Angyalka Valcsics
Angyalka.Valcsics@maine.edu

Problem 2.b. Suppose that $U_1, U_2, \cdots, U_n \overset{iid}{\sim}$ Uniform$(0,1)$ and define

$$Y = \frac{1}{\sqrt[n]{U_1 U_2 \cdots U_n}} = \left(\prod_{j=1}^{n} U_j\right)^{-1/n}$$

2.b.i Determine the probability distribution of $X = \log(Y)$.

Start with using properties of the log function to get

$$X = \log(Y) = \log\left[\left(\prod_{j=1}^{n} U_j\right)^{-1/n}\right] = -\frac{1}{n}\left[\log\left(\prod_{j=1}^{n} U_j\right)\right] = -\frac{1}{n}\left[\sum_{j=1}^{n} \log U_j\right]$$
$$= \frac{1}{n}\left[\sum_{j=1}^{n} -\log U_j\right]$$

From here we should recall the relationship between uniform and exponential random variables shown in example 2.1.4 (Casella Berger). If we take some U_j then find the log-transform $g(U_j) = T = -\log U_j$ we would have $U_j = g^{-1}(t) = e^{-t}$.
Now, since $\frac{d}{du_j}\log u_j = -\frac{1}{u_j} < 0$ for $0 < u_j < 1$, this is a decreasing function. Moreover, it is monotone over the support of U_j. Then with $F_{U_j}(u_j) = u_j$, we have

$$F_T(t) = 1 - F_{U_j}(g^{-1}(t)) = 1 - F_{U_j}(e^{-t}) = 1 - e^{-t} \implies f_T(t) = e^{-t}$$

The sum of n independent exponential random variables has a Gamma(n, 1) distribution with density

$$f_Q(q) = \frac{1}{(n-1)!} q^{n-1} e^{-q}$$

for $q \geq 0$ where $Q = \sum_{j=1}^{n} -\log U_j$. Online reference for Gamma distribution.
Looking back at X, since n is just a positive number, I believe this is a scaling of the Gamma distribution. Let $X = \frac{1}{n}Q$ where $Q = \sum_{j=1}^{n} -\log U_j$ and $Q \sim Gamma(n, 1)$. Then

$$f_Q(q) = \frac{1}{\Gamma(n)} q^{n-1} e^{-q} = \frac{1}{(n-1)!} q^{n-1} e^{-q}$$

Using the transform method, $x = g(q) = (1/n)q$ so $q = g^{-1}(x) = nx$, then

$$f_X(x) = f_Q(g^{-1}(x))\left|\frac{d}{dx} g^{-1}(x)\right| = n\left[\frac{1}{(n-1)!}(nx)^{n-1} e^{-nx}\right]$$
$$= \frac{1}{(n-1)!} n (n)^{n-1} x^{n-1} e^{-nx} = \frac{1}{(n-1)!} n^n x^{n-1} e^{-nx}$$
$$\frac{1}{(n-1)!}\frac{1}{(1/n)^n} x^{n-1} e^{-x/(1/n)} = \frac{1}{(n-1)!(1/n)^n} x^{n-1} e^{-x/(1/n)}$$

Which makes sense since β is a scaling parameter. So $X \sim Gamma(n, 1/n)$

Angyalka Valcsics
Angyalka.Valcsics@maine.edu

2.b.ii What is the probability distribution of X when $n \to \infty$?

Looking at the table of common distributions on page 627 of the textbook, if a random variable is distributed $Gamma(r, \lambda)$ then as $r \to \infty$, X becomes $N(\mu = r\lambda, \sigma^2 = r\lambda^2)$. In this case, $r = n$ and $\lambda = 1/n$. Hence $\mu = r\lambda = n(1/n) = 1$ and $\sigma^2 = r\lambda^2 = n(1/n^2) = 1/n$. So the probability distribution of X as $n \to \infty$ is $N(1, 1/n)$.

2.b.iii Define $V = \sqrt{n}(X - 1)$ and determine the asymptotic ($n \to \infty$) distribution of V.

I tried to do this with a transformation but that proved to be a challenge. Let's start with

$$P(V \leq v) = P(\sqrt{n}(X - 1) \leq v) = P\left(\sqrt{n}\left(\frac{1}{n}\left[\sum_{j=1}^{n} -\log U_j\right] - 1\right) \leq v\right)$$

We are given that $U_1, U_2, \cdots, U_n \overset{iid}{\sim}$ Uniform$(0, 1)$.
Also $E(-\log U_j) = 1$ and $V(-\log U_j) = 1$ since these are distributed $exp(1)$.
See that this is then

$$\sqrt{n}\left(\frac{1}{n}\left[\sum_{j=1}^{n} -\log U_j\right] - 1\right) = \frac{\frac{1}{n}\left[\sum_{j=1}^{n} -\log U_j\right] - 1}{\frac{1}{\sqrt{n}}}$$

which is exactly of the form seen in theorem 7.4 in the Wackerly textbook–the central limit theorem. This means that the distribution function of V converges to the standard normal distribution as $n \to \infty$.