Singular Value Decomposition

**Abstract – This project aims to provide an introduction of Singular Value Decomposition in both graphical and mathematical representation. We will also create a simple calculation based on power iteration method and compare the result with the output of `np.linalg.svd` function from Python Numpy library. This project will also explore a few real-life applications of SVD in the field of Machine Learning and Data Science such as Recommendation System and Image Compression.**

Anh Hoang Chu, *Harrisburg University*

# I. INTRODUCTION

Singular value decomposition (SVD) and Eigendecomposition (EigD) are 2 popular methods of matrix decomposition, which reduces a matrix to lower-rank matrices to simplify more complex operations. EigD decomposes a square matrix into a set of eigen values and eigen vectors. Eigenvectos and eigenvalues is important because they describe long-term behavior of the system. On the other hand, Singular value decomposition decomposes the matrix into singular values and singular vectors like EigD, but SVD also works on non-square matrices which makes it more generally applicable.

EigD and SVD both have many important applications in the field of machine learning and data science, especially in dimensionality reduction to provide the fundamental characteristics of the data. Some real-life applications of SVD include image processing, data reduction, and recommendation systems; while EigD is important for spectral clustering as well as page-rank algorithms, etc.

Some programming language provides built-in software packages to calculate the SVD and EigD. For example, Python has numpy.linalg, scipy.decomp\_svd, sklearn.decomposition libraries which svd function for SVD and eig for EigD. Since users can blindly apply these functions to decompose a matrix into eigenvalues and eigenvectors or singular values and singular vectors, many don’t really understand the steps it takes to compute the output from the input, and as the result apply the functions wrongly, misinterpret the output or fail to modify the function for more specific needs.

The motivation of this project is to understand how to calculate SVD as this will reinforce the fundamental understanding of linear algebra and linear transformation such as matrix stretching and rotating. Moreover, this will provide fundamental knowledge of other important concepts in machine learning such as principal component analysis (PCA), natural language processing, and computer vision etc.

# II. Introduction to Singular value decomposition

Singular Value Decomposition is a method in linear algebra to decompose a matrix (A) into 3 other matrices: 𝐴=𝑈𝑆𝑉𝑇.

SVD performs several transformations of Matrix A: rotate the matrix by transposed of V, apply scale factor S, rotate the matrix by U. Unlike Eigendecomposition, SVD can work on non-square and sparse matrices.

A: m x n matrix to decompose

U: m x m or m x k unitary matrix, columns of U are left singular vectors of A

S: m x n matrix of s non-negative values on diagonal axis, s values of diagonal of Sigma are the singular values of A

VT: n x n or k x n transposed of a unitary matrix, columns of V are singular vectors of A

## SVD graphical illustration

Chart, line chart, bubble chart

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## SVD Math Modelling - Power Iteration algorithm

Given a matrix A, compute the SVD of the matrix

#### Step 1: Compute 𝐴𝑇 and 𝐴𝑇𝐴

* AT = (𝑈Σ 𝑉𝑇)𝑇
* 𝐴𝑇𝐴 = (𝑈Σ 𝑉𝑇)𝑇(𝑈Σ 𝑉𝑇) = 𝑉Σ2𝑉𝑇

#### Step 2: Calculate Eigenvalues and Singular values of 𝐴𝑇𝐴 and sort them in descending order

* Calculate f(c) = 𝐴𝑇𝐴 - 𝑐𝐼 = 0
* Solve f(c) for eigenvalues
* singular values = 𝑠𝑞𝑟𝑡(𝑒𝑖𝑔𝑒𝑛𝑣𝑎𝑙𝑢𝑒𝑠)

#### Step 3: Construct diagonal matrix S by placing singular values in descending order along its diagonal. Compute its inverse, 𝑆−1

#### Step 4: Use the ordered eigenvalues from step 2 and compute the eigenvectors of ATA. Place these eigenvectors along the columns of V and compute its transpose, 𝑉𝑇

#### Step 5. Compute U = 𝐴𝑉𝑆-1

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Next step is to compare the result of Singular Values (S) computed by the above function with Singular Values (S) computed by the built-in np.linalg.svd function in Python

Table

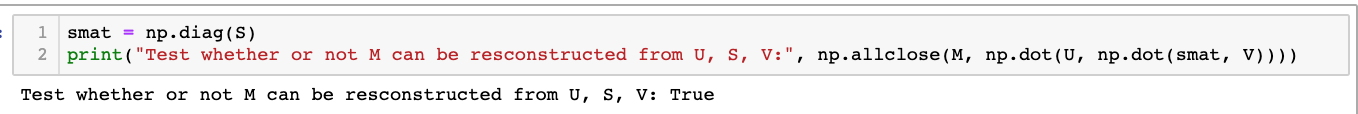
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Even though U and V outputs are different between np.linalg.svd function and manually calculated SVD, we can see that the Singular Values S are the same

S (np.linalg.svd) = S (manual\_svd) = [15.096269, 4.300569, 3.407017]

Moreover, we can reconstruct the original matrix by doing the dot multiplication between U, S and V calculated from both built-in and manual SVD



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# III. APPLICATION OF SVD

SVD has a wide range of real-life applications in Machine Learning and Data Science

* Dimensionality reduction: Data with a large number of features, such as more features (columns) than observations (rows) may be reduced to a smaller subset of features that are most relevant to the prediction problem
* Recommendation System: SVD is used in collaborative approach of Recommendation System
* Image Compression: reduces size of image to an acceptable level of quality to save memory and disk space

## A. SVD in Recommendation System

There are 2 Popular Approaches for Recommendation System.

* Collaborative Filtering: assuming that users who have liked an item in the past will probably like a similar item in the future, this approach investigates past interaction between users and items to identify the similarity of items. SVD is used in Collaborative Filtering Approach
* Content-based Filtering: identifies characteristics of items to recommend similar items. This approach is more suitable when there're available data about items but not enough data about users' preference

Below code snippets illustrate SVD performance in grouping movies with similarity using a standard data science process

### Import data: Import movies and ratings data from Movie Lens dataset

* Ratings data have ratings of each user to movies
* Movies data have information about a movie such as id, title and genre

Graphical user interface, text, application, email

Description automatically generated

### Feature Scaling: Convert ratings pandas dataframe to numpy array for transformation and normalization

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### Build Model: Apply SVD on transformed ratings matrix

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### Evaluate Model

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#### Evaluate built-in function

## Graphical user interface, text, application, email Description automatically generated

The built-in SVD function did a good job in identifying very similar movies for Toy Story with other children, animated, and comedy movies.

#### Evaluate manually computed SVD function

## Text, table Description automatically generated with medium confidenceDue to the size of the input (6k rows in ratings for 3.9k movies), we observe that the manual SVD failed to converge in a reasonable amount of time. Python np.linalg library has much more complex algorithm to optimize the performance of [np.linalg.svd](https://github.com/numpy/numpy/blob/main/numpy/linalg/linalg.py) function. The scope of this project is to only illustrate a very simple algorithm (power iterative method) to calculate U, S, V for a small matrix, this algorithm doesn’t scale well on a matrix with more than 100 rows. To apply manual SVD on this dataset, we have to either reduce the input array or research on other optimization algorithms to apply on this Movie Dataset.

## B. SVD for image compression

Graphical user interface, text, application, email

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Apply above function on an 7x7 black & white image matrix, the manually calculated SVD function successfully compressed the image by different singular values in descending order

Diagram, table

Description automatically generated

A screenshot of a computer

Description automatically generated with low confidence

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