Waves
Pulse Propagation
The Wave Equation

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Last time

- oscillations
- simple harmonic motion (SHM)
- spring systems
- energy in SHM
- introducing waves
- kinds of waves
- wave speed on a string

Warm Up Question

If you stretch a rubber hose and pluck it, you can observe a pulse traveling up and down the hose.

What happens to the speed of the pulse if you stretch the hose more tightly?

(A) It increases.
(B) It decreases.
(C) It is constant.
(D) It changes unpredictably.

[1] Serway & Jewett, page 499, objective question 2.

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If you stretch a rubber hose and pluck it, you can observe a pulse traveling up and down the hose.

What happens to the speed if you fill the hose with water?

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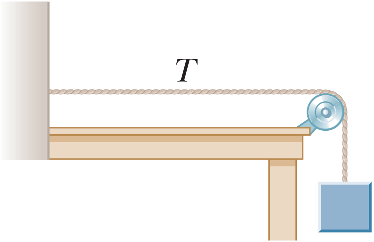
Overview

- pulse propagation
- the wave equation

Example

A uniform string has a mass of m_s and a length of ℓ. The string passes over a pulley and supports an block of mass m_b. Find the speed of a pulse traveling along this string. (Assume the vertical piece of the rope is very short.)



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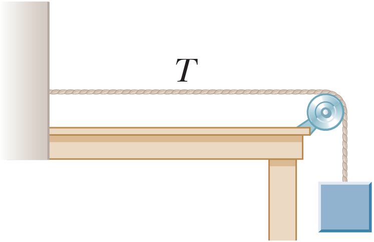


$$v = \sqrt{\frac{m_b g \ell}{m_s}}$$

Pulse Propagation

A wave pulse (in a plane) at a moment in time can be described in terms of x and y coordinates, giving $y(x)$.

We know that the pulse will move with speed v and be displaced, say in the positive x direction, while maintaining its shape.

That means we can also give y as a function of time, $y(x, t)$.

Consider a moving reference frame, S', with the pulse at rest, $y'(x') = f(x')$, no time dependence. Galilean transformation:

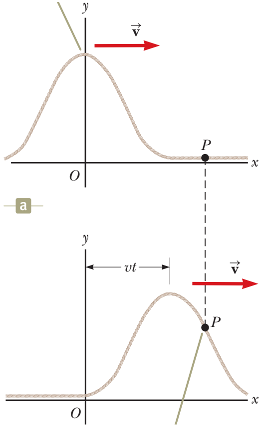
$$x' = x - vt$$

Pulse Propagation

$$x' = x - vt$$

Then in the rest-frame of the string

$$y(x,t) = f(x') = f(x - vt)$$



Pulse Propagation

The shape of the pulse is given by $f(x)$ and can be arbitrary.

Whatever the form of f, if the pulse moves in the $+x$ direction:

$$y(x, t) = f(x - vt)$$

If the pulse moves in the $-x$ direction:

$$y(x, t) = f(x + vt)$$

Wave Pulse Example 16.1

A pulse moving to the right along the x axis is represented by the wave function

$$y(x, t) = \frac{2}{(x - 3.0t)^2 + 1}$$

where x and y are measured in centimeters and t is measured in seconds.

What is the wave speed?

Find expressions for the wave function at $t = 0$, $t = 1.0$ s, and $t = 2.0$ s.

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[1] Serway & Jewett, page 486.

[2] This function is an unnormalized Cauchy distribution, or as physicists say "it has a Lorentzian profile".

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What is the wave speed? **3.0 cm/s**

Find expressions for the wave function at $t = 0$, $t = 1.0$ s, and $t = 2.0$ s.

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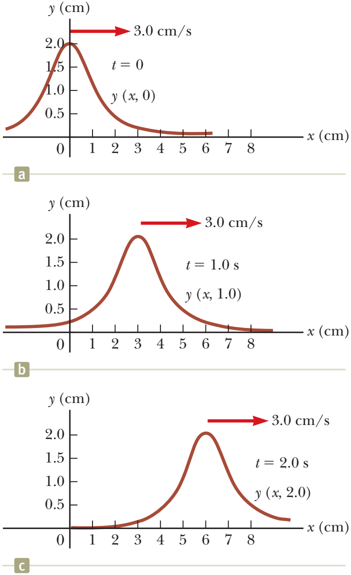
Wave Pulse Example 16.1

$$t = 0, \quad y(x, 0) = \frac{2}{x^2 + 1}$$

$$t = 1, \quad y(x, 1) = \frac{2}{(x - 3.0)^2 + 1}$$

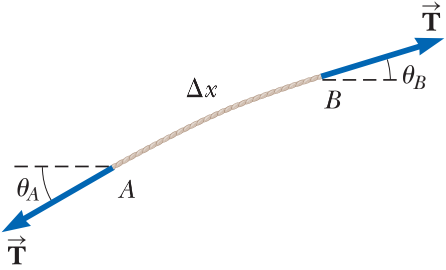
$$t = 2, \quad y(x, 2) = \frac{2}{(x - 6.0)^2 + 1}$$



The Wave Equation

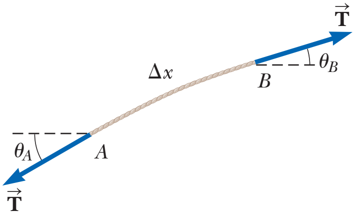
Can we find a general equation describing the displacement (y) of our medium as a function of position (x) and time?

Start by considering a string carrying a disturbance.



The Wave Equation

Consider a small length of string Δx.



As we did for oscillations, start from Newton's 2nd law.

$$\begin{aligned} F_y &= ma_y \\ T\sin\theta_B - T\sin\theta_A &= (\mu\,\Delta x)\frac{\partial^2 y}{\partial t^2} \end{aligned}$$

For small angles

$$\sin\theta \approx \tan\theta$$

The Wave Equation

We can write $\tan\theta$ as the slope of $y(x)$:

$$\tan\theta = \frac{\partial y}{\partial x}$$

Now Newton's second law becomes:

$$\begin{aligned}
T\left(\left.\frac{\partial y}{\partial x}\right|_{x=B} - \left.\frac{\partial y}{\partial x}\right|_{x=A}\right) &= (\mu\,\Delta x)\frac{\partial^2 y}{\partial t^2} \\
\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} &= \frac{\left.\frac{\partial y}{\partial x}\right|_{x=B} - \left.\frac{\partial y}{\partial x}\right|_{x=A}}{\Delta x}
\end{aligned}$$

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We need to take the limit of this expression as we consider an infinitesimal piece of string: $\Delta x \to 0$.

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

where we use the definition of the partial derivative.

The Wave Equation

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

Remember that the speed of a wave on a string is

$$v = \sqrt{\frac{T}{\mu}}$$

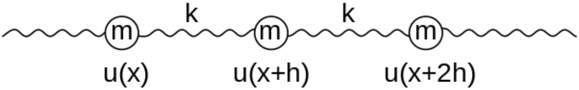
The wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Even though we derived this for a string, it applies much more generally!

The Wave Equation

We can model longitudinal waves like sound waves by a series of masses connected by springs, length h.



u_1 u_2 u_3

u is a function that gives the displacement of the mass at each equilibrium position x, $x + h$, *etc.*

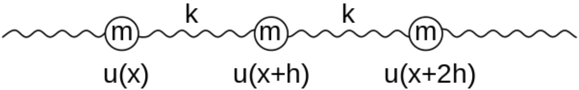
For such a case, the propagation speed is

$$v = \sqrt{\frac{KL}{\mu}}$$

where K is the spring constant of the entire spring chain, L is the length, and μ is the mass density.

[1]Figure from Wikipedia, by Sebastian Henckel.

The Wave Equation



u_1 u_2 u_3

u is a function that gives the displacement of the mass at each equilibrium position x, $x+h$, *etc.*

Consider the mass, m, at equilibrium position $x+h$

$$
\begin{aligned}
F &= ma \\
k(u_3 - u_2) - k(u_2 - u_1) &= m\frac{\partial^2 u}{\partial t^2} \\
\frac{m}{k}\frac{\partial^2 u}{\partial t^2} &= u_3 - 2u_2 + u_1
\end{aligned}
$$

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The Wave Equation

$$\frac{m}{k}\frac{\partial^2 u}{\partial t^2} = u_3 - 2u_2 + u_1$$

We can re-write $\frac{m}{k}$ in terms of quantities for the entire spring chain. Suppose there are N masses.

$m = \frac{\mu L}{N}$ and $k = NK$ and $N = \frac{L}{h}$

$$\frac{\mu}{KL}\frac{\partial^2 u}{\partial t^2} = \frac{u(x+2h) - 2u(x+h) + u(x)}{h^2}$$

Letting $N \to \infty$ and $h \to 0$, the RHS is the definition of the 2nd derivative. Same equation!

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The Wave Equation

The wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

We derived this for a case of transverse waves (wave on a string) and a case of longitudinal waves (spring with mass).

It applies generally!

Solutions to the Wave Equation

Earlier we reasoned that a function of the form:

$$y(x,t) = f(x \pm vt)$$

should describe a propagating wave pulse.

Notice that f does not depend arbitrarily on x and t. It only depends on the two *together* by depending on $u = x \pm vt$.

Does it satisfy the wave equation?

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

(Next lecture...)

Summary

- wave speed on a string
- pulse propagation
- the wave equation

Homework Serway & Jewett:

- Ch 16, onward from page 499. OQs: 5; Probs: 3, 23, 24, 29, 53, 59, 60