Lecture 9 recap

1) Two properties that a linear span must have.

2) Linear span inside linear span theorem.

3) 'Useless' vector

4) Definition of subspace.

5) How to show a subset is not a subspace using 1).

6) How to show a subset is a subspace by writing as linear span.

Lecture 10

Subspaces (cont'd)

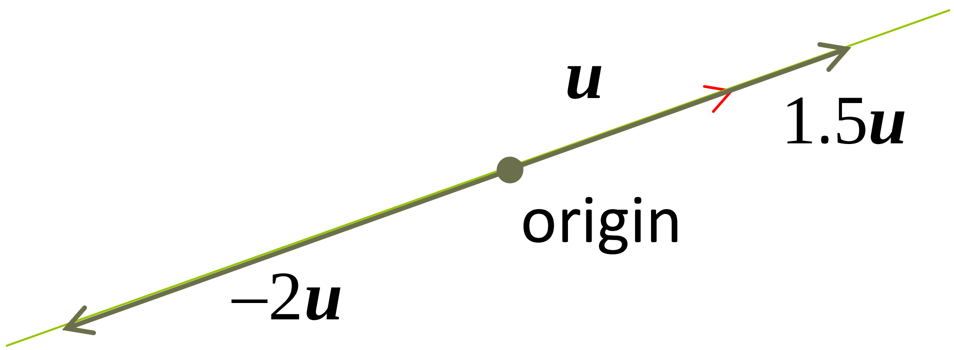
Linear independence

Geometrical examples

Let $\boldsymbol{u}$ be a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

span$\{\boldsymbol{u}\}$ is the set of all linear combinations (or scalar multiples) of $\boldsymbol{u}$.

Geometrically, span$\{\boldsymbol{u}\}$ is a straight line passing through the origin.



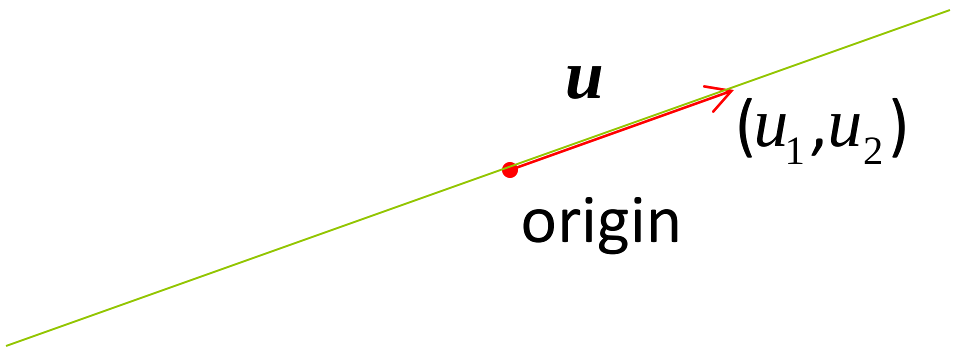
Geometrical examples

Let $\boldsymbol{u}$ be a nonzero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$.

(In $\mathbb{R}^2$) $\boldsymbol{u} = (u_1, u_2)$, span$\{\boldsymbol{u}\} = \{(cu_1, cu_2) \mid c \in \mathbb{R}\}$

(explicit representation)

(implicit representation i.e. equation of line?)



Geometrical examples

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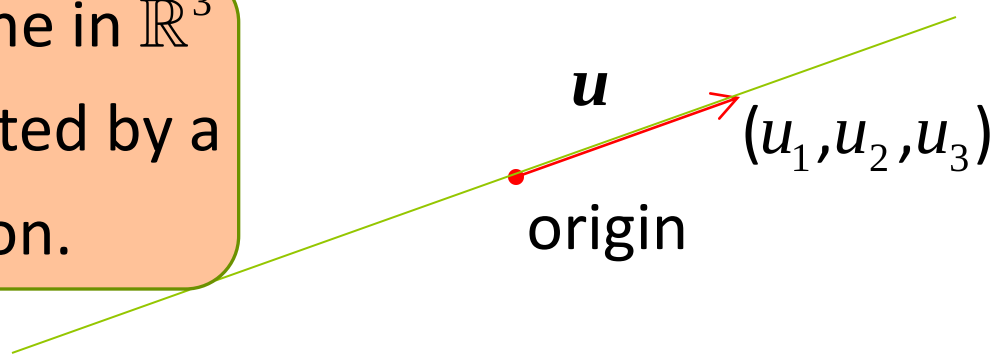
(explicit representation)

(implicit representation i.e. equation of line?)

Remember that a line in $\mathbb{R}^3$ cannot be represented by a single linear equation.



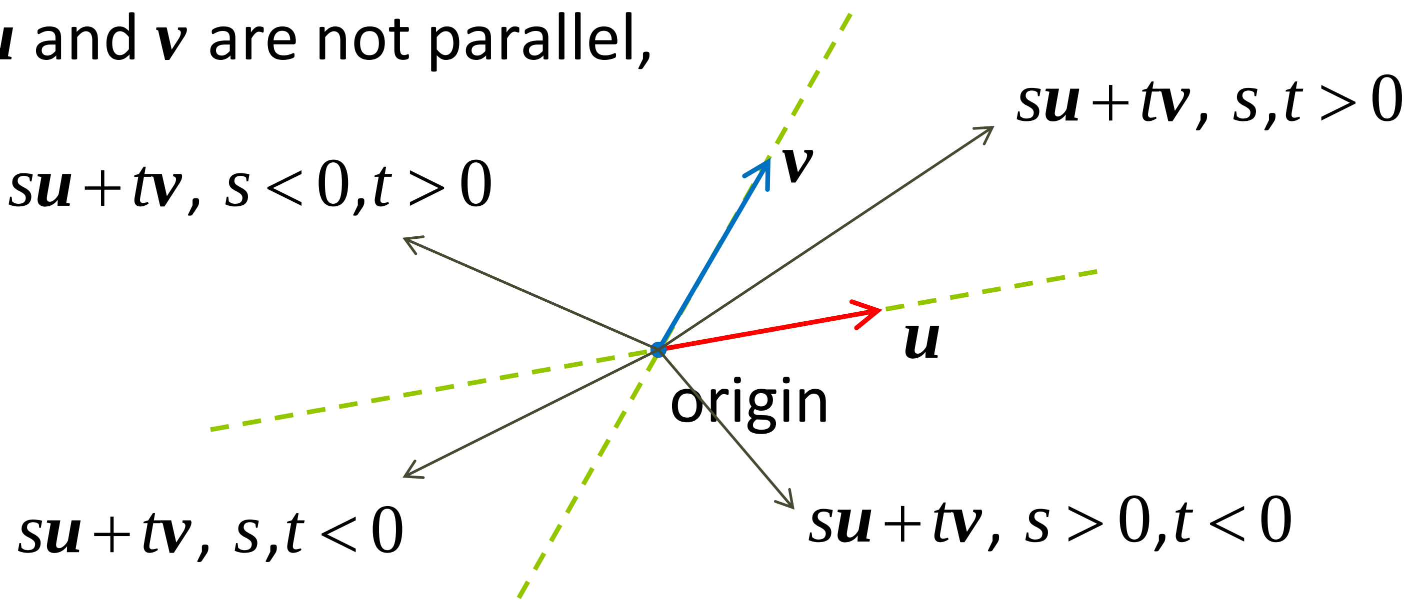
Geometrical examples

Let $\boldsymbol{u},\boldsymbol{v}$ be two nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

span{$\boldsymbol{u},\boldsymbol{v}$} is the set of all linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$=\{s\boldsymbol{u}+t\boldsymbol{v} \mid s,t\in\mathbb{R}\}$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel,



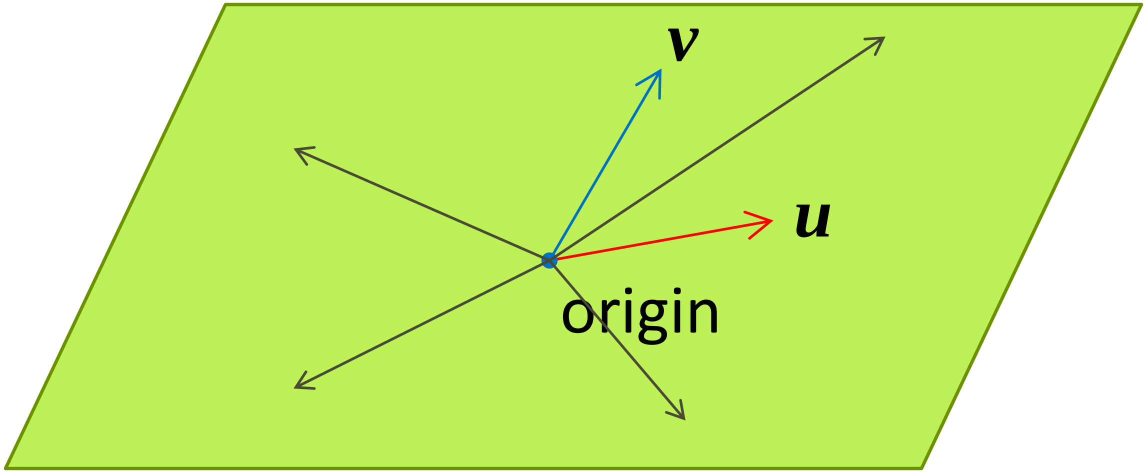
Geometrical examples

Let $\boldsymbol{u},\boldsymbol{v}$ be two nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

span{$\boldsymbol{u},\boldsymbol{v}$} is the set of all linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$=\{s\boldsymbol{u}+t\boldsymbol{v}\mid s,t\in\mathbb{R}\}$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel, span{$\boldsymbol{u},\boldsymbol{v}$} is a plane containing the origin.



Geometrical examples

Let $\boldsymbol{u},\boldsymbol{v}$ be two nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

span{$\boldsymbol{u},\boldsymbol{v}$} is the set of all linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$= \{s\boldsymbol{u} + t\boldsymbol{v} \mid s,t \in \mathbb{R}\}$$

What if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel?

$$\text{span}\{\boldsymbol{u},\boldsymbol{v}\} = \text{span}\{\boldsymbol{u}\}$$

$$= \text{straight line passing}$$

through the origin.

Geometrical examples

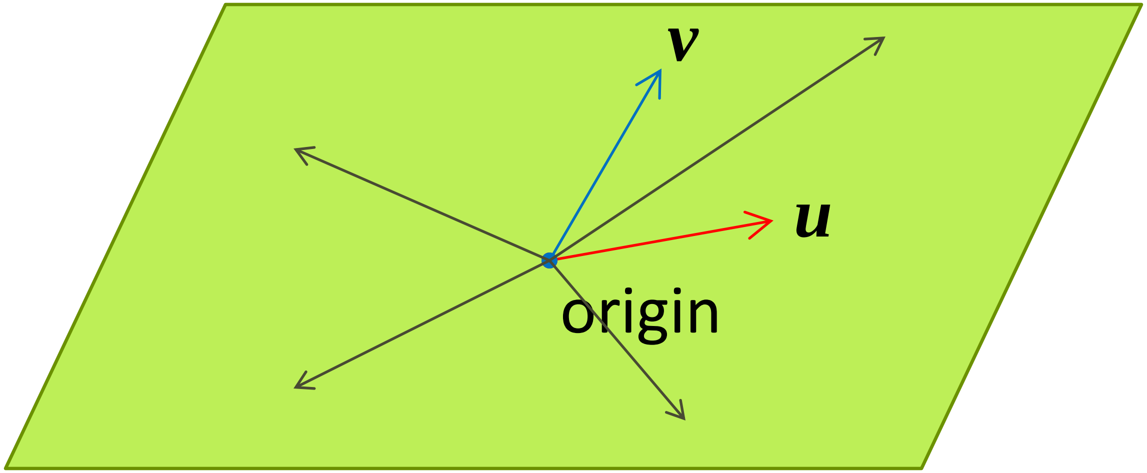
If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel,

(In $\mathbb{R}^2$) span$\{\boldsymbol{u},\boldsymbol{v}\} = \mathbb{R}^2$.

(In $\mathbb{R}^3$) span$\{\boldsymbol{u},\boldsymbol{v}\} = \{s\boldsymbol{u} + t\boldsymbol{v} \mid s,t \in \mathbb{R}\}$ (explicit representation)

(implicit representation, i.e. equation of the plane?)

$\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$



Remark (All subspaces of R^2)

The following are all the subspaces of $\mathbb{R}^2$:

Remark (All subspaces of R^3)

The following are all the subspaces of $\mathbb{R}^3$:

Theorem (Solution set of homogeneous systems)

The solution set of a homogeneous system of linear equations in n variables is a subspace of $\mathbb{R}^n$.

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{cases}$$

Example

Investigate the solution set of the following homogeneous linear system:

$$\begin{cases} x - 2y + 3z = 0 \\ -2x + 4y - 6z = 0 \\ 3x - 6y + 9z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & 4 & -6 & 0 \\ 3 & -6 & 9 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Example

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Geometrically, the solution set is a plane in $\mathbb{R}^3$ containing the origin.

Example

Investigate the solution set of the following homogeneous linear system:

$$\begin{cases} x - 2y + 3z = 0 \\ -2x + 4y - 6z = 0 \\ -3x + 7y - 8z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ -2 & 4 & -6 & 0 \\ -3 & 7 & -8 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Example

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Geometrically, the solution set is a line in $\mathbb{R}^3$ passing through the origin.

Example

Investigate the solution set of the following homogeneous linear system:

$$\begin{cases} x - 2y + 3z = 0 \\ 4x + y + 2z = 0 \\ -3x + 7y - 8z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 4 & 1 & 2 & 0 \\ -3 & 7 & -8 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

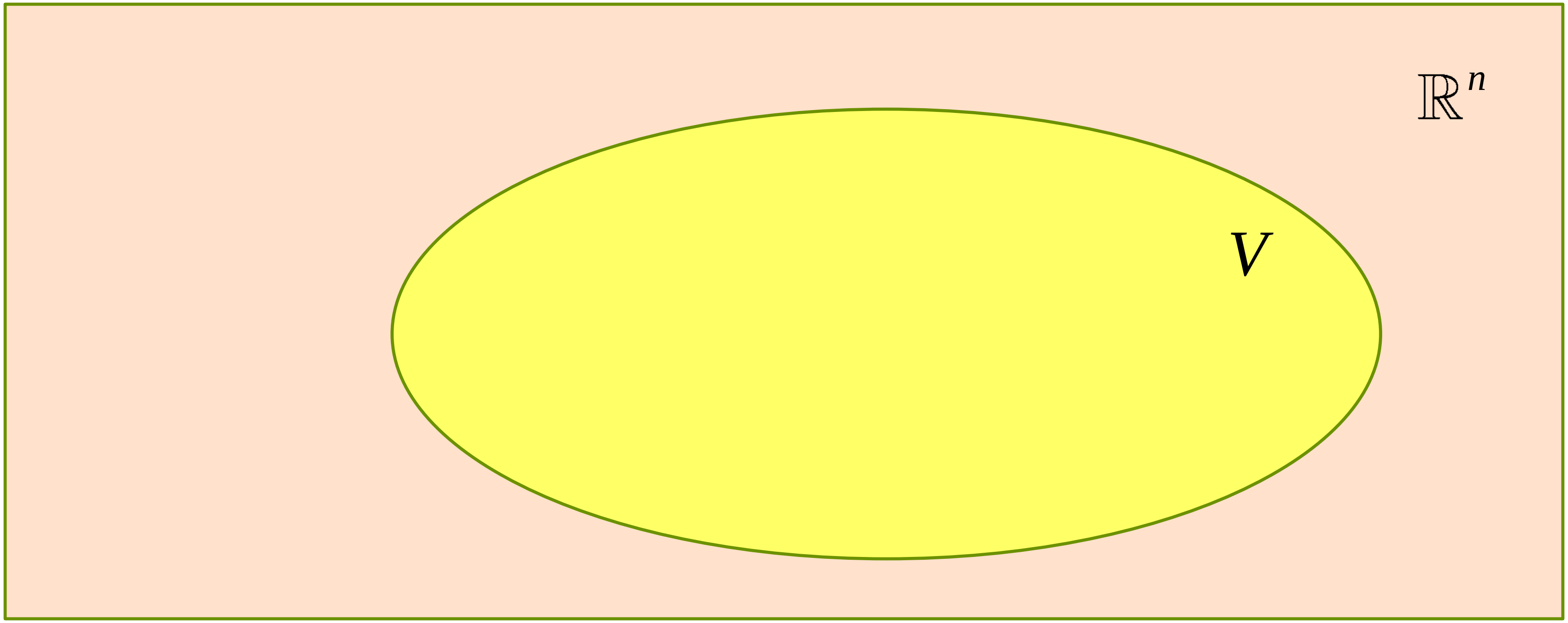
The solution set is the zero space $\{\mathbf{0}\}$.

Abstract definition of subspace

Let V be a non-empty subset of $\mathbb{R}^n$.

Then V is a subspace of $\mathbb{R}^n$ if and only if

for all $\boldsymbol{u}, \mathbf{v} \in V$ and $c, d \in \mathbb{R}$, $c\boldsymbol{u} + d\mathbf{v} \in V$.



Discussion on redundancy

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

We say that $\boldsymbol{u}_k$ is redundant in the span of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$.

I am redundant 

Having me around does

not 'add value' 

$\boldsymbol{u}_k$

Definition (linear independence)

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. Consider the solutions to the following equation (values of $c_1, c_2, \ldots, c_k$)

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \mathbf{0} \qquad (*)$$

1) Clearly, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$ is a solution. This is called the trivial solution to (*).

2) S is called a linearly independent set if (*) has only the trivial solution. In this case, we say that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent vectors.

Definition (linear independence)

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2) S is called a linearly independent set if (*) has only the trivial solution. In this case, we say that $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k$ are linearly independent vectors.

3) S is called a linearly dependent set if (*) has non-trivial solutions. In this case, we say that $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k$ are linearly dependent vectors.

Definition (linear independence)

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. Consider the solutions to the following equation (values of $c_1, c_2, \ldots, c_k$)

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \mathbf{0} \qquad (*)$$

Example (linear independence)

Determine whether $(1,-2,3),(5,6,-1),(3,2,1)$ are linearly independent vectors in $\mathbb{R}^3$.

Vector equation:

Linear system:

Example (linear independence)

Determine whether $(1,-2,3),(5,6,-1),(3,2,1)$ are linearly independent vectors in $\mathbb{R}^3$.

Solving linear system:

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

How many solutions does the linear system have?

$$\begin{cases} a + 5b + 3c = 0 \\ -2a + 6b + 2c = 0 \\ 3a - b + c = 0 \end{cases}$$

Example (linear independence)

Determine whether $(1,-2,3),(5,6,-1),(3,2,1)$ are linearly independent vectors in $\mathbb{R}^3$.

The vectors are linearly dependent.

Solving linear system:

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

How many solutions does the equation have?

$$a(1,-2,3)+b(5,6,-1)+c(3,2,1)=(0,0,0)$$

Example (linear independence)

Determine whether $(1,0,0,1),(0,2,1,0),(1,-1,1,1)$ are linearly independent vectors in $\mathbb{R}^4$.

Vector equation:

Linear system:

Linear independence: 1 or 2 vectors

$S = \{\boldsymbol{u}\}$. When is S a linearly independent set?

When does the equation $c\boldsymbol{u} = \mathbf{0}$ have only the trivial solution $c = 0$?

$S = \{\boldsymbol{u}\}$ is a linearly independent set if and only if $\boldsymbol{u} \neq \mathbf{0}$.

Linear independence: 1 or 2 vectors

$S = \{\boldsymbol{u}, \boldsymbol{v}\}$. When is S a linearly independent set?

When does the equation $c_1\boldsymbol{u} + c_2\boldsymbol{v} = \mathbf{0}$ have non trivial solutions for c_1 and c_2?

$S = \{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly dependent set if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are scalar multiples of each other.

What if a set contains the zero vector?

Let S be a finite set of vectors from $\mathbb{R}^n$. Prove that if $\mathbf{0} \in S$, then S is a linearly dependent set.

Proof:

Theorem (another way to look at linear independence)

Recall the discussion on redundancy.

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$, where $k \geq 2$.

1) S is linearly dependent if and only if at least one $\boldsymbol{u}_i \in S$ can be written as a linear combination of the other vectors in S, that is,

$$\boldsymbol{u}_i = a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + \ldots + a_{i-1}\boldsymbol{u}_{i-1} + a_{i+1}\boldsymbol{u}_{i+1} + \ldots + a_k\boldsymbol{u}_k$$

for some $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \in \mathbb{R}$.

Theorem (another way to look at linear independence)

Recall the discussion on redundancy.

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$, where $k \geq 2$.

2) S is linearly independent if and only if no vector in S can be written as a linear combination of the other vectors in S.

Remark

So a set a vectors is linearly dependent implies that there exists at least one 'redundant' vector in the set.

A set a vectors is linearly independent implies that there is no 'redundant' vector in the set.

Example

$S = \{(2,4),(1,0),(0,3)\}$. Is S a linearly independent set?

$S = \{(1,0,0),(0,2,0),(0,0,-5)\}$. Is S a linearly independent set?

Theorem (guaranteed dependence)

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

If $k > n$, then S is linearly dependent.

Example (guaranteed dependence)

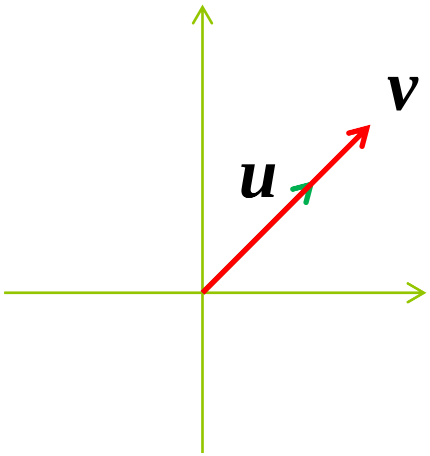
1) A set of three or more vectors in $\mathbb{R}^{\boxed{2}}$ is always linearly dependent.

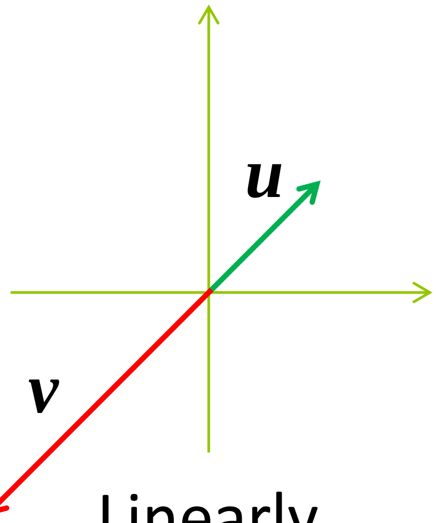
2) A set of four or more vectors in $\mathbb{R}^{\boxed{3}}$ is always linearly dependent.

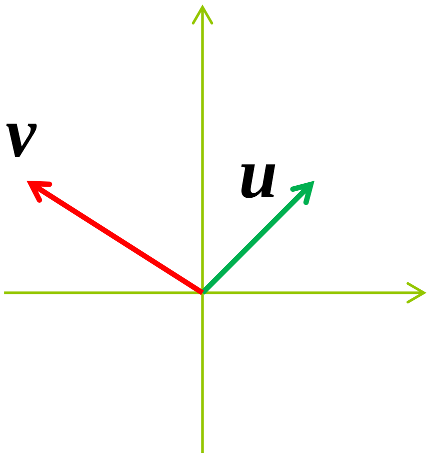
Linear independence (geometrical)

For two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, recall the following:

$S = \{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly dependent set if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are scalar multiples of each other (they lie on the same line).



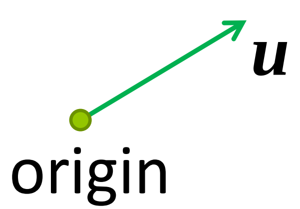


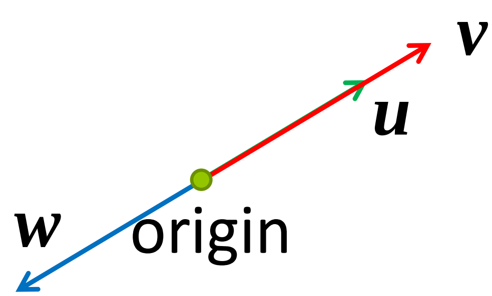


Linear independence (geometrical)

For three vectors in $\mathbb{R}^3$:

$S = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set if and only if they lie on the same line or the same plane (when their initial points are placed at the origin).





$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie on the same line

$\{\boldsymbol{u}\}$ is a linearly independent set

$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly dependent set

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Linear independence (geometrical)

For three vectors in $\mathbb{R}^3$:

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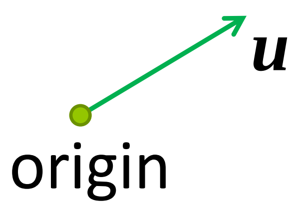
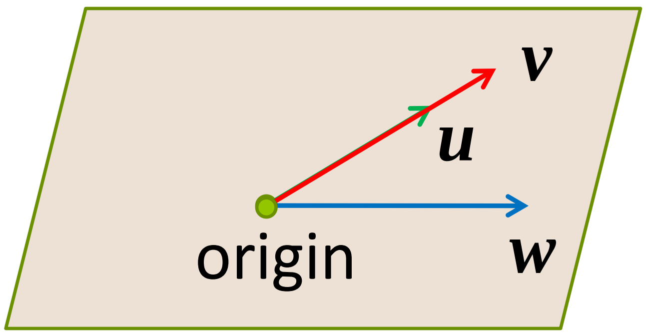




$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie on the same plane

$\{\boldsymbol{u}\}$ is a linearly independent set

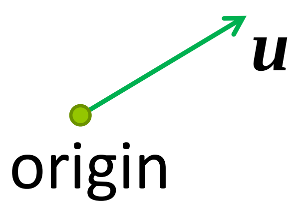
$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly dependent set

$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set

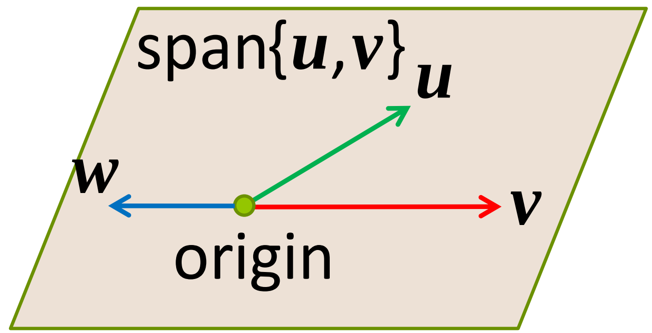
Linear independence (geometrical)

For three vectors in $\mathbb{R}^3$:

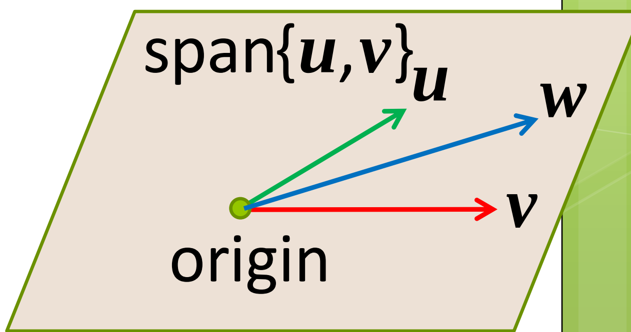
$S = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set if and only if they lie on the same line or the same plane (when their initial points are placed at the origin).



$\{\boldsymbol{u}\}$ is a linearly independent set



$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly independent set



$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set

Linear independence (geometrical)

For three vectors in $\mathbb{R}^3$:

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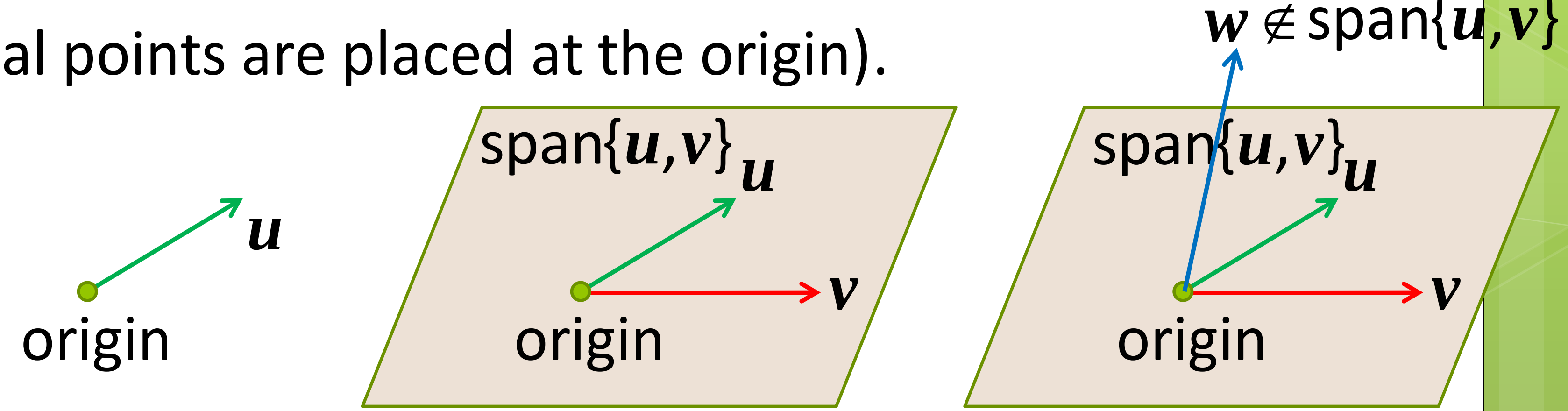

$\{\boldsymbol{u}\}$ is a linearly independent set

$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly independent set

$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly independent set

End of Lecture 10

Lecture 11:

Linear independence (cont'd)

Bases

Dimensions (till Example 3.6.6)