Logistic Regression

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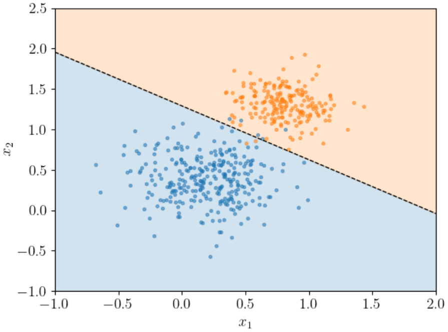
Overview

Classification problem

The goal in classification is to take an input vector x and to assign it to one of K discrete classes C_k where k = 1,...,K.
The input space is thereby divided into decision regions whose boundaries are called decision boundaries or decision surfaces.



Hình 1: Decision boundary

Probabilistic view of classification

Consider first of all the case of two classes. The posterior probability for class C_1 can be written as:

$$p(C_1|x) = \frac{p(x|C_1)p(C_1)}{p(x|C_1)p(C_1) + p(x|C_2)p(C_2)} = \frac{1}{1+e^{-a}} = \sigma(a)$$

where we have defined

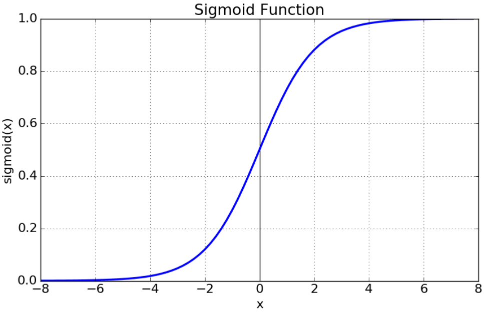
$$a = \log \frac{p(x|C_1)p(C_1)}{p(x|C_2)p(C_2)}$$

and $\sigma(a)$ is the logistic sigmoid function defined by

$$\sigma(a) = \frac{1}{1+e^{-a}}$$

Exercise: Calculate the derivative of sigmoid function.

Probabilistic view of classification (cont.)



Hình 2: Sigmoid function

Logistic regression

The model logistic regression is defined as:

$$p(C_1|\phi) = y(\phi) = \sigma(w^T\phi)$$
$$p(C_2|\phi) = 1 - p(C_1|\phi)$$

For a data set ϕ_n, t_n, where $t_n \in \{0, 1\}$ and $\phi_n = \phi(x_n)$, with n = 1,...,N, the likelihood function can be written

$$p(t|w) = \prod_{n=1}^{N} y_n^{t_n}(1 - y_n)^{1-t_n}$$

where t = $(t_1, ..., t_N)^T$ and $y_n = p(C_1|\phi_n)$

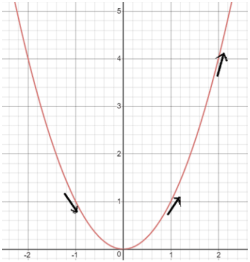
Logistic regression (cont.)

We can define an error function by taking the negative logarithm of the likelihood, which gives the cross_entropy error function in the form

$$L = -\log p(t|w) = -\sum_{n=1}^{N} \{t_n \log y_n + (1 - t_n) \log (1 - y_n)\}$$

where $y_n = \sigma(a_n)$ and $a_n = w^T \phi_n$
Taking the gradient of the error function with respect to w, we obtain

$$\nabla L = \sum_{n=1}^{N} (y_n - t_n)\phi_n$$



Hình 3: Function f(x) = x^2

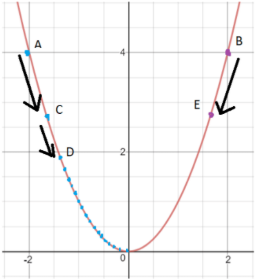
Function f(x) = $x^2 \rightarrow$ f'(x) = 2x. Remarks:

- f'(1) = 2 * 1 < f'(2) = 2 * 2 => the function at x = 2 is steeper than the function at x = 1 => the higher absolute value of gradient, the steeper function is.
- f'(-1) = 2 * (-1) = -2 < 0 => the function decreases (x increases, y decreases)

Gradient descent (cont.)

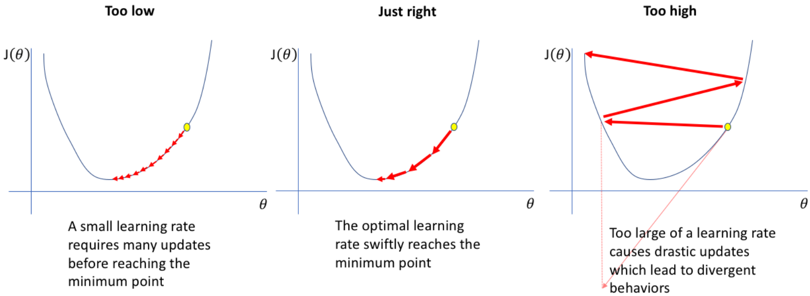
Steps to optimize the function f(x), $R \to R$, $x \to f(x)$:

1. Initialize randomly $x = x_0$
2. Update x = x - learning_rate × f'(x), learning_rate is a positive small number.
3. If f(x) is small enough, stop the algorithm. Otherwise, repeat the second step.



Hình 4: Gradient descent update

How to choose learning rate?



Hình 5: How to choose learning rate

Gradient descent (cont.)

Steps to optimize the function f(x), $R^n \to R$, $x \to f(x)$:

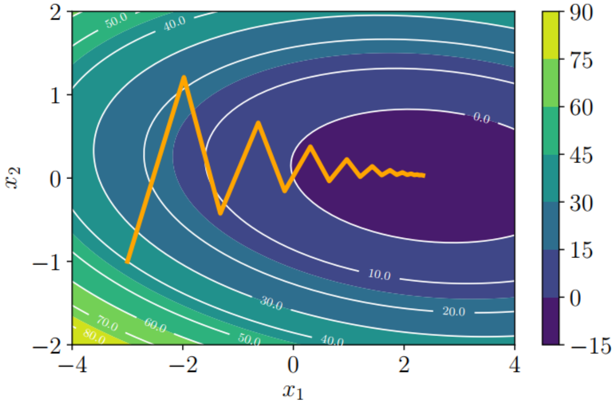
1. Initialize randomly x
2. Update x = x - learning_rate $\times$ $(\frac{df}{dx})^T$, learning_rate is a positive small number.
3. If f(x) is small enough, stop the algorithm. Otherwise, repeat the second step.

For example, f(x): $R^2 \to R$

$$f(x) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Initial $x_0 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, iterate 5 steps of gradient descent algorithm.

Gradient descent (cont.)



Hình 6: Gradient descent algorithm