Lecture 19 Recap

1) Transition matrix between orthonormal bases.

2) Definition of an orthogonal matrix.

3) How to check if a matrix is orthogonal.

4) Equivalent statements to '$\boldsymbol{A}$ is an orthogonal square matrix of order n'.

5) Orthogonal diagonalization. Characterization of matrices that can be orthogonally diagonalized.

6) An algorithm to orthogonally diagonalize a symmetric matrix and some remarks.

Lecture 20

Linear transformations from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$

Definition

A linear transformation is a mapping $T:\mathbb{R}^n \to \mathbb{R}^m$

of the form

$$T\left(\begin{pmatrix} \boxed{x_1} \\ \boxed{x_2} \\ \vdots \\ \boxed{x_n} \end{pmatrix}\right) = \begin{pmatrix} \boxed{a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n} \\ \boxed{a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n} \\ \vdots \\ \boxed{a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n} \end{pmatrix} \qquad \forall \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

n components: vector in $\mathbb{R}^n$

m components: vector in $\mathbb{R}^m$

where $a_{11}, a_{12}, \ldots, a_{mn}$ are real numbers.

Definition

A linear transformation is a mapping $T:\mathbb{R}^n \to \mathbb{R}^m$

of the form

If $m=n$, then T is a linear operator on $\mathbb{R}^n$.

$$T\left(\begin{pmatrix} \boxed{x_1} \\ \boxed{x_2} \\ \vdots \\ \boxed{x_n} \end{pmatrix}\right) = \begin{pmatrix} \boxed{a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n} \\ \boxed{a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n} \\ \vdots \\ \boxed{a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n} \end{pmatrix} \quad \forall \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

so any transformation where each component of the 'output' vector is just some linear expression of the components of the 'input' vector is a linear transformation.

Definition

formula for T

$$T\left(\begin{pmatrix} \boxed{x_1} \\ \boxed{x_2} \\ \vdots \\ \boxed{x_n} \end{pmatrix}\right) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{pmatrix} \quad \forall \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \boxed{x_1} \\ \boxed{x_2} \\ \vdots \\ \boxed{x_n} \end{pmatrix}$$

$T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$

for all $\boldsymbol{x} \in \mathbb{R}^n$

Definition

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \Leftrightarrow T(\boldsymbol{x}) = \boldsymbol{Ax} \;\; \forall \boldsymbol{x} \in \mathbb{R}^n$$

The $m \times n$ matrix $\boldsymbol{A}$ is called the standard matrix for T.

so any transformation T where a matrix $\boldsymbol{A}$ can be found to 'represent' what T does (by pre-multiplying $\boldsymbol{A}$ to any 'input' vector) is a linear transformation.

Example

Let $I:\mathbb{R}^n \to \mathbb{R}^n$ be the transformation defined by

$$I(\boldsymbol{u}) = \boldsymbol{u}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'what you put into I,

you get back the same thing'

For all $\boldsymbol{u} \in \mathbb{R}^n$,

$$\boxed{I(\boldsymbol{u}) = \boldsymbol{u} = \boldsymbol{I}_n\boldsymbol{u}}$$

I is called the

identity transformation.

So I is a linear transformation and $\boldsymbol{I}_n$ is the standard matrix for I.

Example

Let $O : \mathbb{R}^n \to \mathbb{R}^m$ be the transformation defined by

$$O(\boldsymbol{u}) = \mathbf{0}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'whatever you put into O, you get back the zero vector'

For all $\boldsymbol{u} \in \mathbb{R}^n$,

$$\boxed{O(\boldsymbol{u}) = \mathbf{0} = \mathbf{0}_{m\times n}\boldsymbol{u}}$$

O is called the zero transformation.

So O is a linear transformation and $\mathbf{0}_{m\times n}$ is the standard matrix for O.

Example

Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 2x \\ -3y \end{pmatrix}, \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

(Abstract) Definition

For abstract vector spaces:

$T: V \to W$ is a **linear transformation** if and only if

$$T(a\mathbf{u}+b\mathbf{v}) = aT(\mathbf{u})+bT(\mathbf{v})$$

for all $\mathbf{u},\mathbf{v} \in V$ and $a,b \in \mathbb{R}$.

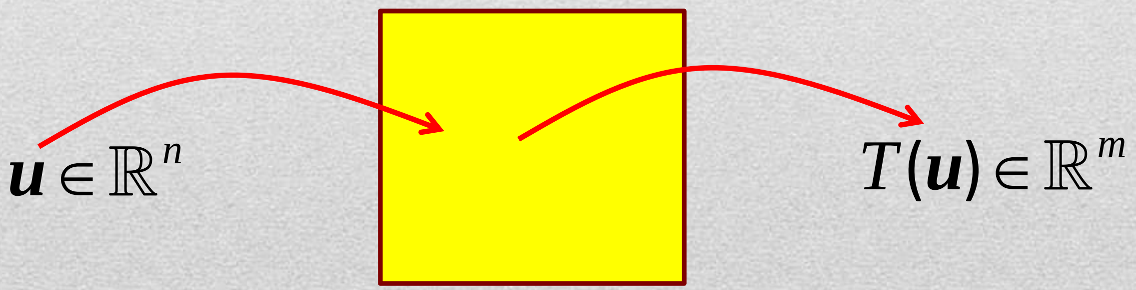
linearly combine first then transform....

$=$ transform first then linearly combine...

A remark about transformations

Transformations are like functions, or sometimes can be considered to be like a 'black box' where you put in something (in our case a vector) and the black box gives you something back in return.

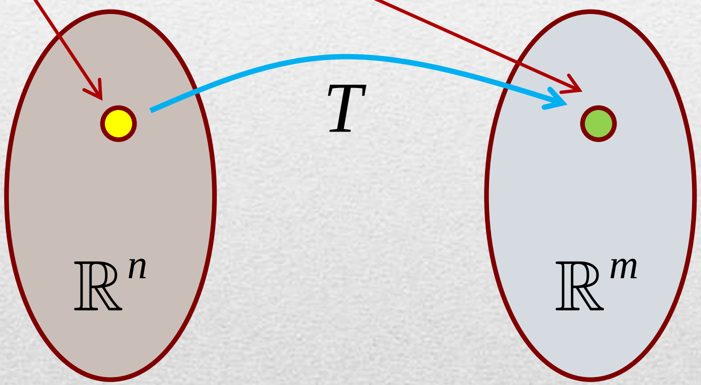
$$T:\mathbb{R}^n \rightarrow \mathbb{R}^m$$



Theorem

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

1) $T(\mathbf{0}) = \mathbf{0}$



'zero' vector must be mapped to 'zero' vector

Theorem

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

2) If $\boldsymbol{u}_1\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k \in \mathbb{R}^n$ and $c_1,c_2,\ldots,c_k \in \mathbb{R}$, then

$$T(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k) = c_1T(\boldsymbol{u}_1) + c_2T(\boldsymbol{u}_2) + \ldots + c_kT(\boldsymbol{u}_k)$$

Recall: $\boxed{T(a\boldsymbol{u}+b\boldsymbol{v})} = \boxed{aT(\boldsymbol{u})+bT(\boldsymbol{v})}$

for all $\boldsymbol{u},\boldsymbol{v} \in V$ and $a,b \in \mathbb{R}$.

linearly combine first then transform....

= transform first then linearly combine...

Example

Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+3 \\ y-3 \end{pmatrix}, \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Example

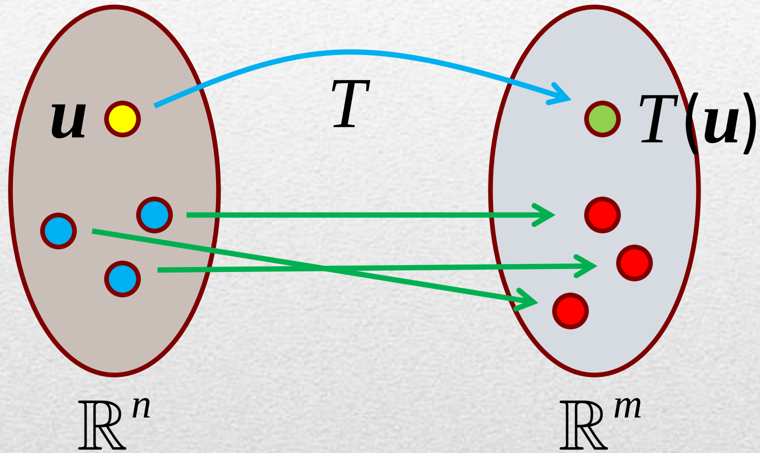
Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation defined by

$$T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x^2 \\ yz \end{pmatrix}, \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

Discussion

Suppose $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.



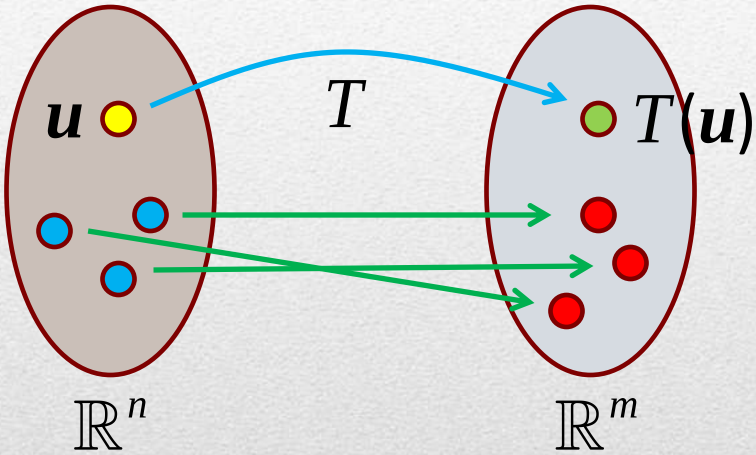
Suppose we know $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$.

What information do we need to know about T before we can determine $T(\boldsymbol{u})$ for every vector $\boldsymbol{u} \in \mathbb{R}^n$?

Discussion

Suppose $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.



Suppose we know $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$.

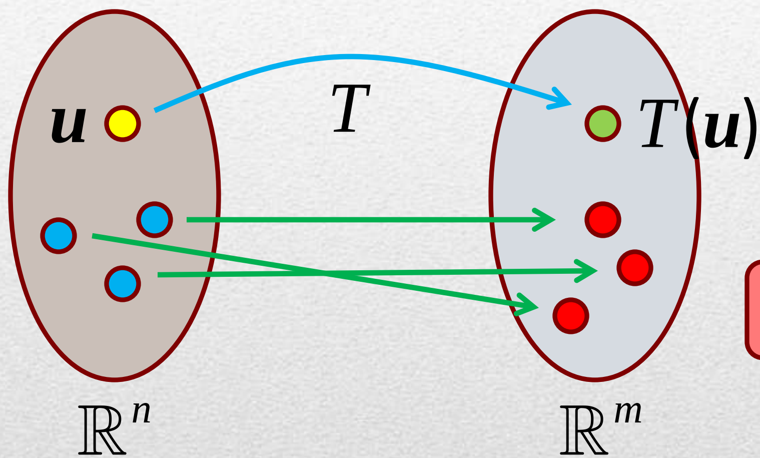
Since $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is a basis for $\mathbb{R}^n$, for any vector $\boldsymbol{u} \in \mathbb{R}^n$, we can write

$$\boldsymbol{u} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_n\boldsymbol{u}_n \quad \text{for some } c_1, c_2, \ldots, c_n \in \mathbb{R}.$$

Discussion

Suppose $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.



Suppose we know $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$.

$\boldsymbol{u} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_n\boldsymbol{u}_n$ for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

$$\Rightarrow T(\boldsymbol{u}) = T(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_n\boldsymbol{u}_n)$$

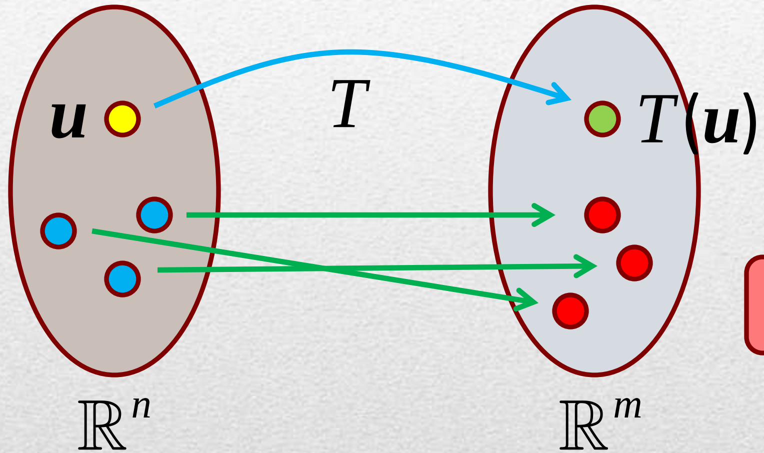
$$= c_1 T(\boldsymbol{u}_1) + c_2 T(\boldsymbol{u}_2) + \ldots + c_n T(\boldsymbol{u}_n)$$

$T(\boldsymbol{u})$ can be completely determined since we know $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$.

Discussion

Suppose $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.



Suppose we know $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$.

The linear transformation T is can be completely determined if we know the 'images' $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$ for any basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ of $\mathbb{R}^n$.

Example

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \qquad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix} \qquad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

(So you only know how T transforms these 3 vectors.)

1) Find the image of the vector $\begin{pmatrix}-1\\4\\6\end{pmatrix}$ under T.

2) Find the formula of T.

Example (solution)

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \qquad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix} \qquad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

(So you only know how T transforms these 3 vectors.)

If the three vectors that we know how T transforms forms a basis for $\mathbb{R}^3$, we will be able to determine T completely.

Example (solution)

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \qquad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix} \qquad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

$\begin{vmatrix}1&1&1\\0&1&1\\2&0&-1\end{vmatrix}\neq 0,$ so $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\-1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$,

and we have enough information to determine T completely.

Example (solution)

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \qquad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix} \qquad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

$$\begin{pmatrix}-1\\4\\6\end{pmatrix}=a\begin{pmatrix}1\\1\\1\end{pmatrix}+b\begin{pmatrix}0\\1\\1\end{pmatrix}+c\begin{pmatrix}2\\0\\-1\end{pmatrix} \Leftrightarrow \begin{cases} a \quad\quad\quad + 2c = -1 \\ a + b \quad\quad\quad = 4 \\ a + b - c = 6 \end{cases}$$

Solving gives: $\boxed{a=3,\ b=1,\ c=-2.}$

Example (solution)

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \qquad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix} \qquad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

$$\begin{pmatrix}-1\\4\\6\end{pmatrix}=3\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}0\\1\\1\end{pmatrix}-2\begin{pmatrix}2\\0\\-1\end{pmatrix}$$ so $$T\left(\begin{pmatrix}-1\\4\\6\end{pmatrix}\right)=3T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)+T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)-2T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)$$

$$=$$

$$\begin{pmatrix}-6\\13\end{pmatrix}=3\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}-1\\2\end{pmatrix}-2\begin{pmatrix}4\\-1\end{pmatrix}$$

Example (solution)

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \qquad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix} \qquad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

2) Find the formula of T.

To find the formula for T, we need to find out how T transforms an arbitrary vector from $\mathbb{R}^3$.

Example (solution)

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \qquad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix} \qquad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=a\begin{pmatrix}1\\1\\1\end{pmatrix}+b\begin{pmatrix}0\\1\\1\end{pmatrix}+c\begin{pmatrix}2\\0\\-1\end{pmatrix} \Leftrightarrow \begin{cases} a \qquad\qquad + 2c = x \\ a + b \qquad\qquad = y \\ a + b - c = z \end{cases}$$

arbitrary vector in $\mathbb{R}^3$

Solving gives: $a = x-2y+2z$
$b=-x+3y-2z \qquad c=y-z$

Example (solution)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x-2y+2z)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-x+3y-2z)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (y-z)\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = (x-2y+2z)\begin{pmatrix} 1 \\ 3 \end{pmatrix} + (-x+3y-2z)\begin{pmatrix} -1 \\ 2 \end{pmatrix} + (y-z)\begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} x-2y+2z+x-3y+2z+4y-4z \\ 3x-6y+6z-2x+6y-4z-y+z \end{pmatrix} = \begin{pmatrix} 2x-y \\ x-y+3z \end{pmatrix}$$

Discussion

Suppose $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $\boldsymbol{A}$ is the standard matrix of T.

Recall the following observation when we were studying matrix multiplication:

ith position

$$\begin{pmatrix} * & \cdots & * & \cdots & * \\ * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ \vdots \\ * \end{pmatrix}$$

ith column

Discussion

Suppose $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and

$\boldsymbol{A}$ is the standard matrix of T. $\boxed{\boldsymbol{A} = \begin{pmatrix} T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & \cdots & T(\boldsymbol{e}_n) \end{pmatrix}}$

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ be the standard basis for $\mathbb{R}^n$.

For $i = 1, 2, \ldots, n$,

$$\underline{T(\boldsymbol{e}_i)} = \boldsymbol{A}\boldsymbol{e}_i = \begin{pmatrix} * & \cdots & * & \cdots & * \\ * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * \end{pmatrix} \begin{pmatrix} \\ \boldsymbol{e}_i \\ \\ \end{pmatrix} = \underline{i\text{th column of } \boldsymbol{A}}$$

So the columns of $\boldsymbol{A}$ gives us the 'images' of the standard basis vectors transformed by T.

Example

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

$$T\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad T\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad T\left(\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

(So you only know how T transforms these 3 vectors.)

1) Find the image of the vector $\begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix}$ under T.

2) Find the formula of T.

Example

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

$$T\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}\qquad T\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix}\qquad T\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}4\\-1\end{pmatrix}$$

We first try to write $\boldsymbol{e}_{1,}\boldsymbol{e}_2,\boldsymbol{e}_3$ as a linear combination of

$$\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\-1\end{pmatrix}$$

Example

We first try to write $\boldsymbol{e}_{1,}\boldsymbol{e}_2,\boldsymbol{e}_3$ as a linear combination of

$$\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}2\\0\\-1\end{pmatrix}$$

$$\begin{array}{c}\left(\begin{array}{ccc|c|c|c}1&0&2&1&0&0\\1&1&0&0&1&0\\1&1&-1&0&0&1\end{array}\right)\\ \begin{array}{cccccc}&&&\boldsymbol{e}_1&\boldsymbol{e}_2&\boldsymbol{e}_3\end{array}\end{array} \xrightarrow{\text{GJE}} \left(\begin{array}{ccc|c|c|c}1&0&0&1&-2&2\\0&1&0&-1&3&-2\\0&0&1&0&1&-1\end{array}\right)$$

Example

We first try to write $\boldsymbol{e}_{1,}\boldsymbol{e}_2,\boldsymbol{e}_3$ as a linear combination of

$$\begin{array}{c} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \\ \qquad\qquad\qquad \boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_3 \end{array} \xrightarrow{\text{GJE}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$\boldsymbol{e}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix} \quad \boldsymbol{e}_2 = -2\begin{pmatrix}1\\1\\1\end{pmatrix} + 3\begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{pmatrix}2\\0\\-1\end{pmatrix} \quad \boldsymbol{e}_3 = 2\begin{pmatrix}1\\1\\1\end{pmatrix} - 2\begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}2\\0\\-1\end{pmatrix}$$

Example

We first try to write $\boldsymbol{e}_{1,}\boldsymbol{e}_2,\boldsymbol{e}_3$ as a linear combination of

$$\begin{array}{c}\left(\begin{array}{ccc|c|c|c} \boxed{1} & \boxed{0} & \boxed{2} & 1 & 0 & 0 \\ \boxed{1} & \boxed{1} & \boxed{0} & 0 & 1 & 0 \\ \boxed{1} & \boxed{1} & \boxed{-1} & 0 & 0 & 1 \end{array}\right) \\ \qquad\qquad\quad \boldsymbol{e}_1 \ \ \boldsymbol{e}_2 \ \ \boldsymbol{e}_3 \end{array} \xrightarrow{\text{GJE}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$\boldsymbol{e}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix} \quad T(\boldsymbol{e}_1) = T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) - T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = \begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

Example

We first try to write $\boldsymbol{e}_{1,}\boldsymbol{e}_2,\boldsymbol{e}_3$ as a linear combination of

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{GJE}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$\begin{array}{ccc} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{array}$$

Similarly,

$$T(\boldsymbol{e}_1)=\begin{pmatrix}2\\1\end{pmatrix} \quad T(\boldsymbol{e}_2)=\begin{pmatrix}-1\\-1\end{pmatrix} \quad T(\boldsymbol{e}_3)=\begin{pmatrix}0\\3\end{pmatrix}$$

So the standard matrix is:

$$\begin{pmatrix}2 & -1 & 0\\1 & -1 & 3\end{pmatrix}$$

Example

Let $T:\mathbb{R}^3 \to \mathbb{R}^4$ be the transformation such that

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x-y \\ x-y+3z \\ -5x+y \\ x+z \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

So the standard matrix for T is

$$\boldsymbol{A} = \begin{pmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ -5 \\ 1 \end{matrix}} & \boxed{\begin{matrix} -1 \\ -1 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 3 \\ 0 \\ 1 \end{matrix}} \end{pmatrix}$$

$$T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \boxed{\begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix}} \qquad T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \boxed{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}} \qquad T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \boxed{\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}}$$

Example

Let $T:\mathbb{R}^3 \to \mathbb{R}^4$ be the transformation such that

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x-y \\ x-y+3z \\ -5x+y \\ x+z \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

So the standard matrix for T is

$$\boldsymbol{A} = \begin{pmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ -5 \\ 1 \end{matrix}} & \boxed{\begin{matrix} -1 \\ -1 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 3 \\ 0 \\ 1 \end{matrix}} \end{pmatrix}$$

Note that: $= \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

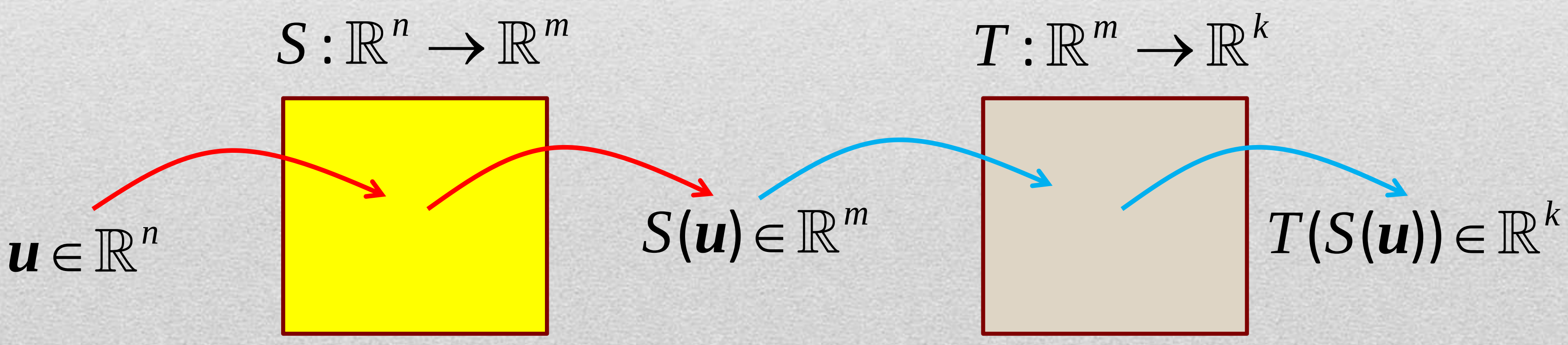
$$\boxed{T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} \quad \forall \boldsymbol{x} \in \mathbb{R}^3}$$

Discussion

Linear transformations are similar to functions.

Functions can be composed, what about linear transformations?

$S:\mathbb{R}^n \to \mathbb{R}^m$ and $T:\mathbb{R}^m \to \mathbb{R}^k$ are linear transformations.



Definition

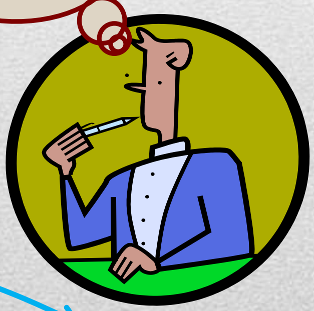
Let $S:\mathbb{R}^n \to \mathbb{R}^m$ and $T:\mathbb{R}^m \to \mathbb{R}^k$ be linear transformations.

The composition of T with S, denoted by $T \circ S$,

is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ such that

$$(T \circ S)(\boldsymbol{u}) = T(S(\boldsymbol{u})) \qquad \forall \boldsymbol{u} \in \mathbb{R}^n.$$

is this a linear transformation?



$S:\mathbb{R}^n \to \mathbb{R}^m$ $\qquad$ $T:\mathbb{R}^m \to \mathbb{R}^k$

$\boldsymbol{u} \in \mathbb{R}^n$ $\qquad$ $S(\boldsymbol{u}) \in \mathbb{R}^m$ $\qquad$ $T(S(\boldsymbol{u})) \in \mathbb{R}^k$

Theorem

If $S:\mathbb{R}^n \to \mathbb{R}^m$ and $T:\mathbb{R}^m \to \mathbb{R}^k$ are linear transformations, then $T \circ S:\mathbb{R}^n \to \mathbb{R}^k$ is again a linear transformation.

Furthermore, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are the standard matrices for S and T respectively, then the standard matrix for $T \circ S$ is $\boldsymbol{BA}$.

That is,

$$T \circ S(\boldsymbol{u}) = \boldsymbol{BAu} \qquad \forall \boldsymbol{u} \in \mathbb{R}^n.$$

Example

Let $S:\mathbb{R}^3 \to \mathbb{R}^2$ be the transformation such that

standard matrix for S:

$$S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

and let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be the transformation such that

standard matrix for T:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ y \\ x \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Example

$$S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$$

Then $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation given by:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ y \\ x \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

$$T \circ S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = T\left(S\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$$

$$= T\left(\begin{pmatrix} x+y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ z \\ x+y \end{pmatrix}$$

$$T \circ S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ z \\ x+y \end{pmatrix}$$

Example

Standard matrix for $T \circ S$:

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

standard matrix for S:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

standard matrix for T:

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example

Standard matrix for $T \circ S$:

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Note that:

$$T \circ S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ z \\ x+y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

End of Lecture 20

Lecture 21:

Ranges and Kernels (end of Section 7.2)