Monty Hall Problem Using Bayes' Rule

The Monty Hall problem is a famous probability puzzle based on a game show scenario. In this problem, a contestant is presented with three doors. Behind one door is a car, and behind the other two doors are goats. The contestant picks a door, say door 1. The host, who knows what is behind each door, then opens another door, say door 3, which reveals a goat. The contestant is then given the option to stick with their original choice or switch to the other unopened door. The question is: should the contestant switch or stay, and what is the probability of winning the car if they switch?

We will use Bayes' rule to analyze this problem and determine the probability of winning if the contestant switches doors.

Definitions

Let:

- C_i: The car is behind door i where $i = 1, 2, 3$
- H_{ij}: The host opens door j after you picked door i

The prior probability that the car is behind any door is:

$$P(C_i) = \frac{1}{3}$$

Conditional Probability

The conditional probability that the host opens door j given that the car is behind door k is:

$$P(H_{ij} \mid C_k) = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{if } j = k \\ \frac{1}{2} & \text{if } i = k \\ 1 & \text{if } i \neq k \text{ and } j \neq k \end{cases}$$

For example, if you pick door 1 ($i = 1$) and the host opens door 3 ($j = 3$), we need to calculate the probability of each scenario.

Calculating Probabilities

We calculate the probability that the host opens door 3:

$$P(H_{13}) = P(H_{13} \mid C_1)P(C_1) + P(H_{13} \mid C_2)P(C_2) + P(H_{13} \mid C_3)P(C_3)$$

Given:

$$P(H_{13} \mid C_1) = \frac{1}{2}$$

$$P(H_{13} \mid C_2) = 1$$
$$P(H_{13} \mid C_3) = 0$$

Substituting these into the equation:

$$P(H_{13}) = \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(0 \times \frac{1}{3}\right)$$
$$= \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

Now, using Bayes' rule to calculate the posterior probability that the car is behind door 1 given that the host opens door 3:

$$P(C_1 \mid H_{13}) = \frac{P(H_{13} \mid C_1)P(C_1)}{P(H_{13})}$$

Substituting the known values:

$$P(C_1 \mid H_{13}) = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}$$

Therefore, the probability that the car is behind door 2 (the other unopened door) is:

$$P(C_2 \mid H_{13}) = 1 - P(C_1 \mid H_{13}) = \frac{2}{3}$$

Conclusion

Thus, the contestant should switch doors, as the probability of winning the car by switching is $\frac{2}{3}$, compared to $\frac{1}{3}$ if they stay with their original choice.