Orthogonal Complements

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In this note, we present a simple proof of a formula for the dimension of the orthogonal complement of a subspace, given a finite-dimensional vector space equipped with a bilinear form.

Lemma. *Given two finite-dimensional vector spaces A, B and a bilinear form $\phi : A \times B \to F$, we have*

$$\dim A + \dim A^{\perp} = \dim B + \dim B^{\perp}.$$

Proof. Consider $\phi_1 : A \to B^*$ and $\phi_2 : B \to A^*$ given by $\phi_1(a) = \phi(a, -)$ and $\phi_2(b) = \phi(-, b)$. For any basis $\mathcal{B}_A, \mathcal{B}_B$ of A, B respectively, we have

$$\operatorname{rk} \phi_1 = \operatorname{rk}([\phi_1]^T_{\mathcal{B}_B^*, \mathcal{B}_A}) = \operatorname{rk}([\phi]_{\mathcal{B}_A, \mathcal{B}_B}) = \operatorname{rk}([\phi_2]_{\mathcal{B}_A^*, \mathcal{B}_B}) = \operatorname{rk} \phi_2.$$

But $\ker \phi_1 = \{a \in A \mid \forall b \in B\, (\phi(a, b) = 0)\} = B^{\perp}$, and similarly if we swap A, B, so by rank-nullity theorem,

$$\dim A - \dim B^{\perp} = \operatorname{rk} \phi_1 = \operatorname{rk} \phi_2 = \dim B - \dim A^{\perp}. \qquad \square$$

Corollary. *Given two finite-dimensional vector spaces V_1, V_2, a bilinear form $\phi : V_1 \times V_2 \to F$, and a subspace $U \subseteq V_2$, we have*

$$\dim U + \dim U^{\perp} = \dim V_1 + \dim(U \cap V_1^{\perp}).$$

Proof. Note that $\{u \in U \mid \forall v \in V_1\, (\phi(v, u) = 0)\} = U \cap V_1^{\perp}$. Now apply the lemma to V_1, U. $\square$

Corollary (Proposition 5.18)**.** *Given a finite-dimensional vector space V, a bilinear form ϕ on V, and a subspace $U \subseteq V$, we have*

$$\dim U + \dim U^{\perp, L} = \dim V + \dim(U \cap V^{\perp, R}),$$
$$\dim U + \dim U^{\perp, R} = \dim V + \dim(U \cap V^{\perp, L}).$$

Proof. The first statement follows from the previous corollary with $V_1 = V_2 = V$. The second statement is analogous. $\square$