Metropolis-Hastings Algorithm in Bayesian Networks

1 Metropolis-Hastings Algorithm

The Metropolis-Hastings (MH) algorithm is used to generate samples from a distribution $\pi(x)$. In the context of Bayesian networks, $\pi(x)$ is the posterior probability of state x given the evidence e, denoted by $P(x \mid e)$.

1.1 Steps of the MH Algorithm

1. **Proposal Step**: A new state x' is proposed based on the current state x using a proposal distribution $q(x' \mid x)$.

2. **Acceptance Step**: The proposed state x' is either accepted or rejected based on an acceptance probability:

$$\alpha(x' \mid x) = \min\left(1, \frac{\pi(x')q(x \mid x')}{\pi(x)q(x' \mid x)}\right)$$

   This acceptance step ensures that the Markov chain defined by the MH algorithm has the desired stationary distribution $\pi(x)$.

2 Transition Kernel

The transition kernel $k(x \to x')$ defines the probability of transitioning from state x to state x'. It is given by:

$$k(x \to x') = q(x' \mid x)\alpha(x' \mid x)$$

2.1 Detailed Balance Equation

The detailed balance equation is a key property of Markov chains, ensuring that the chain converges to the stationary distribution $\pi(x)$. The equation states:

$$\pi(x)k(x \to x') = \pi(x')k(x' \to x)$$

By substituting the transition kernel, we obtain:

$$\frac{\pi(x')}{\pi(x)} = \frac{P(x' \mid e)}{P(x \mid e)}$$

Thus, the Metropolis-Hastings algorithm maintains the balance between forward and reverse transitions, ensuring that the samples generated reflect the correct posterior distribution.

3 Conclusion

The Metropolis-Hastings algorithm provides a method for sampling from complex posterior distributions, such as those encountered in Bayesian networks. By iterating through proposal and acceptance steps, the algorithm generates samples that converge to the true distribution $P(x \mid e)$.