Assignment 3

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Question 1

(a) Why linear programming is suitable for this case study

- Linear programming is suitable in this case study since we are trying to find the optimized solution, in this case, minimizing the cost of acquiring two component A and B, all the while making sure all constraints are met.
- From the given problem, we want to minimize the cost of A (4$/L) and B (12$/L) per week, making the objective function a linear combination of A and B cost. The beverage mix of A and B must satisfy these conditions:
  - Orange constraint: There must be at least 5 Litres of Orange per 100 Litres of beverage mix.
  - Mango constraint: There must be at least 5 Litres of Mango per 100 Litres of beverage mix.
  - Lime constraint: There must be at most 6 Litres of Lime per 100 Litres of beverage mix.
  - Customer constraint: There must be 140 Litres of beverage mix produced per week.

(b) Formulate LP model for the factory

Decision variables: Let x_A and x_B be the amount of product A and product B required per week (in Litres), respectively.

Our objective function is to minimize the total cost of producing the beverage per week:

$$\text{minimize} z = 4x_A + 12x_B$$

Subjected to:

- The orange constraint:

$$\frac{\frac{6}{100}x_A + \frac{4}{100}x_B}{x_A + x_B} \geq \frac{5}{100}$$

$$\implies x_A - x_B \geq 0$$

- The mango constraint:

$$\frac{\frac{4}{100}x_A + \frac{8}{100}x_B}{x_A + x_B} \geq \frac{5}{100}$$

$$\implies -x_A + 3x_B \geq 0$$

- The lime constraint:

$$\frac{\frac{2}{100}x_A + \frac{7}{100}x_B}{x_A + x_B} \leq \frac{6}{100}$$

$$\implies -3x_A + x_B \leq 0$$

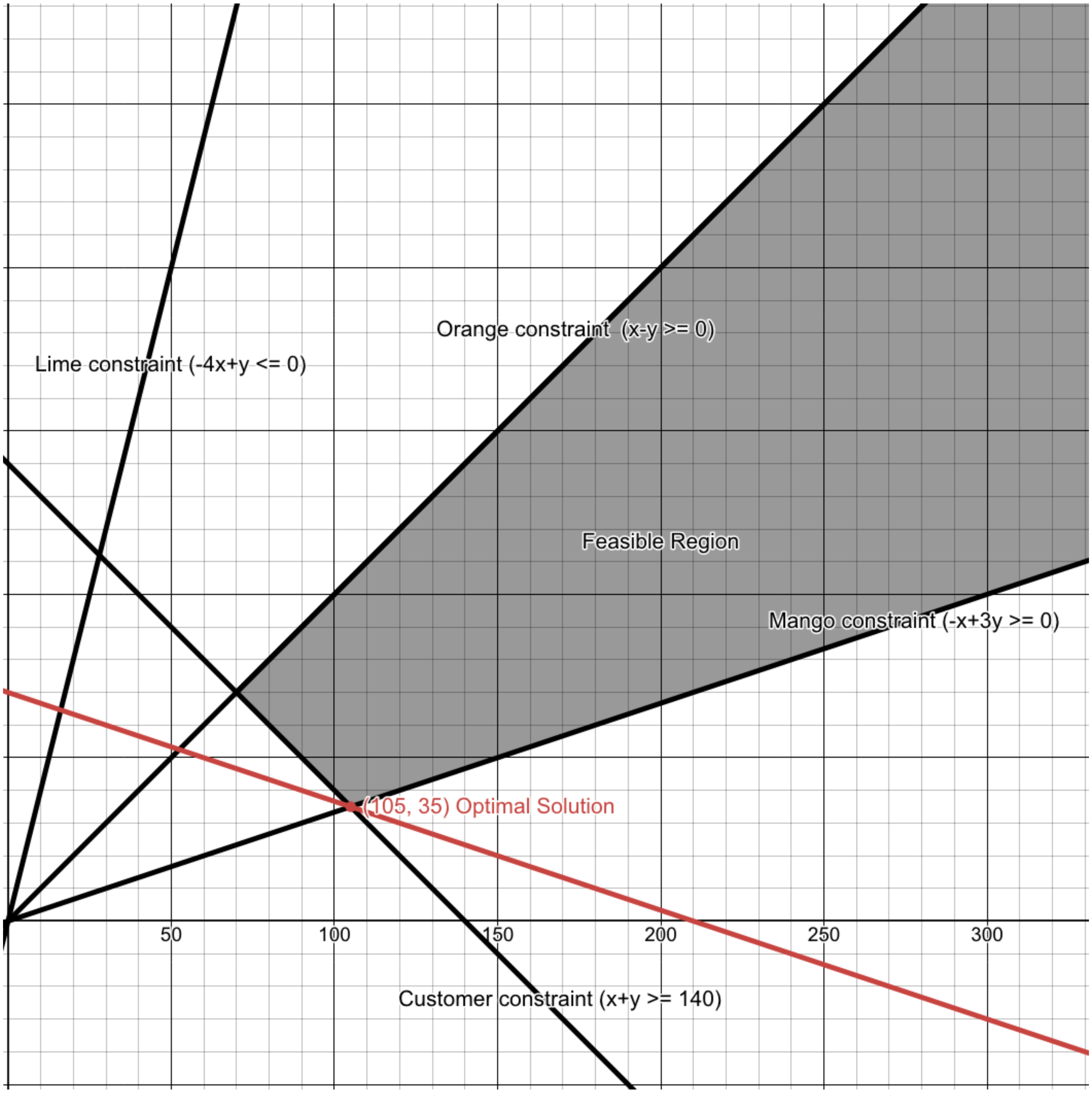
- The customer constraint:

$$x_A + x_B \geq 140$$

- Non-negativity constraint (since cost cannot be negative):

$$x_A, x_B \geq 0$$

(c) Graphical Solution



- We first graph the constraints on 2D-plane. This will give us the feasible region (gray) that a solution is acceptable. To rewrite each constraints to line function, we replaced x_A to x, and x_B to y.
- After graphing the objective function, we can look for the iso-cost lines, which is a line parallel to the objective function and first come into contact with the feasible region. (red)
- From the graph, we can see that the point where $x_A = 105$ and $x_B = 35$ is the optimal solution. It is also the intersection between the customer constraints and mango constraints. Therefor, the optimal cost for product A and B is $4 * 105 + 12 * 35 = 840$ dollars per week.

(d) Range of product A so that optimal solution remains unchanged

We denote c_A as the cost of product A such that the optimal solution remains unchanged.

The objective function $z = c_A x_A + 12x_b$ can be graphed using the line function $y = -\frac{c_A}{12}x - z$.

Visually, we can see that this line can be pivot around the point $(105, 35)$ and limited by the customer constraint (clockwise) and mango constraint (anti-clockwise). The slope for customer and mango constraints can be deduce by:

$$\begin{aligned} x + y \geq 140 &\implies y \geq -x + 140 \\ -x + 3y \geq 0 &\implies y \geq \frac{1}{3}x \end{aligned}$$

Therefor, the slope of the objective function must be within $[-1, \frac{1}{3}]$. Moreover, **cost of product A must be non-negative**:

$$-1 \leq -\frac{c_A}{12} \leq \frac{1}{3}$$

$$\implies 12 \geq c_A \geq -4$$

$$\implies 0 \leq c_A \leq 12$$

The cost for product A in the range of $[0, 12]$ will make the optimal solution remains unchanged.

Question 2

(a) Formulate LP problem

Decision Variables:

- x_{CS}, x_{CA}, x_{CW}: Tons of **Cotton** used for Spring, Autumn, and Winter.
- x_{WS}, x_{WA}, x_{WW}: Tons of **Wool** used for Spring, Autumn, and Winter.
- x_{SS}, x_{SA}, x_{SW}: Tons of **Silk** used for Spring, Autumn, and Winter.

Sales: $60(x_{CS} + x_{WS} + x_{SS}) + 55(x_{CA} + x_{WA} + x_{SA}) + 65(x_{CW} + x_{WW} + x_{SW})$

Purchase price: $30(x_{CS} + x_{CA} + x_{CW}) + 45(x_{WS} + x_{WA} + x_{WW}) + 50(x_{SS} + x_{SA} + x_{SW})$

Production costs: $5(x_{CS} + x_{WS} + x_{SS}) + 3(x_{CA} + x_{WA} + x_{SA}) + 8(x_{CW} + x_{WW} + x_{SW})$

Objective function:

$$\begin{aligned} \text{maximize} z &= \text{Sales} - \text{Purchase Costs} - \text{Production Costs} \\ &= \big(60(x_{CS} + x_{WS} + x_{SS}) + 55(x_{CA} + x_{WA} + x_{SA}) + 65(x_{CW} + x_{WW} + x_{SW})\big) \\ &- \big(30(x_{CS} + x_{CA} + x_{CW}) + 45(x_{WS} + x_{WA} + x_{WW}) + 50(x_{SS} + x_{SA} + x_{SW})\big) \\ &- \big(5(x_{CS} + x_{WS} + x_{SS}) + 3(x_{CA} + x_{WA} + x_{SA}) + 8(x_{CW} + x_{WW} + x_{SW})\big) \\ \implies \text{maximize} z &= (25x_{CS} + 10x_{WS} + 5x_{SS} + 22x_{CA} + 7x_{WA} + 2x_{SA} + 27x_{CW} + 12x_{WW} + 7x_{SW}) \end{aligned}$$

Constraints:

1. Proportional Constraints:

- For **Spring**:

$$x_{CS} \geq 0.55(x_{CS} + x_{WS} + x_{SS}), \quad x_{WS} \geq 0.30(x_{CS} + x_{WS} + x_{SS}), \quad x_{SS} \geq 0.01(x_{CS} + x_{WS} + x_{SS})$$

- For **Autumn**:

$$x_{CA} \geq 0.45(x_{CA} + x_{WA} + x_{SA}), \quad x_{WA} \geq 0.40(x_{CA} + x_{WA} + x_{SA}), \quad x_{SA} \geq 0.02(x_{CA} + x_{WA} + x_{SA})$$

- For **Winter**:

$$x_{CW} \geq 0.30(x_{CW}+x_{WW}+x_{SW}), \quad x_{WW} \geq 0.50(x_{CW}+x_{WW}+x_{SW}), \quad x_{SW} \geq 0.03(x_{CW}+x_{WW}+x_{SW})$$

2. Demand Constraints:

$$x_{CS} + x_{WS} + x_{SS} \leq 3200, \quad x_{CA} + x_{WA} + x_{SA} \leq 3800, \quad x_{CW} + x_{WW} + x_{SW} \leq 4200$$

3. Non-Negativity:

$$x_{CS}, x_{CA}, x_{CW}, x_{WS}, x_{WA}, x_{WW}, x_{SS}, x_{SA}, x_{SW} \geq 0$$

(b) Solve the problem with R

The optimal profit is **203,620$**. The corresponding values of the decision variables are:

$$x_{CS} = 2208, \quad x_{CA} = 2204, \quad x_{CW} = 1974$$
$$x_{WS} = 960, \quad x_{WA} = 1520, \quad x_{WW} = 2100$$
$$x_{SS} = 32, \quad x_{SA} = 76, \quad x_{SW} = 126$$

These values represent the tons of each material (cotton, wool, silk) allocated to Spring, Autumn, and Winter, maximizing the profit.

Question 3

(a) Formulate the payoff matrix for the game and identify possible saddle points

- Each player have a sequence of selected move (first chip, second chip, third chip). In total, there are 6 permutation of chip selection. The payoff matrix is a 6x6 matrix, each cell represent the final payoff of player 1 if the corresponding sequence was played by player 1 (row) and player 2 (column).

	RWB	RBW	WRB	WBR	BRW	BWR
RWB	0	0	0	75	-75	0
RBW	0	0	75	0	0	-75
WRB	0	-75	0	0	0	75
WBR	-75	0	0	0	75	0
BRW	75	0	0	-75	0	0
BWR	0	75	-75	0	0	0

- From a pure strategy viewpoint,
  - the lower value (maximum security level for player 1 or L) is -75
  - the upper value (maximum security level for player 2 or U) of the game U is 75
- Therefor, pure strategy will not result in equilibrium (saddle point)

(b) Linear Programming Formulation

(b.1) For Player 1 Decision Variables:

1. $x_1, x_2, x_3, x_4, x_5, x_6$: Probabilities that Player 1 plays each sequence $RWB, RBW, WRB, WBR, BRW, BWR$
2. v: Minimum payoff for player 1

Objective Function: maximize$(z) = v$

Constraints:

1. **Payoff Constraints:**

$$v + 75x_4 - 75x_5 \leq 0$$

$$v + 75x_3 - 75x_6 \leq 0$$

$$v - 75x_2 + 75x_6 \leq 0$$

$$v - 75x_1 + 75x_5 \leq 0$$

$$v + 75x_1 - 75x_4 \leq 0$$

$$v + 75x_2 - 75x_3 \leq 0$$

2. **Probability Sum Constraint:** Probabilities for Player 1's strategies must sum to 1:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$$

3. **Non-Negativity Constraint:** Probabilities cannot be negative:

$$x_k \geq 0 \quad \forall k \in \{1, 2, 3, 4, 5, 6\}$$

(b.2) For player 2 This formulation is for Player 2, who aims to minimize the maximum payoff (v) that Player 1 can secure. It uses the payoff matrix constraints but flipped to reflect Player 2's perspective.

Decision Variables:

1. $y_1, y_2, y_3, y_4, y_5, y_6$: Probabilities that Player 2 plays each sequence ($RWB, RBW, WRB, WBR, BRW, BWR$).
2. v: maximum payoff for player 1

Objective Function: minimize$(z) = v$

Constraints:

1. **Payoff Constraints:**

$$v - 75y_4 + 75y_5 \geq 0$$

$$v - 75y_3 + 75y_6 \geq 0$$

$$v + 75y_2 - 75y_6 \geq 0$$

$$v + 75y_1 - 75y_5 \geq 0$$

$$v - 75y_1 + 75y_4 \geq 0$$

$$v - 75y_2 + 75y_3 \geq 0$$

2. **Probability Sum Constraint:** Probabilities for Player 2's strategies must sum to 1:

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 1$$

3. **Non-Negativity Constraint:** Probabilities cannot be negative:

$$y_k \geq 0 \quad \forall k \in \{1, 2, 3, 4, 5, 6\}$$

(d) Solve the game

(d.1) Player 1 The optimal mixed strategy and payoff for Player 1 is:

$$x_1 = 0.33,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0.33,\ x_5 = 0.33,\ x_6 = 0,\ v = 0$$

This indicate that player 1 should split their plays between strategy 1 (RWB), 4 (WBR), 5 (BRW) equally (33.33%). The minimum payoff or security level for them would be maximized to $v = 0$. This is a fair game. The expected payoffs can be calculated:

Expected payoff when Player 2 Plays Strategy 1 ($j = 1$):

$$\text{Expected Payoff} = (0.33 \cdot 0) + (0 \cdot 0) + (0 \cdot 0) + (0.33 \cdot -75) + (0.33 \cdot 75) + (0 \cdot 0) = 0$$

Expected payoff when Player 2 Plays Strategy 2 ($j = 2$):

$$\text{Expected Payoff} = (0.33 \cdot 0) + (0 \cdot 0) + (0 \cdot -75) + (0.33 \cdot 0) + (0.33 \cdot 0) + (0 \cdot 75) = 0$$

Expected payoff when Player 2 Plays Strategy 3 ($j = 3$):

$$\text{Expected Payoff} = (0.33 \cdot 0) + (0 \cdot 75) + (0 \cdot 0) + (0.33 \cdot 0) + (0.33 \cdot 0) + (0 \cdot -75) = 0$$

Expected payoff when Player 2 Plays Strategy 4 ($j = 4$):

$$\text{Expected Payoff} = (0.33 \cdot 75) + (0 \cdot 0) + (0 \cdot 0) + (0.33 \cdot 0) + (0.33 \cdot -75) + (0 \cdot 0) = 0$$

Expected payoff when Player 2 Plays Strategy 5 ($j = 5$):

$$\text{Expected Payoff} = (0.33 \cdot -75) + (0 \cdot 0) + (0 \cdot 0) + (0.33 \cdot 75) + (0.33 \cdot 0) + (0 \cdot 0) = 0$$

Expected payoff when Player 2 Plays Strategy 6 ($j = 6$):

$$\text{Expected Payoff} = (0.33 \cdot 0) + (0 \cdot -75) + (0 \cdot 75) + (0.33 \cdot 0) + (0.33 \cdot 0) + (0 \cdot 0) = 0$$

Since expected payoffs are all zeroes, we can expected the total expected payoffs to be 0, no matter which mixed strategy player 2 deploys.

(d.2) Player 2 The optimal mixed strategy and payoff for Player 2 is:

$$y_1 = 0.33,\ y_2 = 0,\ y_3 = 0,\ y_4 = 0.33,\ y_5 = 0.33,\ y_6 = 0$$

This indicates that Player 2 should split their plays between strategy 1 (RWB), 4 (WBR), and 5 (BRW) equally (33.33%). The maximum payoff or security level for their opponent would be minimized to $v = 0$.

Expected Payoff when Player 1 Plays Strategy 1 ($i = 1$):

$$\text{Expected Payoff} = (0.33 \cdot 0) + (0.33 \cdot 0) + (0.33 \cdot 0) = 0$$

Expected Payoff when Player 1 Plays Strategy 2 ($i = 2$):

$$\text{Expected Payoff} = (0.33 \cdot 0) + (0.33 \cdot 0) + (0.33 \cdot 0) = 0$$

Expected Payoff when Player 1 Plays Strategy 3 ($i = 3$):

$$\text{Expected Payoff} = (0.33 \cdot 0) + (0.33 \cdot -75) + (0.33 \cdot 75) = 0$$

Expected Payoff when Player 1 Plays Strategy 4 ($i = 4$):

$$\text{Expected Payoff} = (0.33 \cdot -75) + (0.33 \cdot 0) + (0.33 \cdot 75) = 0$$

Expected Payoff when Player 1 Plays Strategy 5 ($i = 5$):

$$\text{Expected Payoff} = (0.33 \cdot 75) + (0.33 \cdot 0) + (0.33 \cdot -75) = 0$$

Expected Payoff when Player 1 Plays Strategy 6 ($i = 6$):

$$\text{Expected Payoff} = (0.33 \cdot 0) + (0.33 \cdot 75) + (0.33 \cdot -75) = 0$$

Since all expected payoffs are zero, we can expect the total expected payoffs for Player 2 to be 0, no matter which mixed strategy player 1 deploys.

(d.3) Infinite solutions Since for both player 1 and 2, the payoff $v = 0$ means that there exist no mixed strategy where a player can gain an edge on the other, making this a fair game. We can simplify the payoff constraints

1. $75x_4 + -75x_5 \leq 0 \implies x_4 \leq x_5$
2. $75x_3 + -75x_6 \leq 0 \implies x_3 \leq x_6$
3. $-75x_2 + 75x_6 \leq 0 \implies x_6 \leq x_2$
4. $-75x_1 + 75x_5 \leq 0 \implies x_5 \leq x_1$
5. $75x_1 - 75x_4 \leq 0 \implies x_1 \leq x_4$
6. $75x_2 - 75x_3 \leq 0 \implies x_2 \leq x_3$

Which implies:

- $(1), (4), (5) \implies x_1 = x_4 = x_5$
- $(2), (3), (6) \implies x_2 = x_3 = x_6$
- as well as $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1$
- and $x_i \geq 0 \quad \text{for all } i = 1, 2, ..., 6$

There are infinite combination of solutions. For example, $x_1 = x_4 = x_5 = x_2 = x_3 = x_6 = \frac{1}{6}$ is another solution to the problem. Another solution can be $x_1 = \frac{6}{30}$, $x_2 = \frac{4}{30}$, $x_3 = \frac{4}{30}$, $x_4 = \frac{6}{30}$, $x_5 = \frac{6}{30}$, $x_6 = \frac{4}{30}$