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Problem 7.2. Let $X_1, \cdots, X_n$ be a random sample from a gamma(α, β) population.

(a) Find the MLE of β, assuming α is known.

We begin by finding

$$L(\beta|\vec{x}) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x_i^{\alpha-1} e^{-x_i/\beta} = \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \left(\prod_{i=1}^{n} x_i^{\alpha-1}\right)\left(\prod_{i=1}^{n} \exp\left(-x_i/\beta\right)\right)$$

$$= \frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \left(\prod_{i=1}^{n} x_i^{\alpha-1}\right)\left(\exp\left(-\sum_{i=1}^{n} x_i/\beta\right)\right)$$

Then

$$\log(L(\beta|\vec{x})) = \log\left(\frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}} \left(\prod_{i=1}^{n} x_i^{\alpha-1}\right)\left(\exp\left(-\sum_{i=1}^{n} x_i/\beta\right)\right)\right)$$

$$= \log\left(\frac{1}{\Gamma(\alpha)^n \beta^{n\alpha}}\right) + \log\left(\prod_{i=1}^{n} x_i^{\alpha-1}\right) + \log\left(\exp\left(-\sum_{i=1}^{n} x_i/\beta\right)\right)$$

$$= -\log\left(\Gamma(\alpha)^n \beta^{n\alpha}\right) + (\alpha-1)\log\left(\prod_{i=1}^{n} x_i\right) + \left(-\sum_{i=1}^{n} x_i/\beta\right)$$

$$= -\log\left(\Gamma(\alpha)^n\right) - \log\left(\beta^{n\alpha}\right) + (\alpha-1)\log\left(\prod_{i=1}^{n} x_i\right) + \left(-\sum_{i=1}^{n} x_i/\beta\right)$$

$$= -n\log\left(\Gamma(\alpha)\right) - (n\alpha)\log\left(\beta\right) + (\alpha-1)\log\left(\prod_{i=1}^{n} x_i\right) + \left(-\sum_{i=1}^{n} x_i/\beta\right)$$

$$= -n\log\left(\Gamma(\alpha)\right) - (n\alpha)\log\left(\beta\right) + (\alpha-1)\sum_{i=1}^{n}\log\left(x_i\right) + \left(-\sum_{i=1}^{n} x_i/\beta\right)$$

Next, we find the derivative with respect to β and set this equal to zero.

$$\frac{d}{d\beta}\log(L(\beta|\vec{x})) = \frac{d}{d\beta}(-n\alpha)\log\left(\beta\right) + \frac{d}{d\beta}\left(-\sum_{i=1}^{n} x_i/\beta\right) = \frac{-n\alpha}{\beta} - \sum_{i=1}^{n} x_i \frac{-1}{\beta^2}$$

Setting this equal to zero and solving for β, we get

$$\frac{-n\alpha}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2} = 0 \implies \frac{\sum_{i=1}^{n} x_i}{\beta^2} = \frac{n\alpha}{\beta}$$

$$\implies \sum_{i=1}^{n} x_i = n\alpha\beta \implies \hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n\alpha} = \frac{\overline{x}}{\alpha}$$

To show that this is the MLE, we should take the second derivative.

$$\frac{d^2}{d\beta^2}\log(L(\beta|\vec{x})) = \frac{d}{d\beta}\frac{-n\alpha}{\beta} + \sum_{i=1}^{n} x_i \frac{1}{\beta^2} = \frac{n\alpha}{\beta^2} - \sum_{i=1}^{n} x_i \frac{2}{\beta^3} = \frac{n\alpha}{\left(\frac{\sum_{i=1}^{n} x_i}{n\alpha}\right)^2} - \sum_{i=1}^{n} x_i \frac{2}{\left(\frac{\sum_{i=1}^{n} x_i}{n\alpha}\right)^3}$$

$$= \frac{(n\alpha)^3}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2\sum_{i=1}^{n} x_i (n\alpha)^3}{\left(\sum_{i=1}^{n} x_i\right)^3} = \frac{(n\alpha)^3}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2(n\alpha)^3}{\left(\sum_{i=1}^{n} x_i\right)^2} = -\frac{(n\alpha)^3}{\left(\sum_{i=1}^{n} x_i\right)^2} < 0$$

Which is what we wanted to show.

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(b) If α and β are both unknown, there is no explicit formula for the MLEs of α and β, but the maximum can be found numerically. The result on part (a) can be used to reduce the problem to the maximization of a univariate function. Find the MLEs for α and β for the data in Exercise 7.10(c).

Then plugging in $\hat{\beta}$, we need to find the values that maximize

$$\begin{aligned}\log(L(\alpha|\vec{x})) &= -n\log\left(\Gamma(\alpha)\right) - (n\alpha)\log\left(\frac{\overline{x}}{\alpha}\right) + (\alpha-1)\sum_{i=1}^{n}\log\left(x_i\right) + \left(-\sum_{i=1}^{n} x_i/(\overline{x}/\alpha)\right) \\ &= -n\log\left(\Gamma(\alpha)\right) - (n\alpha)\log\left(\frac{\overline{x}}{\alpha}\right) + (\alpha-1)\sum_{i=1}^{n}\log\left(x_i\right) - \left(\frac{\alpha}{\overline{x}}\sum_{i=1}^{n} x_i\right)\end{aligned}$$

Then

$$\frac{dL}{d\alpha} = n\log(\alpha) - n\log(\overline{x}) - n\psi(\alpha) + \sum_{i=1}^{n}\log(x_i)$$

and

$$\frac{d^2L}{d\alpha^2} = \frac{n}{\alpha} - n\psi'(\alpha)$$

I will use R and the Newton-Raphson method for estimating mle.

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```
data <- c(22.0, 23.9, 20.9, 23.8, 25.0, 24.0, 21.7,
          23.8, 22.8, 23.1, 23.1, 23.5, 23.0, 23.0)

sum.ln <- function(x){
   n <- length(x)
   sum = 0
   for (i in 1:n) {
      sum = sum + log(x[i])
   }
   sum
}
c <- sum.ln(data)
d <- n*log(mean(data))
# find first derivative
f.prime <- function(alpha) n*log(alpha) - d - n*digamma(alpha) + c
# find second derivative
f.double <- function(alpha) (n/alpha) - n*trigamma(alpha)

newton.raphson.gamma <- function(x1 = 1, tol = 1e-8){
   x0 = x1 + 1
   while (abs(x1 - x0) > tol) {
      x0 <- x1
      x1 <- x0 - f.prime(x0)/f.double(x0)
   }
   alpha.hat = x1
}

# Newton-Raphson for estimating gamma mle
mle.alpha <- newton.raphson.gamma()
mle.alpha

mle.beta <- mean(data)/mle.alpha
mle.beta

Output:
> mle.alpha
[1] 514.3354
> mle.beta
[1] 0.0449401
```