VietNam National University Ho Chi Minh City

Ho Chi Minh City University of Technology

Calculus 1 (EXERCISE) CC07

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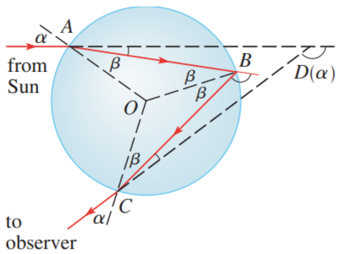
2 Topic 1

1. Study the applied project "The calculus of rainbows" in [1].You need to explain it in details and answer all given questions.

2.1 Problem 1 [1]

The figure shows a ray of sunlight entering a spherical raindrop at A. Some of the light is reflected, but the line AB shows the path of the part that enters the drop. Notice that the light is refracted toward the normal line AO and in fact Snell's Law says that $\sin\alpha = k\sin\beta$, where α is the angle of incidence, β is the angle of refraction, and $k \approx \frac{4}{3}$ is the index of refraction for water.

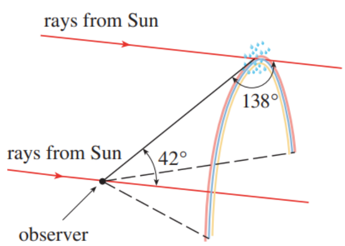


Formation of the primary rainbow

At B some of the light passes through the drop and is refracted into the air, but the line BC shows the part that is reflected. (The angle of incidence equals the angle of reflection.) When the ray reaches C, part of it is reflected, but for the time being we are more interested in the part that leaves the raindrop at C. (Notice that it is refracted away from the normal line.) The angle of deviation $D(\alpha)$ is the amount of clockwise rotation that the ray has undergone during this three-stage process. Thus

$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta$$

Show that the minimum value of the deviation is $D(\alpha) \approx 138°$ and occurs when $\alpha \approx 59.4°$.
The significance of the minimum deviation is that when $\alpha \approx 59.4°$ we have $D'(\alpha) \approx 0°$, so $\Delta D/\Delta\alpha \approx 0$. This means that many rays with $\alpha \approx 59.4°$ become deviated by approximately the same amount. It is the concentration of rays coming from near the direction of minimum deviation that creates the brightness of the primary rainbow. The following figure shows that the angle of elevation from the observer up to the highest point on the rainbow is $180° - 138° = 42°$. (This angle is called the rainbow angle.)



2.2 Solution for problem 1

From Snell's Law, we have $\sin\alpha = k\sin\beta \approx \frac{4}{3}\sin\beta \Leftrightarrow \beta \approx \arcsin\left(\frac{3}{4}\sin\alpha\right)$.

We substitute this into: $\mathrm{D}(\alpha) = \pi + 2\alpha - 4\beta = \pi + 2\alpha - 4\arcsin\left(\frac{3}{4}\sin\alpha\right)$

$\Leftrightarrow$ D'$(\alpha) = 2 - 4\left[1-\left(\frac{3}{4}\sin\alpha\right)^2\right]^{-1/2} = 2 - \frac{3\cos\alpha}{\sqrt{1-\frac{9}{16}\sin^2\alpha}}$.

This is 0 when $\frac{3\cos\alpha}{\sqrt{1-\frac{9}{16}\sin^2\alpha}} = 2 \Leftrightarrow \frac{9}{4}\cos^2\alpha = 1 - \frac{9}{16}\sin^2\alpha \Leftrightarrow \frac{9}{4}\cos^2\alpha = 1 - \frac{9}{16}\left(1-\cos^2\alpha\right) \Leftrightarrow \frac{27}{16}\cos^2\alpha = \frac{7}{16} \Leftrightarrow \cos\alpha = \sqrt{\frac{7}{27}} \Leftrightarrow \alpha = \arccos\sqrt{\frac{7}{27}} \approx 59.4°$,and so the local minimum is D(59.4°) $\approx$ 2.4 radians $\approx 138°$

To see that this an absolute minimum, we check the endpoints, which in this case are $\alpha = 0$ and $\alpha = \frac{\pi}{2}$:

$D(0) = \pi$ radians $= 180°$, and $D\left(\frac{\pi}{2}\right) \approx 166°$.

2.3 Problem 2 [1]

Problem 1 explains the location of the primary rainbow but how do we explain the colors? Sunlight comprises a range of wavelengths, from the red range through orange, yellow, green, blue, indigo, and violet. As Newton discovered in his prism experiments of 1666, the index of refraction is different for each color. (The effect is called dispersion.) For red light the refractive index is $k \approx 1.3318$ whereas for violet light it is $k \approx 1.3435$. By repeating the calculation of Problem 1 for these values of k, show that the rainbow angle is about 42.3° for the red bow and 40.6° for the violet bow. So the rainbow really consists of seven individual bows corresponding to the seven colors.

2.4 Solution for problem 2

If we repeat the same process as problem 1 replacing $\frac{4}{3}$ with k, we will get:

$\mathrm{D}(\alpha) = \pi + 2\alpha - 4\beta = \pi + 2\alpha - 4\arcsin\left(\frac{1}{k}\sin\alpha\right) \Leftrightarrow D'(\alpha) = 2 - \frac{4\cos\alpha}{k\sqrt{1-\left(\frac{\sin\alpha}{k}\right)^2}} \Leftrightarrow \left(\frac{2\cos\alpha}{k}\right)^2 = 1 - \left(\frac{\sin\alpha}{k}\right)^2 \Leftrightarrow 4\cos^2\alpha = k^2 - \sin^2\alpha \Leftrightarrow 3\cos^2\alpha = k^2 - 1 \Leftrightarrow \alpha = \arccos\sqrt{\frac{k^2-1}{3}}$. So for $k \approx 1.3318$(red light) the minimum occurs at $\alpha_1 \approx 1.038$ radians, and so the rainbow angle is about $\pi - D(\alpha_1) \approx 42.3°$. For $k \approx 1.3435$(violet light) the minimum occurs at $\alpha_2 = 1.026$ radians, and so the rainbow angle is about $\pi - D(\alpha_2) \approx 40.6°$

2.5 Problem 3 [1]

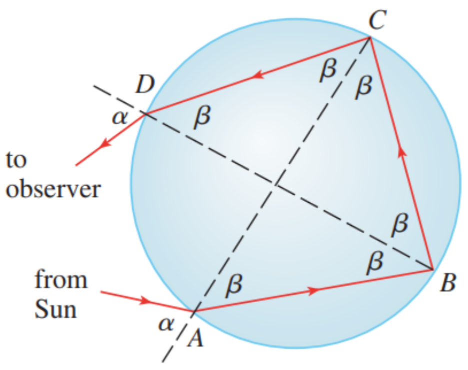
Perhaps you have seen a fainter secondary rainbow above the primary bow. That results from the part of a ray that enters a raindrop and is refracted at A, reflected twice (at B and C), and refracted as it leaves the drop at D . (See the figure at the left.) This time the deviation angle $D(\alpha)$ is the total amount of counterclockwise rotation that the ray undergoes in this four-stage process. Show that

$$D(\alpha) = \pi + 2\alpha - 4\beta$$

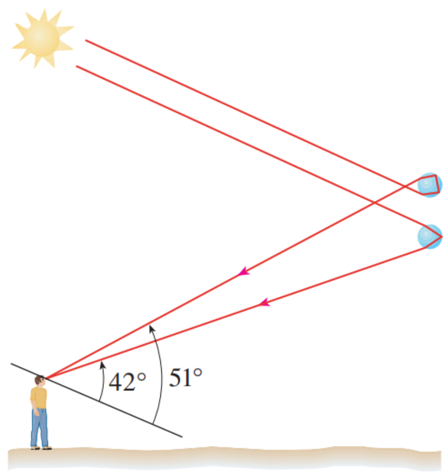
and $D(\alpha)$ has a minimum value when

$$\cos(\alpha) = \sqrt{\frac{k^2-1}{8}}$$

Taking $k = \frac{4}{3}$, show that the minimum deviation is about $129°$ and so the rainbow angle for the secondary rainbow is about $51°$, as shown in the following figure.



Formation of the secondary rainbow



2.6 Solution for problem 3

At each reflection or refraction, the light is bent in a counterclockwise direction: the bend at A is $\alpha - \beta$, the bend at B is $\pi - 2\beta$, the bend at C is again $\pi - 2\beta$, and the bend at D is $\alpha - \beta$. So the total bend is $D(\alpha) = 2(\alpha - \beta) + 2(\pi - 2\beta) = 2\alpha - 6\beta + 2\pi$, as required.
We substitute $\beta = \arcsin\left(\frac{\sin\alpha}{k}\right)$ and differentiate, to get D'(α) $= 2 - \frac{6\cos\alpha}{k\sqrt{1-(\sin\alpha/k)^2}}$, which is 0 when $\frac{3\cos\alpha}{k} = \sqrt{1-\left(\frac{\sin\alpha}{k}\right)^2} \Leftrightarrow 9\cos^2\alpha = k^2 - \sin^2\alpha \Leftrightarrow 8\cos^2\alpha = k^2 - 1 \Leftrightarrow \cos\alpha = \sqrt{\frac{k^2-1}{8}}$.
If $k = \frac{4}{3}$, then the minimum occurs at $\alpha_1 = \arccos\left(\sqrt{\frac{(4/3)^2-1}{8}}\right) \approx 1.254$ radians. Thus, the minimum counterclockwise rotation is $D(\alpha_1) \approx 231°$, which is equivalent to a clockwise rotation of $360° - 231° = 129°$ (see the figure). So the rainbow angle for the secondary rainbow is about $180° - 129° = 51°$, as required. In general, the rainbow angle for the secondary rainbow is $\pi - [2\pi - D(\alpha)] = D(\alpha) - \pi$

2.7 Problem 4 [1]

Show that the colors in the secondary rainbow appear in the opposite order from those in the primary rainbow.

2.8 Solution for problem 4

In the primary rainbow, the rainbow angle gets smaller as k gets larger, as we found in Problem 2, so the colors appear from top to bottom in order of increasing k. But in the secondary rainbow, the rainbow angle gets larger as k gets larger. To see this, we find the minimum deviations for red light and for violet light in the secondary rainbow.

For $k \approx 1.3318$ (red light) the minimum occurs at $\alpha_1 \approx \arccos\left(\sqrt{\frac{1.3318^2 - 1}{8}}\right) \approx 1.255$ radians, and so the rainbow angle is $D(\alpha_1) - \pi \approx 50.6°$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx \arccos\sqrt{\frac{1.3435^2 - 1}{8}} \approx 1.248$ radians, and so the rainbow angle is $D(\alpha_2) - \pi \approx 53.6°$.

Consequently, the rainbow angle is larger for colors with higher indices of refraction, and the colors appear from bottom to top in order of increasing k, the reverse of their order in the primary rainbow.

Note that our calculations above also explain why the secondary rainbow is more spread out than the primary rainbow: in the primary rainbow, the difference between rainbow angles for red and violet light is about 1.7°, whereas in the secondary rainbow it is about 3°.

3 Topic 2

3.1 Problem

In optimization problems, it is very important to solve the equation f(x) = 0. However, it is not always possible to solve this equation exactly. By this reason, we need an approach which enables us to find the root approximately. There are various such approaches. One of them is called NEWTON method. Study the Newton method (see Chapter 4 in [1]). Then write a code using Matlab or Python to illustrate this method. Use your code to solve the following problem:

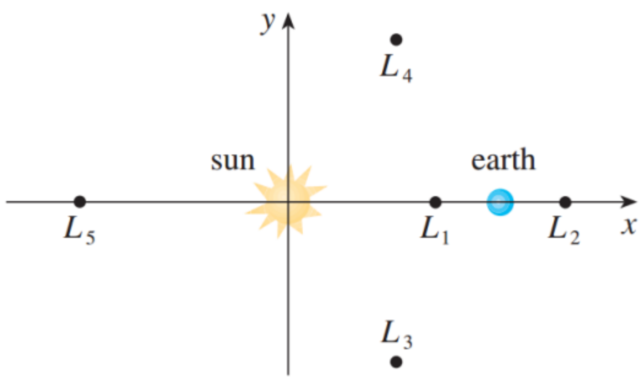
The figure shows the sun located at the origin and the earth at the point (1,0). (The unit here is the distance between the centers of the earth and the sun, called an astronomical unit: $1AU \approx 1.496 \times 10^8 km$.) There are five locations L_1, L_2, L_3, L_4, and L_5 in this plane of rotation of the earth about the sun where a satellite remains motionless with respect to the earth because the forces acting on the satellite (including the gravitational attractions of the earth and the sun) balance each other. These locations are called libration points. (A solar research satellite has been placed at one of these ilibration points.) If m_1 is the mass of the sun, m_2 is the mass of the earth, and $r = \dfrac{m_2}{m_1 + m_2}$, it turns out that the x-coordinate of L_1 is the unique root of the fifth-degree equation

$$p(x) = x^5 - (2+r)x^4 + (1+2r)x^3 - (1-r)x^2 + 2(1-r)x + r - 1 = 0$$

and the x-coordinate of L_2 is the root of the equation

$$p(x) - 2rx^2 = 0$$

Using the value $r \approx 3.04042 \times 10^{-6}$, find the locations of the libration points (a) L_1 and (b) L_2.



3.2 Newton's method

The Newton's method is used for solving equations such as:

$$x + e^x = 4$$

$$tan(x) + sin(x) = 1.2$$

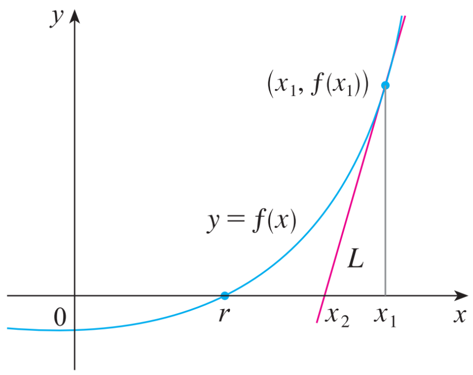
which can not apply any formula The Newton's method is a technique for solving equations of the form $f(x) = 0$ by successive approximation. The idea is to pick an initial guess x_0 such that $f(x_0) = 0$ is reasonably close to 0. We then find the equation of the line tangent to $y = f(x)$ at $x = x_0$ and follow it back to the x axis at a new guess x_1 which is an improved approximation. The formula for this is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Here is how this formula is construct:

Given a graph of $f(x)$

Assume that $x_1 = 5$ and $f(x_1) = f(5) = 1$ and $f'(5) = 2$.



Therefore, we have the tangential equation :

$$f(x_2) - f(5) = f'(5)(x_2 - 5)$$

$$\Leftrightarrow f(x_2) = 1 + 2(x_2 - 5)$$

Then, we get the x-coordinate at $f(x_2) = 0 \Rightarrow 0 \approx 1 + 2(x_2 - 5) \Leftrightarrow x_2 \approx \frac{9}{2}$. As we can see, the location of x_2 on x-coordinate is not match r. $x_2 = \frac{9}{2}$ is just nearly equal r which is $f(x) = 0$. So, the x at $f(x) = 0$ is often unable to find the accurated answer.At this point, the Newton's method is crucial for finding the answer that is closed enough to be accepted

We know that:

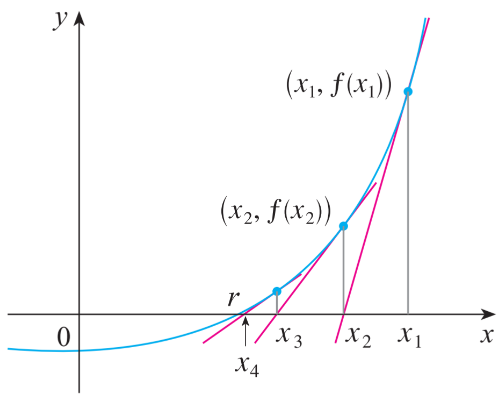
$$f(x_2) \approx f(x_1) + f'(x_1)(x_2 - x_1) \Leftrightarrow 0 \approx f(x_1) + f'(x_1)(x_2 - x_1) \Leftrightarrow x_2 \approx x_1 - \frac{f(x_1)}{f'(x_1)}$$

With x_1 is an initial x which depended on the problem is given

This x_1 is not the final answer because it is not closed enough to the real answer, the tangential equation at x_1 is:

$$f(x_3) \approx f(x_2) + f'(x_2)(x_3 - x_2) \Leftrightarrow 0 \approx f(x_2) + f'(x_2)(x_3 - x_2) \Leftrightarrow x_3 \approx x_2 - \frac{f(x_2)}{f'(x_2)}$$

and so on...



This is the Newton's method and the general formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3.3 Solution

As the graph shows, we let the Earth to be x which is located at (0,1). Therefore we can obtain that $L_1 < x < L_2$. L_1, L_2 is located on the x-axis, so $f_{(L_1)} = f_{(L_2)} = 0$. We then apply the Newton's method as mentioned before:
In L_1 case:
We get the initial $x_0 = 0.9$ because $0 < L_1 < x$ as the graph shows.
We obtain $p(x) = x^5 - (2+r)x^4 + (1+2r)x^3 - (1-r)x^2 + 2(1-r)x + r - 1 = 0$
Taking derivative of $p(x)$ we get:
$p'(x) = 5x^4 - 4x^3(2+r) + 3x^2(1+2r) - 2x(1-r) + 2(1-r)$
with $r \approx 3.04042 \times 10^{-6}$ Applying Newton's method

$$x_1 = x_0 - \frac{p(x_0)}{p'(x_0)} = 0.934489$$

$$x_2 = x_1 - \frac{p(x_1)}{p'(x_1)} = 0.956745$$

$$x_3 = x_2 - \frac{p(x_2)}{p'(x_2)} = 0.971197$$

$$x_4 = x_3 - \frac{p(x_3)}{p'(x_3)} = 0.980485$$

$$x_5 = x_4 - \frac{p(x_4)}{p'(x_4)} = 0.986147$$

$$x_6 = x_5 - \frac{p(x_5)}{p'(x_5)} = 0.989032$$

$$x_7 = x_6 - \frac{p(x_6)}{p'(x_6)} = 0.989909$$

$$x_8 = x_7 - \frac{p(x_7)}{p'(x_7)} = 0.989988$$

As we can see: $x_8 - x_7 = 0.000079 \rightarrow$ which is closed enough to get

$$\Rightarrow x_8 = L_1 \approx 0.989988(AU)$$

This is Python's code for calculating L_1

```python
# This code is for L1 of Topic 2

r = 3.04042e-6
# r = m1/(m1+m2)
# m1 is the mass of the Sun
# m2 is the mass of the Earth
# The unit here is the distance
# between the centers of the earth and the sun, called an astronomical unit: 1AU ~ 1.496*10^8(km)

def f(x):       # f(x) is the equation we are applying the Newton method
    return x**5 - (2 + r)*x**4 + (1 + 2*r)*x**3 - (1 - r)*x**2 + 2*(1 - r)*x + r - 1

def g(x):       # g(x) is the derivative of f(x)
    return 5*x**4 - 4*(2 + r)*x**3 + 3*(1 + 2*r)*x**2 - 2*(1 - r)*x + 2*(1 - r)

def newtonRaphson(x0):
    print('\n\n*** NEWTON RAPHSON METHOD IMPLEMENTATION ***\n')
    i = 0       # just a value to increase the x value when finding the root (x1, x2, x3, x4, ...)
    while True:
        if g(x0) == 0.0:
            print('No solution')
            break
        x1 = x0 - f(x0)/g(x0)
        print(f"x{i} = {round(x1,6)}")
        i += 1      # Increase the value i by 1
        # This function if below check if the difference between x1 and x0 is small enough to stop
        # BECAUSE the x value is always approach to the exact root, so that we just consider a small enough of delta x-> 0

        if abs(x1 - x0) < 1e-5:
            print('\n')
            print(f'The answer root ~ {round(x1,6)}(AU)')
            break
        x0 = x1                     # Update the current x(x1) with the previous x(x0)

x0 = input('\n\nEnter your guess about x value : ')
x0 = float(x0)
newtonRaphson(x0)
```

Listing 1: Python for L_1

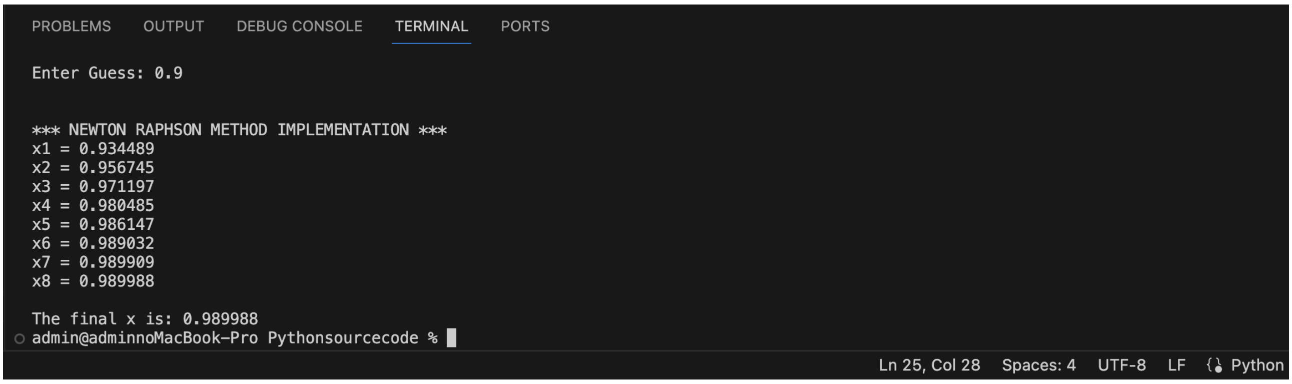


Figure 1: Output for L_1

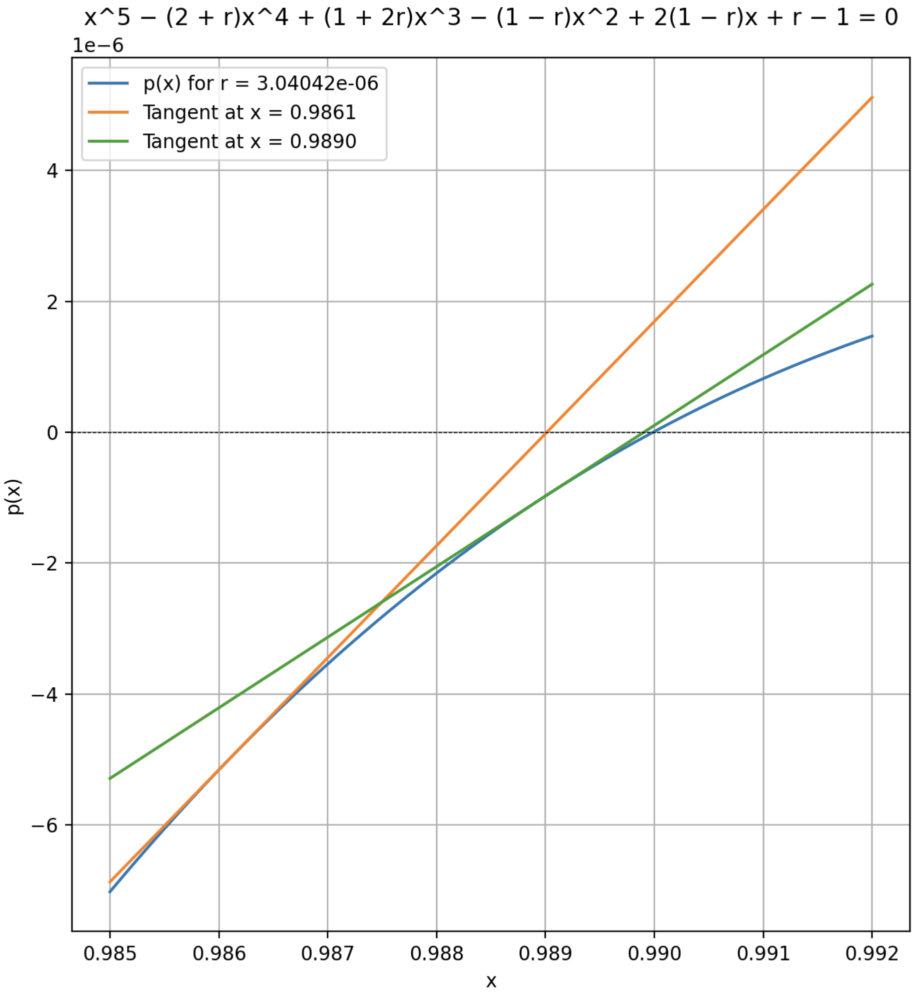


Figure 2: L_1 Graph

```python
import numpy as np
import matplotlib.pyplot as plt

def p(x, r):
    return x**5 - (2 + r)*x**4 + (1 + 2*r)*x**3 - (1 - r)*x**2 + 2*(1 - r)*x + r - 1

def p_prime(x, r):
    return 5*x**4 - 4*(2 + r)*x**3 + 3*(1 + 2*r)*x**2 - 2*(1 - r)*x + 2*(1 - r)

r_value = 3.04042e-6

x_values = np.linspace(0.985, 0.992, 400)

y_values = p(x_values, r_value)

tangent_points = [0.9861, 0.9890]

plt.plot(x_values, y_values, label=f'p(x) for r = {r_value}')

for x_tangent in tangent_points:
    y_tangent = p(x_tangent, r_value)
    slope_tangent = p_prime(x_tangent, r_value)

    tangent_line = lambda x: slope_tangent * (x - x_tangent) + y_tangent
    plt.plot(x_values, tangent_line(x_values), label=f'Tangent at x = {x_tangent:.4f}')

plt.axhline(0, color='black', linewidth=0.5, linestyle='--')

plt.title('x^5 - (2 + r)x^4 + (1 + 2r)x^3     (1 - r)x^2 + 2(1 - r)x + r - 1 = 0')
plt.xlabel('x')
plt.ylabel('p(x)')
plt.legend()
plt.grid(True)
plt.show()
```

Listing 2: Code for L_1 Graph [2] [3]

Applying the same method for L_2 Taking initial $x_0 = 1,1$ because $L_2 > 1$ We have

$$p(x) - 2rx^2 = 0$$

$$\Leftrightarrow x^5 - (2+r)x^4 + (1+2r)x^3 - (1-r)x^2 + 2(1-r)x + r - 1 - 2rx^2 = 0$$

$$\Leftrightarrow x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 = 0$$

$$\Rightarrow f(x) = x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 = 0$$

Taking derivative of $f(x)$ we get:
$f'(x) = 5x^4 - 4x^3(2+r) + 3x^2(1+2r) - 2x(1+r) + 2(1-r)$
with $r \approx 3.04042 \times 10^{-6}$
Applying Newton's method:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.067741$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.045722$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.030868$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.021037$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.014838$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 1.011455$$

$$x_7 = x_6 - \frac{f(x_6)}{f'(x_6)} = 1.010239$$

$$x_8 = x_7 - \frac{f(x_7)}{f'(x_7)} = 1.010081$$

$$x_9 = x_8 - \frac{f(x_8)}{f'(x_8)} = 1.010078$$

As we can see: $x_9 - x_8 = 0.000003 \Rightarrow$ which is closed enough to get

$$\Rightarrow x_9 = L_2 \approx 1.010078(AU)$$

This is Python's code for calculating L_2 :

```python
# This code is for L2 of Topic 2

r = 3.04042e-6
# r = m1/(m1+m2)
# m1 is the mass of the Sun
# m2 is the mass of the Earth
# The unit here is the distance
# between the centers of the earth and the sun, called an astronomical unit: 1AU ~ 1.496*10^8(km)

def f(x):       # f(x) is the equation we are applying the Newton method
    return x**5 - (2 + r)*x**4 + (1 + 2*r)*x**3 - (1 + r)*x**2 + 2*(1 - r)*x + r - 1

def g(x):       # g(x) is the derivative of f(x)
    return 5*x**4 - 4*(2 + r)*x**3 + 3*(1 + 2*r)*x**2 - 2*(1 + r)*x + 2*(1 - r)

def newtonRaphson(x0):
    print('\n\n*** NEWTON RAPHSON METHOD IMPLEMENTATION ***\n')
    i = 0       # just a value to increase the x value when finding the root (x1, x2, x3, x4, ...)
    while True:
        if g(x0) == 0.0:
            print('No solution')
            break
        x1 = x0 - f(x0)/g(x0)
        print(f"x{i} = {round(x1,6)}")
        i += 1                      # Increase the value i by 1
        # This function if below check if the difference between x1 and x0 is small enough to stop
        # BECAUSE the x value is always approach to the exact root, so that we just consider a small enough of delta x-> 0
        if abs(x1 - x0) < 1e-5:
            print('\n')
            print(f'The answer root ~ {round(x1,6)}(AU)')
            break
        x0 = x1                     # Update the current x(x1) with the previous x(x0)

x0 = input('\n\nEnter your guess about x value : ')
x0 = float(x0)
newtonRaphson(x0)
```

Listing 3: Python for L_2

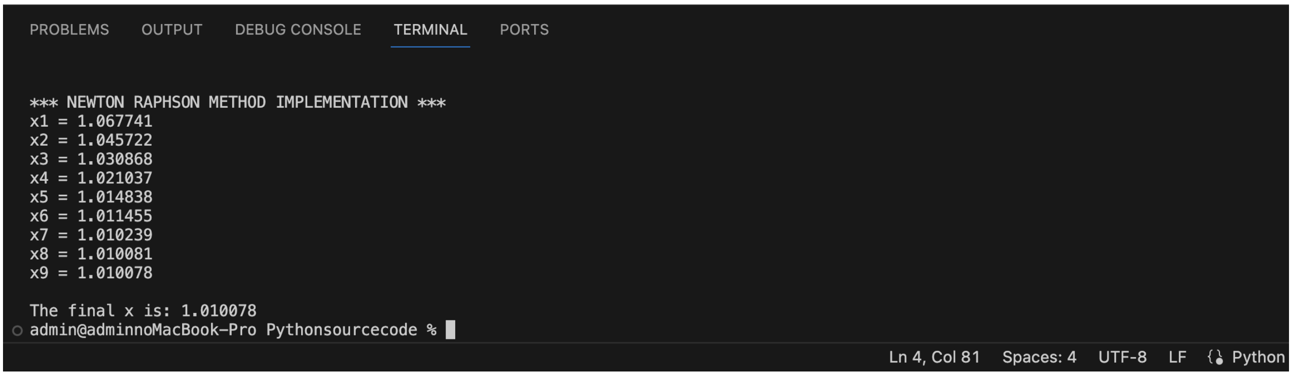
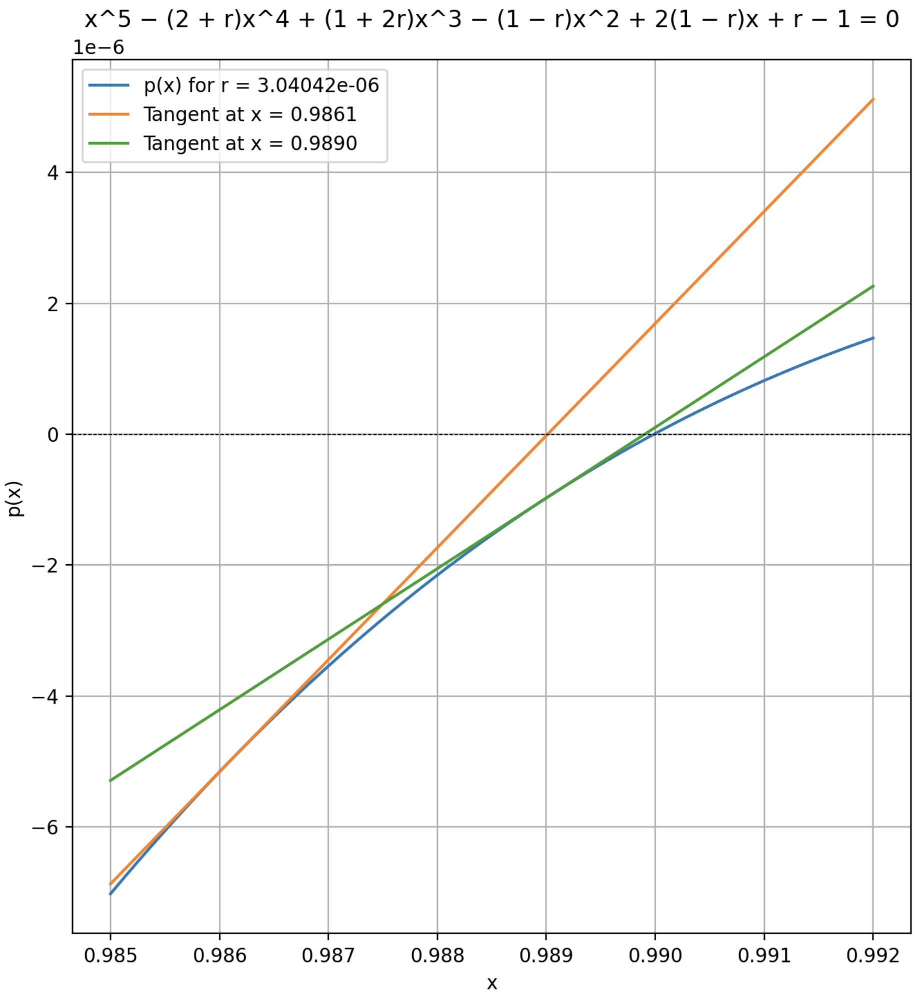


Figure 3: Output for L_2



```python
import numpy as np
import matplotlib.pyplot as plt

def p(x, r):
    return x**5 - (2 + r)*x**4 + (1 + 2*r)*x**3 - (1 + r)*x**2 + 2*(1 - r)*x + r - 1

def p_prime(x, r):
    return 5*x**4 - 4*(2 + r)*x**3 + 3*(1 + 2*r)*x**2 - 2*(1 + r)*x + 2*(1 - r)

r_value = 3.04042e-6

x_values = np.linspace(1.0095, 1.012, 400)

y_values = p(x_values, r_value)

tangent_points = [1.0114, 1.0102]

plt.plot(x_values, y_values, label=f'p(x) for r = {r_value}')

for x_tangent in tangent_points:
    y_tangent = p(x_tangent, r_value)
    slope_tangent = p_prime(x_tangent, r_value)

    tangent_line = lambda x: slope_tangent * (x - x_tangent) + y_tangent
    plt.plot(x_values, tangent_line(x_values), label=f'Tangent at x = {x_tangent:.4f}')

plt.axhline(0, color='black', linewidth=0.5, linestyle='--')
plt.title('x^5     (2 + r)x^4 + (1 + 2r)x^3     (1 + r)x^2 + 2(1     r)x + r     1 = 0')
plt.xlabel('x')
plt.ylabel('p(x)')
plt.legend()
plt.grid(True)
plt.show()
```

Listing 4: Code for L_2 Graph

4 Topic 3

4.1 Problem [1]

(An applied project in [1]) A movie theater has a screen that is positioned 10ft off the floor and is 25ft high. The first row of seats is placed 9ft from the screen and the rows are set 3ft apart. The floor of the seating area is inclined at an angle of $\alpha = 20°$ above the horizontal and the distance up the incline that you sit is x. The theater has 21 rows of seats, so $0 \leq x \leq 60$. Suppose you decide that the best place to sit is in the row where the angle θ subtended by the screen at your eyes is a maximum. Let's also suppose that your eyes are 4ft above the floor, as shown in the figure.

(a) Show that

$$\theta = arccos\left(\frac{a^2 + b^2 - 625}{2ab}\right)$$

where

$$a^2 = (9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2$$

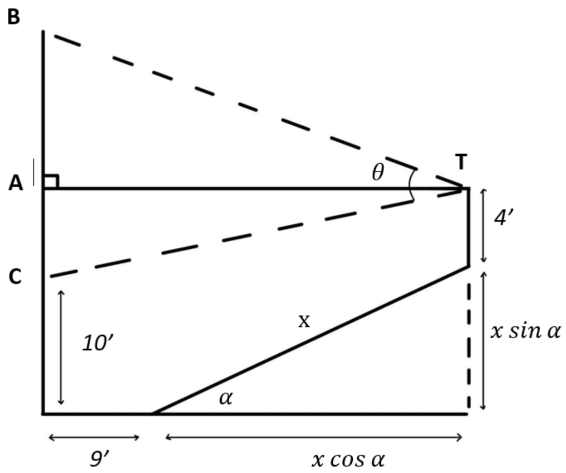
and

$$b^2 = (9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$$

(b) Use a graph of θ as a function of x to estimate the value of x that maximizes θ. In which row should you sit? What is the viewing angle θ in this row? (Use Matlab or Python).

(c) Use the graph of θ to estimate the average value of θ on the interval $0 \leq x \leq 60$. Use Matlab or Python to compute the average value. Compare with the maximum and minimum values of θ

4.2 Solution of question a [1]



$TA = 9 + xcos\alpha$

$AC = 4 + xsin\alpha – 10 = xsin\alpha – 6$

$AB = 35 – (4 + xsin\alpha) = 31 - xsin\alpha$

We have

$TB = \sqrt{TA^2 \ + \ AB^2} = \sqrt{(9 + xcos\alpha)^2 + (31 - xsin\alpha)^2} = a.$

$TC = \sqrt{TA^2 \ + \ AC^2} = \sqrt{(9 + xcos\alpha)^2 + (xsin\alpha \ - \ 6)^2} = b.$

Using the Law of Cosines on ΔTBC, we get:

$$BC^2 = TB^2 + TC^2 - 2TB \cdot TC \cdot cos\theta$$

$$\Leftrightarrow 25^2 = a^2 + b^2 - 2ab\cos\theta$$

$$\Leftrightarrow \theta = arccos\left(\frac{a^2 + b^2 - 625}{2ab}\right) \ (as\ required)$$

4.3 Python code for question b and c [1] [3]

```
import numpy as np
import matplotlib.pyplot as plt

def calculate_theta(x):
    a = np.sqrt(np.power(9 + x * np.cos(np.pi / 9), 2) + np.power(31 - x * np.sin(np.pi / 9), 2)) #a = sprt((9+x.cos(n/9))^2)
    b = np.sqrt(np.power(9 + x * np.cos(np.pi / 9), 2) + np.power(x * np.sin(np.pi / 9) - 6, 2))
    theta = np.arccos((np.power(a, 2) + np.power(b, 2) - 625) / (2 * a * b))
    return theta  # Return theta in radians

# Part (b): Graph of theta as a function of x
x_values = np.linspace(0, 60, 1000)
theta_values = calculate_theta(x_values)

plt.plot(x_values, np.degrees(theta_values))  # Convert theta to degrees for plotting
plt.title('Graph of theta as a function of x')
plt.xlabel('x')
plt.ylabel('theta (degrees)')
plt.axhline(0, color='black', linewidth = 1)  # Horizontal line at y=0
plt.axvline(0, color='black', linewidth = 1)  # Vertical line at x=0
plt.legend(loc = 'upper right', title = 'theta = arccos((a^2+b^2-625)/2ab)') #title for curve
plt.show()

# Estimate x that maximizes theta
max_theta_index = np.argmax(theta_values)
x_max_theta = x_values[max_theta_index]
max_theta = np.degrees(theta_values[max_theta_index])
print(f"The value of x that maximizes theta is approximately {x_max_theta:.2f}, => so that we should sit at 9th row")
print(f"In this row, theta is maximized and its value is approximately {max_theta:.2f} degrees.")

# Part (c): Estimate the average value of theta on the interval 0 <= x <= 60
average_theta = np.degrees(np.mean(theta_values))
print(f"The average value of theta on the interval 0 <= x <= 60 is approximately {average_theta:.2f} degrees.")
```

Listing 5: Python for question b and c

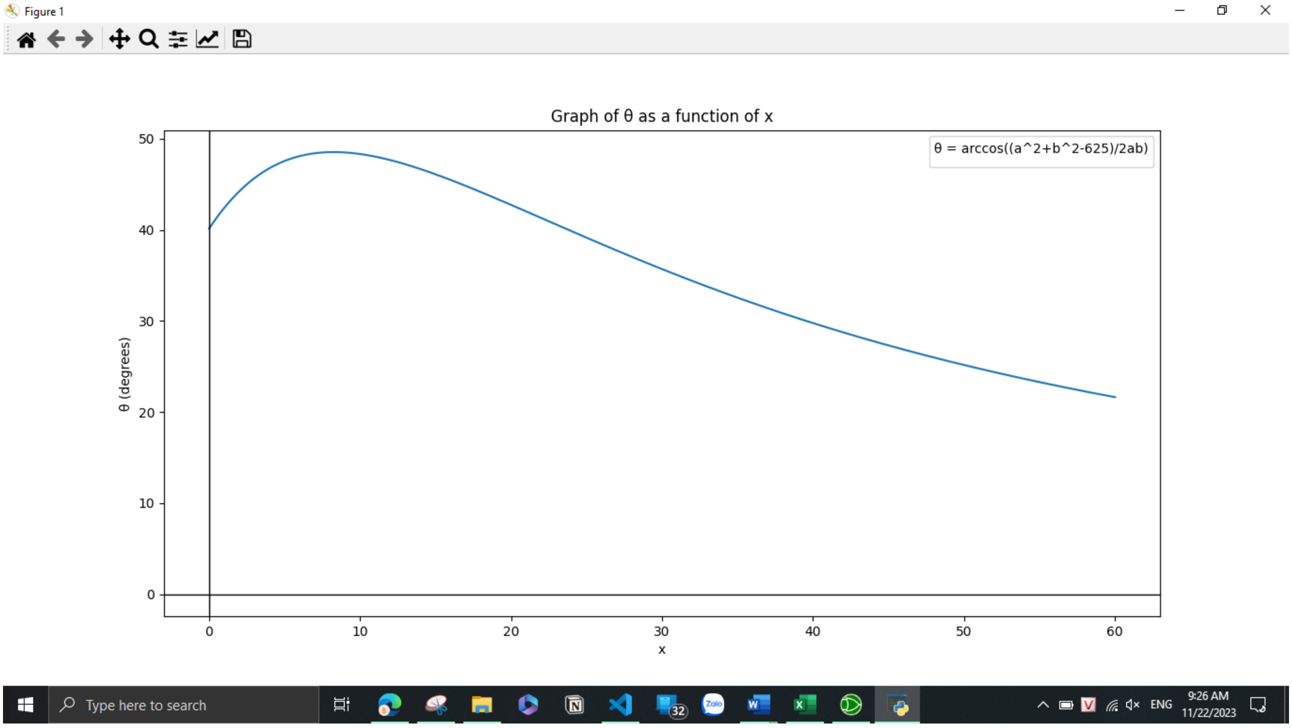
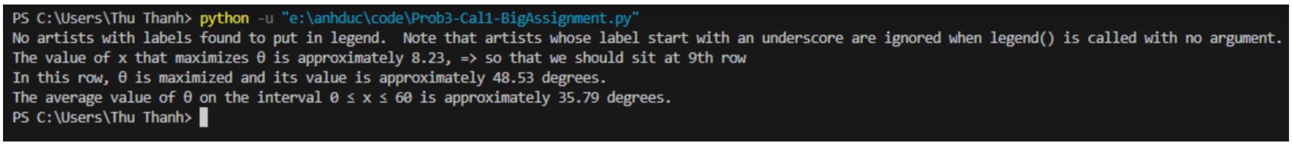


Figure 4: Graph of θ as a function of x



```
PS C:\Users\Thu Thanh> python -u "e:\anhduc\code\Prob3-Cal1-BigAssignment.py"
No artists with labels found to put in legend.  Note that artists whose label start with an underscore are ignored when legend() is called with no argument.
The value of x that maximizes θ is approximately 8.23, => so that we should sit at 9th row
In this row, θ is maximized and its value is approximately 48.53 degrees.
The average value of θ on the interval 0 ≤ x ≤ 60 is approximately 35.79 degrees.
PS C:\Users\Thu Thanh>
```

Figure 5: Output from terminal

Suppose that the first row is at x = 0.

Therefore, the best place to sit is in the fourth row with x = 9ft (closest to 8.23ft) and the viewing angle θ about 0.85 radian or 49 degrees.

5 Topic 4

5.1 Problem [1]

If water (or other liquid) drains from a tank, we expect that the flow will be greatest at first (when the water depth is greatest) and will gradually decrease as the water level decreases. But we need a more precise mathematical description of how the flow decreases in order to answer the kinds of questions that engineers ask: How long does it take for a tank to drain completely? How much water should a tank hold in order to guarantee a certain minimum water pressure for a sprinkler system?
Let h(t) and V (t) be the height and volume of water in a tank at time t. If water drains through a hole with area a at the bottom of the tank, then Torricelli's Law says that:

$$\frac{dV}{dt} = -a\sqrt{2gh}$$

Where g is the acceleration due to gravity. So the rate at which water flows from the tank is proportional to the square root of the water height.

- Suppose the tank is cylindrical with height 6ft and radius 2ft and the hole is circular with radius 1 inch. If we take g = 32ft/s^2, show that h satisfies the differential equation: $\frac{dh}{dt} = -\frac{1}{72}\sqrt{h}$

- Solve this equation to find the height of the water at time t, assuming the tank is full at time t = 0.

- How long will it take for the water to drain completely?

5.2 Solution for the 1st problem

$1\,inch = \frac{1}{12}ft$

Calculate the area of the hole at the bottom: $a = \pi.r^2 = \pi.\left(\frac{1}{12}\right)^2 = \frac{1}{144}\pi$

Volume of the cylinder: $V = h.\pi.R^2$

The derivative of the volume with respect to time following the formula:

$$\frac{dV}{dt} = \frac{dV}{dh}.\frac{dh}{dt}$$

$$\iff -a\sqrt{2gh} = \pi.R^2.\frac{dh}{dt} \qquad \text{(Applying Torricelli's Law)}$$

$$\iff \frac{dh}{dt} = \frac{-a\sqrt{2gh}}{\pi.R^2} = \frac{-\frac{\pi}{144}\sqrt{2.32 \times h}}{\pi.6^2} = -\frac{1}{72}\sqrt{h} \qquad \text{(Requirement)}$$

5.3 Solution for the 2nd problem

As obtained: $\dfrac{dh}{dt} = -\dfrac{1}{72}\sqrt{h}$

$$\iff h'(t) = -\frac{1}{72}\sqrt{h(t)} \qquad \text{(Rewrite)}$$

$$\iff \frac{h'(t)}{2\sqrt{h(t)}} = -\frac{1}{144}$$

$$\iff \int_0^t \frac{h'(t)}{2\sqrt{h(t)}}dt = -\int_0^t \frac{1}{144}dt \qquad \text{(Taking integral on both sides)}$$

$$\iff \sqrt{h_t} - \sqrt{h_o} = -\frac{1}{144}(t-0) \qquad (h_o\text{: height at the beginning; } h_t\text{: height at the time t})$$

$$\iff \sqrt{h_t} = \sqrt{6} - \frac{t}{144} \qquad (\text{Condition: } 0 \leq \text{t} \leq 144\sqrt{6})$$

$$\iff h_t = \left(\sqrt{6} - \frac{t}{144}\right)^2 \qquad (h_t \text{ expression with respect to time})$$

As the time t goes up in value, h_t decreases until it falls down to zero, this proved to be suitable with the theory considering the water in the cylinder is draining. Therefore, the expression is valid.

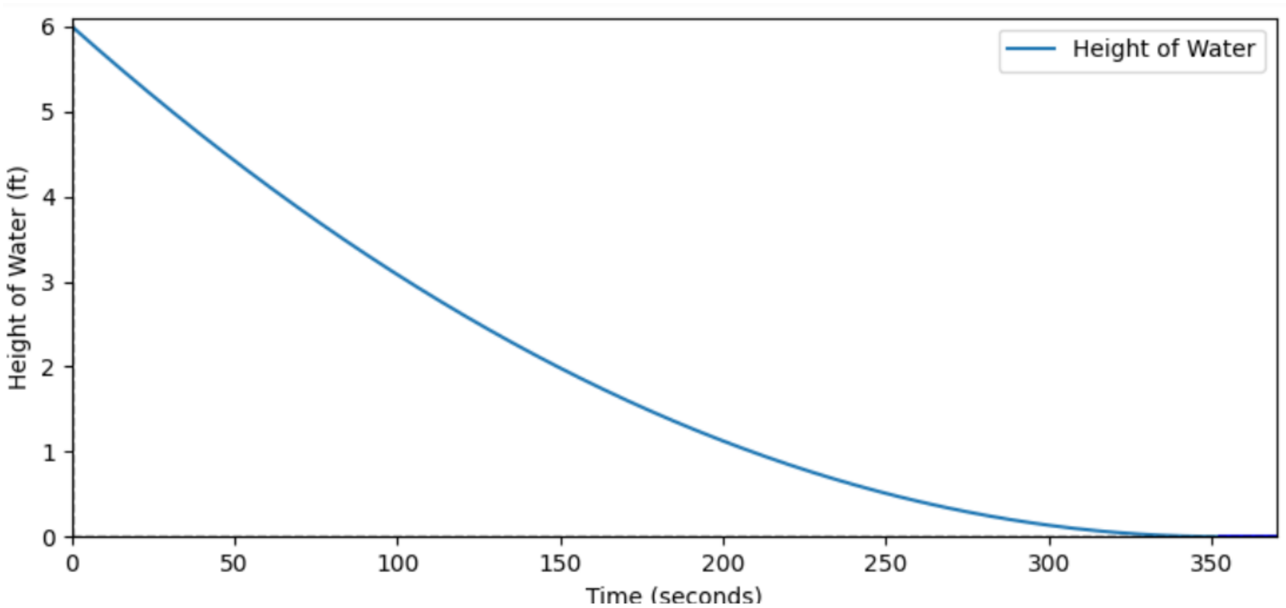


Figure 6: Graph of water's height with respect to time

5.4 Solution for the 3rd problem

For the water to drain completely, $h_t = 0$:

$$\iff \sqrt{6} - \frac{t}{144} = 0$$

$$\iff t = 144\sqrt{6} = 352.73 \qquad \text{(s)}$$

Conclusively, it takes at least 352.73 seconds, almost 6 minutes, to empty the cylinder. If t $>$ 352.73 seconds, h_t remains 0:

$h_t = 0 \qquad (t > 352.73)$

6 Bibliography

References

[1] James Stewart Calculus 8th edition

[2] https://www.codesansar.com/numerical-methods/newton-raphson-method-python-program.htm

[3] https://matplotlib.org/stable/tutorials/pyplot.htmlsphx-glr-tutorials-pyplot-py