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Problem 3. Suppose that $X_1, \cdots, X_n$ is a random sample from a population with density function

$$f(x|\theta) = \begin{cases} \dfrac{1}{2\theta^3} x^2 e^{-x/\theta} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$.

(a) Find the rejection region for the most powerful test of size α for the test $H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta > \theta_0$.

First, we will find the likelihood function,

$$L(\theta|\vec{x}) = \prod_{i=1}^{n} \frac{1}{2\theta^3} x_i^2 e^{-x_i/\theta} = \left(\frac{1}{2\theta^3}\right)^n \prod_{i=1}^{n} x_i^2 e^{-\sum_{i=1}^n x_i/\theta} = 2^{-n}\theta^{-3n} \prod_{i=1}^{n} x_i^2 e^{-\sum_{i=1}^n x_i/\theta}$$

Next, we find the unrestricted MLE of θ by finding the log likelihood, taking the derivative, and setting it equal to zero.

$$\log L(\theta|\vec{x}) = \log\left(2^{-n}\theta^{-3n} \prod_{i=1}^{n} x_i^2 e^{-\sum_{i=1}^n x_i/\theta}\right)$$

$$= -n\log(2) - 3n\log(\theta) + 2\left(\sum_{i=1}^{n} \log(x_i)\right) - \frac{\sum_{i=1}^n x_i}{\theta}$$

Then

$$\frac{d}{d\theta} \log L(\theta|\vec{x}) = -\frac{3n}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta^2} \stackrel{set}{=} 0$$

We have that the unrestricted MLE of θ is

$$\frac{\sum_{i=1}^n x_i}{\theta^2} = \frac{3n}{\theta}$$

$$\sum_{i=1}^{n} x_i = 3n\theta$$

$$\hat{\theta} = \frac{\sum_{i=1}^n x_i}{3n} = \frac{1}{3}\overline{X}$$

Of course, under the null hypothesis we have that the likelihood is maximized by θ_0. Then the LRT statistic is

$$\lambda(\vec{x}) = \frac{\sup_{\Theta_0} L(\theta|\vec{x})}{\sup_{\Theta} L(\theta|\vec{x})} = \frac{L(\theta_0|\vec{x})}{L(\hat{\theta}|\vec{x})} = \frac{2^{-n}\theta_0^{-3n}\prod_{i=1}^n x_i^2 e^{-\sum_{i=1}^n x_i/\theta_0}}{2^{-n}\hat{\theta}^{-3n}\prod_{i=1}^n x_i^2 e^{-\sum_{i=1}^n x_i/\hat{\theta}}}$$

$$= \left(\frac{\theta_0}{\hat{\theta}}\right)^{-3n} \exp\left(-\sum_{i=1}^{n} x_i/\theta_0 + \sum_{i=1}^{n} x_i/\hat{\theta}\right) = \left(\frac{\hat{\theta}}{\theta_0}\right)^{3n} \exp\left(-\sum_{i=1}^{n} x_i/\theta_0 + \sum_{i=1}^{n} x_i/\hat{\theta}\right)$$

$$= \left(\frac{\overline{X}}{3\theta_0}\right)^{3n} \exp\left(-n\overline{X}\left(\frac{1}{\theta_0} + \frac{3}{\overline{X}}\right)\right) = \left(\frac{\overline{X}}{3\theta_0}\right)^{3n} \exp\left(-\frac{n\overline{X}}{\theta_0} + 3n\right)$$

The LRT rejects H_0 if $\lambda(\vec{x}) \leq c$, so we set

$$\left(\frac{\overline{X}}{3\theta_0}\right)^{3n} \exp\left(-\frac{n\overline{X}}{\theta_0} + 3n\right) \leq c$$

$$\exp\left(-\frac{n\overline{X}}{\theta_0} + 3n\right) \leq \frac{c}{\left(\dfrac{\overline{X}}{3\theta_0}\right)^{3n}}$$

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$$-\frac{n\overline{X}}{\theta_0} + 3n \leq \log \frac{c}{\left(\dfrac{\overline{X}}{3\theta_0}\right)^{3n}}$$

$$\frac{2n\overline{X}}{\theta_0} \geq -2\log \frac{c}{\left(\dfrac{\overline{X}}{3\theta_0}\right)^{3n}} + 6n$$

Consequently, we should reject H_0 when

$$\frac{2n\overline{X}}{\theta_0} \geq \tau_\alpha$$

This problem is similar to 10.95 in the Wackerly text, where they give a hint that we should make use of the χ^2 distribution. We know that under the null hypothesis, $X_i \sim \text{Gamma}(3,\, \theta_0)$. The mgf for X is $M_X(t) = (1/(1-\theta_0 t))^3$. Then the mgf for $\overline{X}$, using theorem 5.2.7, is

$$M_{\overline{X}}(t) = (M_X(t/n))^n = \left[\left(\frac{1}{1-\theta_0(t/n)}\right)^3\right]^n = \left(\frac{1}{1-(\theta_0/n)t}\right)^{3n} \sim gamma(3n, \theta_0/n)$$

That is,

$$\frac{2n\overline{X}}{\theta_0} \sim gamma(3n, 2)$$

Now, from the table of common distributions in the back of the text, we can also say that

$$T = \frac{2n\overline{X}}{\theta_0} \sim \chi^2_{6n}$$

Finally, we reject if $T > \chi^2_{6n,\alpha}$. The fact that this is the most powerful test follows from the Neyman-Pearson Lemma.

(b) Suppose that $\theta_1 > \theta_0$. Determine an expression for the power of the test in part (a.) and plot this power function for all $\theta > \theta_0$.

We know that our test rejects the null hypothesis when $T > \chi^2_{6n,\alpha}$ so

$$\text{power } = P\left(T_0 > \chi^2_{6n,\alpha} | \theta_1 > \theta_0\right)$$

Then bringing in the value for θ_1, we get

$$P\left(\frac{2n\overline{X}}{\theta_0} > \chi^2_{6n,\alpha} | \theta_1 > \theta_0\right)$$

$$P\left(2n\overline{X} > \chi^2_{6n,\alpha}\theta_0 | \theta_1 > \theta_0\right)$$

$$P\left(\frac{2n\overline{X}}{\theta_1} > \chi^2_{6n,\alpha}\frac{\theta_0}{\theta_1} | \theta_1 > \theta_0\right)$$

$$P\left(T_1 > \chi^2_{6n,\alpha}\frac{\theta_0}{\theta_1} | \theta_1 > \theta_0\right)$$

Where $T_1 \sim \chi^2_{6n}$ under the alternative hypothesis. Since $\theta_1 > \theta_0$, $1 > \theta_0/\theta_1$. Then we are asked to plot this power function for all $\theta > \theta_0$. I was not sure what to set θ_0, α, or n to so I just picked some fixed values and made the power function a function of θ. I should note that I fixed $\theta_0 = 5$, $\alpha = 0.05$, and $n = 25$ for the power plots.

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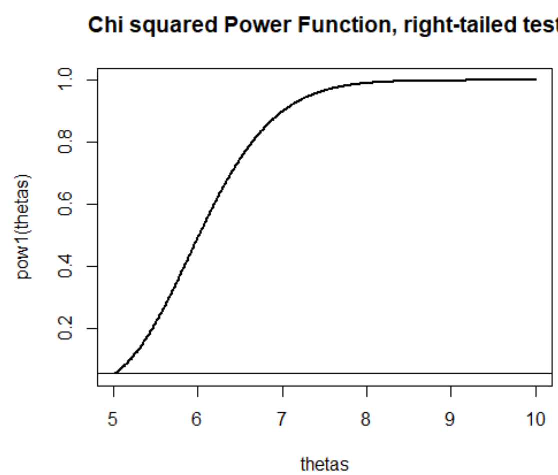
```
# Suppose
theta0 <- 5
# H1: \theta > \theta0
# This parameter space under H1 contains values > theta0
thetas <- seq(theta0+0.01, 10, by = 0.01)
n=25
# then the right-tailed power function is

pow1 <- function(theta) 1 - pchisq((theta0/theta)*qchisq(1-0.05, 6*n), 6*n)

plot(thetas, pow1(thetas), type = "l", lwd = 2,
     main = "Chi squared Power Function, right-tailed test")
min(pow1(thetas))
# 0.05192409
abline(h = min(pow1(thetas)))
```



(c) Repeat parts (a.) and (b.) but with a two-sided alternative $H_1 : \theta \neq \theta_0$.

Based on our test statistic computed in part (a), we should reject the null hypothesis when $T > \chi^2_{6n,\alpha/2}$ or $T < \chi^2_{6n,1-\alpha/2}$.

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```
# However for this test, H1: \theta \neq \theta0
# This parameter space under H1 does not contain theta0
thetas <- seq(-theta0, theta0*3, by = 0.03)

pow2 <- function(theta) {
   x1 <- qchisq(0.025, 6*n)
   x2 <- qchisq(1 - 0.025, 6*n)
   pchisq((theta0/theta)*x1, 6*n) + (1 - pchisq((theta0/theta)*x2, 6*n))
}
plot(thetas, pow2(thetas), type = "l", lwd = 2,
     main = "Chi squared Power Function, 2-tailed")
min(pow2(thetas))
# 0.0498818
abline(h = min(pow2(thetas)))
```