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**Problem 4.33.** Compare the $L_q$'s of the two-priority model, two-rate case with those of the two-priority model, one-rate case when the $\mu$ of the latter equals $min(\mu_1, \mu_2)$.

First, assume that $\mu_1 \leq \mu_2$. Then

$$L_q^{(1)} = \frac{\lambda_1 \rho}{\mu - \lambda_1} = \frac{\lambda_1 [\lambda_1/\mu_1 + \lambda_2/\mu_1]}{1-\rho_1}$$
$$= \frac{\lambda_1 \left[\rho_1/\mu_1 + \lambda_2/\mu_1^2\right]}{1-\rho_1} \geq \frac{\lambda_1 \left[\rho_1/\mu_1 + \lambda_2/\mu_2^2\right]}{1-\rho_1}$$
$$= \frac{\lambda_1 [\rho_1/\mu_1 + \rho_2/\mu_2]}{1-\rho_1}$$

Similarly,

$$L_q^{(2)} = \frac{\lambda_2 \rho}{(\mu - \lambda_1)(1-\rho)} = \frac{\lambda_2 [\lambda_1/\mu_1 + \lambda_2/\mu_1]}{(\mu_1 - \lambda_1)(1 - \lambda_1/\mu_1 - \lambda_2/\mu_1)}$$
$$= \frac{(\lambda_2/\mu_1) [\lambda_1/\mu_1 + \lambda_2/\mu_1]}{(1-\rho_1)(1 - \lambda_1/\mu_1 - \lambda_2/\mu_1)}$$
$$= \frac{(\lambda_2/\mu_1) \left[\rho_1/\mu_1 + \lambda_2/\mu_1^2\right]}{(1-\rho_1)(1 - \lambda_1/\mu_1 - \lambda_2/\mu_1)} \geq \frac{(\lambda_2/\mu_1) \left[\rho_1/\mu_1 + \lambda_2/\mu_2^2\right]}{(1-\rho_1)(1 - \lambda_1/\mu_1 - \lambda_2/\mu_1)} = \frac{\lambda_2(\rho_1/\mu_1 + \rho_2/\mu_2)}{(1-\rho_1)(1-\rho)}$$

**Problem 4.36.** For the non preemptive priority queue, suppose that $\mu_i = \mu$ for all $i$. Under this condition, show that the average wait in queue across all customers is the same s the average with in queue for the M/M/1 queue.

Using the given hypothesis, we have

$$W_q^{(i)} = \frac{\sum_{k=1}^{r} \lambda_k/\mu^2}{\left(1 - \sum_{k=1}^{i-1} \lambda_k/\mu\right)\left(1 - \sum_{k=1}^{i} \lambda_k/\mu\right)}$$
$$= \frac{\lambda}{\left(\mu - \sum_{k=1}^{i-1} \lambda_k\right)\left(\mu - \sum_{k=1}^{i} \lambda_k\right)}$$

Hence

$$\overline{W}_q = \sum_{i=1}^{r} \frac{\lambda_i W_q^{(i)}}{\lambda} = \sum_{i=1}^{r} \frac{\lambda_i}{\left(\mu - \sum_{k=1}^{i-1} \lambda_k\right)\left(\mu - \sum_{k=1}^{i} \lambda_k\right)}$$

For $i = 1$ and $i = 2$ we get

$$\frac{\lambda_1}{\mu(\mu - \lambda_1)} + \frac{\lambda_2}{(\mu - \lambda_1)(\mu - \lambda_1 - \lambda_2)} = \frac{\lambda_1 + \lambda_2}{\mu(\mu - \lambda_1 - \lambda_2)}$$

Moving on in this way we would get

$$\overline{W}_q = \sum_{i=1}^{r} \frac{\lambda_i}{\left(\mu - \sum_{k=1}^{i-1} \lambda_k\right)\left(\mu - \sum_{k=1}^{i} \lambda_k\right)} = \frac{\lambda}{\mu(\mu - \lambda)}$$

which is what we expect for the M/M/1 queue.

**Problem 4.42.** Patients arrive at the emergency room of a hospital according to a Poisson process. A patient is classified into ine if three types: high, medium, or low priority. On and average day, there may be approximately 10 arrivals per hour. One-fifth of the arrivals are high priority, three-tenths are medium, and one-half are low. All registration matters are handled by a single clerk. The time to register a patient is exponentially distributed with a mean of 5.5 min (regardless of classification). Assume that there is no preemption. What is the average time to complete the registration for each type of patients? What is the average time for an arbitrary patient. Verify that the latter can also be obtained using an M/M/1 model.

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| | A | B | C | D |
|---|---|---|---|---|
| 2 | | lambda | rho | sigma |
| 3 | 1 | 2 | 10.90909091 | 0.1833333333 |
| 4 | 2 | 3 | 10.90909091 | 0.4583333333 |
| 5 | 3 | 5 | 10.90909091 | 0.9166666667 |
| 6 | | | | |
| 7 | | W_{q}^{(1)} | 0.1028911565 | |
| 8 | | W_{q}^{(2)} | 0.1899529042 | |
| 9 | | W_{q}^{(3)} | 1.861538462 | |
| 10 | | | | |
| 11 | | Overall: | | M/M/1 |
| 12 | | 1.008333333 | | 1.1 |
| 13 | | Additional wait: | | |
| 14 | | 0.09166666667 | | |
| 15 | | Total: | | |
| 16 | | 1.1 | | |
| 17 | | | | |

**Problem 7.20.** Consider an M/M/1/$\infty$ queue for which $\lambda = 1$ and $\mu = 3$. Every customer going through the system pays an amount \$15 dollars, but costs the system \$6 per unit time it spends in the system.

(a) What is the average profit rate of this system.
We have that

$$Z = 15\lambda - 6L$$

Hence since

$$L = \frac{\rho}{1-\rho}$$

this is

$$Z = 15\lambda - 6L = 15\lambda - \frac{6\rho}{1-\rho} = 15 - 3 = 12 \text{ dollars}$$

(b) See problem in text.

Now we have that

$$Z = 15\lambda(1 - p_k) - 6L$$

where

$$p_k = \frac{(1-\rho)p^k}{1-\rho^{k+1}}$$

and

$$L = \frac{\rho(1-(k+1)\rho^k + k\rho^{k+1}}{(1-\rho^{k+1})(1-\rho)}$$

Hence

$$Z = \frac{15(1-\rho^k)}{1-\rho^{k+1}} - \frac{6\rho(1-(k+1)\rho^k + k\rho^{k+1})}{(1-\rho^{k+1}(1-\rho))}$$

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| | | | |
|---|---|---|---|
| | | rho | 1-rho |
| | | 0.3333333333 | 0.6666666667 |
| | | | |
| | k | Z | |
| | 1 | 5.166666667 | |
| | 2 | 6.581196581 | |
| | 3 | 6.92962963 | |
| | 4 | 7.000510152 | <-- |
| | 5 | 7.008298286 | |
| | 6 | 7.005495754 | |
| | 7 | | |
| | 8 | | |

**Problem 7.25.** Consider a tool crib where the manager believes that costs are only associated with idle time; that is, the time that the clerks are idle (waiting for mechanics to come) and the time that the mechanics are idle (waiting in the queue for a clerk to become available). Using costs \$30 per hour per clerk and \$70 per hour idle per mechanic, with $1/\lambda = 50$ s, $1/\mu = 60$ s. Find the optimal number of clerks. Comment.

We have that the expected cost of idle time is

$$E[C_I(c)] = 30(c - \lambda/\mu) + 70L_q$$

where $\lambda/\mu = 6/5 = 1.2$.

| | | | |
|---|---|---|---|
| c | L_q | E[C(idle_c)] | |
| 2 | 0.675 | 71.25 | |
| 3 | 0.0941 | 60.587 | <--- |
| 4 | 0.0159 | 85.113 | |

**Problem 7.26.** Consider an M/M/1 queue where the mean service rate $\mu$ is under the control of management. There is a customer waiting cost per unit time spent in the system of $C_w$ and a service rate that is proportional to the square of the mean service rate, the constant of proportionality being $C_s$. Find the

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optimal value of $\mu$ in terms of $\lambda$, $C_s$, and $C_w$. Solve when $\lambda = 10$, $C_s = 2$, and $C_w = 20$.

The cost equation is

$$K(\mu) = C_s\mu^2 + C_W\lambda W = C_S\mu^2 + C_W\left[\frac{\lambda}{\mu-\lambda}\right]$$

$$\frac{\partial K(\mu)}{\partial \mu} = 2C_S\mu - \frac{C_W\lambda}{(\mu-\lambda)^2}$$

and to verify we take the second derivative

$$\frac{\partial^2 K(\mu)}{\partial^2 \mu} = 2C_S\mu - \frac{2C_W\lambda}{(\mu-\lambda)^3}$$

Then

$$\mu^3 - 2\mu^2\lambda + \mu\lambda^2 - C_W\lambda/2C_S$$

$$\mu^3 - 20\mu^2 + 100\mu - 50$$