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Problem 6.1. Calculate the embedded transition probabilities for an M/G/1 where service is uniformly distributed on (a,b).

We have that

$$k_n = \int_a^b \frac{e^{-\lambda t}(\lambda t)^n}{n!}\frac{1}{b-a}dt$$

Problem 6.3. (a) Find L, L_q, W_q, and W for an M/G/1 queue with service times that are beta-distributed.

Suppose that $G \sim Beta(\alpha, \beta)$. Then

$$E[S] = \frac{1}{\mu} = \frac{\alpha}{\alpha+\beta}$$

$$\sigma_B^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\rho = \lambda\frac{\alpha}{\alpha+\beta}$$

Now

$$L = \rho + \frac{\rho^2 + \lambda\sigma_B^2}{2(1-\rho)} = \lambda\frac{\alpha}{\alpha+\beta} + \frac{\left(\lambda\frac{\alpha}{\alpha+\beta}\right)^2 + \lambda^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}{2(1-\lambda\frac{\alpha}{\alpha+\beta})}$$

$$L_q = L - \rho = \lambda\frac{\alpha}{\alpha+\beta} + \frac{\left(\lambda\frac{\alpha}{\alpha+\beta}\right)^2 + \lambda^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}{2(1-\lambda\frac{\alpha}{\alpha+\beta})} - \lambda\frac{\alpha}{\alpha+\beta}$$

$$= \frac{\left(\lambda\frac{\alpha}{\alpha+\beta}\right)^2 + \lambda^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}{2(1-\lambda\frac{\alpha}{\alpha+\beta})}$$

$$W_q = \frac{L_q}{\lambda} = \frac{\left(\lambda\frac{\alpha}{\alpha+\beta}\right)^2 + \lambda^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}{2\lambda(1-\lambda\frac{\alpha}{\alpha+\beta})}$$

$$W = \frac{L}{\lambda} = \lambda\frac{\alpha}{(\lambda)\alpha+\beta} + \frac{\left(\lambda\frac{\alpha}{\alpha+\beta}\right)^2 + \lambda^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}{2\lambda(1-\lambda\frac{\alpha}{\alpha+\beta})}$$

(b) Find L, L_q, W_q, and W for an M/G/1 queue with service times that follow a type-2 Erlang distribution and where the arrival rate is $\lambda = 1/3$.

Now we have that

$$E[S] = \frac{1}{\mu}$$

$$\sigma_B^2 = \frac{1}{2\mu^2}$$

$$\rho = \frac{\lambda}{\mu} = \frac{1}{3\mu}$$

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Hence

$$L = \frac{1}{3\mu} + \frac{\frac{1}{9\mu^2} + \frac{1}{9}\frac{1}{2\mu^2}}{2(1-\frac{1}{3\mu})}$$

$$= \frac{1}{3\mu} + \frac{\frac{1}{9\mu^2} + \frac{1}{18\mu^2}}{\frac{6\mu-2}{3\mu}}$$

$$= \frac{1}{3\mu} + \frac{6\mu}{18\mu^2(6\mu-2)} + \frac{3\mu}{18\mu^2(6\mu-2)}$$

$$= \frac{1}{3\mu} + \frac{9\mu}{18\mu^2(6\mu-2)} = \frac{1}{3\mu} + \frac{1}{2\mu(6\mu-2)}$$

$$\frac{1}{3\mu} + \frac{1}{12\mu^2-4\mu} = \frac{12\mu^2-\mu}{3\mu(12\mu^2-4\mu)} = \frac{12\mu-1}{3(12\mu^2-4\mu)}$$

$$L_q = L - \rho = \frac{12\mu-1}{3(12\mu^2-4\mu)} - \frac{1}{3\mu} = \frac{3\mu(12\mu-1) - 3(12\mu^2-4\mu)}{9\mu(12\mu^2-4\mu)}$$

$$\frac{3\mu(12\mu-1) - 3\mu(12\mu-4)}{3\cdot 3\mu(12\mu^2-4\mu)} = \frac{1}{12\mu^2-4\mu)}$$

$$W = \frac{L}{\lambda} = \frac{3(12\mu-1)}{3(12\mu^2-4\mu)} = \frac{12\mu-1}{12\mu^2-4\mu}$$

$$W_q = \frac{L_q}{\lambda} = \frac{3}{12\mu^2-4\mu}$$

Problem 6.5. (a) Consider a customer arriving to an M/G/1 queue in steady state. Under the condition that the arriving customer finds the server busy, show that at the time of the arrival

$$E[\text{residual service time}|\text{server busy}] = \frac{E[S^2]}{2E[S]}$$

where S is a random service time.

From Wikipedia we are given that the limiting pdf for residual time, which is defined as the remaining time that a newly arriving customer to a non-empty queue has to wait until being served, is

$$\phi(x) = \frac{1-F(x)}{E[S]}$$

where $F(x)$ is the cumulative distribution function of the holding times S_i. Then

$$E[R] = \int_{t=0}^{\infty} t\cdot\phi(t)dt = \frac{1}{E[S]}\int_{t=0}^{\infty} t(1-F(t))dt$$

$$= \frac{1}{E[S]}\int_{t=0}^{\infty}\frac{1}{2}t^2 f(t)dt = \frac{1}{2E[S]}\int_{t=0}^{\infty} t^2 f(t)dt$$

$$= \frac{E[S^2]}{2E[S]}$$

(b) Similarly, for an arbitrary arrival (without conditioning on the status of the server), show that at the time of arrival

$$E[\text{residual service time}] = \frac{\lambda E[S^2]}{2}$$

Suppose that $R(t)$ is the residual service time, the remaining service time, of the customer in service at time t. This is zero when no customer is in service and S_i when the i^{th} customer begins service. Then $R(T)$ looks like this

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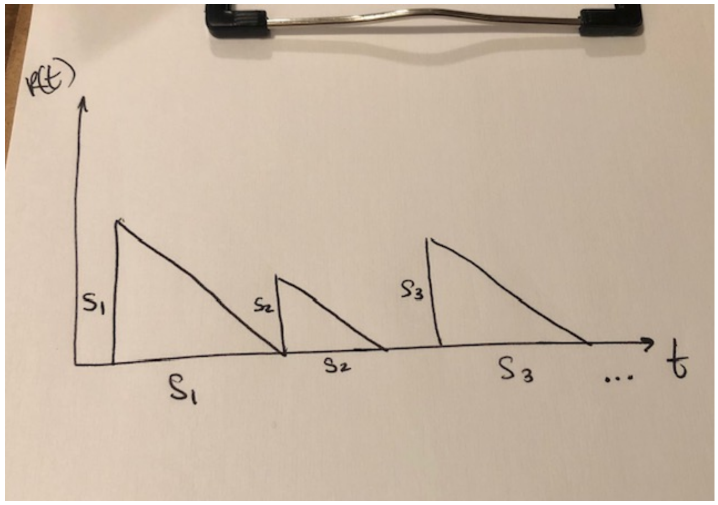

Hence, because these values have equal opportunity of occurring over the interval $[0, T]$,

$$E[R] = \frac{1}{T}\int_0^T R(t)dt$$

where T is some large value.

Now using the image above, this area is the sum of the area of the triangles $(1/2 \cdot b \cdot h)$

$$E[R] = \frac{1}{T}\int_0^T R(t)dt = \frac{1}{T}\sum_{i=1}^{S(T)} \frac{1}{2} S_i^2$$

where $S(T)$ is the number of service completions by time T. Multiplying numerator and denominator by this,

$$E[R] = \frac{S(T)}{2T}\frac{1}{S(T)}\sum_{i=1}^{S(T)} S_i^2$$

As $T \to \infty$, $\dfrac{S(T)}{T} \to \lambda$. Also $\dfrac{1}{S(T)}\sum_{i=1}^{S(T)} S_i^2$ is the second moment of S. Then we have that

$$E[R] = \frac{E[S^2]}{2\lambda}$$

Problem 6.11. Consider an M/D/1 queue with service time equal to b time units. Suppose that the system size is measured when time is a multiple of b, and let X_n denote the system size at time $t = n \cdot b$. Show that the stochastic process $\{X_n, n = 0, 1, 2, \cdots\}$ is a Markov chain, and find its transition matrix.

Since the system size is measured at time $t = n \cdot b$, where b is the service time, I think this is essentially counting how many customers are left in the system after the n^{th} service. So we could say the same as page 261 in the text.

$$X_{n+1} = \begin{cases} X_n - 1 + A_{n+1} & X_n > 0 \\ A_{n+1} & X_n = 0 \end{cases}$$

Where X_n denote the system size at time $t = n \cdot b$ and A_{n+1} is the number of customers who arrive during the service time of the $(n+1)^{th}$ customer. To argue that this is a Markov chain we only need to show that A_{n+1} is independent of the past states. Since the number of customers who arrive during the $(n+1)^{th}$ service time does not depend on the earlier departure points, this holds.

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Problem 6.20. A certain assembly-line operation is assumed to be of the M/G/1 type, with input rate 5/h and service times with mean 9 min and variance 90 min^2. Find L, L_q, W_q, and W. Is the operation improved or degraded if the service times are forced to be exponential with the same mean?

See the QTS output below.

M/G/1: POISSON ARRIVALS TO A SINGLE GENERAL SERVER

Input Parameters:	
Arrival rate (λ)	0.083333
Mean service time (1/μ)	0.111111
Service time variance (σ^2)	90.
Solve	
Results:	
Mean interarrival time (1/λ)	12.0
Service rate (μ)	9.0
Service time Coefficient of Variation (σ*μ)	85.381497
Server utilization (ρ)	0.93%
Fraction of time system is empty (p_0)	0.990741
Mean system size (L)	0.324723
Mean size of queue (Lq)	0.315464
Mean system waiting time (W)	3.896677
Mean waiting time in the queue (Wq)	3.785566
Expected length of the busy period (B)	0.11215

M/M/1: POISSON ARRIVALS TO A SINGLE EXPONENTIAL SERVER

Input Parameters:	
Arrival rate (λ)	0.083333
Mean service time (1/μ)	0.111111
Plot Parameters:	
Maximum size for probability chart	15
Total time horizon for probability plotting	10.
Solve	
Results:	
Mean interarrival time (1/λ)	12
Service rate (μ)	9
Server utilization (ρ)	0.93%
Mean number of customers in the system (L)	0.009345794
Mean number of customers in the queue (Lq)	8.65351E-05
Expected non-empty queue size (Lq')	1.009345794
Mean waiting time (W)	0.112149533
Mean waiting time in the queue (Wq)	0.001038422
Mean length of busy period (B)	0.112149533

M_M_1 | SizeDistributionChart | TimeDistributionChart | PlotData
Ready

The operation is improved if service times are forced to be exponential.

Problem 6.21. Customers arrive at an ATM machine according to a Poisson process with rate $\lambda = 60/h$. The following transaction times are observed (in seconds): 28, 71, 70, 70, 51, 62, 36, 25, 35, 87, 69, 27, 56, 25, 36.

(a) Would an M/M/1 queue be an appropriate model for this system (why or why not)?

For an M/M/1 queue, the transaction times mean and standard deviation should be equal. Since this is not the case, an M/M/1 model would not be appropriate.

```
> transaction_time_sec = c(28, 71, 70, 70, 51, 62, 36,
25, 35, 87, 69, 27, 56, 25, 36)
> mean(transaction_time_sec)
[1] 49.86667
> sd(transaction_time_sec)
[1] 20.75664
```

(b) Estimate the average number of people waiting in line at the ATM.

```
mu = 1/mean(transaction_time_sec) # 0.02
sig2 = sd(transaction_time_sec) # 20.757

lam_sec = 1/60 # 0.0167
rho = lam_sec / mu
rho # 0.831

L_q = (rho^2 + (lam_sec^2)*(sig2))/(2*(1-rho)) # 2.062
```

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(c) The bank wishes to keep the average line length (number in queue) less than or equal to one. What is the average transaction time needed to achieve this goal (assuming that the variance of the transaction times is held constant)?

Set

$$
\begin{aligned}
L_q = \frac{\rho^2 + \lambda^2\sigma_B^2}{2(1-\rho)} &= 1 \\
\frac{\mu^2}{\mu^2}\frac{\rho^2 + \lambda^2\sigma_B^2}{2(1-\rho)} &= 1 \\
\frac{\lambda^2 + \lambda^2\mu^2\sigma_B^2}{2(\mu^2 - \lambda\mu)} &= 1 \\
\lambda^2 + \lambda^2\mu^2\sigma_B^2 &= 2(\mu^2 - \lambda\mu) \\
(2 - \lambda^2\sigma_B^2)\mu^2 - 2\lambda\mu - \lambda^2 &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mu &= \frac{2\lambda \pm \sqrt{2\lambda^2 + 4\lambda^2(2 - \lambda\sigma_B^2)}}{2(2 - \lambda^2\sigma_B^2)} \\
&= \frac{2\lambda \pm 2\lambda\sqrt{1 + 2 - \lambda\sigma_B^2)}}{2(2 - \lambda^2\sigma_B^2)} \\
&= \frac{\lambda \pm \lambda\sqrt{3 - \lambda\sigma_B^2)}}{2 - \lambda^2\sigma_B^2}
\end{aligned}
$$

Using R, we take the positive value

```
> (lam_sec + lam_sec*sqrt(3 - (lam_sec^2)*sig2))/(2 - (lam_sec^2)*sig2)
[1] 0.022819
> (lam_sec - lam_sec*sqrt(3 - (lam_sec^2)*sig2))/(2 - (lam_sec^2)*sig2)
[1] -0.006104144

> 1/((lam_sec + lam_sec*sqrt(3 - (lam_sec^2)*sig2))/(2 - (lam_sec^2)*sig2))
[1] 43.82313
```

That is, they should aim to keep the average service time below approximately 43 seconds.

Problem 6.32. When an AIDS case is first diagnosed in the United States by a physician, it is required that a report be filed (serviced) on the case with the CDC in Atlanta. It takes a random amount of time (with an expected value of approximately 3 months) for a doctor to finish the report and send it into the CDC (unknown distribution). An OR team has analyzed the report arrival stream and believes that new patients come to the doctors all over the nation as a Poisson process. If 50,000 new reports are completed each year, what is the mean number of reports in process by doctors at any one instant (assuming steady state)?

I think we could expect the mean number of reports in the system to be $\lambda t = 50000 \cdot 0.25 = 12,500$ reports.

Problem 6.35. (a) Find the solution r_0, in closed form, of $\beta(z) = z$, the generating function equation for a G/M/1 queue, when the interarrival times follow a hyperexponential distribution with density function

$$a(t) = q\lambda_1 e^{-\lambda_1 t} + (1-q)\lambda_2 e^{-\lambda_2 t}$$

The laplace transform is

$$A^*(s) = \int_0^\infty e^{-st}a(t)dt$$

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$$= \int_0^\infty e^{-st}\left[q\lambda_1 e^{-\lambda_1 t} + (1-q)\lambda_2 e^{-\lambda_2 t}\right] dt$$
$$= \int_0^\infty q\lambda_1 e^{-(s+\lambda_1)t} + \int_0^\infty (1-q)\lambda_2 e^{-(s+\lambda_2)t} dt$$
$$= \frac{q\lambda_1}{s+\lambda_1} + \frac{(1-q)\lambda_2}{s+\lambda_2}$$

Now we must solve

$$z = A^*[\mu(1-z)] = \frac{q\lambda_1}{\mu(1-z)+\lambda_1} + \frac{(1-q)\lambda_2}{\mu(1-z)+\lambda_2}$$
$$z = \frac{[q\lambda_1]\,[\mu(1-z)+\lambda_2]}{[\mu(1-z)+\lambda_1]\,[\mu(1-z)+\lambda_2]} + \frac{[(1-q)\lambda_2]\,[\mu(1-z)+\lambda_1]}{[\mu(1-z)+\lambda_1]\,[\mu(1-z)+\lambda_2]}$$
$$z = \frac{[q\lambda_1]\,[\mu(1-z)+\lambda_2] + [(1-q)\lambda_2]\,[\mu(1-z)+\lambda_1]}{[\mu(1-z)+\lambda_1]\,[\mu(1-z)+\lambda_2]}$$
$$z = \frac{q\lambda_1\mu(1-z) + q\lambda_1\lambda_2 + (1-q)\lambda_2\mu(1-z) + (1-q)\lambda_2\lambda_1}{[\mu(1-z)+\lambda_1]\,[\mu(1-z)+\lambda_2]}$$
$$z\,[\mu(1-z)+\lambda_1]\,[\mu(1-z)+\lambda_2] = q\lambda_1\mu(1-z) + q\lambda_1\lambda_2 + (1-q)\lambda_2\mu(1-z) + (1-q)\lambda_2\lambda_1$$

Let's divide through by $z-1$ or $-(1-z)$,

$$z\left[\mu^2(1-z)(1-z) + \lambda_1\mu(1-z) + \lambda_2\mu(1-z) + \lambda_1\lambda_2\right] = q\lambda_1\mu(1-z) + q\lambda_1\lambda_2 + (1-q)\lambda_2\mu(1-z) + (1-q)\lambda_2\lambda_1$$
$$\mu^2(1-z)(1-z)z + \lambda_1\mu(1-z)z + \lambda_2\mu(1-z)z + \lambda_1\lambda_2 z = q\lambda_1\mu(1-z) + q\lambda_1\lambda_2 + (1-q)\lambda_2\mu(1-z) + (1-q)\lambda_1\lambda_2$$
$$-\mu^2(z - z^{2)} - \lambda_1\mu z - \lambda_2\mu z + \frac{\lambda_1\lambda_2 z}{z-1} = -q\lambda_1\mu + \frac{q\lambda_1\lambda_2}{z-1} - (1-q)\lambda_2\mu + \frac{(1-q)\lambda_1\lambda_2}{z-1}$$
$$-\mu^2 z + \mu^2 z^2 - (\lambda_1\mu + \lambda_2\mu)z + \frac{\lambda_1\lambda_2 z}{z-1} = -q\lambda_1\mu + \frac{q\lambda_1\lambda_2}{z-1} - (1-q)\lambda_2\mu + \frac{(1-q)\lambda_1\lambda_2}{z-1}$$
$$-\mu^2 z + \mu^2 z^2 - (\lambda_1\mu + \lambda_2\mu)z + \frac{\lambda_1\lambda_2 z}{z-1} = -q\lambda_1\mu - (1-q)\lambda_2\mu + \frac{\lambda_1\lambda_2}{z-1}$$
$$\mu^2 z^2 - (\lambda_1\mu + \lambda_2\mu + \mu^2)z = -q\lambda_1\mu - (1-q)\lambda_2\mu + \frac{\lambda_1\lambda_2}{z-1} - \frac{\lambda_1\lambda_2 z}{z-1}$$
$$\mu^2 z^2 - (\lambda_1\mu + \lambda_2\mu + \mu^2)z = -q\lambda_1\mu - (1-q)\lambda_2\mu - \lambda_1\lambda_2$$
$$\mu^2 z^2 - (\lambda_1\mu + \lambda_2\mu + \mu^2)z + q\lambda_1\mu + (1-q)\lambda_2\mu + \lambda_1\lambda_2 = 0$$

Now, we use the quadratic equation

$$z = \frac{\lambda_1\mu + \lambda_2\mu + \mu^2 \pm \sqrt{(\lambda_1\mu + \lambda_2\mu + \mu^2)^2 - 4\mu^2(q\lambda_1\mu + (1-q)\lambda_2\mu + \lambda_1\lambda_2)}}{2\mu^2}$$

Hence

$$r_0 = \frac{\lambda_1\mu + \lambda_2\mu + \mu^2 - \sqrt{(\lambda_1\mu + \lambda_2\mu + \mu^2)^2 - 4\mu^2(q\lambda_1\mu + (1-q)\lambda_2\mu + \lambda_1\lambda_2)}}{2\mu^2}$$

(b) Find the steady-state waiting-time distribution for a G/M/1 queue with $\mu = 8$ and interarrival density function

$$a(t) = (0.3)3e^{-3t} + (0.7)10e^{-10t}$$

We have that $\lambda_1 = 3$, $\lambda_2 = 10$, and $q = 0.3$. Then

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```
num = lambda1*mu + lambda2*mu + mu^2 - sqrt((lambda1*mu + lambda2*mu + mu^2)^2 -
4*mu^{2}*(q*lambda1*mu + (1-q)*lambda2*mu  + lambda1*lambda2))
denom = 2*mu^2
r0 = num/denom
r0 # 0.7963544
```

Finally,

$$w_q = 1 - 0.796e^{-1.629z}$$

Problem 6.36. Show that if $G = M$, the root r_0 of 6.55 is ρ and that 6.60 yields the familiar M/M/1 result.

For an M/M/1 queue, we have

$$A^*(s) = \frac{\lambda}{\lambda + s}$$

then

$$B(z) = A^*(\mu(1-z)) = \frac{\lambda}{\lambda + \mu(1-z)} = z \implies$$
$$\lambda = z\lambda + \mu z - \mu z^2$$

Divide through by μ,

$$\rho = z\rho + z - z^2$$
$$z^2 - (\rho + 1)z + \rho = 0$$
$$(z-1)(z+\rho) = 0$$

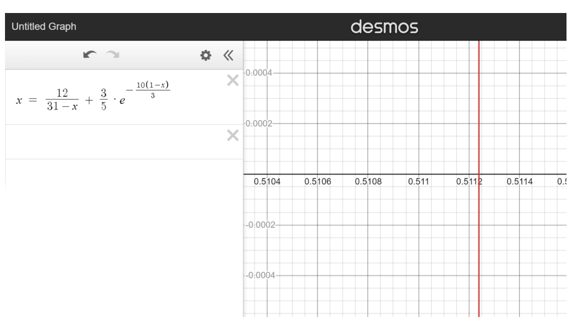
Hence the root r_0 of 6.55 is ρ. Clearly this yields the familiar result of $q_n = (1-\rho)\rho^n$.

Problem 6.41. Consider a G/M/1 system whose solution depends on the real root of the nonlinear equation

$$z = \frac{12}{31 - z} + \frac{3}{5}e^{-10(1-z)/3}$$

Use successive substitution to find the root rounded to three decimal places. Then determine the line-delay CDF under the assumption that the service rate is 1.

I will just find an estimation for the root graphically.



Let's use $z = 0.5112 \approx 0.511$ as an estimation for r_0.

Now the line-delay CDF with $\mu = 1$ is

$$W_q(t) = 1 - 0.511e^{-(1-0.511)t} = 1 - 0.511e^{-0.489t}$$

for $t \geq 0$.

Problem 6.42. Consider a G/M/1 queue where the interarrival times are uniformly distributed from 0 to 6 min and the expected service time is 2 min.

(a) Find the average queue length as found by an arriving customer.

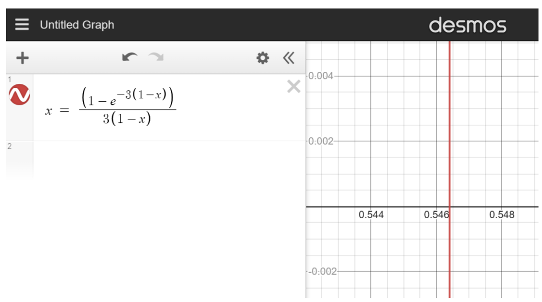
First we need to determine the root r_0,

$$
\begin{aligned}
A^*(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^\infty e^{-st} \frac{1}{6-0} dt \\
&= \frac{1-e^{-6s}}{6s}
\end{aligned}
$$

Then

$$z = A^*[\mu(1-z)] = \frac{1-e^{-6\mu(1-z)}}{6\mu(1-z)}$$

We can solve for this iteratively but I will just go it graphically. Clearly, $z = 1$ is a trivial solution. We also have



Let's just use 0.546 as an approximation for r_0. Then

$$L_q^A = \frac{r_0^2}{1-r_0} = \frac{(0.546)^2}{1-(0.546)} = 0.6566$$

(b) Find the average time spent in queue for each customer.

We have that

$$W_q = \frac{r_0}{\mu(1-r_0)} = \frac{(0.546)}{0.5(1-(0.546))} = 2.4053$$

(c) Find the probability that a customer's waiting time in queue is greater than 3 min.

Next,

$$W_q^C(3) = r_0 \cdot e^{-\mu(1-r_0)\cdot 3} = (0.546) \cdot e^{-0.5(1-(0.546))\cdot 3} = 0.2763$$