Project Report

1. Introduction to Libor Market Model

The Libor Market Model (LMM) is an interest rate model based on evolving Libor market forward rates. In contrast to models that evolve the instantaneous short rate (i.e. Hull-White, Black-Derman-Toy models) or instantaneous forward rates (i.e. Heath-Jarrow-Morton model), which are not directly observable in the market, the objects modeled using LMM are market-observable quantities (LIBOR forward rates). This makes LMM popular with market practitioners. Another feature that makes the LMM popular is that it is consistent with the market standard approach for pricing caps using Black's formula.

The LMM can be used to price any financial product whose pay-off can be decomposed into a set of forward rates (i.e. caplets, Bermudan swaptions, and other complicated interest rate derivatives). It assumes that the evolution of each forward rate is lognormal. Each forward rate has a time dependent volatility and time dependent correlations with the other forward rates. After specifying these volatilities and correlations, the forward rates can be evolved using Monte Carlo simulation. Expectations of cash flows are then calculated in order to determine the fair value of an instrument.

Why is LMM so popular? There are two main reasons. The first reason lies in the agreement between the LMM model and well-established market formulas for two basic derivative products. Indeed, the lognormal forward rate model prices caps with Blacks cap formula, which is the standard formula employed in the cap market. Meanwhile, the lognormal forward swap rate model prices swaptions with Black swaption formula, which again is the standard formula employed in the swaption market. However, as we know, if forward rates are lognormal then swap rates cannot be lognormal. Since the caps and swaptions markets are the two main markets in the interest-rate-options world, it is important for a model to be compatible with such market formulas.

The second reason is that Libor Market Model can capture the term structure of cap volatilities better than short rate models. In real world, the cap volatilities have a hump shape. The volatilities always increase and reach the peak in 1-3 years and then decrease for long maturities. Figure 1.1 shows the volatility curves generated by different models. Applying specific instantaneous volatility functions in LMM, we can generate volatility structures that closely match the volatility structures of current market caps, whereas using HW or BDT models we can only get monotonically decreasing volatility curves.

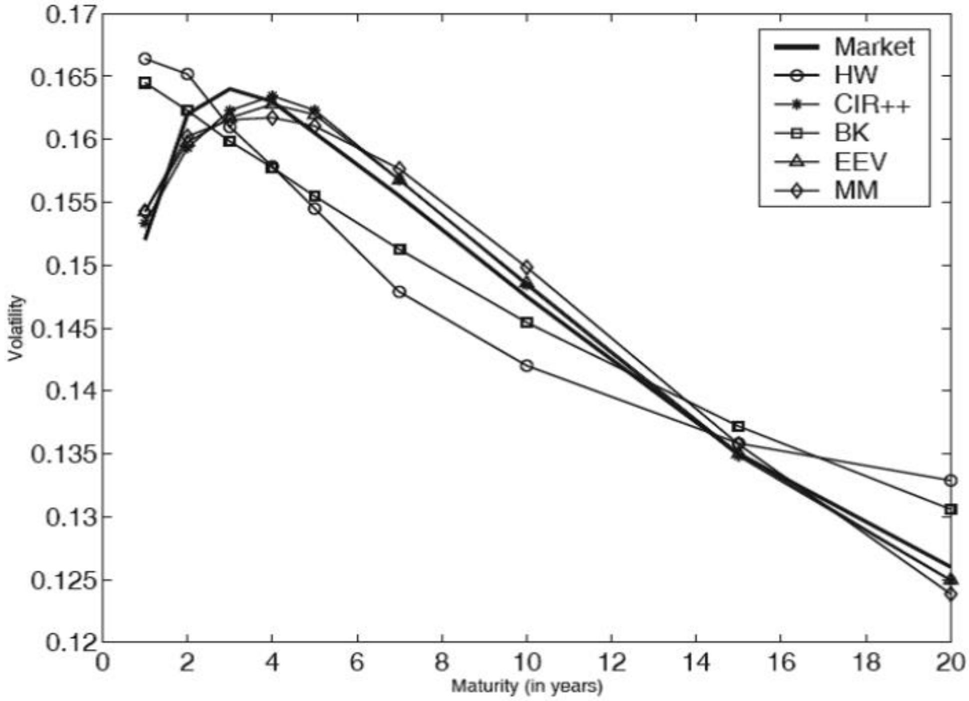

Figure 1.1 Volatility Curves Generated by Different Models

Also, LMM can generate time homogeneous volatilities curves in that regardless of when we start from the volatility curves have the same shape-just depending upon time to maturity. This pattern is exactly what we observe in market. However, if we calibrate short rate models to existing cap or caplet volatilities then we typically see time heterogeneity. The hump shape appears for only current cap prices or maybe even those starting in one year from now, but they rarely capture the full time homogeneity.

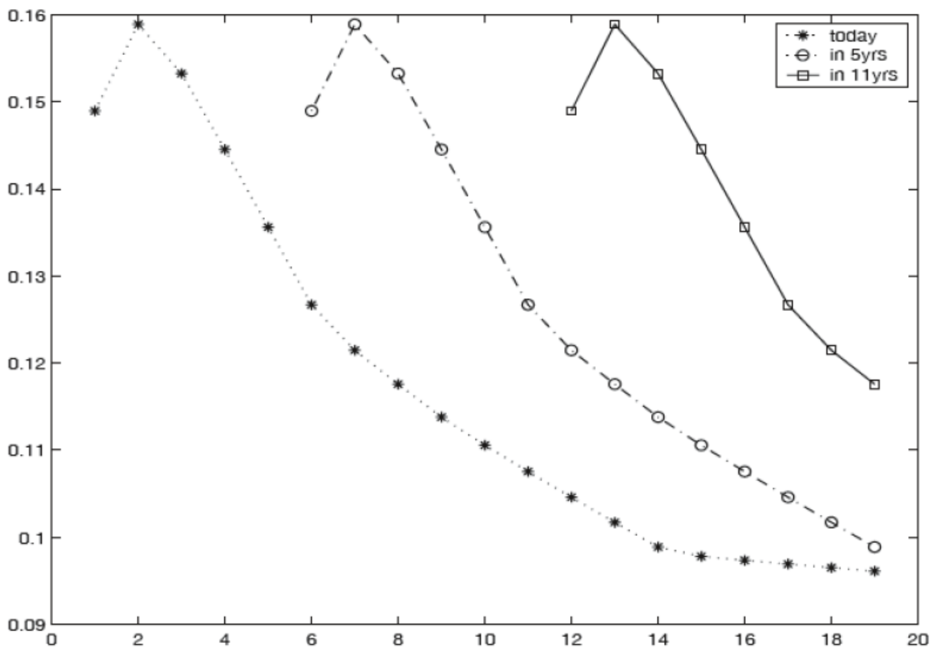

Figure 1.2 Time Homogenous Volatilities (observed from market)

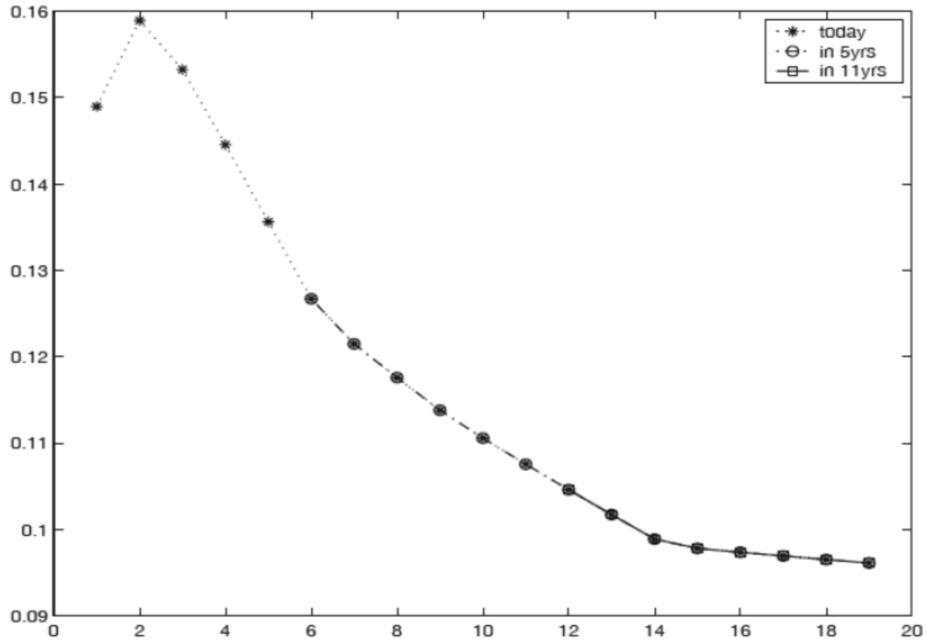

Figure 1.3 Time Heterogeneous Volatilities (from short rate models)

This report is organized as follows. We describe the data we need to value the caplets, swaptions, and the third product in Part 2. In Part 3, we discuss the details of calibration of Black-Derman-Toy model (BDT) and the valuation procedure. In Part 4, we will talk about how we implement LMM and the valuation results. Finally, we will compare the valuation results from BDT and LMM and conclude in Part 5.

2. Data Collection

The product we choose is the ***Callable Step-Up Notes issued by UBS***. The key dates for the product are shown in Table 2.1. The maturity of this product is 3.5 years, and we collect 4-year data for this data because 4-year data would give us more Black swaption volatilities, which are necessary when calibrating Libor Market Model.

Table 2.1 Key Dates of The Product

Pricing Date	10/25/2019
Issue Date	10/29/2019
Maturity Date	4/29/2023

The product pays semi-annual interests starting on April 29, 2020 and ending on the maturity date, subject to business day convention. There are 7 interest payment dates in total. The quarterly redemption dates start from January 29, 2020 and end on January 29, 2023, subject to business day convention. There are 13 redemption dates in total. Obviously, some early redemption dates are on interest payment dates and some are not. To value caplets, European swaptions, Bermudan Swaption and this product, we need to collect forward rates, cap volatilities and swaption volatilities. The details of these data are as follows:

Forward Rates: three-month forward rates from October 29, 2019 to October 30, 2023 that are reset every three months.

Table 2.2 Forward Rates (from Bloomberg)

Date	T	τ	P (0, T)	Forward Rate
10/29/2019	0		1	1.9281300%
01/29/2020	0.255556	0.255556	0.995096717	1.8227298%
04/29/2020	0.508333	0.252778	0.990532881	1.6450148%
07/29/2020	0.761111	0.252778	0.986431072	1.5784890%
10/29/2020	1.016667	0.255556	0.982467879	1.5517885%
01/29/2021	1.272222	0.255556	0.978587113	1.5313470%
04/29/2021	1.522222	0.250000	0.974855010	1.4778073%
07/29/2021	1.775000	0.252778	0.971226926	1.4298691%
10/29/2021	2.030556	0.255556	0.967690877	1.5289503%
01/31/2022	2.291667	0.261111	0.963842966	1.5092042%
04/29/2022	2.536111	0.244444	0.960300259	1.4902652%
07/29/2022	2.788889	0.252778	0.956696327	1.4708837%
10/31/2022	3.050000	0.261111	0.953036058	1.5273430%
01/30/2023	3.302778	0.252778	0.949370743	1.5196126%
04/28/2023	3.547222	0.244444	0.945857254	1.5121288%
07/31/2023	3.808333	0.261111	0.942137378	1.5049115%
10/30/2023	4.061111	0.252778	0.938566993	1.5633871%

Cap Volatilities: Black cap volatilities set on October 29, 2019 with maturities equal to and less than 4 years.

Table 2.3 Black Cap Volatilities (from Bloomberg)

Expiry	Cap Vols
1	33.38%
2	41.25%
3	44.31%
4	45.92%

Swaption Volatilities: Black swaption volatilities set on October 29, 2019 with the sum of maturities and swap tenors equal to and less than 4 years.

Table 2.4 Black Swaption Volatilities (from Bloomberg)

Expiry \ Tenor	1Yr	2Yr	3Yr
6Mo	38.70	47.40	48.64
9Mo	45.55	46.24	46.80

1Yr	44.32	44.96	46.83
2Yr	45.59	45.62	-
3Yr	45.10	-	-

3. Black-Derman-Toy Model Implementation

3.1. Caplet Stripping

Before constructing the short rates tree, we need to use ‘strip’ algorithm first to get all the caplet volatilities. We first determine the forward swap rate for each cap. Then we set the volatilities of caps maturing in 1-4 years equal to the corresponding market cap volatilities, and set first two caps volatility equal to the market cap volatility maturing in one year. Then we use linear interpolation for the other gaps in the market caps volatilities. Then we obtain Black caplet and cap prices. To set Black caplet prices equal to market caplet prices one by one, we can back out the implied volatilities for each caplet. The result is shown in Table 3.1.

Table 3.1 Implied Volatilities

Date	Market Cap Vols	Implied Vols
2019/10/29		
2020/01/29		
2020/04/29	0.33380000	0.333800000
2020/07/29	0.33380000	0.333800000
2020/10/29	0.33380000	0.333800000
2021/01/29	0.35363671	0.395311514
2021/04/29	0.37304219	0.428180472
2021/07/29	0.39266329	0.462401366
2021/10/29	0.41250000	0.497097910
2022/01/31	0.42033760	0.456726394
2022/04/29	0.42767493	0.469986683
2022/07/29	0.43526240	0.483263041
2022/10/31	0.44310000	0.497093017
2023/01/30	0.44712500	0.478346145
2023/04/28	0.45101731	0.485360633
2023/07/31	0.45517500	0.49260913
2023/10/30	0.45920000	0.499962069

3.2. Tree Construction

We decide to use monthly time steps and each time step in the tree is different since we use actual/360 convention.

First we use log linear interpolation to get all of monthly zero-coupon-bond prices. Next, we construct trees for $r_{i,j}$ and $\pi_{i,j}$.

Specifically,

$$r_{0,0} = -\frac{\ln(\hat{P}_1)}{\Delta t_1}$$

$$r_{i,j} = U_i * exp\,((2 * j - i) * \sigma_{(i)} * \sqrt{\Delta t_i})$$

$$\pi_{0,0} = 1$$

$$\pi_{i,j} = \begin{cases} 0.5 * \pi_{i-1,j} * \exp(-r_{i-1,j} * \Delta t_i) + 0.5 * \pi_{i-1,j-1} * \exp(-r_{i-1,j-1} * \Delta t_i) & j < i \\ 0.5 * \pi_{i-1,j-1} * \exp(-r_{i-1,j-1} * \Delta t_i) & j = i \end{cases}$$

Set the values of $\sigma_{(i)}$ piecewise constant according to implied volatilities from the previous stripping procedure.

$$\sigma_{(1)} = \cdots = \sigma_{(5)} = \sigma_1$$

$$\sigma_{(6)} = \cdots = \sigma_{(8)} = \sigma_2$$

$$\sigma_{(9)} = \cdots = \sigma_{(11)} = \sigma_3$$

...

The intuition of keeping short rate volatilities equal to Black volatilities is from Kazziha and Rebonato (1998). They show that in BDT model, we have:

$$var[lnr(T)] = \sigma(T)^2 T$$

This remarkable result implies that despite the process being mean reverting the unconditional variance of the short rate is simply the instantaneous variance at time T. It does not depend upon the previous short rate values or the previous instantaneous volatility parameters.

Next, we calculate P_i as

$$P_i = \sum_j \pi_{i,j}$$

Then we use Solver to back out U_i one by one by equating $\hat{P}_i = P_i$.

3.3. Valuation with BDT

3.3.1. Caplets

Now we can use the short rate tree to value the caplets for the calibration of volatilities. Due to the monthly time steps, we need to construct zero-coupon-bond trees for every three-month period to calculate the value of caplets.

$$P_{i,j} = \begin{cases} 1 & maturity \\ 0.5 * (P_{i+1,j+1} + P_{i+1,j}) * exp(-r_{i,j} * \Delta t_{i+1}) & otherwise \end{cases}$$

Then we back out three-month Libor based on these three-month zero-coupon-bond prices.

$$L_{i,j} = (\frac{1}{P_{i,j}} - 1)/(T_{i+3} - T_i)$$

The value of caplet is

$$V_{i,j} = \sum \pi_{i,j} * (T_{i+3} - T_i) * P_{i,j} * \max(L_{i,j} - K, 0)$$

K is the exercise rate of the caplet. Then we back out the $\sigma_{(i)}$ as piecewise constant.

We can see that the changes of zero-coupon-bond prices are relatively small. The summary of this procedure is shown in Table 3.2.

Table 3.2 Caps and Caplets Summary

Market value	Tree value	Diff	New Vols	Original Vols	Change in Vols
0.00030687	0.00030687	3.072E-12	0.315180899	0.333800000	-5.58%
0.000297731	0.000297731	5.435E-12	0.324927903	0.333800000	-2.66%
0.000355021	0.000355947	9.261E-07	0.339098896	0.333800000	1.59%
0.000517475	0.000517468	6.637E-09	0.391631996	0.395311514	-0.93%
0.000624044	0.00062494	8.96E-07	0.430383259	0.428180472	0.51%
0.000704806	0.000705748	9.424E-07	0.46317528	0.462401366	0.17%
0.000792573	0.000792572	8.271E-10	0.49425941	0.497097910	-0.57%
0.000942901	0.000943738	8.364E-07	0.458241612	0.456726394	0.33%
0.000938939	0.000939768	8.297E-07	0.471402517	0.469986683	0.30%

0.001024651	0.001024027	6.241E-07	0.483263041	0.483263041	0.00%
0.00111421	0.001113332	8.779E-07	0.497093017	0.497093017	0.00%
0.001170584	0.00117126	6.758E-07	0.480047219	0.478346145	0.36%
0.001177814	0.001177814	5.107E-11	0.488483454	0.485360633	0.64%
0.00130472	0.00130472	1.618E-11	0.494225561	0.492609130	0.33%
0.001310272	0.001310272	3.241E-11	0.502113117	0.499962069	0.43%

To check how much the zero-coupon-bond prices have changed, we show as the following.

Table 3.3 Market P(0,T) vs New P(0,T)

Date	P(0,T)	New P(0,T)	Change in P(0,T)
2019/10/29	1	1	0.00000%
2020/01/29	0.995097	0.995099219	0.00025%
2020/04/29	0.990533	0.990544568	0.00118%
2020/07/29	0.986431	0.986449644	0.00188%
2020/10/29	0.982468	0.982480392	0.00127%
2021/01/29	0.978587	0.97860554	0.00188%
2021/04/29	0.974855	0.974868649	0.00140%
2021/07/29	0.971227	0.971238388	0.00118%
2021/10/29	0.967691	0.967711278	0.00211%
2022/01/31	0.963843	0.963857704	0.00153%
2022/04/29	0.9603	0.960309549	0.00097%
2022/07/29	0.956696	0.956705543	0.00096%
2022/10/31	0.953036	0.953045199	0.00096%
2023/01/30	0.949371	0.949370937	0.00002%
2023/04/28	0.945857	0.945840614	-0.00176%
2023/07/31	0.942137	0.942127174	-0.00108%
2023/10/30	0.938567	0.938542848	-0.00257%

We can see that zero-coupon-bond prices have not changed too much. So, we don't need to calibrate U_is one more time.

3.3.2. European Swaptions

After calibration to both zero-coupon-bond and caplet prices, we can use short rate tree to value the European swaptions and check how BDT performs in valuation. All the swaptions we choose for this project receive ***quarterly floating rate payments*** and pay ***quarterly fixed rate payments***. The principles of these swaptions are $1.

We take the first one as an example here and the algorithm is as following. K is the swap rate.

At $T = 2$ and $i = 24$,

$$V_{i,j} = 1 + K * (T_i - T_{i-3})$$

On coupon payment dates, $i = 21, 18, 15, 12$,

$$V_{i,j} = exp\left(-r_{i,j}\Delta t_{i+1}\right) * 0.5 * \left(V_{i+1,j+1} + V_{i+1,j}\right) + K * (T_i - T_{i-3})$$

On other dates,

$$V_{i,j} = exp\left(-r_{i,j}\Delta t_{i+1}\right) * 0.5 * \left(V_{i+1,j+1} + V_{i+1,j}\right)$$

Table 3.4 BDT European Swaption Values vs Black Values

Maturity x Tenor	Black Values	BDT Values	Relative Errors
1 x 1	0.002612665	0.002647372	1.328414%
1 x 2	0.005277603	0.005465775	3.565482%
1 x 3	0.008193645	0.008306282	1.374687%
2 x 1	0.003732364	0.003897377	4.421139%
2 x 2	0.007422131	0.007836156	5.578250%
3 x 1	0.00443166	0.004775264	7.753393%
0.5 x 1	0.001720309	0.00164314	-4.485764%
0.5 x 2	0.004056727	0.003594602	-11.391573%
0.5 x 3	0.006150022	0.005585559	-9.178227%
0.75 x 1	0.002394608	0.002155141	-10.000259%
0.75 x 2	0.004753126	0.004551649	-4.238831%

* $relative\ error = (BDT\ value - Black\ value)/Black\ value$

We can see that for European swaptions maturing less than one year, the error between market Black value and tree value is large since the trading volume of these swaptions are typically low, resulting in the unprecise market volatilities that cannot capture the true volatility of those swaptions. For European swaptions with longer maturities, the error is also large, we think it is due to the volatility structure. In the Black formula, we impose a constant volatility to the caps and swaptions, but these volatilities are in fact time-varying. So for European swaptions with longer maturities, the prices are more sensitive to the volatility structure. However, for European swaptions with maturities 1 or 2 year, the error is very small and BDT tree performs pretty well.

3.3.3. UBS Product

To value the product, we construct the tree according to the algorithm as following.

At maturity, $T = 3.5$, $i = 42$ and $k = 14$,

$$V_{i,j} = 1000 + CPN_k$$

On just Redemption dates, $i = 39, 33, 27, \dots, 3$ and $k = \frac{i}{3}$,

$$V_{i,j} = min(ex\,p\big(-r_{i,j}\Delta t_{i+1}\big) * 0.5 * (V_{i+1,j+1} + V_{i+1,j}), 1000 + CPN_k)$$

On both Redemption and Interest Payment Dates, $i = 36, 30, 24, \dots, 6$ and $k = \frac{i}{3}$,

$$V_{i,j} = min(ex\,p\big(-r_{i,j}\Delta t_{i+1}\big) * 0.5 * (V_{i+1,j+1} + V_{i+1,j}), 1000) + CPN_k$$

On other dates,

$$V_{i,j} = ex\,p\big(-r_{i,j}\Delta t_{i+1}\big) * 0.5 * (V_{i+1,j+1} + V_{i+1,j})$$

And the list of coupons is as following, $k = \frac{i}{3}$, and dates are subject to 30/360 convention.

$$\begin{aligned}
CPN_k &= 1000 * 0.025 * (T_i - T_{i-6}) \quad && if\ i = 42, 36 \\
CPN_k &= 1000 * 0.025 * (T_i - T_{i-6}) \quad && if\ i = 39, 33 \\
CPN_k &= 1000 * 0.022 * (T_i - T_{i-6}) \quad && if\ i = 30, 24, \dots, 6 \\
CPN_k &= 1000 * 0.022 * (T_i - T_{i-6}) \quad && if\ i = 27, 21, \dots, 3
\end{aligned}$$

Then finally the value of the product is just $V_{0,0}$. The value we get from the BDT tree is $997.95845. We will discuss the value later.

3.3.4. Bermudan Swaption

We choose to value a Bermudan swaption with exercise rate 1.5% and allows the holder to enter a new 2-year swap at year 1 and year 2. The principle is $1. All the payments are quarterly. The algorithm is as following.

The first step is to build two coupon bond trees maturing at year 4 and year 3. $K = 1.5\%$.

At maturity,

$$V_{i,j} = 1 + K * (T_i - T_{i-3})$$

On coupon payment dates,

$$V_{i,j} = ex\,p\big(-r_{i,j}\Delta t_{i+1}\big) * 0.5 * \big(V_{i+1,j+1} + V_{i+1,j}\big) + K * (T_i - T_{i-3})$$

On other dates,

$$V_{i,j} = ex\,p\big(-r_{i,j}\Delta t_{i+1}\big) * 0.5 * \big(V_{i+1,j+1} + V_{i+1,j}\big)$$

Then we build a tree for Bermudan swaption.

At $T = 2$,

$$BS_{i,j} = max(0, 1 - (V_{i,j} - K * (T_i - T_{i-3})))$$

At other dates between $T = 1$ and $T = 2$,

$$BS_{i,j} = exp\left(-r_{i,j}\Delta t_{i+1}\right) * 0.5 * \left(BS_{i+1,j+1} + BS_{i+1,j}\right)$$

At $T = 1$,

$$BS_{i,j} = max(exp\left(-r_{i,j}\Delta t_{i+1}\right) * 0.5 * \left(BS_{i+1,j+1} + BS_{i+1,j}\right), 1 - (V_{i,j} - K * (T_i - T_{i-3})))$$

Finally, we calculate

$$BS = \sum_j \pi_{i,j} * BS_{i,j}$$

The value for this Bermudan swaption is \$0.007898239.

4. Libor Market Model Implementation

4.1. Determine Instantaneous Volatilities and Correlation Matrix

4.1.1. Determine Instantaneous Forward Rate Volatilities

To determine the time-dependent instantaneous forward rate volatilities, we decide to use a restricted function from which we can calculate the volatilities of all forward rates. The function we choose is Function 7:

$$\sigma_k(t) = \phi_k[a + b(T_{k-1} - t)]e^{-c(T_{k-1}-t)} + d$$

where T_{k-1} is the reset date of the *k*th forward rate

The reason is that this linear-exponential parametric form is a time-homogeneous form, which means that the instantaneous volatility of a forward rate is not a function of the current date but only a function of the time until the rate is set. The parameter ϕ_k allows for the exact fitting of specific forward rate volatilities to Black caplet volatilities.

This parametric form is able to produce both a monotonically decreasing and “humped” structure of the instantaneous volatility, so we have to make some restrictions on the values of *a*, *b*, *c* and *d* to ensure we can get the hump when calibrate volatilities. To achieve this, we should interpret parameters a, b, c and d in financial terms first. For very long maturities (i.e. when $T_{k-1} - t$ is large), the instantaneous volatility approaches d. Therefore we can interpret d as the volatility of forward rates very far in the future. Generally, this parameter should have a positive value. The parameter c can be interpreted as the rate at which the volatility approaches the limit d. For forward rates with very short maturities (i.e. when $T_{k-1} - t$ approaches zero), we can see the instantaneous volatility approaches $a+d$. This leads us to conclude that this quantity should also be greater than zero. We can gain some insight into the location of the hump in the volatility structure by evaluating $\sigma_k(t)' = 0$. From this we find that the hump is located at $T_{k-1} - t = \frac{b-ca}{bc}$. Therefore, a hump will only be present for $b > 0$. The initial positive slope allows us to infer the further restriction that $a < b/c$ in this case.

After determining the parametric function and the restrictions for parameters, we calibrate the volatilities. Our goal here is to find ϕ_k's, a, b, c and d that can satisfy the following equations:

$$\sigma_{Black,k}^2 = \frac{\phi_k^2}{T_{k-1}} \int_t^{T_{k-1}} \left([a + b(T_{k-1} - t)]e^{-c(T_{k-1}-t)} + d\right)^2 dt$$

where $k = 1,2,3, \dots ,15$ (since there are 15 forward rates from t=0 to t=4)
T_{k-1} is the reset date of the kth forward rate
t=0 (since we only have the Black volatilities at t=0)
$\sigma_{Black,k}$ are the Black caplet volatilities we get from 3.1. Caplet Stripping

Finally, we get parameter values shown in Table 4.1.

Table 4.1 Parameter Values for Function 7

a	-0.5055523	ϕ_1	1.6991961
b	0.552561	ϕ_2	1.094371
c	2.166913	ϕ_3	0.9015544
d	0.515997	ϕ_4	0.9622669
		ϕ_5	0.9810567
		ϕ_5	1.0209754
		ϕ_7	1.0707941
		ϕ_8	0.9670482
		ϕ_9	0.9826958
		ϕ_{10}	1.0016035
		ϕ_{11}	1.0230051
		ϕ_{12}	0.9787271
		ϕ_{13}	0.9885418
		ϕ_{14}	0.9995903
		ϕ_{15}	1.0110996

4.1.2. Determine Forward Rate Correlations

Similar to what we do to determine instantaneous volatilities, we should also choose a parametric function for the correlations. In general, we would like our instantaneous correlations to exhibit three features:

1. $\rho_{i,j}$ should be positive for any i, j.

2. When moving away from a “1” diagonal entry along either a column or a row, we should have a monotonically decreasing pattern. This is because joint movements of far away rates are less correlated than those of close rates.

3. When we move along the zero coupon curve, the larger the tenor the more correlated changes in the adjacent ranges become, so sub-diagonals should be increasing or the map $i \to \rho_{i+p,i}$ will be increasing for a fixed ρ.

The correlation function we choose is:

$$\rho_{i,j} = \exp[-\frac{|i-j|}{M-1}\left(-\ln\alpha + \eta\frac{i^2+j^2+ij-3Mi-3Mj+3i+3j+2M^2-M-4}{(M-2)(M-3)}\right)]$$

where $\alpha > 0,\ 0 \le \eta \le -ln\alpha$

M is the number of the forward rates and it is 15 in this project

This exponential parametric form is not time dependent (i.e., it does not depend on the current date nor the times until the two forward rates are set, but only on the difference between the two times). Correlations generated by this function is positive since it is a exponential form. Meanwhile, when $i = j$, the correlation is 1 as required. When $|i-j|$ is larger, the correlation is smaller. One more advantage of this function is that we can generate positive semi-definite matrix with it. Therefore, we decide to use it.

It should be noted that the above specification of the instantaneous correlation between forward rates does not entirely describe the “terminal correlation” between the rates. The terminal correlation is determined by a combination of both the instantaneous volatility and correlation structures.

To calibrate correlations, we apply Rebonato’s Formula:

$$\sigma^2_{Black}(T_0)T_0 \approx \sum_{i,j=1}^{n} \frac{w_i(0)w_j(0)F_i(0)F_j(0)}{S^2_{T_0,T_n}(0)} \int_0^{T_0} \rho_{i,j}\sigma_i(t)\sigma_j(t)dt$$

where T_0 and T_n are the exercise time and maturity of the European swaption

$\sigma_{Black}(T_0)$ is the swaption volatility of the European swaption

$S_{T_0,T_n}(0)$ is the at the money swap rate of the swaption

$$w_i(0) = \frac{\tau_i \prod_{k=0}^{i} \frac{1}{1+\tau_k F_k(0)}}{\sum_{p=1}^{n} \tau_p \prod_{k=0}^{p} \frac{1}{1+\tau_k F_k(0)}}$$

This formula assumes that the weights and forward rates do not change through time, which contradicts with real world. Although we expect this approximation to be poor, it performs pretty well since the LMM valuation results are nice. So we finally decide to use it without adjustments.

We choose eleven Black swaption volatilities (exactly the eleven European swaptions we value in 3.3.2) to do the calibration. The parameter values for the correlation function are:

$$\alpha = 0.2365966$$
$$\eta = 8.534145e - 09$$

Then we use the parametric function to generate the correlation matrix (shown in Table 4.2). As we can see, this is a positive semi-definite that satisfy all the three features we mentioned above (although the feature 3 seems not to be satisfied in the table, sub-diagonals are actually increasing if we look at the 6th digit after the decimal, which we do not show due to limited room).

Table 4.2 Correlation Matrix

j \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976	0.5392	0.4864	0.4388	0.3959	0.3572	0.3222	0.2907	0.2623	0.2366
2	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976	0.5392	0.4864	0.4388	0.3959	0.3572	0.3222	0.2907	0.2623
3	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976	0.5392	0.4864	0.4388	0.3959	0.3572	0.3222	0.2907
4	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976	0.5392	0.4864	0.4388	0.3959	0.3572	0.3222
5	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976	0.5392	0.4864	0.4388	0.3959	0.3572
6	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976	0.5392	0.4864	0.4388	0.3959
7	0.5392	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976	0.5392	0.4864	0.4388
8	0.4864	0.5392	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976	0.5392	0.4864
9	0.4388	0.4864	0.5392	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976	0.5392
10	0.3959	0.4388	0.4864	0.5392	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624	0.5976
11	0.3572	0.3959	0.4388	0.4864	0.5392	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343	0.6624
12	0.3222	0.3572	0.3959	0.4388	0.4864	0.5392	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139	0.7343
13	0.2907	0.3222	0.3572	0.3959	0.4388	0.4864	0.5392	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022	0.8139
14	0.2623	0.2907	0.3222	0.3572	0.3959	0.4388	0.4864	0.5392	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000	0.9022
15	0.2366	0.2623	0.2907	0.3222	0.3572	0.3959	0.4388	0.4864	0.5392	0.5976	0.6624	0.7343	0.8139	0.9022	1.0000

4.2. Forward Rates Simulation

After specifying which forward rates to evolve and their instantaneous volatilities and correlations, we now perform Monte Carlo simulation and generate forward rates. Here, we choose long jumps since the UBS product has an early exercise feature, and the jump step is 3 month. To deal with the problem caused by time dependent drifts, we determine to use the iteration simulation procedure.

We choose the longest dates zero coupon bond $P(0,4)$ as the numeraire. This means that the forward rate $F_{15}(S)$ is driftless. So its value at time T $(T > S)$ is determined analytically:

$$F_{15}(T) = F_{15}(S)\exp(-\frac{1}{2}\int_S^T \sigma_{15}^2(t)dt + \sum_{i=1}^{m}\int_S^T \sigma_{15,i}(t)dW_i$$

where m is the number of the forward rates which are reset after t=S
$\sigma(t)$ is a square root of the covariance matrix, where the covariance matrix will be full of expressions of the form $\rho_{i,j}\sigma_i(t)\sigma_j(t)$. It is generated by Cholesky factorization

Then the next forward rate $F_{14}(S)$ has a drift that depends only upon $F_{15}(S)$, so its drift can be calculated using the known $F_{15}(S)$ and $F_{15}(T)$. As we move through our forward rates, we can calculate the predictor corrector drift iteratively:

$$\int_S^T \mu_k^{new}(F,S)dt = -\frac{1}{2}\sum_{j=k+1}^{15}\left(\frac{F_j(S)\tau_j}{1+F_j(S)\tau_j} + \frac{F_j(T)\tau_j}{1+F_j(T)\tau_j}\right)\rho_{k,j}\sigma_j(t)\sigma_k(t)$$

where $k = 1,2,3,\dots,14$ in this report

Here we update the forward rates one at a time, and then use them to calculate the drift for the next, one period shorter, forward rate. Generally, the formula to simulate the forward rates (except F_{15}, the formula of which has been shown) is as follows:

$$C_{i,j} = \int_S^T \rho_{i,j}\sigma_i(t)\sigma_j(t)$$

$$F_k(T) = F_k(S)\exp(\int_S^T \mu_k^{new}(F,S)dt - \frac{1}{2}C_{k,k} + \sum_{i=1}^{m}\sigma_{k,i}\,\phi_i)$$

where m is the number of the forward rates which are reset after t=S
$\sigma(t)$ is a square root of the covariance matrix, where the covariance matrix will be full of expressions of the form $\rho_{i,j}\sigma_i(t)\sigma_j(t)$. It is generated by Cholesky factorization

$k = 1,2,3, \dots, 14$ in this report

ϕ_i s here are not the parameters in determining the instantaneous volatilities, but the i.i.d. standard distributed random numbers

We choose 10,000 as the number of simulations and generate 10,000 paths of forward rates. One of them is shown in Table 4.3.

Table 4.3 One Simulation Output

	$t=0$	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}
F_1	0.01823	0.01615														
F_2	0.01645	0.01464	0.01448													
F_3	0.01578	0.01313	0.01281	0.01323												
F_4	0.01552	0.01512	0.01600	0.01918	0.02040											
F_5	0.01531	0.01545	0.01778	0.02160	0.02508	0.02867										
F_6	0.01478	0.01346	0.01642	0.02080	0.02278	0.03486	0.03531									
F_7	0.01430	0.01382	0.01859	0.02310	0.02310	0.03641	0.03022	0.03095								
F_8	0.01529	0.01615	0.02240	0.02297	0.02390	0.03184	0.02998	0.02947	0.03103							
F_9	0.01509	0.01686	0.02283	0.02222	0.02504	0.03483	0.03306	0.03203	0.03577	0.03994						
F_{10}	0.01490	0.01587	0.01665	0.01547	0.01697	0.02257	0.02734	0.02652	0.03326	0.04241	0.04034					
F_{11}	0.01471	0.01849	0.02067	0.01911	0.02298	0.03946	0.04959	0.04474	0.05382	0.08352	0.07109	0.08437				
F_{12}	0.01527	0.01938	0.02118	0.02232	0.02729	0.04316	0.04847	0.04152	0.05086	0.08131	0.06812	0.09560	0.08923			
F_{13}	0.01520	0.01755	0.01863	0.02121	0.02369	0.03868	0.04588	0.03912	0.04840	0.06458	0.04891	0.07766	0.06537	0.06376		
F_{14}	0.01512	0.01339	0.01195	0.01315	0.01506	0.02352	0.02464	0.02430	0.02761	0.03352	0.02620	0.04277	0.04068	0.03713	0.03600	
F_{15}	0.01505	0.01526	0.01298	0.01372	0.01359	0.01866	0.01683	0.01391	0.01790	0.01843	0.01359	0.02599	0.02284	0.01970	0.01944	0.01899

4.3. Valuation

4.3.1. Caplets

There are 15 caplets needed to be valued (with maturities at *t*=0.5, 0.75, 1, …, 4). The underlying of these caplets are F_is that are reset at T_{i-1}s. We can value these "simple" products directly since we have simulate the forward rates in 4.2. The formula of pricing caplets is as follows:

$$caplet_i\ at\ reset\ date = \tau_i * \frac{1}{1+\tau_i F_i(T_{i-1})} * max(0, F_i(T_{i-1}) - K)$$

$$caplet_i\ at\ initial = P(0,4)[caplet_i\ at\ reset\ date * \prod_{j=i}^{15}(1+\tau_j F_j(T_{i-1}))]$$

We calculate the caplet prices of each path and finally the means of these 10,000 paths. The prices from LMM are shown in Table 4.4.

Table 4.4 LMM Caplet Values vs Black Caplet Values

Maturity	LMM Values	Black Values	Relative Errors
0.5	0.000300591	0.00030687	-2.046143%
0.75	0.000298058	0.000297731	0.109831%
1	0.000358285	0.000355021	0.919382%
1.25	0.000513725	0.000517475	-0.724673%
1.5	0.000621569	0.000624044	-0.396607%
1.75	0.000703114	0.000704806	-0.240066%
2	0.000789088	0.000792573	-0.439707%
2.25	0.000941265	0.000942901	-0.173507%
2.5	0.000939579	0.000938939	0.068162%
2.75	0.001035732	0.001024651	1.081441%
3	0.001160276	0.00111421	4.134409%
3.25	0.00120996	0.001170584	3.363791%
3.5	0.001234827	0.001177814	4.840578%
3.75	0.001377894	0.00130472	5.608406%
4	0.00147695	0.001310272	12.720870%

$*\ relative\ error = (LMM\ value - Black\ value)/Black\ value$

As we can see, the differences between LMM values and Black values are very small. This finding helps us to verify that all the work in 4.1 and 4.2 are valid.

4.3.2. European Swaptions

The European swaptions we are valuing here are still the 11 swaptions that we price in 3.3.2. Again, we assume all the swaptions receive *quarterly floating rate payments* and pay *quarterly fixed rate payments*. The principle is $1. We will use T_E and T_N to denote the exercise date and the maturity of the swaption. For example, for a swaption with exercise date at *t*=2 and tenor of 2 years, T_E is 2 and T_N is 4. The formula for pricing European swaption is as follows:

$$V = P(0, T_M) \frac{max(0,1 - P(T_E, T_N) - \sum_{j=1}^{4(T_N - T_E)} K\tau_{E+j} P(T_E, T_{E+j}))}{P(T_E, T_M)}$$

$$\text{where} \quad T_M = 4 \text{ and } M = 15$$

$$P(T_E, T_j) = \frac{1}{\prod_{k=1}^{j-E}(1 + \tau_{E+k} F_{E+k}(T_E))}$$

Like what we do in pricing caplets, we calculate the swaption values of each path and finally the means of these 10,000 paths. The swaption values from LMM are shown in Table 4.5.

Table 4.5 LMM Values vs Black Values

Maturity x Tenor	LMM Values	Black Values	Relative Errors
1 x 1	0.002668804	0.002612665	2.148726%
1 x 2	0.005172405	0.005277603	-1.993291%
1 x 3	0.007325141	0.008193645	-10.599727%
2 x 1	0.003844272	0.003732364	2.998314%
2 x 2	0.007343109	0.007422131	-1.064681%
3 x 1	0.004891546	0.00443166	10.377285%
0.5 x 1	0.001757269	0.001720309	2.148451%
0.5 x 2	0.003562364	0.004056727	-12.186253%
0.5 x 3	0.00508258	0.006150022	-17.356718%
0.75 x 1	0.002198317	0.002394608	-8.197208%
0.75 x 2	0.004402269	0.004753126	-7.381605%

We can see that the differences between LMM swaption values and Black swaption values are relatively small except those with longer maturities or tenor (i.e. 1 x 3, 3 x 1) and those with very short maturities (i.e. *maturity = 0.5 or 0.75*) since we capture the volatility and correlation structure using parametric calibration functions when implementing LMM method. We will discuss the details in the last section.

4.3.3. UBS Product

Recall that the UBS product has quarterly redemption feature. To deal with this feature, we decide to use Longstaff and Schwartz (2001) method, which estimates the conditional expected product value at the next time step by simulating lots of paths and then carry out a regression of the future realized value as a function of current quantities.

To simplify the valuation, we generate some new data lists. One is the cash flows if redemption happens. It is the value of the cash flow an investor would get if the product is redeemed on the redemption date. The other is the coupons given the coupon rates and interest periods.

Table 4.6 Redemption Values and Coupons

T	Potential Redemption Dates	Coupon Rates	Values If Redeemed	Coupons
T_0	2020/01/29	0.022	1005.5000	0.0000
T_1	2020/04/29	0.022	1011.0000	11.0000
T_2	2020/07/29	0.022	1005.5000	0.0000
T_3	2020/10/29	0.022	1011.0000	11.0000
T_4	2021/01/29	0.022	1005.5000	0.0000
T_5	2021/04/29	0.022	1011.0000	11.0000
T_6	2021/07/29	0.022	1005.5000	0.0000
T_7	2021/10/29	0.022	1011.0000	11.0000
T_8	2022/01/31	0.022	1005.6222	0.0000
T_9	2022/04/29	0.022	1011.0000	11.0000
T_{10}	2022/07/29	0.025	1006.2500	0.0000
T_{11}	2022/10/31	0.025	1012.6389	12.6389
T_{12}	2023/01/30	0.025	1006.2500	0.0000
T_{13}	2023/04/28	0.025	1012.3611	12.3611

In the detailed formula for valuing this callable step-up note, we will use $V_{R,i}$ to denote the value an investor would receive if the note is redeemed at $t = T_i$, and use C_i to denote the coupon value at $t = T_i$. We will start our valuation from the maturity of this product, and then calculate the values one time step earlier and repeat until we get the values at $t = T_0$. At T_{13}, the value of this product (no matter which path) is:

$$V_{13} = 1000 * \left(1 + 0.025 * \frac{1}{360}\right) = 1012.3611$$

This is also the value of $V_{R,13}$. Next, we backward the values at T_{12}. At $t = T_{12}$, we should compute the expected value of continuation value conditional on the forward rates and swap rates at T_{12}, then compare it with the redemption value. We assume if the continuation value is less than the redemption value, UBS will not redeem the note, otherwise, redemption happens at T_{12}. Namely, V_{13} is the minimum of the redemption value and continuation value. To get the conditional expected value of continuation value, we run the following regression:

$$\frac{V_{13}}{1+F_{13}(T_{12})} = b_0 + b_1 x_1^{12} + b_2 x_2^{12} + b_3 x_3^{12} + b_4 x_4^{12}$$

$$\text{where} \quad x_1^{12} = F_{12}(T_{12})$$

$$x_2^{12} = \frac{1 - P(T_{12}, T_{15})}{\sum_{j=13}^{15} \tau_j P(T_j, T_{15})}$$

$$x_3^{12} = (x_1^{12})^2$$

$$x_4^{12} = (x_2^{12})^2$$

We have 10,000 data sets since we choose to simulate 10,000 paths of forward rates. After estimating the values of b_0, b_1, b_2, b_3 and b_4, we use these parameters to compute the continuation value and get the value at T_{12}:

$$CV_{12} = \left(\widehat{b_0} + \widehat{b_1} x_1^{12} + \widehat{b_2} x_2^{12} + \widehat{b_3} x_3^{12} + \widehat{b_4} x_4^{12}\right) + C_i$$

$$V_{12} = min(CV_{12}, V_{R,12})$$

Now, we can repeat this process 12 more times until we get V_0. Now we have 10,000 V_0's. To get the value of this product, we do the following procedure for each path:

$$V = P(0,4) * V_0 * \prod_{i=1}^{15} (1 + \tau_i F_i(T_0))$$

Finally, we take the mean of values from all the paths and the result is \$1,000.348, which is really close to the real-world price, \$1,000. We think the difference is so tiny and a credit spread is not necessary to cover the difference.

4.3.4. Bermudan Swaption

Like the UBS product, the Bermudan swaption also has a call feature, and we again use Longstaff and Schwartz method. Recall that the Bermudan swaption we choose has a strike rate 1.5% and allows the holder to enter a new 2-year swap at year 1 (T_3) and year 2 (T_7). The principle is \$1.

The valuation process is very similar to the process for UBS product valuation. We calculate the Bermudan swaption value at *t*=2 (denoted by V_2) and the value at *t*=1 if early exercise happens at *t*=1 (denoted by EV_2) first:

$$V_2 = P(2,4)\frac{max(0,1 - P(2,4) - \sum_{j=1}^{8} K\,\tau_{7+j}P(2,T_{7+j}))}{P(2,4)}$$

$$EV_1 = P(1,4)\frac{max(0,1 - P(1,3) - \sum_{j=1}^{8} K\,\tau_{3+j}P(2,T_{3+j}))}{P(1,4)}$$

Now we need the continuation value at *t*=1 to get the swaption value at *t*=1. To get the conditional expected value of continuation value, we run the following regression:

$$V_2P(1,2) = b_0 + b_1x_1^{12} + b_2x_2^{12} + b_3x_3^{12} + b_4x_4^{12}$$
$$\text{where} \quad x_1^{12} = F_4(1)$$
$$x_2^{12} = \frac{1 - P(1,4)}{\sum_{j=4}^{15} \tau_j P(T_j, T_{15})}$$
$$x_3^{12} = (x_1^{12})^2$$
$$x_4^{12} = (x_2^{12})^2$$

After estimating the values of b_0, b_1, b_2, b_3 and b_4, we use these parameters to compute the continuation value and get the value at $t = 1$ or T_3:

$$CV_1 = \widehat{b_0} + \widehat{b_1}x_1^{12} + \widehat{b_2}x_2^{12} + \widehat{b_3}x_3^{12} + \widehat{b_4}x_4^{12}$$
$$V_1 = max(CV_1, EV_1)$$

Now we have 10,000 V_1s. To get the value of this swaption, we do the following procedure for each path:

$$V = P(0,4) * V_1 * \prod_{i=4}^{15}(1 + \tau_i F_i(T_3))$$

Finally, we take the mean of values from all the paths and the result is $0.00818214.

5. Discussion and Conclusion

First, we compare the two methods in terms of pricing caplets. In BDT method, we solve the parameters σ_is in the tree by equating caplet values from the tree and the market values. So, BDT method prices the caplets exactly same as the Black value. The error, if any, is small enough after we calibrate σ_is and U_is correctly. In LMM method, we can see that the errors, taking market caplet values as benchmark, are quite small. However, we can see the differences (Figure 5.1) between caplet prices are increasing as the maturity increases but still very small.

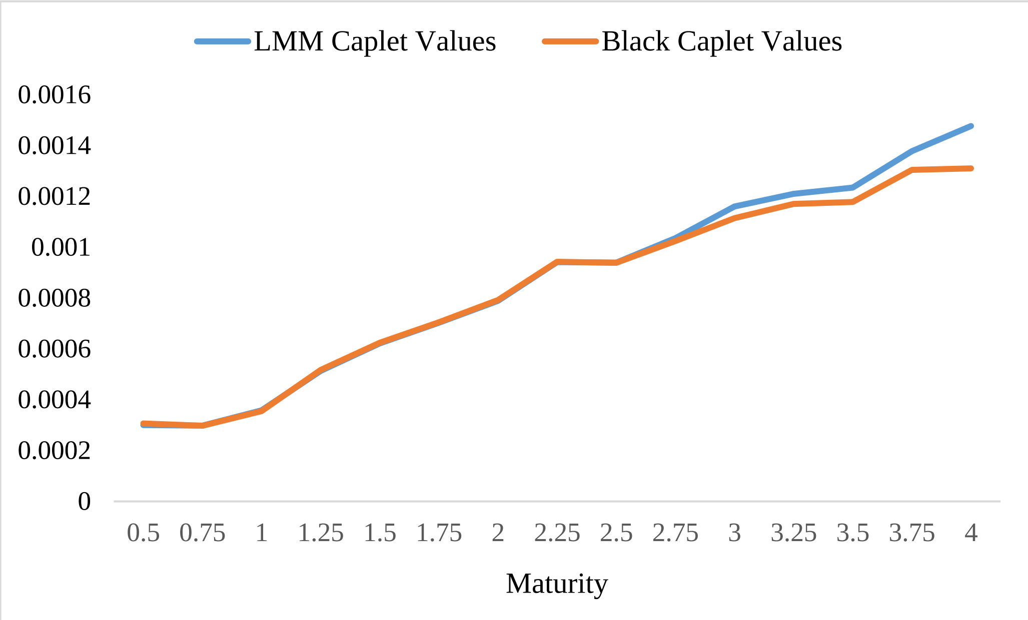

Figure 5.1 LMM Caplet Values vs Black Caplet Values

When we look at European swaption values, we can see that the differences between LMM method and the Black Model are now larger.

Table 5.2 LMM Swaption Errors vs BDT Swaption Errors

Maturity x Tenor	Relative Error 1	Relative Error 2
1 x 1	2.148726%	1.328414%
1 x 2	-1.993291%	3.565482%
1 x 3	-10.599727%	1.374687%
2 x 1	2.998314%	4.421139%
2 x 2	-1.064681%	5.578250%
3 x 1	10.377285%	7.753393%
0.5 x 1	2.148451%	-4.485764%
0.5 x 2	-12.186253%	-11.391573%
0.5 x 3	-17.356718%	-9.178227%
0.75 x 1	-8.197208%	-10.000259%
0.75 x 2	-7.381605%	-4.238831%

* $relative\ error\ 1 = (LMM\ value - Black\ value)/Black\ value$

$relative\ error\ 2 = (BDT\ value - Black\ value)/Black\ value$

From Table 5.2, we can see neither LMM nor BDT prices European swaptions with very short maturities (i.e. *maturity =0.5 or 0.75*) well. The relative errors are nearly all higher than 5%. The reason may be that these swaptions have small transaction volume are illiquid in the market. As a result, even the swaption volatilities from Bloomberg may be not precise. Therefore, even the Black values of these swaptions may be not accurate, and we cannot judge the performances of BDT or LMM when valuing these swaptions with very short maturities since even the benchmark is open to doubt.

As for those European swaptions with relative long maturities and tenors (i.e. 1 x 1, 1 x 2, 2 x 1, 2 x 2), the LMM (with relative errors less than 3%) performs a little better than BDT model (with relative errors from 1.3% to 5.6%). This is not surprising since the correlation, which is important in pricing swaptions, is calibrated in LMM method but is not taken into consideration in BDT method. To improve the BDT tree, a plausible way is to use more time steps. But this may bring more errors since we need more interpolation on zero coupon bonds.

However, for European swaptions with longer maturities or tenors (i.e. 1 x 3, 3 x 1), the LMM method gives bad results. The absolute values of relative errors are even larger than 10%. This result also coincides with the finding that as maturity increases, the error between LMM caplet value and Black caplet value increases. From these two findings, we infer that the problem is from the forward rate volatilities. The parametric function we choose for instantaneous volatilities may not capture the volatilities of long term forward rates. As a result, even when valuing simple derivatives like caplets and swaptions, we cannot get accurate prices for those with long maturities. Maybe we can solve this problem by using a new volatility function or make some adjustments in calibration.

As for the UBS note, the public issue price is $1,000. The value of this product is $997.958 from monthly-step BDT tree and $1000.34 from LMM method. The LMM method seems to perform better. This difference may be caused by the different ways to deal with call feature. When implementing the Libor Market Model, we use Longstaff and Schwartz method to solve the call feature, which estimates the conditional expected product value at the next time step by simulating lots of paths and then carry out a regression of the future realized value as a function of current quantities. In BDT model, at each potential date, we just compare the redemption value and the expected discounted value. However, to get the expected discounted values for comparison, we only have a small number of information, whereas in LMM, we can use lots of paths of data. Maybe if we choose shorter time step (i.e. 1 day, 5 days), we can improve the performance of BDT, but we need more interpolation to get more zero coupon bond prices. The interpolation may have opposite effects and we are not sure whether we can improve the performance of BDT in the trade-off.

The Bermudan swaption we are pricing has a strike rate 1.5% and allows the holder to enter a new 2-year swap at year 1 or year 2. The value from BDT tree is $0.007898239 and the value from LMM is $0.00818214. However, we don’t have a market benchmark to compare the performances of these models. According to the performances of these two models in pricing the UBS product which has a call feature, we think the Bermudan swaption value from LMM may be more accurate since the Longstaff and Schwartz method can deal with the call feature better.

To conclude, compared with Black-Derman-Toy model, the Libor Market Model can calibrate to swaption prices and generate correct volatility structures and possible non-parallel shifts in yield curves in future. But LMM is also difficult to implement. When valuing caplets, we can get accurate results in LMM if the maturities of the caplets are relatively short. As the maturity of the caplet increases, the error increases but still is not significant. BDT performs perfectly, of course, since we equate BDT caplet values and Black caplet values. When valuing European swaptions, LMM performs better than BDT model except for those swaptions with higher maturities or tenors. When valuing products with call feature like the UBS product and Bermudan swaptions, LMM performs better than BDT because we can use Longstaff and Schwartz method, which can solve the call feature well, in the Libor Market Model.