Finitely Generated Abelian Groups

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In this note, we present Milne's proof for the fundamental theorem of finitely generated abelian groups.

Lemma. *Given integers* $a_1, \dots, a_k \geq 0$ *with* $\gcd(a_1, \dots, a_k) = 1$*, and an abelian group* G *generated by* $\{x_1, \dots, x_k\}$*, there exists a generating set* $\{y_1, \dots, y_k\}$ *such that* $y_1 = x_1^{a_1} \cdots x_k^{a_k}$.

Proof. Induction on $s = a_1 + \cdots + a_k$. If $s = 1$ there is nothing to prove.

If $s > 1$ then we may assume that $a_1 \geq a_2 > 0$. Then $\{x_1, x_2 + x_1, x_3, \dots, x_k\}$ generate G, and $(a_1 - a_2) + a_2 + \cdots + a_k < s$. Thus by induction hypothesis there exists a generating set $\{y_1, \dots, y_k\}$ such that

$$\begin{aligned} y_1 &= (a_1 - a_2)x_1 + a_2(x_1 + x_2) + a_3 x_3 + \cdots + a_k x_k \\ &= a_1 x_1 + a_2 x_2 + \cdots + a_k x_k, \end{aligned}$$

as desired. $\square$

Theorem. *Every finitely generated abelian group* G *is the direct product of cyclic groups.*

Proof. Induction on the minimal number of generators for G. If G is generated by one element then it is cyclic by definition.

Otherwise, among all generating sets $\{x_1, \dots, x_n\}$ of G of minimal size $n > 1$, choose one for which $o(x_1)$ is minimal (possibly ∞). If $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} = e$ for some $x_1^{m_1} \neq e$, by division algorithm and replacing some x_i with their inverses, we may choose $0 < m_1 < o(x_1)$ and $m_2, \dots, m_n \geq 0$.

Let $d = \gcd(m_1, \dots, m_n) \geq 1$, and $a_i = m_i/d$. By the lemma there exists a generating set $\{y_1, \dots, y_n\}$ with $y_1 = x_1^{a_1} \cdots x_n^{a_n}$. But $y_1^d = e$, so $o(y_1) \leq d \leq m_1 < o(x_1)$, contradicting our choice of x_i.

Thus $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} = e$ implies $x_1^{m_1} = e$. Hence $\langle x_1 \rangle \cap \langle x_2, \dots, x_n \rangle = \{e\}$, so $G = \langle x_1 \rangle \times \langle x_2, \dots, x_n \rangle$. By induction hypothesis, $\langle x_2, \dots, x_n \rangle$ is a direct product of cyclic groups, so G is also a direcct product of cyclic groups. $\square$