Combinatorial Set Theory

Ang Yan Sheng

Notes for a second course on set theory, with a focus on combinatorics. This follows the independent study module UIS4921 Combinatorial Set Theory.

The texts referenced for this set of notes are:

– A. Hajnal and P. Hamburger (1999), *Set Theory*, trans. Cambridge Univ. Press.

– K. Hrbacek and T. Jech (1999), *Introduction to Set Theory* (3rd ed.).

– K. Kunen (1980), *Set Theory: An Introduction to Independence Proofs* (2nd ed.).

Prerequisites We assume that the reader has taken a first course in axiomatic set theory, up to the theory of ordinals and cardinals.

Notation Unless otherwise stated, Greek letters vary over ordinals.

1 Preliminaries

We assume ZFC throughout this set of notes, except in the first section on cofinality.

1.1 Cofinality

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For an ordinal β, we say that $S \subseteq \beta$ is *cofinal in* β if S has no upper bound in β, ie. Cofinal set

$$\forall b < \beta \, \exists s \in S \, [s \geq b].$$

For $\alpha, \beta \in \mathbf{ORD}$, a function $f : \alpha \to \beta$ is called a *cofinal map* if $\mathrm{ran}(f)$ is cofinal in β. In this case, we also say that f *maps* α *cofinally to* β. Cofinal map

The *cofinality* of an ordinal β, written $\mathrm{cf}(\beta)$, is the minimal ordinal α for which there is a cofinal map from α to β. Cofinality $\mathrm{cf}(\beta)$

Example 1.1. *If β is a successor ordinal, then $\mathrm{cf}(\beta) = 1$.*

Example 1.2. *The image of every map from a finite ordinal to ω is a finite set, and thus has a finite upper bound. Hence $\mathrm{cf}(\omega) = \omega$.*

Proposition 1.3. $\mathrm{cf}(\beta) \leq |\beta|$.

Proof. Any bijection $f : |\beta| \to \beta$ is cofinal. □

Proposition 1.4. *There exists a cofinal map $f : \mathrm{cf}(\beta) \to \beta$ which is strictly increasing, ie. $\xi < \eta < \mathrm{cf}(\beta)$ implies $f(\xi) < f(\eta)$.*

Proof. Take any cofinal map $g : \mathrm{cf}(\beta) \to \beta$, and define $f : \mathrm{cf}(\beta) \to \beta$ recursively by

$$f(\eta) = \max(g(\eta), \sup\{f(\xi) + 1 \, : \, \xi < \eta\}).$$

By construction, $\eta > \xi \implies f(\eta) \geq f(\xi) + 1 > f(\xi)$. Also, $f(\eta) \geq g(\eta)$, so $\mathrm{ran}(g)$ unbounded in β implies $\mathrm{ran}(f)$ unbounded in β, so f is cofinal. □

Alt (sketch). Take any cofinal map $g : \mathrm{cf}(\beta) \to \beta$, and let

$$S = \{\eta < \mathrm{cf}(\beta) \, : \, \forall \xi < \eta \, [g(\xi) < g(\eta)]\}.$$

Note that $g \restriction S : S \to \beta$ is cofinal and strictly increasing, so $\mathrm{cf}(\beta) \leq \mathrm{otp}(S) \leq \mathrm{cf}(\beta)$ implies $\mathrm{otp}(S) = \mathrm{cf}(\beta)$. Hence any isomorphism $f : \mathrm{cf}(\beta) \to S$ gives a strictly increasing cofinal map $g \circ f : \mathrm{cf}(\beta) \to \beta$. □

Corollary 1.5. $\mathrm{cf}(\beta)$ *is the minimal order type of a cofinal set in* β.

Next we show that the cofinality of an ordinal is preserved under strictly increasing cofinal maps.

Proposition 1.6. *Let $f : \alpha \to \beta$ be a strictly increasing cofinal map. Then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.*

Proof. $\mathrm{cf}(\alpha) \geq \mathrm{cf}(\beta)$: Fix $g : \mathrm{cf}(\alpha) \to \alpha$ cofinal. For any $\eta < \beta$, choose $\xi < \alpha$ such that $f(\xi) \geq \eta$. Now there exists $\zeta < \mathrm{cf}(\alpha)$ with $g(\zeta) \geq \xi$, so

$$f(g(\zeta)) \geq f(\xi) \geq \eta.$$

Hence $f \circ g : \mathrm{cf}(\alpha) \to \beta$ is also cofinal.

$\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$: For any $h : \mathrm{cf}(\beta) \to \beta$ cofinal, define $h_0 : \mathrm{cf}(\beta) \to \alpha$ by

$$h_0(\xi) = \min\{\eta < \beta \, : \, f(\eta) > h(\xi)\}.$$

For any $\zeta < \alpha$, choose $\xi < \mathrm{cf}(\beta)$ such that $h(\xi) \geq f(\zeta)$. Then $f(h_0(\xi)) > h(\xi) \geq f(\zeta)$ implies $h_0(\xi) > \zeta$, so h_0 is cofinal. □

Corollary 1.7. *Let $S \subseteq \beta$ be a cofinal set. Then* $\mathrm{cf}(\mathrm{otp}(S)) = \mathrm{cf}(\beta)$.

Thus the cofinality of an ordinal is preserved when passing to cofinal subsets.

Corollary 1.8. $\mathrm{cf}(\mathrm{cf}(\beta)) = \mathrm{cf}(\beta)$.

Proposition 1.9. *(AC) Let $\kappa \geq \omega$ be a cardinal. Then $\mathrm{cf}(\kappa)$ is the minimal cardinality of a family F of subsets of κ such that $|S| < \kappa$ for all $S \in F$, and $\bigcup F = \kappa$.*

Proof. If $f : \mathrm{cf}(\kappa) \to \kappa$ is a cofinal map, then $F = \{f(\xi) : \xi < \mathrm{cf}(\kappa)\}$ is a family of size $\mathrm{cf}(\kappa)$ satisfying the given conditions.

Conversely, if F is a family satisfying the given conditions, then $T = \{\max(S) : S \in F\}$ is cofinal (otherwise T, and hence $\bigcup F = \kappa$, has an upper bound in κ, contradiction). Let $g : |F| \to F$ be a bijection, and define $h : |F| \to \kappa$ by $h(\xi) = \max(g(\xi))$. Then $\mathrm{ran}(h) = T$ implies h is cofinal, hence $|F| \geq \mathrm{cf}(\kappa)$. □

The following result can be seen as a generalisation of Cantor's Theorem that $2^\kappa > \kappa$. Under AC, it has implications for cardinal exponentiation.

Proposition 1.10 (König's Lemma). *Let $\langle \kappa_i : i \in I \rangle$ and $\langle \lambda_i : i \in I \rangle$ be sequences of cardinals with $\kappa_i < \lambda_i$ for all $i \in I$. Define*

$$X = \bigcup \{\{i\} \times \kappa_i : i \in I\}, \quad Y = \prod_{i \in I} \lambda_i.$$

Then there does not exist a surjection from X onto Y.

Proof. Fix any $f : X \to Y$. For each $i \in I$, let $\pi_i : Y \to \lambda_i$ denote projection on the i-th coordinate, ie. $\pi_i(y) = y(i)$. Now define $g \in Y$ by

$$g(i) = \min(\lambda_i \backslash \mathrm{Im}_{\pi_i \circ f}(\{i\} \times \kappa_i)).$$

Note that $\mathrm{Im}_{\pi_i \circ f}(\{i\} \times \kappa_i))$ is well-orderable with

$$|\mathrm{Im}_{\pi_i \circ f}(\{i\} \times \kappa_i))| \leq \kappa_i < \lambda_i,$$

hence $g(i) \in \lambda_i$ is well-defined for all $i \in I$, so g is well-defined. Also,

$$g(i) \notin \mathrm{Im}_{\pi_i \circ f}(\{i\} \times \kappa_i) \implies g \notin \mathrm{Im}_f(\{i\} \times \kappa_i)$$

for all $i \in I$, so $g \notin \mathrm{Im}_f(X)$. Thus f is not a surjection. □

Corollary 1.11. *(AC) If $\kappa \geq \omega$, then $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.*

Proof. Take $I = \mathrm{cf}(\kappa)$, $\lambda_i = \kappa$, and $\langle \kappa_i : i \in I \rangle$ unbounded in κ. □

Corollary 1.12. *(AC) If $\lambda \geq \omega$ and $\kappa \geq 2$, then $\mathrm{cf}(\kappa^\lambda) > \lambda$.*

Proof. Note that $(\kappa^\lambda)^{\mathrm{cf}(\kappa^\lambda)} > \kappa^\lambda = \kappa^{\lambda \otimes \lambda} = (\kappa^\lambda)^\lambda$. □

An ordinal β is called *regular* if β is a limit ordinal and $\mathrm{cf}(\beta) = \beta$. Regular cardinal

Example 1.13. *If β is a limit ordinal, then $\mathrm{cf}(\beta)$ is regular.*

Example 1.14. *ω is regular.*

Proposition 1.15. *Every regular ordinal is a cardinal.*

Proof. If β is regular, then $\beta = \mathrm{cf}(\beta) \leq |\beta|$. □

Proposition 1.16. *(AC) Let $\kappa \geq \omega$ be a cardinal. Then κ is regular if and only if the union of any family of $< \kappa$ sets, each of size $< \kappa$, has size $< \kappa$:*

$$|F| < \kappa \wedge \forall S \in F\,[|S| < \kappa] \implies \left|\bigcup F\right| < \kappa.$$

Corollary 1.17. *(AC) κ^+ is regular.*

Proof. $\kappa \otimes \kappa = \kappa < \kappa^+$. □

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Proposition 1.16 can be modified to give a combinatorial version of the downward Löwenheim-Skolem theorem, as follows. We call any $f : A^n \to A$ an *n-ary function* on A, and we say that $B \subseteq A$ is *closed* under f if $\mathrm{Im}_f(B^n) \subseteq B$. (In particular, a 0-ary function f is a constant in A, and B is closed under f if $f \in B$.) A *finitary function* is an n-ary function for some $n \geq 0$.

n-ary function

Finitary function

Given a set $\mathcal{S}$ of finitary functions on A, and a subset $B \subseteq A$, the *closure* of B under $\mathcal{S}$ is the smallest $C \subseteq A$ that contains B and is closed under all functions in $\mathcal{S}$. Note that C is equal to the intersection of all subsets of A containing B that are closed under all functions in $\mathcal{S}$.

Proposition 1.18. *(AC) Let $\kappa \geq \omega$ be a cardinal, and let A be any set. If $\mathcal{S}$ is a family of $\leq \kappa$ finitary functions on A, and $B \subseteq A$ with $|B| \leq \kappa$, then the closure of B under $\mathcal{S}$ has cardinality $\leq \kappa$.*

Proof. For any $D \subseteq A$ and $f \in \mathcal{S}$ n-ary, write $f * D = \mathrm{Im}_f(D^n)$. Then

$$|D| \leq \kappa \implies |D^n| \leq \kappa \implies |f * D| \leq \kappa.$$

Define a sequence $B = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$ by

$$C_{n+1} = C_n \cup \bigcup_{f \in \mathcal{S}} f * C_n.$$

Then by induction we have $|C_n| \leq \kappa$ by Proposition 1.16. Hence $C = \bigcup_n C_n$ is the closure of B under $\mathcal{S}$, and $|C| \leq \kappa$ by Proposition 1.16. □

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We say that $\kappa > \omega$ is *weakly inaccessible* if it is a regular limit cardinal. Under AC, we say that κ is *strongly inaccessible* if it is regular and $\lambda < \kappa \implies 2^\lambda < \kappa$.

Weakly inaccessible

Strongly inaccessible

The terminology is suggestive: if κ is a weak inaccessible, then it cannot be reached by the cardinal successor operation on cardinals $< \kappa$, or by summing $< \kappa$ cardinals that are $< \kappa$. In addition, if κ is a strong inaccessible then it cannot be reached by the power set operation on cardinals $< \kappa$.

It is clear that every strong inaccessible is also a weak inaccessible. Note that under GCH, the two notions are equivalent.

Example 1.19. *Some authors do not require inaccessible ordinals to be uncountable, in which case ω is strongly inaccessible.*

Proposition 1.20. *If α is a limit ordinal, then $\mathrm{cf}(\omega_\alpha) = \mathrm{cf}(\alpha)$.*

Hence if ω_α is a weak inaccessible, then $\omega_\alpha = \alpha$. However, this is not a sufficient condition.

Example 1.21. *Let $\alpha = \sup(\omega, \omega_\omega, \omega_{\omega_\omega}, \ldots)$. Then $\omega_\alpha = \alpha$, but $\mathrm{cf}(\alpha) = \omega$.*

Hence inaccessible cardinals are quite large. In fact, the existence of an inaccessible implies the consistency of ZFC, so ZFC cannot prove that inaccessibles exist.

1.2 Filters

From here on we will assume ZFC unless otherwise indicated.

For a nonempty set A, a *filter* on S is a collection F of subsets of S satisfying: Filter

1. $A \in F$ and $\emptyset \notin F$;
2. $X, Y \in F \implies X \cap Y \in F$;
3. $X \subseteq Y \subseteq A \wedge X \in F \implies Y \in F$.

Intuitively, a filter captures the notion of "large" sets: the intersection of two large sets is large, and a superset of a large set is also large.

Example 1.22. $F = \{A\}$ *is the* trivial filter *on* A.

Example 1.23. *Let* $X \subseteq A$ *be nonempty; then* $F = \{Y \subseteq A : X \subseteq Y\}$ *is the* principal filter *on* A *generated by* X. *Note that every filter on a finite set is principal.*

Example 1.24. *Let* A *be an infinite set. Then* $F = \{X \subseteq A : |A \backslash X| < \omega\}$ *is the filter of all* cofinite *sets of* S. *Note that this is not a principal filter.*

Example 1.25. $F = \{X \subseteq [0,1] : \mu(X) = 1\}$ *is a filter on* $[0,1]$, *where* μ *is the Lebesgue measure on* $[0,1]$.

A family G of sets has the *finite intersection property* if every finite sub-family has nonempty intersection, ie. Finite intersection property

$$\forall H \subseteq G \, [|H| < \omega \implies \bigcap H \neq \emptyset].$$

Clearly, every filter has the finite intersection property. Conversely, we can construct a filter from a family with the finite intersection property, as follows.

Proposition 1.26. *If* $G \subseteq \mathcal{P}(F)$ *has the finite intersection property, then there is a filter* F *on* S *such that* $G \subseteq F$.

Proof. Let F be the collection of all $X \subseteq S$ such that $X \supseteq \bigcap H$ for some finite $H \subseteq G$. It is easy to verify that F is the desired filter. $\square$

In fact, since every filter extending G must contain all finite intersections of sets in G, the filter F constructed above is the inclusion-minimal filter on S that extends G. Thus we say that the filter F is *generated by* G. Filter generated by a set

It is easy to see that filters are closed under finite intersections. Some filters are closed under larger intersections: we say that a filter F is *κ-complete* if for every family $S \subseteq F$ with $|S| < \kappa$, we have $\bigcap S \in F$. Thus every filter is ω-complete, and the measure filter in Example 1.25 is ω_1-complete. κ-complete [20 Jan

The dual concept to the filter is the *ideal*, which is a collection I of subsets of S such that $\emptyset \in I$, $A \notin I$, and I is closed under finite union and downward inclusion. Note that every filter F on A induces the *dual ideal* $F^* = \{A \backslash X : X \in F\}$, and conversely every ideal I on A induces the *dual filter* $I^* = \{A \backslash X : X \in I\}$. Similar to the filter, an ideal captures the notion of "small" sets in A. Ideal Dual ideal Dual filter

Many properties of filters, such as κ-completeness, have natural duals for ideals. We will focus on filters instead of ideals, since the two concepts are dual to each other. However, it will be easier to state a few results in the language of ideals.]

1.3 Club and stationary sets

Let X be a set of ordinals, and let α be a limit ordinal. Then α is called a *limit point* of X if $\sup(X \cap \alpha) = \alpha$. Limit point

Proposition 1.27. *α is a limit point of X if and only if α is the supremum of some increasing sequence $\langle \alpha_\xi : \xi < \gamma \rangle$ in X.*

Proof. ($\Rightarrow$): Let $f : \mathrm{otp}(X \cap \alpha) \to X \cap \alpha$ be an isomorphism, so $\langle f(\xi) : \xi < \mathrm{otp}(X \cap \alpha) \rangle$ is increasing, and $\sup\{f(\xi) : \xi < \mathrm{otp}(X \cap \alpha)\} = \sup(X \cap \alpha) = \alpha$.

($\Leftarrow$): We have $\alpha \geq \sup(X \cap \alpha) \geq \sup\{\alpha_\xi : \xi < \gamma\} = \alpha$, so $\sup(X \cap \alpha) = \alpha$. □

Let μ be a limit ordinal. A set $X \subseteq \mu$ is called *closed* if it contains all its limit points less than μ. X is called a *club set* in μ if it is closed and unbounded in μ. Club set

Example 1.28. *If $\mathrm{cf}(\mu) = \omega$, then any countable cofinal set in μ is club in μ.*

Hence the study of club sets only becomes interesting when $\mathrm{cf}(\mu) > \omega$.

Proposition 1.29. *If $\mathrm{cf}(\mu) > \omega$ and A is unbounded in μ, then*

$$B = \{\alpha < \mu : \alpha \text{ limit point of } A\}$$

is club in μ.

Proof. Let β be any limit point of B. Then $\beta = \sup\{\alpha_\xi : \xi < \gamma\}$ for some $\alpha_\xi \in B$. Hence $\sup(A \cap \alpha_\xi) = \alpha_\xi$, so

$$\sup(A \cap \beta) = \sup(\bigcup_{\xi<\gamma} A \cap \alpha_\xi) = \sup_{\xi<\gamma}(\sup(A \cap \alpha_\xi)) = \sup_{\xi<\gamma} \alpha_\xi = \beta,$$

where we used the fact that $\sup(\bigcup_\xi A_\xi) = \sup_\xi(\sup A_\xi)$. Hence $\beta \in B$, so B is closed.

Now for any $\alpha < \mu$, define $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \cdots$ recursively by

$$\alpha_{n+1} = \min\{a \in A : a > \alpha_n\}.$$

Then $\eta = \sup_n \alpha_n$ satisfies $\alpha < \eta < \mu$ and $\eta \in B$, so B is unbounded. □

Proposition 1.30. *If $\mathrm{cf}(\mu) > \omega$, then the intersection of any family of less than $\mathrm{cf}(\mu)$ club sets in μ is club.*

Proof. Let $\langle C_\alpha : \alpha < \lambda \rangle$ be a sequence of $\lambda < \mathrm{cf}(\mu)$ club sets in μ, and let $D = \bigcap_{\alpha<\lambda} C_\alpha$.

If $\sup(D \cap \xi) = \xi$ then $\sup(C_\alpha \cap \xi) = \xi \implies \xi \in C_\alpha$ for all $\alpha < \lambda$, so $\xi \in D$. Hence D is closed.

Define $f_\alpha : \mu \to \mu$ ($\alpha < \lambda$) and $g : \mu \to \mu$ by

$$f_\alpha(\xi) = \min\{x \in C_\alpha : x > \xi\}, \qquad g(\xi) = \sup\{f_\alpha(\xi) : \alpha < \lambda\}.$$

Note that $\lambda < \mathrm{cf}(\mu)$ implies $g(\xi) < \mu$ for all $\xi < \mu$.

Now define $\xi = \xi_0 < \xi_1 < \xi_2 < \cdots$ recursively by $\xi_{n+1} = g(\xi_n)$, so that $\eta = \sup_n \xi_n$ satisfies $\xi < \eta < \mu$. Then $\xi_n < f_\alpha(\xi_n) \leq \xi_{n+1}$, so C_α is unbounded in η. Thus $\eta \in C_\alpha$ for all $\alpha < \lambda$, so η is an element in D greater than ξ. Hence D is unbounded. □

In particular, if $\mathrm{cf}(\mu) > \omega$, the collection of all club sets in μ has the finite intersection property. The filter generated by this collection is called the *club filter* on μ, given by Club filter club(μ)

$$\mathrm{club}(\mu) = \{X \subseteq \mu : \exists C \subseteq X\, [C \text{ is club in } \mu]\}.$$

Corollary 1.31. *If* $\mathrm{cf}(\mu) > \omega$*, then* $\mathrm{club}(\mu)$ *is a* $\mathrm{cf}(\mu)$*-complete filter.*

For a sequence $\langle X_\alpha : \alpha < \mu \rangle$ of subsets of μ, the *diagonal intersection* is defined as Diagonal intersection

$$\triangle_{\alpha<\mu} X_\alpha = \{\xi < \mu : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}.$$

Proposition 1.32. *Let* $\kappa > \omega$ *be regular, and let* C_α *be club in* κ *for all* $\alpha < \kappa$*. Then* $D = \triangle_{\alpha<\kappa} C_\alpha$ *is club in* κ*.*

Proof. If δ is a limit point of D, then for all $\zeta < \delta$,

$$\delta = \sup\{\xi : \zeta < \xi < \delta \wedge \xi \in \bigcap_{\alpha<\xi} C_\alpha\} \leq \sup(\delta \cap \bigcap_{\alpha<\zeta} C_\alpha) \leq \delta,$$

so $\delta = \sup(\delta \cap \bigcap_{\alpha<\zeta} C_\alpha)$ implies $\delta \in \bigcap_{\alpha<\zeta} C_\alpha$ for all $\zeta < \delta$. Hence $\delta \in \bigcap_{\alpha<\delta} C_\alpha$, so $\delta \in D$. Thus D is closed.

Define $g : \kappa \to \kappa$ by

$$g(\xi) = \min\{x \in \bigcap_{\alpha<\xi} C_\alpha : x > \xi\}.$$

Now define $\xi = \xi_0 < \xi_1 < \xi_2 < \cdots$ recursively by $\xi_{n+1} = g(\xi_n)$, so that $\eta = \sup_n \xi_n$ satisfies $\xi < \eta < \kappa$. Then for all $n \leq m < \omega$, we have $\xi_m \in \bigcap_{\alpha<\xi_m} C_\alpha \subseteq \bigcap_{\alpha<\xi_n} C_\alpha$, so $\eta \in \bigcap_{\alpha<\xi_n} C_\alpha$ for all $n < \omega$. Hence η is an element in D greater than ξ. Hence D is unbounded. ☐

Corollary 1.33. *If* $\kappa > \omega$ *is regular, then* $\mathrm{club}(\kappa)$ *is closed under diagonal intersections.*

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We have the following version of the downward Löwenheim-Skolem theorem.

Proposition 1.34. *Let* $\kappa > \omega$ *be regular, and let* $\mathcal{A}$ *be a family of* $< \kappa$ *finitary functions on* κ*. Then* $C = \{\gamma < \kappa : \gamma$ *closed under* $\mathcal{A}\}$ *is club in* κ*.*

Proof. If $\delta < \kappa$ satisfies $\delta = \sup(C \cap \delta)$, then for all n-ary $f \in \mathcal{A}$ and $\zeta < \delta$, there exists $\gamma > \zeta$ in C, so $\mathrm{Im}_f(\zeta^n) \subseteq \mathrm{Im}_f(\gamma^n) \subseteq \gamma < \delta$. Hence δ is closed under $\mathcal{A}$, so $\delta \in C$. Hence C is closed.

Define $g : \kappa \to \kappa$ by $g(\xi) = \max(\xi + 1, \zeta)$, where ζ is the supremum of the closure of ξ under $\mathcal{A}$. Note that this closure has size at most $\max(|\mathcal{A}|, |\xi|) < \kappa$, by Proposition 1.18, so g is well-defined and $g(\xi) > \xi$. Also, for all n-ary $f \in \mathcal{A}$, we have $\mathrm{Im}_f(\xi^n) \subseteq g(\xi)$.

Thus for all $\xi < \kappa$, we may define $\xi = \xi_0 < \xi_1 < \xi_2 < \cdots$ recursively by $\xi_{n+1} = g(\xi_n)$, so that $\eta = \sup_n \xi_n$ satisfies $\xi < \eta < \kappa$, and η is closed under $\mathcal{A}$. Hence $\eta \in C$, so C is unbounded. ☐

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A set $S \subseteq \mu$ is *stationary* in μ if $S \cap C \neq \emptyset$ for all C club in μ. Note that S is stationary if and only if $\kappa \backslash S \notin \mathrm{club}(\kappa)$, ie. $S \notin \mathrm{club}^*(\kappa)$. For this reason, $\mathrm{club}^*(\kappa)$ is sometimes referred to as the *non-stationary ideal* on κ. Stationary set Non-stationary ideal

Recalling Example 1.25, we can set up an analogy with the measure filter: club sets correspond to sets of full measure, and stationary sets correspond to sets of positive measure.

Proposition 1.35. *If* $\mathrm{cf}(\mu) > \omega$*, then every club set in* μ *is stationary in* μ*.*

Proposition 1.36. *If* $\mathrm{cf}(\mu) > \omega$*, then the union of any family of less than* $\mathrm{cf}(\mu)$ *non-stationary sets is non-stationary.*

Proposition 1.37. *If* $\mathrm{cf}(\mu) > \omega$*, and* S, C *are stationary and club in* μ*, respectively, then* $S \cap C$ *is stationary in* μ*.*

Proof. For any C' club in μ, we have $C \cap C'$ is club in μ, so $S \cap C \cap C' \neq \emptyset$. □

Proposition 1.38. *If* λ *is regular and* $\mathrm{cf}(\mu) > \lambda$*, then* $\{\gamma < \mu : \mathrm{cf}(\gamma) = \lambda\}$ *is stationary in* μ*.*

Proof. For any C club in μ, let η be the λ^{th} element of C. Then $C \cap \eta$ is cofinal in η, so $\mathrm{cf}(C \cap \eta) = \mathrm{cf}(\lambda) = \lambda$. □

Let $S \subseteq \alpha$. A function $f : S \to \alpha$ is called *regressive* if $f(\xi) < \xi$ for all $\xi \in S$. Regressive

The following theorem, central to the study of stationary sets, is due to Fodor.

Theorem 1.39 (Pressing-Down Lemma). *Let* $\kappa > \omega$ *be regular, and let* S *be stationary in* κ*. For any regressive* $f : S \to \kappa$*, there exists* $\alpha < \kappa$ *such that* $f^{-1}(\{\alpha\})$ *is stationary in* κ*.*

Proof. Assume for sake of contradiction that for all $\alpha < \kappa$, there exists a club set $C_\alpha \subseteq \kappa$ such that $f^{-1}(\{\alpha\}) \cap C_\alpha = \emptyset$. Let $D = \triangle_{\alpha<\kappa} C_\alpha$.

Since S is stationary and D is club in κ, there exists $\gamma \in D \cap S$. Then for all $\alpha < \gamma$, we have $\gamma \in C_\alpha \implies f(\gamma) \neq \alpha$, contradicting $f(\gamma) < \gamma$. □

2 Solovay's Theorem

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Proposition 2.1 (Ulam). *Let $\kappa = \lambda^+$, and $\mathcal{I}$ be a κ-complete ideal on κ such that each singleton is in $\mathcal{I}$. Then there are pairwise disjoint $X_\alpha \subseteq \kappa$ for $\alpha < \kappa$ such that $X_\alpha \notin \mathcal{I}$.*

Proof. Note that every subset of κ of size $\leq \lambda$ is in $\mathcal{I}$.

For each $\rho < \kappa$, choose an injection $f_\rho : \rho \to \lambda$. Now for any $\alpha < \kappa$,

$$(\alpha + 1) \cup \bigcup_{\xi < \lambda} \{\rho > \alpha \,:\, f_\rho(\alpha) = \xi\} = \kappa.$$

Since $\alpha + 1 \in \mathcal{I}$, if $\{\rho > \alpha \,:\, f_\rho(\alpha) = \xi\} \in \mathcal{I}$ for all $\xi < \lambda$ then $\kappa \in \mathcal{I}$ by κ-completeness of $\mathcal{I}$, contradiction. Hence there exists $h : \kappa \to \lambda$ such that

$$\{\rho > \alpha \,:\, f_\rho(\alpha) = h(\rho)\} \notin \mathcal{I}.$$

Now there exists $\zeta < \lambda$ such that $|h^{-1}(\{\zeta\})| = \kappa$. For each $\alpha \in h^{-1}(\{\zeta\})$, take $X_\alpha = \{\rho > \alpha \,:\, f_\rho(\alpha) = \zeta\}$. By injectivity of f_ρ, we have $\langle X_\alpha \,:\, \alpha \in h^{-1}(\{\zeta\})\rangle$ are pairwise disjoint, so X_α satisfy the given conditions. $\square$

Corollary 2.2. *Let $\kappa > \omega$ be regular. Then there is a family of κ disjoint stationary subsets of κ.*

Proof. If κ is successor then apply Proposition 2.1 to club$^*(\kappa)$.

Otherwise, κ is weakly inaccessible. Now $\omega_{\xi+1}$ is regular for all $\xi < \kappa$, so $\langle\{\gamma < \kappa \,:\, \mathrm{cf}(\gamma) = \omega_{\xi+1}\} \,:\, \xi < \kappa\rangle$ is a family of κ disjoint stationary sets. $\square$

Corollary 2.3. *Let $\kappa = \lambda^+$. If $S \subseteq \kappa$ is stationary, then there is a family of κ disjoint stationary subsets of S.*

Proof. Apply Proposition 2.1 to $\{X \subseteq \kappa \,:\, X \cap S \text{ non-stationary}\}$. $\square$

The statement of the above corollary also holds if κ is weakly inaccessible, by a theorem of Solovay. We first prove some intermediate results.

Proposition 2.4. *Let $\kappa > \lambda > \omega$ be regular cardinals, and let $W \subseteq \{\alpha < \kappa \,:\, \mathrm{cf}(\alpha) = \lambda\}$ be stationary in κ. Then there is a family of κ disjoint stationary subsets of W.*

Proof. We first assume that there exists a regressive $f : W \to \kappa$ such that $\{\alpha \in W \,:\, f(\alpha) \geq \eta\}$ is stationary for every $\eta < \kappa$. Then by the pressing-down lemma, the set

$$S = \{\eta < \kappa \,:\, f^{-1}(\{\eta\}) \text{ stationary}\}$$

is unbounded in κ, so $|S| = \kappa$, and $\langle f^{-1}(\{\eta\}) \,:\, \eta \in S\rangle$ is a family of pairwise disjoint stationary subsets of W.

We now show the existence of such an f. For each $\alpha \in W$, take an increasing sequence $\langle a^\alpha_\xi \,:\, \xi < \lambda\rangle$ with limit α. Define $f_\xi : \lambda \to \kappa$ by $f_\xi(\alpha) = a^\alpha_\xi$ for all $\xi < \lambda$.

Now assume for sake of contradiction that for each $\xi < \lambda$ there exists $\eta_\xi < \kappa$ such that $\{\alpha \in W \,:\, f_\xi(\alpha) \geq \eta_\xi\}$ is not stationary, so there exists C_ξ club in κ such that $f_\xi(\alpha) < \eta_\xi$ for all $\alpha \in C_\xi \cap W$. Let $\eta = \sup_{\xi<\lambda} \eta_\xi < \kappa$, and $C = \bigcap_{\xi<\lambda} C_\xi$ club in κ. Then $f_\xi(\alpha) < \eta$ for all $\alpha \in C \cap W$ and $\xi < \lambda$, so $\alpha < \eta$ for all $\alpha \in C \cap W$, contradiction. Hence there exists $\xi < \lambda$ such that f_ξ has the desired property. $\square$

Corollary 2.5. *Let $\kappa > \omega$ be regular, and let $W \subseteq \{\alpha < \kappa \,:\, \mathrm{cf}(\alpha) < \alpha\}$ be stationary in κ. Then there is a family of κ disjoint stationary subsets of W.*

Proof. By the pressing-down lemma on $\mathrm{cf} : W \to \kappa$, there exists $\lambda < \kappa$ such that $\{\alpha \in W : \mathrm{cf}(\alpha) = \lambda\}$ is stationary. □

Proposition 2.6. *Let $\kappa > \omega$ be regular, and let $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \alpha\}$ be stationary in κ. Then $T = \{\alpha \in S : S \cap \alpha$ non-stationary in $\alpha\}$ is stationary in κ.*

Proof. We may assume that $\omega \notin S$. Let C be club in κ, and let C' be the set of limit points of C, which is also club. Let $\alpha = \min(S \cap C')$, so $C \cap \alpha$ is club in α, thus $C' \cap \alpha$ is club in α. Now $(S \cap \alpha) \cap (C' \cap \alpha) = S \cap C' \cap \alpha = \emptyset$, so $S \cap \alpha$ is not stationary in α. Thus $\alpha \in C \cap T$, so T intersects all clubs in κ. □

Theorem 2.7. *Let $\kappa > \omega$ be regular, and let $A \subseteq \kappa$ be stationary. Then there is a family of κ disjoint stationary subsets of A.*

Proof. If $A \cap \{\alpha < \kappa : \mathrm{cf}(\alpha) < \alpha\}$ is stationary then we are done. Otherwise $B = A \cap \{\alpha < \kappa : \mathrm{cf}(\alpha) = \alpha\}$ is stationary, and hence so is

$$W = \{\alpha \in B : B \cap \alpha \text{ non-stationary in } \alpha\}.$$

If $f : W \to \kappa$ is regressive and $\{\alpha \in W : f(\alpha) \geq \eta\}$ is stationary for every $\eta < \kappa$, then by the pressing-down lemma, the set

$$S = \{\eta < \kappa : f^{-1}(\{\eta\}) \text{ stationary}\}$$

is unbounded in κ, so $|S| = \kappa$, and $\langle f^{-1}(\{\eta\}) : \eta \in S\rangle$ is a family of pairwise disjoint stationary subsets of W.

We now show the existence of such an f. For each $\alpha \in W$, $W \cap \alpha$ is non-stationary in α, so there is a club in α that does not intersect W, which we can enumerate as $\langle a_\xi^\alpha : \xi < \alpha\rangle$. Define $f_\xi : \kappa \backslash (\xi + 1) \to \kappa$ by $f_\xi(\alpha) = a_\xi^\alpha$ for all $\xi < \kappa$.

Now assume for sake of contradiction that there exists $\eta : \kappa \to \kappa$ such that for each $\xi < \lambda$, we have $\{\alpha \in W \backslash (\xi + 1) : f_\xi(\alpha) \geq \eta(\xi)\}$ is not stationary. Then there exists $C_\xi \subseteq \kappa \backslash (\xi + 1)$ club in κ such that $f_\xi(\alpha) < \eta(\xi)$ for all $\alpha \in C_\xi \cap W$.

Let $C = \triangle_{\xi<\kappa} C_\xi$, so $\alpha \in C \cap W$ implies $f_\xi(\alpha) < \eta(\xi)$ for all $\xi < \alpha$. By downward Löwenheim-Skolem (Proposition 1.34), the set $D = \{\gamma \in C : \forall \xi < \gamma\, [\eta(\xi) < \gamma]\}$ is club in κ, so we may take $\gamma, \Gamma \in D \cap W$, $\gamma < \Gamma$. Now $a_\xi^\Gamma = f_\xi(\Gamma) < \eta(\xi) < \gamma$ for all $\xi < \gamma$, so $a_\gamma^\Gamma = \sup_{\xi<\gamma} a_\xi^\Gamma = \gamma$, contradicting $\{a_\xi^\alpha : \xi < \alpha\} \cap W = \emptyset$. Hence there exists $\xi < \lambda$ such that f_ξ has the desired property. □

3 Canonical Sequences

27 Jan

Given $\kappa > \omega$ regular, and $f, g \in {}^\kappa\kappa$, we say that f *dominates* g (mod club(κ)), and write $f \prec g$, if $\{\alpha : f(\alpha) < g(\alpha)\} \in \text{club}(\kappa)$. Similarly, we write $f \preceq g$ if $\{\alpha : f(\alpha) \leq g(\alpha)\} \in \text{club}(\kappa)$.

Proposition 3.1. *$\prec$ is a well-founded (strict) partial order.*

Proof. The fact that $\prec$ is a partial order follows from the fact that club(κ) is a filter.

Since club(κ) is ω_1-complete, if $f_0 \succ f_1 \succ f_2 \succ \cdots$ then there exists a club set $C \subseteq \kappa$ on which $f_0(x) > f_1(x) > f_2(x) > \cdots$, contradiction. □

A sequence $\langle f_\alpha : \alpha < \kappa^+\rangle$ of elements in ${}^\kappa\kappa$ is called a *canonical sequence* if it is well-ordered by $\prec$ (ie. $f_\alpha \prec f_\beta$ for all $\alpha < \beta < \kappa^+$), and for any other sequence $\langle g_\alpha : \alpha < \kappa^+\rangle$ in ${}^\kappa\kappa$ well-ordered by $\prec$, we have $f_\alpha \preceq g_\alpha$ for all $\alpha < \kappa^+$. Canonical sequence

Proposition 3.2. *If $\langle f_\alpha : \alpha < \kappa^+\rangle$ and $\langle g_\alpha : \alpha < \kappa^+\rangle$ are canonical sequences in ${}^\kappa\kappa$, then $\{\alpha : f(\alpha) = g(\alpha)\} \in \text{club}(\kappa)$.*

We now show the existence of canonical sequences by construction.

Theorem 3.3. *Let $\kappa > \omega$ be regular. Then there is a canonical sequence in ${}^\kappa\kappa$.*

Proof. For $\alpha < \kappa$, let $f_\alpha(\xi) = \alpha$ for all $\xi < \kappa$. For $\kappa \leq \alpha < \kappa^+$, let $h_\alpha : \kappa \to \alpha$ be bijective, and let $f_\alpha(\xi) = \text{otp}(\text{Im}_{h_\alpha}(\xi))$ for all $\xi < \kappa$. We now show that $\langle f_\alpha : \alpha < \kappa^+\rangle$ is a canonical sequence.

Clearly $f_\alpha \prec f_\beta$ for all $\alpha < \beta < \kappa$. Also, if $\alpha < \kappa \leq \beta < \kappa^+$, then $\eta := \sup(\text{Im}_{h_\beta^{-1}}(\alpha+1)) < \kappa$, so $f_\alpha(\xi) = \alpha < f_\beta(\xi)$ for all $\xi \geq \eta$. Hence $f_\alpha \prec f_\beta$.

Also, if $\langle g_\alpha : \alpha < \kappa^+\rangle$ in ${}^\kappa\kappa$ is well-ordered by $\prec$, then for all $\alpha < \kappa$, we have

$$C = \bigcap_{\zeta < \zeta' \leq \alpha} \{\xi : g_\zeta(\xi) < g_{\zeta'}(\xi)\}$$

is the intersection of $< \kappa$ many club sets, and hence is club in κ. Thus for all $\xi \in C$, we have $\langle g_\zeta(\xi) : \zeta \leq \alpha\rangle$ is an increasing sequence, so $g_\alpha(\xi) \geq \alpha = f_\alpha(\xi)$. Thus $f_\alpha \preceq g_\alpha$.

Now for $\kappa \leq \alpha < \beta < \kappa^+$, let $\eta_0 = h_\beta^{-1}(\alpha)$, and define

$$C_{\alpha,\beta} = \{\xi < \kappa : \xi > \eta_0 \wedge [\xi \text{ closed under } h_\beta^{-1} h_\alpha]\}.$$

Note that $C_{\alpha,\beta}$ is club in κ. Now for all $\xi \in C_{\alpha,\beta}$, we have $\text{Im}_{h_\alpha}(\xi) \subseteq \text{Im}_{h_\beta}(\xi)$, and $\text{Im}_{h_\beta}(\xi)$ contains $\eta_0 \geq \sup(\text{Im}_{h_\alpha}(\xi))$. Hence $f_\alpha(\xi) = \text{otp}(\text{Im}_{h_\alpha}(\xi)) < \text{otp}(\text{Im}_{h_\beta}(\xi)) = f_\beta(\xi)$ for all $\xi \in C$, so $f_\alpha \prec f_\beta$.

If $\langle g_\alpha : \alpha < \kappa^+\rangle$ in ${}^\kappa\kappa$ is well-ordered by $\prec$, we perform transfinite induction on α to show that $f_\alpha \preceq g_\alpha$ for all $\kappa \leq \alpha < \kappa^+$. For $\alpha = \kappa$, we may assume that $h_\kappa = \text{id}_\kappa$, so $f_\kappa = \text{id}_\kappa$. Now

$$\mathop{\triangle}_{\zeta<\kappa} \{\xi < \kappa : g_\kappa(\xi) > \zeta\} = \{\xi < \kappa : g_\kappa(\xi) \geq \xi\}$$

is club in κ, so $f_\kappa \preceq g_\kappa$.

Now take $\kappa < \alpha < \kappa^+$, and assume that $f_\beta \preceq g_\beta$ for all $\kappa \leq \beta < \alpha$. Thus $f_\beta \prec g_\alpha$, and we want to show that $f_\alpha \preceq g_\alpha$. Thus it suffices to show that for any

$g \in {}^\kappa\kappa$, if $\{\xi : g(\xi) < f_\alpha(\xi)\}$ is stationary then there exists $\kappa \leq \beta < \alpha$ such that $\{\xi : g(\xi) \leq f_\beta(\xi)\}$ is stationary.

Let $\langle \alpha_\zeta : \zeta < \mathrm{cf}(\alpha) \rangle$ be an increasing sequence with limit α. Define $j : \kappa \to \mathrm{cf}(\alpha)$ by $j(\eta) = \min\{\xi : h_\alpha(\eta) < \alpha_\xi\}$. If $\mathrm{cf}(\alpha) < \kappa$, let

$$C = \{\xi : \xi \text{ closed under } j\} \cap \bigcap_{\zeta < \mathrm{cf}(\alpha)} C_{\alpha_\zeta, \alpha},$$

so C is club in κ, and for all $\xi \in C$ we have $f_\alpha(\xi) = \sup_{\eta < \xi} f_{\alpha_\eta}(\xi)$. $\square$

TBC

4 Trees

24 Feb

4.1 Preliminaries

A *tree* is a partial order $\langle T, <\rangle$ such that for all $x \in T$, we have

Tree
$\mathrm{pred}_T(x)$

$$\mathrm{pred}_{\langle T,<\rangle}(x) = \mathrm{pred}_T(x) = \{y \in T : y < x\}$$

is well-ordered by $<$.

Given a tree T, the *height of* $x \in T$ is defined as

Height of $x \in T$
$\mathrm{ht}_T(x)$

$$\mathrm{ht}_T(x) = \mathrm{otp}(\mathrm{pred}_T(x)).$$

For any ordinal α, the α-th *level* of T is

Level
$\mathrm{Lev}_\alpha(T)$

$$\mathrm{Lev}_\alpha(T) = \{x \in T : \mathrm{ht}(x, T) = \alpha\}.$$

The *height of* T is

Height of T
$\mathrm{ht}(T)$

$$\mathrm{ht}(T) = \min\{\alpha : \mathrm{Lev}_\alpha(T) = \emptyset\}.$$

A *subtree* of T is any $T' \subseteq T$ that is downward closed, ie.

Subtree

$$\forall x \in T' \,\forall y \in T\, [y < x \implies y \in T'],$$

or equivalently, $\forall x \in T' \,[\mathrm{pred}_T(x) \subseteq T']$.

Example 4.1. *Every ordinal δ is a tree under the usual ordering. For all $\alpha < \delta$, we have $\mathrm{ht}_\delta(\alpha) = \alpha$, $\mathrm{Lev}_\alpha(\delta) = \alpha$, and $\mathrm{ht}(\delta) = \delta$.*

Example 4.2. *The* complete I-ary tree of height δ *is the partial order $\langle {}^{<\delta}I, \subseteq\rangle$. We have $\mathrm{Lev}_\alpha({}^{<\delta}I) = {}^{\alpha}I$, and $\mathrm{ht}({}^{<\delta}I) = \delta$. In particular, ${}^{<\delta}2$ is called the* complete binary tree of height δ.

Complete I-ary tree
Complete binary tree

Proposition 4.3. *For any tree $\langle T, <\rangle$, we have $\mathrm{ht}(T) = \sup\{\mathrm{ht}(x) + 1 : x \in T\}$.*

Proposition 4.4. *Let T be a tree, and let $T' \subseteq T$ be a subtree. Then for all $x \in T'$, we have $\mathrm{ht}_T(x) = \mathrm{ht}_{T'}(x)$.*

Proposition 4.5. *Let $\langle T, <\rangle$ be a tree, $x \in T$, and $\alpha < \mathrm{ht}_T(x)$. Then there exists a unique $y < x$ such that $\mathrm{ht}_T(y) = \alpha$.*

Proof. Let $S = \mathrm{pred}_T(x) \cong \mathrm{ht}_T(x)$, and note that for all $y \in S$,

$$\mathrm{ht}_T(y) = \mathrm{otp}(\mathrm{pred}_T(y)) = \mathrm{otp}(\mathrm{pred}_S(y)).$$

Hence $\mathrm{ht}_T \restriction S : S \to \mathrm{ht}_T(x)$ is an isomorphism, and the result follows. □

Proposition 4.6. *Let $\langle T, <\rangle$ be a tree, and let $S \subseteq \mathrm{ht}(T)$ be a set of ordinals. Then $\langle \bigcup_{\alpha \in S} \mathrm{Lev}_\alpha(T), <\rangle$ is a tree.*

Recall that for a partial order, a *chain* (resp. *antichain*) is a subset of pairwise comparable (resp. incomparable) elements.

Chain
Antichain

A *path* through a tree T is a chain which intersects every $\mathrm{Lev}_\alpha(T)$ for $\alpha < \mathrm{ht}(T)$.

Path
3 Mar

Proposition 4.7. *Let T be a tree, and let C be a chain in T such that $\sup_{x \in C} \mathrm{ht}_T(x) = \mathrm{ht}(T)$. Then $\bigcup_{x \in C} \mathrm{pred}_T(x)$ is a path through T.*

A tree T is *well-pruned* if $|\mathrm{Lev}_0(T)| = 1$ and

Well-pruned tree

$$\forall x \in T \,\forall \alpha\, [\mathrm{ht}_T(x) < \alpha < \mathrm{ht}(T) \implies \exists y \in \mathrm{Lev}_\alpha(T)\, [x < y]].$$

Theorem 4.8. *Let $\lambda \geq \omega$ be a cardinal, and suppose that T is a tree with* $\mathrm{ht}(T) = \lambda$ *and*

$$\forall \alpha < \lambda \, [\sum_{\xi < \alpha} |\operatorname{Lev}_\xi(T)| < \lambda].$$

Then T has a well-pruned subtree of height λ.

Proof. Consider $T' = \{x \in T : |\{y \in T : y > x\}| = \lambda\}$. This is clearly downwards closed in T. Now for any $x \in T'$ and $\mathrm{ht}_T(x) < \alpha < \lambda$, let $S = \{y \in T : y > x\} \cap \operatorname{Lev}_\alpha(T)$. Then

$$\begin{aligned} \bigcup_{y \in S} \{z \in T : z > y\} &= \bigcup_{\substack{y > x \\ \mathrm{ht}_T(y) = \alpha}} \{z \in T : z > y\} \\ &= \{z \in T : z > x \wedge \mathrm{ht}_T(z) > \alpha\} \\ &= \{z \in T : z > x\} \setminus \left(\bigcup_{\xi < \alpha + 1} \operatorname{Lev}_\xi(T) \right), \end{aligned}$$

which has size λ. Now $|S| < \lambda$, so there exists $y \in S$ with $|\{z \in T : z > y\}| = \lambda$, so $y \in T'$.

A similar argument shows that $\operatorname{Lev}_0(T') \neq \emptyset$, so for any $x_0 \in \operatorname{Lev}_0(T')$, the set $\{y \in T' : y \geq x_0\}$ is a well-pruned subtree of T. ☐

Corollary 4.9. *Let $\lambda \geq \omega$ be a cardinal, and suppose that T is a tree of height λ with all its levels finite. Then T has a well-pruned subtree of height λ.*

Proof.

$$\sum_{\xi < \alpha} |\operatorname{Lev}_\xi(T)| \begin{cases} \leq \omega \otimes \alpha, & \alpha \geq \omega \\ < \omega, & \alpha < \omega \end{cases} \implies \sum_{\xi < \alpha} |\operatorname{Lev}_\xi(T)| < \lambda.$$

☐

Proposition 4.10 (Kőnig's lemma). *Let T be a tree of height ω such that all its levels are finite. Then there is a path through T.*

Proof. Take a well-pruned subtree $T' \subseteq T$ of height ω. Let $\operatorname{Lev}_0(T') = \{x_0\}$. If we have $x_n \in \operatorname{Lev}_n(T')$, there exists $x_{n+1} \in \operatorname{Lev}_{n+1}(T')$ such that $x_{n+1} > x_n$. Now $\langle x_n : n < \omega \rangle$ is a path through T', and hence a path through T. ☐

Generalisations of Kőnig's lemma are of central interest in the study of trees. For starters, we have the following result for trees with finite levels.

Theorem 4.11. *Let $\lambda \geq \omega$ be a cardinal, and T be a tree with* $\mathrm{ht}(T) = \lambda$*. such that all its levels are finite. Then there is a path through T.*

Proof. If $\mathrm{cf}(\lambda) = \omega$, take $\alpha_0 < \alpha_1 < \cdots$ such that $\sup_n \alpha_n = \lambda$. Consider $T' = \bigcup_n \operatorname{Lev}_{\alpha_n}(T)$; by Kőnig's lemma there is a path through T', say $x_n \in \operatorname{Lev}_{\alpha_n}(T)$ with $x_0 < x_1 < \cdots$. Then $\bigcup_n \mathrm{pred}_T(x_n)$ is a path through T.

If $\mathrm{cf}(\lambda) > \omega$, take a well-pruned subtree $T' \subseteq T$ of height λ. Let

$$S_n = \{\alpha < \lambda : |\operatorname{Lev}_\alpha(T')| = n\},$$

so $\bigcup_{n<\omega} S_n = \lambda$. Thus $\sup_{n<\omega} \sup S_n = \lambda$, hence $\sup S_m = \lambda$ for some $m < \omega$.

Let $\beta = \min S_m$. For all $\alpha \in S_m$ and $x \in \mathrm{Lev}_\alpha(T')$, there exists a unique $y \in \mathrm{Lev}_\beta(T')$ such that $x \geq y$. Hence the set

$$\begin{aligned} f &= \geq \restriction \mathrm{Lev}_\alpha(T') \times \mathrm{Lev}_\beta(T') \\ &= \{(x, y) \in \mathrm{Lev}_\alpha(T') \times \mathrm{Lev}_\beta(T') : x \geq y\} \end{aligned}$$

is a function $f : \mathrm{Lev}_\alpha(T') \to \mathrm{Lev}_\beta(T')$, which must be bijective since $|\mathrm{Lev}_\alpha(T')| = |\mathrm{Lev}_\beta(T')| = m$.

Fix $x_\beta \in \mathrm{Lev}_\beta(T')$. For all $\alpha \in S_m$, choose $x_\alpha \in \mathrm{Lev}_\alpha(T')$ such that $x_\alpha \geq x_\beta$. Now if $\alpha > \alpha' \geq \beta$ then

$$\mathrm{pred}_{T'}(x_\alpha) \cap \mathrm{Lev}_{\alpha'}(T') \subseteq \{y \in \mathrm{Lev}_{\alpha'}(T') : y \geq x_\beta\} = \{x_{\alpha'}\},$$

so $x_\alpha > x_{\alpha'}$. Thus $\langle x_\alpha : \alpha \in S_m \rangle$ is a chain in T with $\sup_{\alpha \in S_m} \mathrm{ht}_T(x_\alpha) = \sup S_m = \lambda$, so $\bigcup_{\alpha \in S_m} \mathrm{pred}_T(x_\alpha)$ is a path through T. □

4.2 κ-trees and Aronszajn trees

For a regular κ, a *κ-tree* is a tree T of height κ such that all levels have size less than κ. A *κ-Aronszajn tree* is a κ-tree with no chain of size κ. Hence Kőnig's lemma is equivalent to the statement that there are no ω-Aronszajn trees.

κ-tree

κ-Aronszajn tree

Proposition 4.12. *Let κ be regular, and T be a κ-tree. Then T is a κ-Aronszajn tree if and only if there is no path through T.*

Proposition 4.13. *Let κ be regular. Then every κ-tree has a well-pruned κ-subtree.*

Theorem 4.14. *Let $\kappa = \lambda^+$, where λ is regular and $2^{<\lambda} = \lambda$. Then there is a κ-Aronszajn tree.*

Proof. We first show that for α regular, we have $(2^{<\alpha})^{<\alpha} = 2^{<\alpha}$. Given $\gamma < \alpha$ and any $f : \gamma \to 2^{<\alpha}$, there exists $g : \gamma \to \alpha$ with $f(\xi) < 2^{g(\xi)}$ for all $\xi < \gamma$. Now $\sup \mathrm{Im}\, g = \beta < \alpha$, so $f \in {}^{\gamma}(2^\beta)$ for some $\beta < \alpha$. Hence

$$2^{<\alpha} \leq (2^{<\alpha})^{<\alpha} = \sup_{\gamma<\alpha} (2^{<\alpha})^\gamma = \sup_{\gamma<\alpha} \sup_{\beta<\alpha} (2^\beta)^\gamma \leq 2^{<\alpha},$$

and we are done.

Consider $T = \{s \in {}^{<\kappa}\lambda : s \text{ injective}\}$. Note that T is a subtree of ${}^{<\kappa}\lambda$. Also, the union of any path through T would be an injective function from κ to λ, contradiction. Hence there are no paths through T.

If $s, t \in {}^{\alpha}\lambda$, define $s \sim t$ if $|\{\xi < \alpha : s(\xi) \neq t(\xi)\}| < \lambda$. Note that this is an equivalence relation on ${}^{<\kappa}\lambda$. We will construct $\langle s_\alpha : \alpha < \kappa \rangle$ such that:

(i) $s_\alpha : \alpha \to \lambda$ is injective;
(ii) $s_\alpha \sim s_\beta \restriction \alpha$ for all $\alpha < \beta < \kappa$; and
(iii) $\lambda \setminus \mathrm{ran}(s_\alpha)$ is stationary in λ.

Once we construct such a sequence, let

$$T^* = \bigcup_{\alpha<\kappa} \{t \in \mathrm{Lev}_\alpha(T) : t \sim s_\alpha\}.$$

Then (ii) implies that T^* is a subtree of T, and $\mathrm{ht}_{T^*}(t) = \mathrm{ht}_T(t)$. Hence $\mathrm{Lev}_\alpha(T^*) \subseteq \mathrm{Lev}_\alpha(T)$, so

$$|\operatorname{Lev}_\alpha(T)| = |\{t \in \operatorname{Lev}_\alpha(T) \,:\, t \sim s_\alpha\}| \le \lambda^{<\lambda} = (2^{<\lambda})^{<\lambda} = \lambda.$$

Now T^* has no paths since T has no paths, so T^* is a κ-Aronszajn tree.

It remains to define $\langle s_\alpha \,:\, \alpha < \kappa\rangle$ inductively. For $\alpha < \kappa$, by (iii) we may take $s_{\alpha+1} = s_\alpha \cup \{\langle \alpha, \min(\lambda \setminus \operatorname{ran}(s_\alpha))\rangle\}$.

For $\gamma < \kappa$ limit, given $\langle s_\alpha \,:\, \alpha < \gamma\rangle$, fix an increasing sequence $\langle \alpha_\xi \,:\, \xi < \mathrm{cf}(\gamma)\rangle$ with supremum γ. We will construct $\langle t_\xi \,:\, \xi < \mathrm{cf}(\gamma)\rangle$ such that:

(i') $t_\xi : \alpha_\xi \to \lambda$ is injective;
(ii') $t_\xi = t_\zeta \restriction \alpha_\xi$ for all $\xi < \zeta < \gamma$; and
(iii') $t_\xi \sim s_{\alpha_\xi}$.

Once we construct such a sequence, take $s'_\gamma = \bigcup_{\xi < \mathrm{cf}(\gamma)} t_\xi$. Then s'_γ satisfies (i), and $\langle \alpha_\xi \,:\, \xi < \mathrm{cf}(\gamma)\rangle$ cofinal in γ implies (ii).

If $\mathrm{cf}(\gamma) < \lambda$ then $\lambda \setminus \operatorname{ran}(s'_\gamma) = \bigcap_{\xi < \mathrm{cf}(\gamma)} (\lambda \setminus \operatorname{ran}(t_\xi)$ is the intersection of a family of $< \lambda$ stationary sets in λ, and is thus stationary. Thus we may set $s_\gamma = s'_\gamma$.

If $\mathrm{cf}(\gamma) = \lambda$, then

$$A = \mathop{\triangle}_{\xi < \lambda} (\lambda \setminus \operatorname{ran}(t_\xi)) = \{\delta < \lambda \,:\, \forall \beta < \delta\, [\delta \notin \operatorname{ran}(t_\beta)]\}$$

is stationary in λ, and hence can be written as a disjoint union of two stationary sets, say $A = B \cup C$. Now $|A| = |B| = \lambda$, so there exists a bijection $g : A \to B$.

Now define

$$s_\gamma(\theta) = \begin{cases} s'_\gamma(\theta) & s'_\gamma(\theta) \notin A \\ g(s'_\gamma(\theta)) & s'_\gamma(\theta) \in A. \end{cases}$$

Note that s_γ satisfies (i). Also, if $\xi < \lambda$ and $\theta < \alpha_\xi$, then

$$s'_\gamma(\theta) = t_\xi(\theta) \in A \implies s'_\gamma(\theta) \le \xi.$$

Hence $|\{\theta < \alpha_\xi \,:\, s'_\gamma(\theta) \in A\}| \le \xi < \lambda$, so $s_\gamma \restriction \alpha_\xi \sim s'_\gamma \restriction \alpha_\xi = t_\xi \sim s_\xi$. Hence s_γ satisfies (ii). Finally, $\operatorname{ran}(s_\gamma) \subseteq \lambda \setminus C$, so $\lambda \setminus \operatorname{ran}(s_\gamma)$ is stationary in λ, so s_γ satisfies (iii).

It remains to construct $\langle t_\xi \,:\, \xi < \mathrm{cf}(\gamma)\rangle$ inductively. Take $t_0 = s_{\alpha_0}$. Given t_ξ for $\xi < \gamma$, let $t'_{\xi+1} = t_\xi \cup s_{\alpha_{\xi+1}} \restriction (\alpha_{\xi+1} \setminus \alpha_\xi)$. This function might fail to be injective, since it attains values in $S_\xi = \operatorname{ran}(t_\xi) \cap \operatorname{Im}_{s_{\alpha_{\xi+1}}}(\alpha_{\xi+1} \setminus \alpha_\xi)$ twice. Now

$$\begin{aligned} S_\xi &\subseteq \left[\operatorname{ran}(t_\xi) \setminus \operatorname{ran}(s_{\alpha_\xi})\right] \cup \left[\operatorname{ran}(s_{\alpha_\xi}) \cap \operatorname{Im}_{s_{\alpha_{\xi+1}}}(\alpha_{\xi+1} \setminus \alpha_\xi)\right] \\ &\subseteq \left[\operatorname{ran}(t_\xi) \setminus \operatorname{ran}(s_{\alpha_\xi})\right] \cup \left[\operatorname{ran}(s_{\alpha_\xi}) \setminus \operatorname{ran}(s_{\alpha_{\xi+1}} \restriction \alpha_\xi)\right], \end{aligned}$$

so $|S_\xi| < \lambda$. Fix any injective $h : S_\xi \to \lambda \setminus \operatorname{ran}(t'_{\xi+1})$, and define

$$t_{\xi+1}(\theta) = \begin{cases} t'_{\xi+1}(\theta) & \theta < \alpha_\xi \vee s_{\alpha_{\xi+1}}(\theta) \notin S_\xi \\ h(s_{\alpha_{\xi+1}}(\theta)) & \theta \ge \alpha_\xi \wedge s_{\alpha_{\xi+1}}(\theta) \in S_\xi. \end{cases}$$

Now $t_{\xi+1}$ satisfies (i') and (ii'), and

$$\begin{aligned} &\{\theta < \alpha_{\xi+1} \,:\, t_{\xi+1}(\theta) \ne s_{\alpha_{\xi+1}}(\theta)\} \\ &= \{\theta < \alpha_\xi \,:\, t_\xi(\theta) \ne s_{\alpha_{\xi+1}}(\theta)\} \cup \{\alpha_\xi \le \theta < \alpha_{\xi+1} \,:\, s_{\alpha_{\xi+1}}(\theta) \in S_\xi\}, \end{aligned}$$

which has size $< \lambda$ since $t_\xi \sim s_{\alpha_\xi} \sim s_{\alpha_{\xi+1}} \restriction \alpha_\xi$. Hence $t_{\xi+1}$ satisfies (iii') as well.

For $\eta < \gamma$ limit, given $\langle t_\xi : \xi < \eta \rangle$, take $t_\eta = \bigcup_{\xi<\eta} t_\xi$. Then (i') and (ii') are clear, and $\{\theta < \alpha_\eta : t_\eta(\theta) \neq s_{\alpha_\eta}(\theta)\}$ is contained in

$$\bigcup_{\xi<\eta} \{\theta < \alpha_\xi : t_\xi(\theta) \neq s_{\alpha_\xi}(\theta)\} \cup \{\theta < \alpha_\xi : s_{\alpha_\xi}(\theta) \neq s_{\alpha_\eta}(\theta)\},$$

and thus is a union of $\eta < \mathrm{cf}(\gamma) \leq \lambda$ many sets of size $< \lambda$. Hence $|\{\theta < \alpha_\eta : t_\eta(\theta) \neq s_{\alpha_\eta}(\theta)\}| < \lambda$, so $t_\eta \sim s_{\alpha_\eta}$, so (iii') holds. □

Corollary 4.15. *There is a ω_1-Aronszajn tree.*

4.3 The Suslin problem

For any cardinal $\kappa \geq \omega$, a *κ-Suslin tree* is a tree T such that $|T| = \kappa$, and every chain and every antichain of T has size $< \kappa$. κ-Suslin tree

Proposition 4.16. *Let κ be regular. Then every κ-Suslin tree is a κ-tree.*

Proof. Let T be a κ-Suslin tree. Every level is an antichain, so all levels have size $< \kappa$. If $x \in \mathrm{Lev}_\kappa(T)$ then $\mathrm{pred}_T(x)$ is a chain of size κ, so $\mathrm{ht}(T) \leq \kappa$. But $\kappa = |T| = \sum_{\alpha<\mathrm{ht}(T)} |\mathrm{Lev}_\alpha(T)|$, so $\mathrm{ht}(T) = \kappa$. □

Corollary 4.17. *Let κ be regular. Then every κ-Suslin tree is a κ-Aronszajn tree.*

Corollary 4.18. *Let κ be regular. Then every κ-Suslin tree has a well-pruned κ-Suslin subtree.*

The study of Suslin trees is related, perhaps surprisingly, to natural questions arising from topology.

A topological space X is said to have the *countable chain condition*, or c.c.c., if there is no uncountable family of pairwise disjoint nonempty open subsets of X. Countable chain condition

Proposition 4.19. *If X is separable, then X is c.c.c..*

A *Suslin line* is a total ordering $\langle X, < \rangle$ such that in the order topology, X is c.c.c. but not separable. *Suslin's hypothesis* (SH) is the statement "there are no Suslin lines." It is known that SH is independent of ZFC+GCH. Suslin line Suslin's hypothesis

Proposition 4.20. *If X is a Suslin line, then X^2 is not c.c.c..*

Proof. For $a, b \in X$ with $a < b$, write $(a, b) = \{x \in X : a < x < b\}$.

Assume that we have $a_\alpha, b_\alpha, c_\alpha \in X$ for $\alpha < \omega_1$ such that

1. $a_\alpha < b_\alpha < c_\alpha$;
2. $(a_\alpha, b_\alpha) \neq \emptyset$, $(b_\alpha, c_\alpha) \neq \emptyset$; and
3. $(a_\alpha, c_\alpha) \cap \{b_\xi : \xi < \alpha\} = \emptyset$.

Let $U_\alpha = (a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha) \neq \emptyset$. Now if $\xi < \alpha$, then

$$\begin{aligned} \text{either } b_\xi < a_\alpha &\implies (a_\xi, b_\xi) \cap (a_\alpha, b_\alpha) = \emptyset, \\ \text{or } b_\xi > c_\alpha &\implies (b_\xi, c_\xi) \cap (b_\alpha, c_\alpha) = \emptyset. \end{aligned}$$

Hence $\langle U_\alpha : \alpha < \omega_1 \rangle$ are pairwise disjoint open sets in X^2, so X^2 is not c.c.c..

It remains to construct $a_\alpha, b_\alpha, c_\alpha$ inductively. Let W be the set of isolated points of X. By c.c.c., W is countable. Now given a_ξ, b_ξ, c_ξ for $\xi < \alpha$, note that

$$X \backslash (\overline{W \cup \{b_\xi : \xi < \alpha\}})$$

is open, and nonempty since X is not separable. Hence it contains some nonempty (a_α, c_α). This interval contains no isolated points, so it is infinite. In particular, we can choose $b_\alpha \in (a_\alpha, c_\alpha)$ such that $(a_\alpha, b_\alpha) \neq \emptyset$ and $(b_\alpha, c_\alpha) \neq \emptyset$. Hence all conditions for $a_\alpha, b_\alpha, c_\alpha$ are satisfied, and we are done. □

Similar to pruning trees, we can always pick a subset of a Suslin line that is more well-behaved, as in the following result.

Proposition 4.21. *If a Suslin line exists, then there is a Suslin line X such that*

1. *X is dense in itself (ie. $a < b \implies (a, b) \neq \emptyset$); and*
2. *Every nonempty open set in X is not separable.*

Proof. Let Y be a Suslin line. For $x, y \in Y$, write $x \sim y$ if $x \leq y$ and (x, y) is separable, or if $y \leq x$ and (y, x) is separable. This is easily checked to be an equivalence relation.

Let $X = Y/\sim$. Now if $I \in X$ and $x, y \in I$ with $x < y$, then $(x, y) \subseteq I$. Hence we may define a total order on X by $I < J$ if $\exists x \in I\, \exists y \in J\, [x < y]$. Note that this is equivalent to $\forall x \in I\, \forall y \in J\, [x < y]$.

We now show that every $I \in X$ is separable. Let $\mathcal{M}$ be a maximal collection of pairwise disjoint nonempty open intervals (x, y) with $x, y \in I$. Since Y is c.c.c., write $\mathcal{M} = \{(x_n, y_n) : n \in \omega\}$. Since $x_n \sim y_n$, we can choose D_n countable and dense in (x_n, y_n). Write $D = \bigcup_n D_n$.

If $z \in I$ and $z \in (x, y) \subseteq I$, then (x, y) intersects some (x_n, y_n) by maximality of $\mathcal{M}$. Hence $z \in \overline{D}$, except possibly if z is the first or last element of I. Hence D with the first and last elements of I (if they exist) forms a countable dense subset of I, so I is separable.

If $I < J$ and (I, J) is separable, let $\mathcal{K} \subseteq X$ be a countable collection containing I and J, and dense in (I, J). Let $\mathcal{D} = \bigcup_{K \in \mathcal{K}} D_K$, where D_K is a countable dense subsets of K in Y, so $\bigcup \mathcal{D}$ is dense in $S = \bigcup\{L : I \leq L \leq J\}$. Now for any $x \in I$ and $y \in J$, we have $(x, y) \subseteq S$, which is separable, so $x \sim y$, contradiction. Hence (I, J) is not separable. In particular, $(I, J) \neq \emptyset$, so both conditions are satisfied.

Finally, if $\langle (I_\alpha, J_\alpha) : \alpha < \omega_1 \rangle$ is a sequence of disjoint open intervals in X, then pick $x_\alpha \in I_\alpha$ and $y_\alpha \in J_\alpha$, so $\langle (x_\alpha, y_\alpha) : \alpha < \omega_1 \rangle$ is a sequence of disjoint open intervals in Y, contradicting Y is c.c.c.. Hence X is c.c.c., and we are done. □

The following theorem establishes the link between Suslin lines and Suslin trees.

Theorem 4.22. *There is an ω_1-Suslin tree if and only if there is a Suslin line.*

Proof. ($\Rightarrow$): Let T be an ω_1-Suslin tree. We may assume that T is well-pruned.

For a chain $C \subseteq T$, write $\mathrm{ht}_T(C) = \sup\{\mathrm{ht}_T(x) : x \in C\}$. Since T is ω_1-Aronszajn, we have $\mathrm{ht}_T(C) < \omega_1$. Note that $C \cap \mathrm{Lev}_\alpha(T)$ is a singleton if $\alpha < \mathrm{ht}_T(C)$, and is empty if $\alpha \geq \mathrm{ht}_T(C)$. For $\alpha < \mathrm{ht}_T(C)$, write $C(\alpha)$ for the unique element in C at height α.

If C is maximal then T well-pruned implies C has no maximal element, so $\mathrm{ht}_T(C)$ is limit.

Let L be the set of all maximal chains in T. Fix a total order $\prec$ on T, and define a lexicographical order $\lhd$ on L as follows. If $C, D \in L$ are distinct, then let $d(C, D) = \min\{\alpha : C(\alpha) \neq D(\alpha)\} < \min(\mathrm{ht}_T(C), \mathrm{ht}_T(D))$, and define $C \lhd D$ if $C(d(C, D)) \prec D(d(C, D))$. We now show that $\langle L, \lhd \rangle$ is a Suslin line.

Suppose $\{(C_\xi, D_\xi) : \xi < \omega_1\}$ is a family of disjoint nonempty open intervals. Then pick $E_\xi \in (C_\xi, D_\xi)$, and note that $m = \max(d(C_\xi, E_\xi), d(D_\xi, E_\xi)) < \mathrm{ht}_T(E_\xi)$. Since $\mathrm{ht}_T(E_\xi)$ is limit, there exists $m < \alpha_\xi < \mathrm{ht}_T(E_\xi)$. Note that $E \in L$, $E(\alpha_\xi) = E_\xi(\alpha_\xi)$ implies $E \in (C_\xi, D_\xi)$. Hence $\{E_\xi(\alpha_\xi) : \xi < \omega_1\}$ is an antichain in T, contradicting T is Suslin. Thus L is c.c.c..

Suppose L is separable, say $\langle C_\xi : \xi < \omega\rangle$ is dense in L. Then $\delta = \sup_{\xi<\omega} \mathrm{ht}_T(C_\xi) < \omega_1$, so $S = \{C \in L : \mathrm{ht}_T(C) < \delta\}$ is dense in L. Fix $x \in \mathrm{Lev}_\delta(T)$, and pick $\alpha > \delta$ such that there exists distinct $y, z, w \in \mathrm{Lev}_\alpha(T)$, which are contained in $D, E, F \in L$, respectively. If $D \lhd E \lhd F$ then (D, F) is nonempty, but $x \in D \cap F$ implies $(D, F) \cap S = \emptyset$, contradiction. Hence L is not separable.

($\Leftarrow$): Let $\langle L, \lhd\rangle$ be a Suslin line. We may assume that L is dense in itself and no nonempty open subset of L is separable. Let $\mathcal{J} = \{(a, b) : a, b \in L \wedge a \lhd b\}$. Then $\leq\, = \,\supseteq$ is a partial order on $\mathcal{J}$.

Assume that we have $\mathcal{I}_\beta \subseteq \mathcal{J}$ for $\beta < \omega_1$ such that

1. The elements of $\mathcal{I}_\beta$ are pairwise disjoint;
2. $\bigcup \mathcal{I}_\beta$ is dense in L; and
3. If $\alpha < \beta < \omega_1$, $I \in \mathcal{I}_\alpha$ and $J \in \mathcal{I}_\beta$, then

$$[I \cap J = \emptyset] \vee [J \subseteq I \wedge I \backslash \overline{J} \neq \emptyset].$$

Let $T = \bigcup_\beta \mathcal{I}_\beta$. Note that 1-3 implies that T is a tree, with $\mathrm{Lev}_\beta(T) = \mathcal{I}_\beta$. Any antichain of T is a collection of pairwise disjoint intervals of L, and thus is countable. Also, if $\langle I_\xi : \xi < \alpha\rangle$ is a chain in T then $\{I_\xi \backslash \overline{I_{\xi+1}} : \xi < \alpha\}$ is a collection of pairwise disjoint open sets in L, so $\alpha < \omega_1$. Hence T is an ω_1-Suslin tree. [10 Mar]

It remains to construct $\langle \mathcal{I}_\beta : \beta < \omega_1\rangle$ by induction. Let $\mathcal{I}_0$ be a maximal disjoint sub-family of $\mathcal{J}$, so $\bigcup \mathcal{I}_0$ is dense in L. Given $\mathcal{I}_\alpha$, for all $I = (a, b) \in \mathcal{I}_\alpha$, choose $c \in I$ and let $\mathcal{K}_I = \{(a, c), (c, b)\}$. Then $\bigcup \mathcal{K}_I$ is dense in I, and

$$I \backslash \overline{(a, c)} \supseteq (c, b) \neq \emptyset, \quad I \backslash \overline{(c, b)} \supseteq (a, c) \neq \emptyset,$$

so $\mathcal{I}_{\alpha+1} = \bigcup_{I \in \mathcal{I}_\alpha} \mathcal{K}_I$ satisfies 1-3.

If $\gamma < \omega_1$ is limit and given $\langle \mathcal{I}_\beta : \beta < \gamma\rangle$ satisfying 1-3, let $\mathcal{I}_\gamma$ be a maximal disjoint family of

$$\mathcal{K} = \{K \in \mathcal{J} : \forall \alpha < \gamma \forall I \in \mathcal{I}_\alpha [I \cap K = \emptyset \vee (K \subseteq I \wedge I \backslash \overline{K} \neq \emptyset)]\}.$$

Then $\mathcal{I}_\gamma$ satisfies 1 and 3. To show 2, we claim that every $J \in \mathcal{J}$ contains some $K \in \mathcal{K}$. Note that $\mathcal{I}_\alpha$ is countable for all $\alpha < \gamma$, so the set E of all endpoints of intervals in $\bigcup_{\alpha<\gamma} \mathcal{I}_\alpha$ is countable. Now J is not separable, there exists $K_1 \in \mathcal{J}$ with $K_1 \subseteq J$ and $K_1 \cap E = \emptyset$. Hence for all $I \in \bigcup_{\alpha<\gamma} \mathcal{I}_\alpha$, either $I \cap K_1 = \emptyset$ or $K_1 \subseteq I$. Now if $c_1 \in K_1 = (a_1, b_1)$ then $K = (a_1, c_1) \in \mathcal{K}$ and $K \subseteq J$, and we are done. □

Proposition 4.23.

Proof.

4.4 Kurepa trees

3 Mar

On the other extreme, we can investigate trees with many paths. For κ regular, a *κ-Kurepa tree* is a κ-tree with at least κ^+ paths. κ-Kurepa tree

Example 4.24. *The complete binary tree ${}^{<\omega}2$ is an ω-Kurepa tree. More generally, for κ strongly inaccessible, ${}^{<\kappa}2$ is a κ-Kurepa tree.*

Let κ-KH denote the statement "there is a κ-Kurepa tree." *Kurepa's hypothesis* (KH) is the statement ω_1-KH. It is known that KH is independent of ZFC+GCH. Kurepa's hypothesis

For κ regular, a *κ-Kurepa family* is any $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ of size at least κ^+, such that $\forall \alpha < \kappa\, [|\{A \cap \alpha : A \in \mathcal{F}\}| < \kappa]$. κ-Kurepa family

Note that the ordering of κ is not essential to the definition of a κ-Kurepa family, as the next result shows.

Proposition 4.25. *Let κ be regular. For $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{F}| \geq \kappa^+$ and $X \subseteq \kappa$, write $\mathcal{F}_X = \{A \cap X : A \in \mathcal{F}\}$. Then $\mathcal{F}$ is a κ-Kurepa family if and only if*

$$\forall X \subseteq \kappa\, [|X| < \kappa \implies |\mathcal{F}_X| < \kappa].$$

Proof. ($\Leftarrow$): Let $X = \alpha$ for each $\alpha < \kappa$.

($\Rightarrow$): Take $\alpha = \sup X < \kappa$ for each $X \subseteq \kappa$ with $|X| < \kappa$. $\square$

Theorem 4.26. *Let κ be regular. Then κ-KH holds if and only if there is a κ-Kurepa family.*

Proof. ($\Rightarrow$): Let T be a κ-Kurepa tree, and let $\mathcal{F}$ be the set of all paths through T. Then $|\mathcal{F}| \geq \kappa^+$. Also, for all $X \subseteq T$ with $|X| < \kappa$, we have $\sup\{\mathrm{ht}_T(x) : x \in X\} = \alpha < \kappa$, so every element of $\mathcal{F}_x$ is of the form $\mathrm{pred}_T(z) \cap X$ for some $z \in \mathrm{Lev}_\alpha(T)$. Hence $|\mathcal{F}_x| \leq |\mathrm{Lev}_\alpha(T)| < \kappa$. Now $|T| = \kappa$, so any bijection from T to κ takes $\mathcal{F}$ to a κ-Kurepa family.

($\Leftarrow$): Let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ be a κ-Kurepa family. For any set $S \subseteq \kappa$, let $\chi_S : \kappa \to 2$ be its characteristic function. Let

$$T = \bigcup_{\alpha<\kappa} \{\chi_S{\restriction}\alpha : S \in \mathcal{F}\} \subseteq {}^{<\kappa}2.$$

This is a subtree of ${}^{\kappa}2$ of height κ with $|\mathcal{F}| \geq \kappa^+$ paths. Now

$$|\mathrm{Lev}_\alpha(T)| = |\{\chi_S{\restriction}\alpha : S \in \mathcal{F}\}| = |\{\chi_{S'} : S' \in \mathcal{F}_\alpha\}| < \kappa,$$

so T is a κ-Kurepa tree. $\square$

5 Ramsey Theory and Partition Calculus

5.1 Preliminaries

3 Feb

The following classical result can be seen as the starting point of Ramsey theory. Let K_n denote the complete graph on n vertices.

Proposition 5.1. *If each edge of a K_6 is coloured either red or blue, then there is a monochromatic induced K_3.*

Proof. Consider an edge-colouring of the complete graph on vertices $v_1, v_2, \ldots, v_6$. By pigeonhole principle, among the 5 edges $(v_1, v_2), (v_1, v_3), \ldots, (v_1, v_6)$, there must exists 3 of them that are of the same colour. Without loss of generality assume that $(v_1, a), (v_1, b), (v_1, c)$ are red.

Now if (a, b) is red then we have a red K_3 induced on $\{v_1, a, b\}$, and similarly if (b, c) or (c, a) is red. Otherwise, we have a blue K_3 induced on $\{a, b, c\}$. □

Ramsey theory and partition calculus are concerned with the generalisations of the above result to infinite sets. We will now restate the above result in a more general setting.

For any set S and $r \in \mathbb{N}$, define $[S]^r = \{X \subseteq S : |X| = r\}$. Given a cardinal κ, we are concerned with partitioning $[\kappa]^r$ into subsets $\langle A_\xi : \xi < s\rangle$, which can be seen as *colour classes*. A subset $H \subseteq \kappa$ is called *homogeneous* (or *monochromatic*) for the partition if $[H]^r \subseteq A_\xi$ for some $\xi < s$. Now given a cardinal λ, we write $\kappa \to (\lambda)^r_s$ if for any partition as above, there exists a homogeneous set with size at least λ. The negation of this statement is written $\kappa \not\to (\lambda)^r_s$. $[S]^r$ Homogeneous $\kappa \to (\lambda)^r_s$

More generally, given $\langle \lambda_\xi : \xi < s\rangle$, we write $\kappa \to (\lambda_\xi)^r_{\xi<s}$ if for any partition of $[\kappa]^r$ into colour classes $\langle A_\xi : \xi < s\rangle$, there exists $H \subseteq \kappa$ and $\xi < s$ such that $[H]^r \subseteq A_\xi$ and $|H| \geq \lambda_i$. If s is finite, we usually write $\kappa \to (\lambda_0, \ldots, \lambda_{s-1})^r_s$ instead. $\kappa \to (\lambda_\xi)^r_{\xi<s}$

Note that the first result in this chapter can be succintly written as $6 \to (3)^2_2$.

Proposition 5.2 (Finite Ramsey's Theorem). *For any $1 \leq k, r, s < \omega$, there exists $n < \omega$ such that $n \to (k)^r_s$.*

Proof.

Theorem 5.3 (Ramsey's Theorem). *For any $1 \leq r, s < \omega$, we have*

$$\omega \to (\omega)^r_s.$$

Proof. We proceed by induction on r. Note that $\omega \to (\omega)^1_s$ states that for any partition of ω into finitely many subsets, one of the subsets must be infinite.

Assume the claim holds for r, and suppose that $[\omega]^{r+1} = \bigcup_{i=0}^{s-1} A_i$. For any $a \in \omega$, and any $S \subseteq \omega \backslash \{a\}$ infinite, let

$$B_i = \{X \in [S]^r : X \cup \{a\} \in A_i\} \quad \text{for } i < s.$$

By induction hypothesis there exists $i = i(a, S) < s$ and $H = H(a, S) \subseteq S$ infinite such that $[H]^r \subseteq B_i$, ie. $X \cup \{a\} \in A_i$ for all $X \in [H]^r$.

Now define a_n, i_n, H_n $(n < \omega)$ inductively as follows:

$$\begin{aligned} a_0 &= 0, & a_{n+1} &= \min H_n, \\ i_0 &= i(0, \omega \backslash \{0\}), & i_{n+1} &= i(a_{n+1}, H_n \backslash \{a_{n+1}\}), \\ H_0 &= H(0, \omega \backslash \{0\}), & H_{n+1} &= H(a_{n+1}, H_n \backslash \{a_{n+1}\}). \end{aligned}$$

Then $a_0 < a_1 < a_2 < \cdots$, and $\{a_n, a_{k_1}, \ldots, a_{k_r}\} \in A_{i_n}$ for all $n < k_1 < k_2 < \cdots < k_r$. Now there exists $j < s$ such that $M = \{m < \omega : i_m = j\}$ is infinite, so $H = \{a_m : m \in M\}$ is infinite and $[H]^{r+1} \subseteq a_j$. Thus the claim holds for $r + 1$, and we are done. $\square$

5.2 2-partitions

We have the following negative results.

Proposition 5.4 (Erdős). *For any cardinal $\kappa \geq \omega$, we have*

$$2^\kappa \not\to (3)^2_\kappa.$$

Proof. Let $\prec$ be the lexicographical order on ${}^\kappa 2$. Define $d : [{}^\kappa 2]^2 \to \kappa$ by

$$d(\{s, s'\}) = \min\{\alpha < \kappa : s(\alpha) \neq s'(\alpha)\}.$$

Clearly if $d(\{s, s'\}) = d(\{s', s''\}) = \alpha$, then $s(\alpha) = s''(\alpha)$, so $d(\{s, s''\}) \neq \alpha$. Hence there is no $H \subseteq \kappa$ of size 3 which is homogeneous for the partition induced by d. $\square$

Proposition 5.5 (Sierpiński). *For any cardinal $\kappa \geq \omega$, we have*

$$2^\kappa \not\to (\kappa^+)^2_2.$$

Proof. Let $\prec$ be the lexicographical order on ${}^\kappa 2$, and let $\lhd$ be a well-ordering of ${}^\kappa 2$. Consider the partition

$$\begin{aligned} A_0 &= \{\{x, y\} \in [{}^\kappa 2]^2 : x \prec y \wedge x \lhd y\}, \\ A_1 &= \{\{x, y\} \in [{}^\kappa 2]^2 : x \prec y \wedge x \rhd y\}. \end{aligned}$$

Assume for sake of contradiction that $H \subseteq [{}^\kappa 2]^2$ is a homogeneous set of size at least κ^+ for this partition, say $[H]^2 \subseteq A_0$. Then taking the initial segment of $(H, \lhd)$ that is order-isomorphic to κ^+, we see that there exists a sequence $\langle s_\xi : \xi < \kappa^+ \rangle$ in ${}^\kappa 2$ that is strictly increasing with respect to $(H, \prec)$, ie. $\alpha < \beta < \kappa^+ \implies s_\alpha \prec s_\beta$.

Define $d : [{}^\kappa 2]^2 \to \kappa$ by

$$d(\{s, s'\}) = \min\{\alpha < \kappa : s(\alpha) \neq s'(\alpha)\}.$$

and define $f : \kappa^+ \to \kappa$ by

$$f(\xi) = \min\{d(\{s_\xi, s_{\xi'}\}) : \xi < \xi' < \kappa^+\}.$$

Note that $\forall \alpha < f(\xi) \forall \xi < \xi' < \kappa^+ [s_{\xi'}(\alpha) = s_\xi(\alpha)]$, so f is nondecreasing. Hence κ^+ regular implies that f is eventually constant, say $f(\xi) = C$ for all $\Xi \leq \xi < \kappa^+$.

For any $\Xi \leq \xi < \xi' < \kappa^+$, note that $s_\xi(\alpha) = s_{\xi'}(\alpha)$ for all $\alpha < C$. Thus $s_\xi \prec s_{\xi'}$ implies $s_\xi(C) \leq s_{\xi'}(C)$. Hence $\langle s_\xi(C) : \Xi \leq \xi < \kappa^+ \rangle$ is nondecreasing, and hence eventually constant, say on $\Xi' \leq \xi < \kappa^+$. Then $f(\Xi') \geq C + 1$, contradiction. The case for $[H]^2 \subseteq A_1$ is analogous. $\square$

Corollary 5.6. $2^\omega \not\to (\omega_1)^2_2$.

In particular, the naive extension of Ramsey's theorem to ω_1, namely "$\omega_1 \to (\omega_1)^r_s$", is false.

Note that the corollary has a simpler proof, using the fact that ω_1 does not embed into $\mathbb{R}$: if $f : \omega_1 \to \mathbb{R}$ is order-preserving, then the open intervals $\langle (f(\xi), f(\xi+1)) : \xi < \omega_1 \rangle$ contain pairwise distinct rationals, contradicting $|\mathbb{Q}| = \omega$.

Theorem 5.7 (Erdős-Dushnik-Miller). *For any cardinal $\kappa \geq \omega$, we have*

$$\kappa \to (\kappa, \omega)^2_2.$$

Proof. Let $[\kappa]^2 = A_0 \cup A_1$. For $i < 2$ and $x \in \kappa$, write

$$F_i(x) = \{y \in \kappa \backslash \{x\} : \{x, y\} \in A_i\}.$$

We split into the following two cases.

Case 1: For all $A \subseteq \kappa$ with $|A| = \kappa$, there exists $x \in A$ with $|F_1(x) \cap A| = \kappa$.

We will inductively construct a set of size ω homogeneous in colour 1. Let $X_0 = \kappa$, and define

$$x_n = \min\{x \in X_n : |F_1(x) \cap X_n| = \kappa\}, \qquad X_{n+1} = F_1(x_n) \cap X_n \quad (n < \omega).$$

For $m < n < \omega$, we can prove by induction that:

1. $|X_n| = \kappa$, so $\langle x_n : n < \omega \rangle$ and $\langle X_n : n < \omega \rangle$ are well-defined;
2. $x_m \notin X_{m+1} \supseteq X_n$, so $\langle x_n : n < \omega \rangle$ are pairwise distinct;
3. $x_n \in X_n \subseteq X_{m+1} \subseteq F_1(x_m)$, so $\{x_m, x_n\} \in A_1$.

Hence $H = \{x_n : n < \omega\}$ satisfies $|H| = \omega$ and $[H]^2 \subseteq A_1$.

Case 2: There exists $A \subseteq \kappa$ with $|A| = \kappa$, and $|F_1(x) \cap A| < \kappa$ for all $x \in \kappa$.

We will inductively construct a set of size κ homogeneous in colour 0. Without loss of generality, we may assume that $A = \kappa$. For $S \subseteq \kappa$, write

$$G_1(S) = S \cup \bigcup \{F_1(x) : x \in S\}.$$

If κ is regular, define

$$x_\xi = \min \left(\kappa \backslash \bigcup_{\eta < \xi} G_1(\{x_\eta\})\right) \quad (\xi < \kappa).$$

Then $H = \{x_\xi : \xi < \kappa\}$ satisfies $|H| = \kappa$ and $[H]^2 \subseteq A_0$.

If κ is singular, let $\langle \kappa_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ be an increasing sequence of regular cardinals with supremum κ.

For any λ regular with $\mathrm{cf}(\kappa) < \lambda < \kappa$, and any $B \subseteq \kappa$ with size λ, we have

$$B = \bigcup_{\xi < \mathrm{cf}(\kappa)} \{x \in B : |F_1(x)| \leq \kappa_\xi\},$$

so there exists $\xi < \mathrm{cf}(\kappa)$ such that $C = \{x \in B : |F_1(x)| \leq \kappa_\xi\}$ has size λ, and

$$|G_1(C)| \leq \lambda \oplus (\lambda \otimes \kappa_\xi) < \kappa.$$

Now we inductively construct $\langle X_\xi : \xi < \mathrm{cf}(\kappa) \rangle$, $\langle S_\xi : \xi < \mathrm{cf}(\kappa) \rangle$ as follows. For $\xi < \mathrm{cf}(\kappa)$, choose $X_\xi \subseteq \kappa$ such that

$$X_\xi \subseteq \kappa \backslash \bigcup_{\eta < \xi} G_1(S_\eta), \quad |X_\xi| = \kappa_\xi.$$

Now by the statement for regular cardinals proved above, X_ξ has a subset B of size κ_ξ, homogeneous in colour 0. By the above discussion, B has a subset S_ξ of size κ_ξ, such that $|G_1(S_\xi)| < \kappa$.

Note that:

1. $\kappa \backslash \bigcup_{\eta<\xi} G(S_\eta)$ has size κ, so $\langle X_\xi : \xi < \mathrm{cf}(\kappa)\rangle$, $\langle S_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ are well-defined;
2. If $\eta < \xi < \mathrm{cf}(\kappa)$ then $S_\eta \cap S_\xi \subseteq G_1(S_\eta) \cap X_\xi = \emptyset$, so $\langle S_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ are pairwise disjoint;
3. S_ξ is homogeneous in colour 0, so if $x, y \in S_\xi$ then $\{x, y\} \in A_0$;
4. If $\eta < \xi < \mathrm{cf}(\kappa)$, $x \in S_\eta$ and $y \in S_\xi$, then $y \notin G_1(S_\eta)$ implies $\{x, y\} \in A_0$.

Hence $H = \bigcup_{\xi<\mathrm{cf}(\kappa)} S_\xi$ satisfies $|H| = \sup_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi = \kappa$, and $[H]^2 \subseteq A_0$. ☐

Corollary 5.8. $\omega_1 \to (\omega_1, \omega_0)^2_2$.

If $\langle \lambda_\xi : \xi < s\rangle$ is given such that $\lambda_\xi = \tau$ for all $1 \leq \xi < \tau$, we will write

$$\kappa \to (\lambda_0, (\tau)_{s-1})^2 \quad \text{to mean} \quad \kappa \to (\lambda_\xi)^r_{\xi<s},$$

where $s - 1 = \mathrm{otp}(s\backslash\{0\})$.

Theorem 5.9 (Erdős-Radó). *Let $\lambda < \rho$ be regular cardinals such that*

$$\rho'^{\lambda'} < \rho \quad \textit{for all } \rho' < \rho,\ \lambda' < \lambda.$$

Then for all $\tau < \lambda$, we have

$$\rho \to (\rho, (\lambda)_{\tau-1})^2.$$

Proof. Let $A = \{\alpha < \rho : \mathrm{cf}(\alpha) = \lambda\}$, so A is stationary in ρ. Let $f : [A]^2 \to \tau$ be any partition with τ colours.

Now for any $\alpha \in A$, construct a sequence $\langle \beta^\alpha_\nu : \nu < \xi_\alpha\rangle$ inductively as follows. Let

$$\begin{aligned} A^\alpha_\nu = \{\beta < \alpha : {} & f(\{\beta, \alpha\}) > 0 \\ & \wedge \forall \mu < \nu\, [\beta > \beta^\alpha_\mu] \\ & \wedge \forall \mu < \nu\, [f(\{\beta^\alpha_\mu, \beta\}) = f(\{\beta^\alpha_\mu, \alpha\})]\}. \end{aligned}$$

If A^α_ν is empty then we let $\xi_\alpha = \nu$, and the sequence terminates. Otherwise, let

$$\beta^\alpha_\nu = \min A^\alpha_\nu.$$

We split into the following two cases.

<u>Case 1</u>: $\xi_\alpha \geq \lambda$ for some $\alpha \in A$.

Since $\tau < \lambda$ and λ is regular, the function $g : \lambda \to \tau\backslash\{0\}$ given by $g(\mu) = f(\{\beta^\alpha_\mu, \alpha\})$ must take constant value C on some $H \subseteq \lambda$ of size λ. Then for all $\mu, \nu \in H$, $\mu < \nu$, we have

$$f(\{\beta^\alpha_\mu, \beta^\alpha_\nu\}) = f(\{\beta^\alpha_\mu, \alpha\}) = g(\mu) = C,$$

so H is homogeneous in colour $C > 0$.

<u>Case 2</u>: $\xi_\alpha < \lambda$ for all $\alpha \in A$.

Let $\delta_\alpha = \sup\{\beta^\alpha_\nu : \nu < \xi_\alpha\}$. Since $\mathrm{cf}(\alpha) = \lambda$ for all $\alpha \in A$, we have $\delta_\alpha < \alpha$. By Pressing-down Lemma, there exists $B \subseteq A$ stationary in κ such that $\delta_\alpha = \delta$ for all $\alpha \in B$. In particular, we have $|B| = \rho$.

Now the number of increasing sequences $\langle \beta_\nu : \nu < \xi \rangle$ with $\xi < \lambda$ and $\beta_\nu < \delta$ is at most $\sum_{\xi<\lambda} |\delta|^{|\xi|}$. Since ρ is regular and each summand is less than ρ, the sum is less than ρ. Hence there exists $\xi < \lambda$, a sequence $\langle \beta_\nu : \nu < \xi \rangle$, and a subset $H \subseteq B$, such that $|H| = \rho$, and

$$\xi_\alpha = \xi, \quad \beta^\alpha_\nu = \beta_\nu \quad \text{for all } \alpha \in H,\ \nu < \xi.$$

For any $\alpha, \alpha' \in H$, $\alpha < \alpha'$, assume that $f(\{\alpha, \alpha'\}) > 0$. Then for all $\nu < \xi$,

$$f(\{\beta^\alpha_\nu, \alpha\}) = f(\{\beta^\alpha_0, \beta^\alpha_\nu\}) = f(\{\beta^{\alpha'}_0, \beta^{\alpha'}_\nu\}) = f(\{\beta^{\alpha'}_\nu, \alpha'\}).$$

Thus $\alpha \in A^{\alpha'}_\xi$, contradicting $\xi_{\alpha'} = \xi$. Hence H is homogeneous in colour 0. □

Corollary 5.10. *If $\kappa \geq \omega$, then*

$$(2^\kappa)^+ \to ((2^\kappa)^+, (\kappa^+)_\kappa)^2.$$

Proof. Take $\rho = (2^\kappa)^+$, $\lambda = \kappa^+$, $\tau = \kappa$. □

Corollary 5.11. $(2^\omega)^+ \to (\omega_1)^2_\omega$.

5.3 r-partitions

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We now combine ideas from the proofs of Ramsey's theorem and the above Erdős-Radó theorem, to generalise to partition relations for $r > 2$.

Let ρ be a cardinal, γ an ordinal, $r < \omega$, and $f : [\rho]^{r+1} \to \gamma$. The set $H \subseteq \rho$ is called *end-homogeneous* with respect to f if for each $V \in [\rho]^r$ and $\max V < \alpha, \beta < \rho$, we have End-homogeneous

$$f(V \cup \{\alpha\}) = f(V \cup \{\beta\}).$$

Proposition 5.12 (Stepping-up Lemma). *Let $\kappa > \omega$ be a cardinal, $\gamma < \kappa$, $\rho = (2^{<\kappa})^+$, and $r < \omega$. Then for any $f : [\rho]^{r+1} \to \gamma$, there exists $H \subseteq \rho$ that is end-homogeneous with respect to f such that* $\operatorname{otp} H = \kappa + 1$.

Proof. Similar to the proof of Theorem 5.9, we define the sequence $\langle \beta^\alpha_\nu : \nu < \xi_\alpha \rangle$ for each $\alpha < \rho$ inductively as follows. Let

$$\begin{aligned} A^\alpha_\nu = \{\beta < \alpha : \forall \mu < \nu\, [\beta > \beta^\alpha_\mu] \\ \wedge \forall V \in [\{\beta^\alpha_\mu : \mu < \nu\}]^r\, [f(V \cup \{\beta\}) = f(V \cup \{\alpha\})]\}. \end{aligned}$$

If A^α_ν is empty then we let $\xi_\alpha = \nu$, and the sequence terminates. Otherwise, let

$$\beta^\alpha_\nu = \min A^\alpha_\nu.$$

If $\xi_\alpha \geq \kappa$ for some $\alpha < \rho$, then

$$H = \{\beta^\alpha_\nu : \nu < \kappa\} \cup \{\alpha\}$$

has order type $\kappa + 1$ and is end-homogeneous with respect to f, and we are done.

Otherwise, note that if $\nu < \xi_\alpha$ and $\alpha' = \beta^\alpha_\nu$, then induction gives

$$\forall \mu < \nu\, [\beta^{\alpha'}_\mu = \beta^\alpha_\mu] \quad \text{and} \quad \xi_{\alpha'} = \nu.$$

Thus if we define the relation

$$\alpha' \prec \alpha \iff [\alpha' < \alpha] \wedge [\exists \nu < \xi_\alpha\, [\alpha' = \beta^\alpha_\nu]],$$

then $T = \langle \rho, \prec \rangle$ is a tree (called the *canonical partition tree* for f), with $\operatorname{ht}_T(\alpha) = \xi_\alpha$ for all $\alpha < \rho$. Canonical partition tree

Now the assumption $\forall \alpha < \rho\,[\xi_\alpha < \kappa]$ implies $\mathrm{ht}(T) \le \kappa$. Moreover, if $\alpha, \alpha' \in \mathrm{Lev}_T(\xi)$ with $\alpha' < \alpha$, $\mathrm{pred}_T(\alpha) = \mathrm{pred}_T(\alpha')$ (ie. $\forall \nu < \xi\,[\beta^\alpha_\nu = \beta^{\alpha'}_\nu]$), and

$$\forall V \in [\{\beta^\alpha_\nu : \nu < \xi\}]^r\,[f(V \cup \{\alpha'\}) = f(V \cup \{\alpha\}),$$

then $\alpha' \in A^\alpha_\xi$, contradicting $\xi_\alpha = \xi$. Hence for every $\xi < \kappa$, the function $V \mapsto f(V \cup \{\alpha\})$ uniquely determines $\alpha \in \mathrm{Lev}_T(\xi)$, so

$$|\{\alpha' : \mathrm{pred}_T(\alpha') = \mathrm{pred}_T(\alpha)\}| \le |\gamma|^{|[\xi]^r|} \le 2^{|\gamma||\xi+\omega|}.$$

Hence we can bound the size of each level by

$$|\mathrm{Lev}_T(\xi)| \le 2^{|\gamma||\xi+\omega|} \prod_{\eta<\xi} |\mathrm{Lev}_T(\eta)|.$$

Now $\sum_{\eta<\xi} |\gamma||\eta+\omega| \le |\xi||\gamma||\xi+\omega| = |\gamma||\xi+\omega|$, so transfinite induction gives $|\mathrm{Lev}_T(\xi)| \le 2^{|\gamma||\xi+\omega|}$. Hence

$$(2^{<\kappa})^+ = \rho = |T| = \sum_{\xi<\kappa} |\mathrm{Lev}_T(\xi)| \le \kappa \otimes 2^{<\kappa} = 2^{<\kappa},$$

since $2^{<\kappa} \ge \kappa$, contradiction. □

To state the Erdős-Radó theorem for $r > 2$, we define an iterated exponential as follows. For each cardinal λ let $\exp_i(\lambda)$

$$\exp_0(\lambda) = \lambda, \qquad \exp_{i+1}(\lambda) = 2^{\exp_i(\lambda)} \quad \text{for } i < \omega.$$

Theorem 5.13 (Erdős-Radó). *Let $\kappa \ge \omega$ be a cardinal, $\gamma < \mathrm{cf}(\kappa)$, and $r < \omega$. Then*

$$[\exp_r(2^{<\kappa})]^+ \to (\kappa)^{r+2}_\gamma.$$

Proof. Note that $2^{<\lambda^+} = 2^\lambda$. Hence the sequence given by

$$\kappa_0 = \kappa, \qquad \kappa_{i+1} = (2^{<\kappa_i})^+ \quad \text{for } i < \omega$$

satisfies $\kappa_{i+1} = [\exp_i(2^{<\kappa})]^+$. Thus we need to show that

$$\kappa_r \to (\kappa)^{r+1}_\gamma$$

for all $r < \omega$. This holds for $r = 0$, since $\kappa \to (\kappa)^1_\gamma$.

Assume that the statement is true for some r. For any $f : [\kappa_{r+1}]^{r+1} \to \gamma$, by Stepping-up lemma there exists $H \subseteq \kappa_{r+1}$ end-homogeneous with respect to f such that $\mathrm{otp}\, H = \kappa_r + 1$. Write $H = K \cup \{\alpha\}$, where $\alpha = \max H$.

Now define $f' : [K]^r \to \gamma$ by $f'(V) = f(V \cup \{\alpha\})$. By induction hypothesis there exists $L \subseteq K$ of size κ which is homogeneous with respect to f, say in colour $\nu < \gamma$. Then for any $V \cup \{\mu\} \in [L]^{r+1}$, with $\mu > \max V$, we have

$$f(V \cup \{\mu\}) = f(V \cup \{\alpha\}) = f'(V) = \nu,$$

by end-homogeneity of H. Hence L is homogeneous with respect to f, and the statement holds for $r + 1$. □

6 Set Mappings

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Given a set X, a *set mapping on X* is a function $F : X \to \mathcal{P}(X)$ such that $x \notin F(x)$ for all $x \in X$. F has *order* λ if $|F(x)| < \lambda$ for all $x \in X$. A subset $S \subseteq X$ is *free with respect to F* if $x \notin F(y)$ for all $x, y \in S$.

Set mapping
Order
Free set

Proposition 6.1. *Let F be a set mapping on X of order λ, and let $Y \subseteq X$. Then the function $F_Y : Y \to \mathcal{P}(Y)$ defined by $F_Y(W) = Y \cap F(W)$ is a set mapping on Y of order λ.*

Moreover, any $S \subseteq Y$ that is free with respect to F_Y is also free with respect to F.

Example 6.2. *For any κ, the set mapping on κ defined by $F(\alpha) = \alpha$ has order κ, and no two-element subset of κ is free with respect to F.*

Proposition 6.3 (Lázár). *Let $\kappa \geq \omega$ be regular, $\lambda < \kappa$, and F be a set mapping of order λ on κ. Then there exists $S \subseteq \kappa$ of size κ that is free with respect to F.*

Proof. Define a sequence $\langle X_\alpha : \alpha < \lambda \rangle$ of subsets of κ inductively as follows: given X_β for $\beta < \alpha$, let X_α be a maximal subset of $\kappa \backslash \bigcup_{\beta<\alpha} X_\beta$ that is free with respect to F; note that this exists by the Teichmüller-Tukey lemma.

If $|X_\alpha| < \kappa$ for all $\alpha < \lambda$, let $Y = \bigcup_{\alpha<\lambda} X_\alpha \cup \{F(y) : y \in X_\alpha\}$. Note that by regularity of κ,

$$|Y| \leq \sum_{\alpha<\lambda} |X_\alpha| \oplus (|X_\alpha| \otimes \lambda) < \kappa.$$

Pick any $x \in \kappa \backslash Y$; then for all $\alpha < \lambda$,

$$x \in \kappa \backslash \bigcup_{\beta<\alpha} X_\beta \quad \text{and} \quad \forall y \in X_\alpha\, [x \notin F(y)],$$

so by maximality of X_α we have $F(x) \cap X_\alpha \neq \emptyset$. Since the X_α are pairwise disjoint, we must have $|F(x)| \geq \lambda$, contradiction. □

If F is a set mapping on X, the subset $Y \subseteq X$ is said to be *closed with respect to F* if $F(u) \subseteq Y$ for all $u \in Y$. We have the following analogue to the downward Löwenheim-Skolem theorem for set mappings.

Closed wrt set mapping

Proposition 6.4. *Let $\lambda \geq \omega$, and F be a set mapping on X of order λ. If $S \subseteq X$ satisfies $|S| \leq \lambda$. Then there exists $S \subseteq Y \subseteq X$ with $|C| \leq \lambda$ such that Y is closed with respect to F.*

Proof. Define $\langle S_n : n < \omega \rangle$ by $S_0 = S$ and $S_{n+1} = S_n \cup \{F(z) : z \in S_n\}$; note that $|S_{n+1}| \leq \lambda \oplus (\lambda \otimes \lambda) = \lambda$ by induction. Now take $Y = \bigcup_{n<\omega} S_n$, which is clearly closed with respect to F, and $|Y| \leq \omega \otimes \lambda = \lambda$. □

Theorem 6.5 (Fodor). *Let $\lambda \geq \omega$, and F be a set mapping on X of order λ. Then X can be represented as the union of at most λ free sets with respect to F.*

Proof. We may assume that $X = \kappa$ for some cardinal $\kappa > \lambda$. For each $\gamma < \kappa$, take Y_γ of size at most λ closed with respect to F such that $\gamma \in Y_\gamma$.

Now define a sequence $\langle X_\alpha : \alpha < \kappa \rangle$ of subsets of κ inductively as follows: given X_β for $\beta < \alpha$, let $Z_\alpha = \bigcup_{\beta<\alpha} X_\beta$, $\gamma_\alpha = \min(\kappa \backslash Z_\alpha)$, and $X_\alpha = Y_{\gamma_\alpha} \backslash Z_\alpha$. Note that the X_α are pairwise disjoint, and $|X_\alpha| \le \lambda$. Furthermore, γ_α is increasing in α, so $\bigcup_{\alpha<\kappa} X_\alpha = \kappa$.

For $\nu < \lambda$, let S_ν consist of the ν-th elements of each X_α (if it exists). Then $\bigcup_{\nu<\lambda} S_\nu = \kappa$, and $|S_\nu \cap X_\alpha| \le 1$ for all $\alpha < \kappa$ and $\nu < \lambda$. It now suffices to partition each S_ν into λ many sets free with respect to F.

Define $\Phi_\nu : S_\nu \to \lambda$ as follows. If $x \in S_\nu \cap X_\alpha$ and Φ_ν has been defined on $S_\nu \cap \bigcup_{\beta<\alpha} X_\beta$, let

$$\Phi_\nu(x) = \min(\lambda \backslash \{\Phi_\nu(y) : y \in F(x) \cap S_\nu\}).$$

Note that $x \in X_\alpha$ implies $y \in F(x) \subseteq Y_{\gamma_\alpha}$, so if $y \in X_\beta$ then $\beta \le \alpha$, and $x \ne y$ implies $\beta < \alpha$. Thus Φ_ν is well-defined, and $\Phi^{-1}(\mu)$ is free with respect to F for all $\mu < \lambda$. Thus we have the desired partition $S_\nu = \bigcup_{\mu<\lambda} \Phi^{-1}(\mu)$. □

Note that Fodor's theorem implies Lázár's result, except when $\kappa = \omega$; however, this case is an easy consequence of Ramsey's theorem.

We now generalise Lázár's result for singular cardinals.

Theorem 6.6 (Hajnal set mapping theorem). *Let $\kappa \ge \omega$, $\lambda < \kappa$, and F be a set mapping of order λ on κ. Then there exists $S \subseteq \kappa$ of size κ that is free with respect to F.*

Proof. By Lázár's theorem, We may assume that κ is singular. Let $\tau = \max(\lambda, \mathrm{cf}(\kappa))^+$; note that τ is regular, so $\tau < \kappa$.

Let $\langle \kappa_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ be an increasing sequence of regular cardinals with supremum κ such that $\kappa_0 > \tau$.

We now define sequences $\langle S_\alpha : \alpha < \mathrm{cf}(\kappa) \rangle$ and $\langle S_{\alpha,\nu} : \alpha < \mathrm{cf}(\kappa), \nu < \tau \rangle$ of subsets of κ inductively as follows. Given S_β and $S_{\beta,\nu}$ for $\beta < \alpha$, $\nu < \tau$, such that

$$S_\beta = \bigcup_{\nu<\tau} S_{\beta,\nu}, \qquad |S_\beta| = |S_{\beta,\nu}| = \kappa_\beta,$$

let

$$Z_\alpha = \bigcup_{\beta<\alpha} S_\beta \cup \{F(y) : y \in S_\beta\}.$$

Note that $|Z_\alpha| \le \sum_{\beta<\alpha} \kappa_\beta \oplus (\kappa_\beta \otimes \lambda) < \kappa_\alpha < \kappa$. Hence we may pick a subset $Y_\alpha \subseteq \kappa \backslash Z_\alpha$ of size α; by Lázár's theorem we may assume that Y_α is free with respect to F.

Now define $\nu : Y_\alpha \to \tau$ by

$$\nu(y) = \sup\{\nu < \tau : \exists \beta < \alpha\, [S_{\beta,\nu} \cap F(y) \ne \emptyset]\}.$$

Since the $S_{\beta,\nu}$ are pairwise disjoint and $|F(y)| < \lambda$, we have $\nu(y) < \tau$. Thus there exists $S_\alpha \subseteq Y_\alpha$ and $\nu_\alpha < \tau$ with

$$|S_\alpha| = \kappa_\alpha, \qquad \forall y \in S_\alpha\, [\nu(y) = \nu_\alpha].$$

To complete the inductive definition, take any $|S_{\alpha,\nu}| = \kappa_\alpha$ with $S_\alpha = \bigcup_{\nu<\tau} S_{\alpha,\nu}$.

Let $\nu^* = \sup_{\alpha<\mathrm{cf}(\kappa)} \nu_\alpha \oplus 1 < \tau$.Suppose $x \in S_{\alpha,\nu^*}$, $y \in S_{\beta,\nu^*}$. If $\alpha = \beta$ then $x, y \in Y_\alpha$, so $x \notin F(y)$ and $y \notin F(x)$. If $\beta < \alpha$, then $x \in X_\alpha$ and $F(y) \subseteq Z_\beta \subseteq Z_\alpha$ implies $x \notin F(y)$. Also, $\nu_\alpha < \nu^*$ implies $F(x) \cap S_{\beta,\nu^*} = \emptyset$, so $y \notin F(x)$.

Hence $S = \bigcup_{\alpha<\mathrm{cf}(\kappa)} S_{\alpha,\nu^*}$ satisfies $|S| = \sum_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha = \kappa$, and S is free with respect to F. □