Finitely Generated Abelian Groups

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In this note, we present a proof for the fundamental theorem of finitely generated abelian groups.

Theorem. *Every finitely generated abelian group G is the direct product of cyclic groups.*

Proof. Induction on the minimal number of generators for G. If G is generated by one element then it is cyclic by definition.

Suppose that the smallest generating set for G has size $n > 1$. If there is some generating set $\{g_1, \dots, g_n\}$ such that $g_1^{i_1} \cdots g_n^{i_n} = e$ implies $i_1 = \cdots = i_n = 0$, then $g_1^{i_1} \cdots g_n^{i_n} \mapsto (i_1, \dots, i_n)$ is an isomorphism between G and $\mathbb{Z}^n$.

Otherwise, for every generating set $\{g_1, \dots, g_n\}$ there exists $i_1, \dots, i_n$, not all zero, such that $g_1^{i_1} \cdots g_n^{i_n} = e$. By negating all exponents if necessary, we see that at least one of these exponents is positive.

Consider the set of all positive integers which can appear as exponents in some relation between elements of some generating set for G. We have shown that this set is nonempty, hence it must have a minimal element, say m. Thus there is a generating set $\{a_1, \dots, a_n\}$ such that (after rearrangement of elements) $a_1^m a_2^{s_2} \cdots a_n^{s_n} = e$.

Now $s_2 = qm + r$ for some $0 \le r < m$, so $(a_1 a_2^q)^m a_2^r \cdots a_n^{s_n} = e$. Since $\{a_1 a_2^q, a_2, \dots, a_n\}$ also generates G, $r > 0$ would contradict the minimality of m. Hence $r = 0$, and $m \mid s_2$. Similarly, $m \mid s_k$ for all $2 \le k \le n$.

Set $b_1 = a_1 a_2^{s_2/m} \cdots a_n^{s_n/m}$. Then $\{b_1, a_2, \dots, a_n\}$ generates G, and $b_1^m = e$.

If $b_1^{r_1} = a_2^{r_2} \cdots a_n^{r_n}$ for some $r_1, \dots, r_n$ then $r_1 = q'm + r'$ for some $0 \le r < m$. Hence $a_1^m a_2^{s_2} \cdots a_n^{s_n} = e$ gives

$$\begin{aligned} e &= b_1^{r_1} \prod_{2 \le k \le n} a_k^{-r_k} \\ &= a_1^{r_1} \prod_{2 \le k \le n} a_k^{r_1 s_k/m - r_k} \\ &= a_1^{r'} \prod_{2 \le k \le n} a_k^{r_1 s_k/m - r_k - q' s_k}, \end{aligned}$$

and $r' > 0$ would contradict the minimality of m. Hence $r = 0$, and $m \mid r_1$. Thus $b_1^{r_1} \in \langle a_2, \dots, a_n \rangle = H$ implies that $b_1^{r_1} = e$.

Hence $b_1 H = G$ and $\langle b_1 \rangle \cap H = \{e\}$, so $G = \langle b_1 \rangle \times H$. By induction hypothesis, H is a direct product of cyclic groups, so G is also a direct product of cyclic groups. $\square$