \mid \text{Raining}) = P(\text{Stock market goes up})$$

and

$$P(\text{Raining} \wedge \text{Stock market goes up}) = P(\text{Raining}) \cdot P(\text{Stock market goes up})$$

Conditional Independence

Let:

- A = It's raining
- B = The lawn is wet
- C = The sprinkler is on

The events A and B are conditionally independent given C if:

$$P(\text{Lawn is wet} \mid \text{Raining, Sprinkler is on}) = P(\text{Lawn is wet} \mid \text{Sprinkler is on})$$

and

$$P(\text{Raining} \wedge \text{Lawn is wet} \mid \text{Sprinkler is on}) = P(\text{Raining} \mid \text{Sprinkler is on}) \cdot P(\text{Lawn is wet} \mid \text{Sprinkler is on})$$

Probability Calculation with Independence Assumption

$$P(A \mid \neg B) = \frac{P(A \wedge \neg B)}{P(\neg B)}$$

Let $A = (A \wedge \neg B) \vee (A \wedge B)$. Using the law of total probability:

$$P(A) = P(A \wedge \neg B) + P(A \wedge B)$$

Therefore,

$$P(A \wedge \neg B) = P(A) - P(A \wedge B)$$

Since $\neg B$ is independent of A:

$$P(\neg B \mid A) = P(\neg B)$$

Thus,

$$P(A \wedge \neg B) = P(A) \cdot P(\neg B)$$

Finally,

$$P(A \mid \neg B) = \frac{P(A) \cdot P(\neg B)}{P(\neg B)} = P(A) \cdot (1 - P(B))$$

Assumption: A and B are independent.