Bicuspidal Geodesics on Punctured Hyperbolic Surfaces

FYP Midterm Talk

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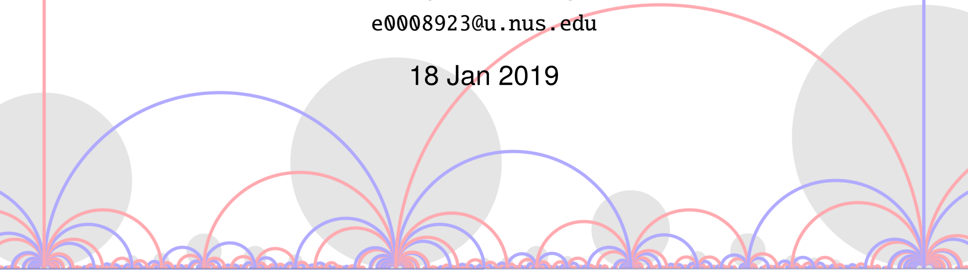
18 Jan 2019



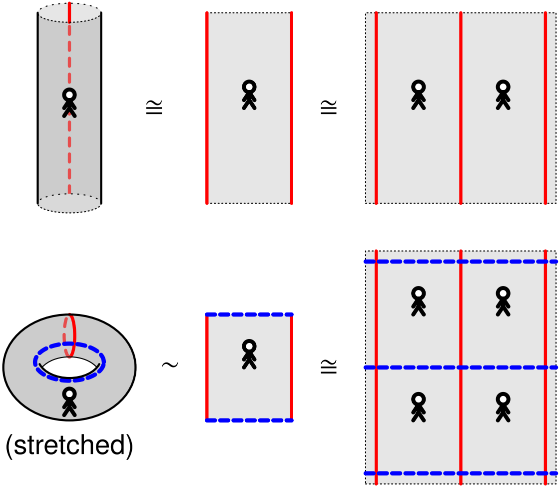
Geometry on surfaces

- Length and angle
  (= Riemannian metric)
- Straight lines
  (= Geodesics)
- How many _______ geodesics are there?
  (Change the question until it is interesting)

Flat surfaces

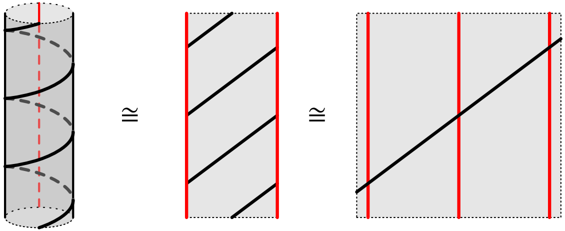
Gaussian curvature $\equiv 0$

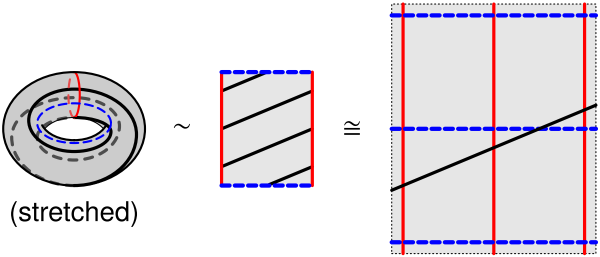
Can be evenly covered by a flat plane



Geodesics on flat surfaces

Geodesics on surface $\leftrightarrow$ Geodesics on plane





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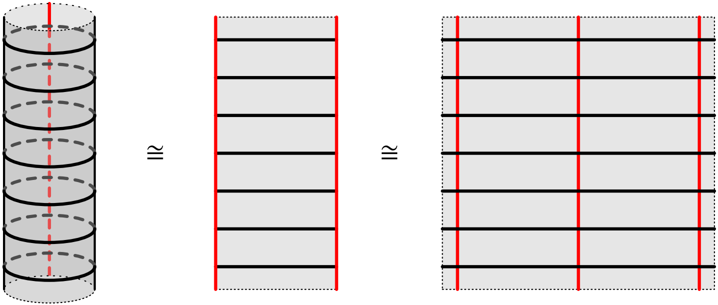
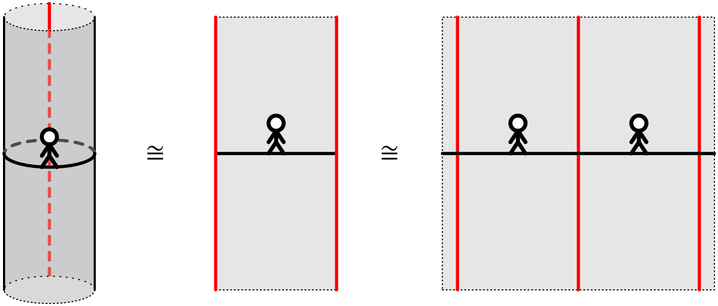
Counting geodesics on the cylinder

How many **closed** geodesics are there?



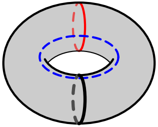
Infinitely many

How many **closed** geodesics are there **from a point**?

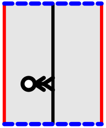


Exactly two

How many **closed** geodesics are there **from a point**?



$\sim$



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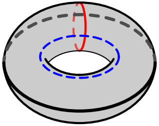
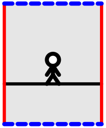


$\sim$



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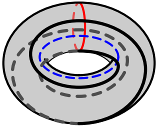
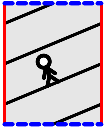


$\sim$



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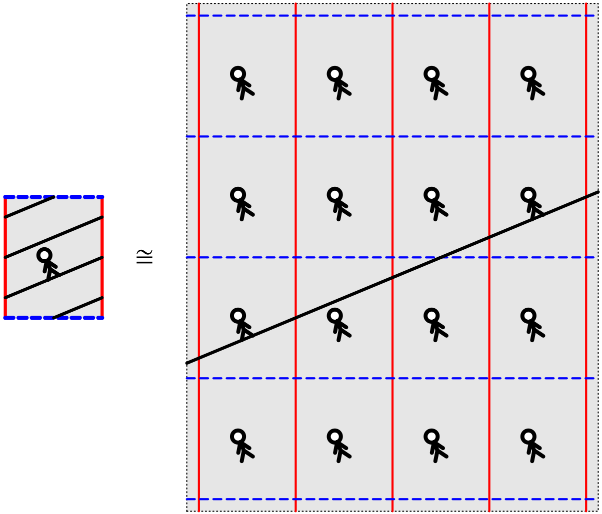
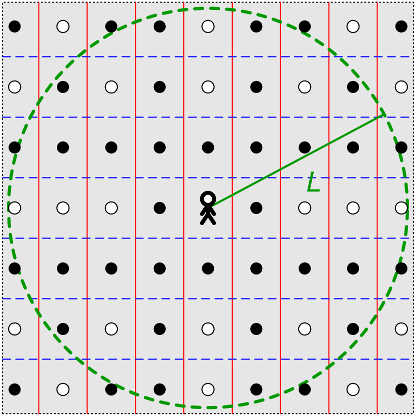
Counting geodesics on the flat torus



One per pair of coprime integers (m, n)

How many **closed** geodesics are there **from a point** with **length at most** L?



Flat torus of area 1

$$\approx \pi L^2 \times \mathbb{P}\left(\begin{array}{c}\text{two random}\\ \text{integers}\\ \text{are coprime}\end{array}\right)$$

Counting geodesics on the flat torus

$$\begin{aligned}
\mathbb{P}(\text{both divisible by } p) &= \frac{1}{p^2} \\
\mathbb{P}(\text{not both divisible by } p) &= 1 - \frac{1}{p^2} \\
\mathbb{P}\,(\text{coprime}) &= \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) \\
&= \frac{1}{\zeta(2)}
\end{aligned}$$

$$\#\left\{ \begin{array}{c} \text{closed geodesics from a point} \\ \text{with length at most } L \end{array} \right\} \sim \frac{1}{\zeta(2)} \pi L^2$$

Counting geodesics on the flat torus

$$\#\left\{\begin{array}{c}\text{closed geodesics from a point}\\ \text{with length at most } L\end{array}\right\} \sim \frac{1}{\zeta(2)}\pi L^2$$

Theorem (Eskin-Masur-Zorich 2003)

Let $\mathcal{H}$ be the space of all flat surfaces of area 1 with prescribed conical singularities. Then for almost every surface S in $\mathcal{H}$,

$$\#\left\{\begin{array}{c}\text{maximal cylinders of closed geodesics}\\ \text{with length at most } L\end{array}\right\} \sim c_{\mathcal{H}}\pi L^2,$$

where $c_{\mathcal{H}}$ is a constant that can be explicitly computed from $\mathcal{H}$.

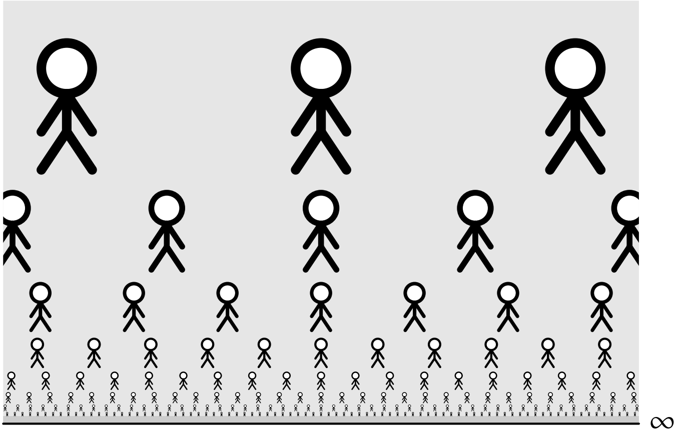
For $\mathcal{H} = \{\text{flat tori}\}$, we can compute $c_{\mathcal{H}} = \frac{6}{\pi^2}$.

Corollary

$$\zeta(2) = \frac{\pi^2}{6}.$$

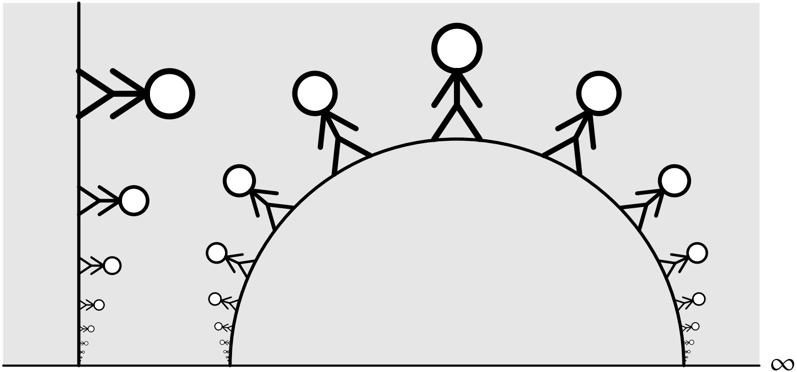
Gaussian curvature $\equiv -1$

Can be evenly covered by the hyperbolic plane

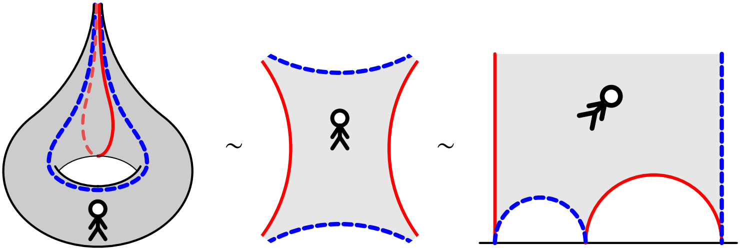


Poincaré half-plane model

Geodesics on the hyperbolic plane



Hyperbolic surfaces



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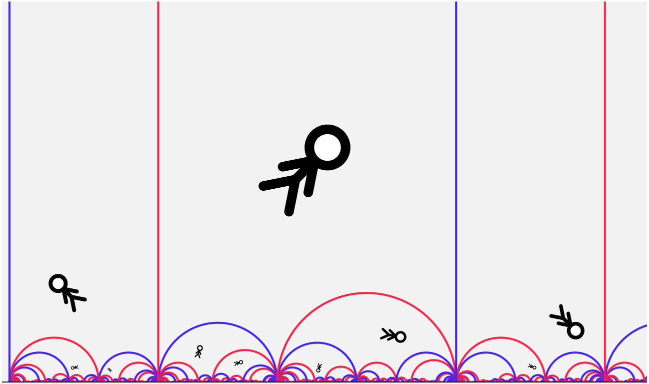
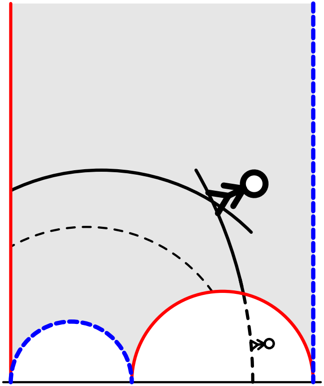


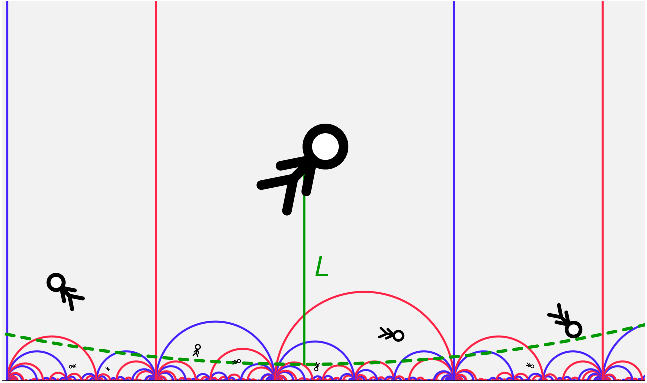
How many **closed** geodesics are there **from a point**?



Sometimes, none!

(Usually, there's one nearby.)

How many **closed** geodesics are there
with **length at most** L?



Guess: exponential in L?

Counting geodesics on hyperbolic surfaces

Prime Geodesic Theorem (Sarnak 1980)

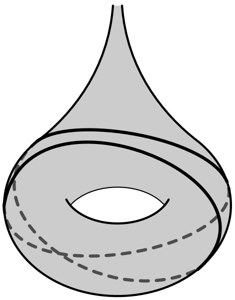
On a closed hyperbolic surface with finite area,

$$\#\left\{\begin{array}{c}\text{closed geodesics}\\ \text{with length at most } L\end{array}\right\} \sim \frac{e^L}{L}.$$

Compare with the prime number theorem:

$$\#\{\text{primes} \leq n\} \sim \frac{n}{\log n}$$

Another type of geodesic



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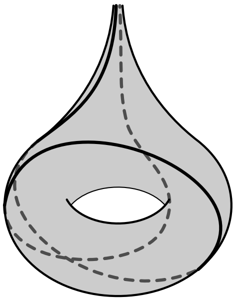
closed
geodesics



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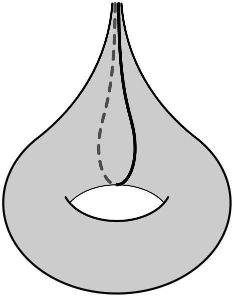
bicuspidal
geodesics

Counting bicuspidal geodesics

How many **bicuspidal** geodesics are there
with **length at most** L?

Zero!



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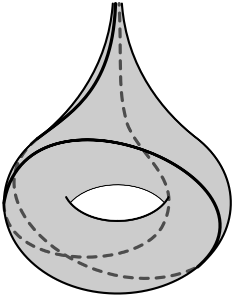

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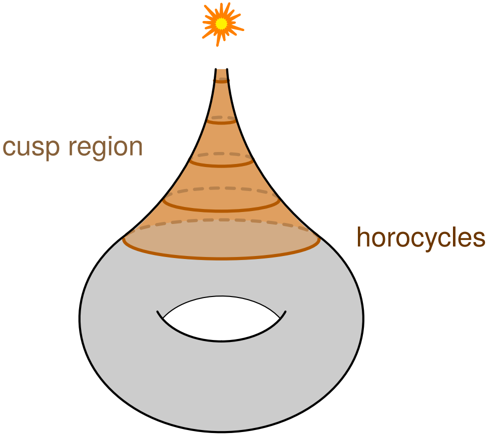
But clearly some are “longer” than others…


BOOM!

An explosion occurs at the cusp



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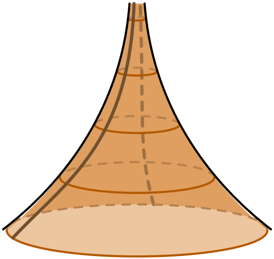
Cusp regions



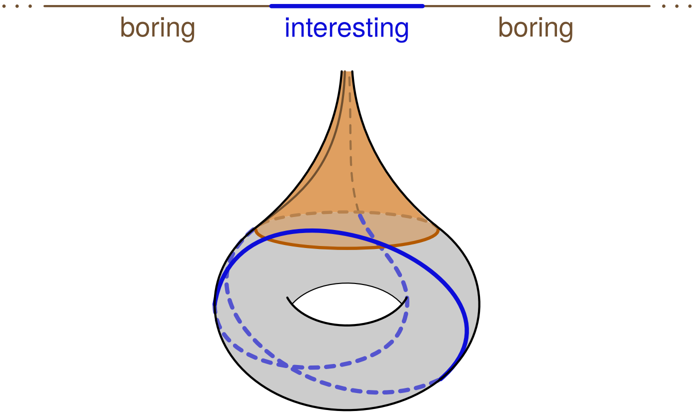
Bounded by horocycle of length 2

All cusp regions are congruent

$$\text{Wave front} \perp \text{Propagation}$$
$$\Downarrow$$
$$\text{Horocycles} \perp \text{Geodesics}$$

Collar theorem

The cusp regions on a hyperbolic surface are pairwise disjoint.



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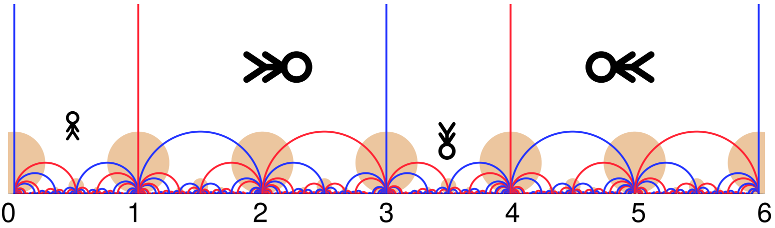
Definition

Normalised length = length of **interesting** part

How many **bicuspidal** geodesics are there with **normalised length at most** L?

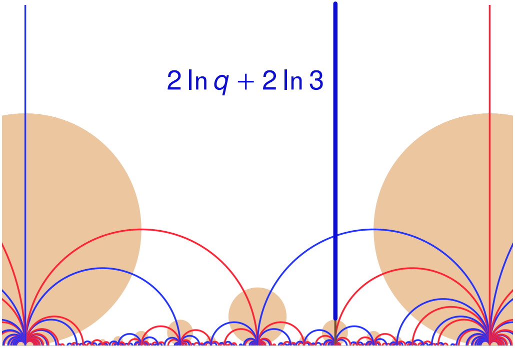
$$S = \mathbb{H} \Big/ \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \right\rangle$$



Map repeats every 6 units

Images of cusp: $\mathbb{Q} \cup \{\infty\}$

Example: the modular torus



$\frac{p}{q}$

$$\left\{ \begin{array}{c} \text{bicuspidal geodesics with} \\ \text{normalised length at most } L \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{rational numbers in } [0, 6) \\ \text{with denominator at most } n \end{array} \right\}$$

$$(L = 2\ln n + 2\ln 3)$$

Example: the modular torus

$$
\begin{aligned}
&\#\left\{\begin{array}{c}\text{rational numbers in } [0,6) \\ \text{with denominator at most } n\end{array}\right\} \\
&= \#\left\{0 \leq \frac{p}{q} < 6 \;:\; q \leq n\right\} \\
&= \#\{p, q \text{ coprime} \;:\; 0 \leq p < 6q,\, q \leq n\} \\
&\sim \frac{1}{2}(n)(6n) \times \mathbb{P}\left(\begin{array}{c}\text{two random integers} \\ \text{are coprime}\end{array}\right) \\
&= \frac{3}{\zeta(2)} n^2.
\end{aligned}
$$

Main question

How many **bicuspidal** geodesics are there
with **normalised length at most** L?

Main Theorem

Let S be a hyperbolic surface with genus g and p punctures.
Let C_1 and C_2 be any cusp regions on S. Then

$$\#\left\{\begin{array}{c}\text{bicuspidal geodesics from } C_1 \text{ to } C_2 \\ \text{with normalised length at most } L\end{array}\right\} \sim c_S e^L,$$

where $c_S = \dfrac{2}{(2g-2+p)\pi^2}$.

Example: the modular torus

$$
\#\left\{\begin{array}{c}\text{bicuspidal geodesics with} \\ \text{normalised length at most } L\end{array}\right\} \sim \frac{2}{(2g-2+p)\pi^2}
$$

$$
\begin{aligned}
&= \frac{2}{\pi^2} e^L \qquad (g = 1,\ p = 1) \\
&= \frac{18}{\pi^2} n^2 \qquad (L = 2 \ln n + 2 \ln 3) \\
&\sim \frac{3}{\zeta(2)} n^2
\end{aligned}
$$

Corollary

$$
\zeta(2) = \frac{\pi^2}{6}.
$$

References

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Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants.
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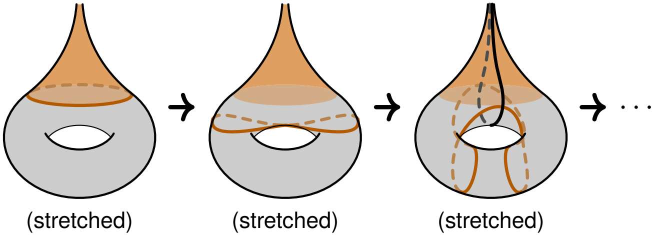
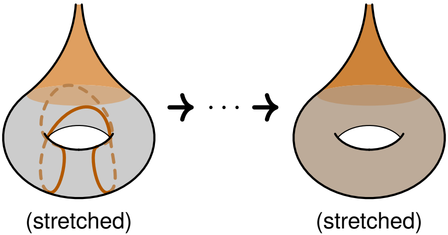
Proof sketch



$$\#\left\{\begin{array}{c}\text{returns to cusp region}\\ \text{after time } L\end{array}\right\} = \#\left\{\begin{array}{c}\text{bicuspidal geodesics with}\\ \text{normalised length at most } L\end{array}\right\}$$

Proof sketch



Theorem (Zagier 1981, Sarnak 1981)

Long horocycles equidistribute with respect to area.

$$\text{proportion of wave front in cusp region} \longrightarrow \frac{\text{Area(cusp region)}}{\text{Area(surface)}}$$

Proof sketch

$$\#\left\{\begin{array}{c}\text{returns to cusp region}\\ \text{after time } L\end{array}\right\} = \#\left\{\begin{array}{c}\text{bicuspidal geodesics with}\\ \text{normalised length at most } L\end{array}\right\}$$

Technical estimates $\updownarrow$

$$\begin{array}{c}\text{proportion of wave front}\\ \text{in cusp region}\end{array} \longrightarrow \frac{\mathrm{Area}(\text{cusp region})}{\mathrm{Area}(\text{surface})}$$