LECTURE 04 RECAP

Definition of matrices, entries, size of a matrix.

Special types of matrices.

Matrix operations and some laws.

Matrix multiplication and differences with real number multiplication.

Different ways to represent matrix multiplication.

Representing a linear system by a matrix equation.

Matrix transpose and some laws.

LECTURE 05

Inverses of square matrices

Elementary matrices

FOR REAL NUMBERS...

If x is a real number, it is easy to solve

$$2x = 5,$$

since corresponding to '2', there is another number '$\frac{1}{2}$' such that $2 \times \frac{1}{2} = 1$, allowing us to have

$$\frac{1}{2} \times 2x = \frac{1}{2} \times 5$$

$$\Rightarrow 1 \times x = \frac{5}{2}$$

$$\Rightarrow x = \frac{5}{2}$$

WHAT ABOUT FOR MATRICES?

If $\boldsymbol{X}$ is a matrix, how can we solve

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} 2 & 1 & -1 \\ 3 & -5 & 0 \\ 0 & 1 & 4 \end{pmatrix}?$$

Is there a matrix $\begin{pmatrix} \boxed{?} \end{pmatrix}$ such that

$$\begin{pmatrix} \boxed{?} \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} \boxed{?} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 3 & -5 & 0 \\ 0 & 1 & 4 \end{pmatrix}$$

WHAT ABOUT FOR MATRICES?

$$\left(\boxed{?}\right)\begin{pmatrix}2&-1&3\\0&1&3\\3&1&2\end{pmatrix}\boldsymbol{X}=\left(\boxed{?}\right)\begin{pmatrix}2&1&-1\\3&-5&0\\0&1&4\end{pmatrix}$$

$$\Rightarrow\left(?\right)\boldsymbol{X}=\left(\boxed{?}\right)\begin{pmatrix}2&1&-1\\3&-5&0\\0&1&4\end{pmatrix}$$

$?=\boldsymbol{I}$ (identity matrix)

$$\Rightarrow\boldsymbol{X}=\left(\boxed{?}\right)\begin{pmatrix}2&1&-1\\3&-5&0\\0&1&4\end{pmatrix}$$

DEFINITION (INVERTIBLE MATRICES)

Let $\boldsymbol{A}$ be a square matrix of order n.

$\boldsymbol{A}$ is said to be an invertible matrix if there exists another square matrix $\boldsymbol{B}$ of the same order such that

$$\boxed{\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}_n}$$

If such a $\boldsymbol{B}$ exists, it is called an inverse of $\boldsymbol{A}$.

$\boldsymbol{A}$ is said to be singular if it has no inverse.

BEFORE WE PROCEED

The definition of an invertible matrix is an existential one.

"I tried very hard to find $\boldsymbol{B}$ but I could not..."

...does not mean $\boldsymbol{B}$ does not exists!

Could there be more than one such $\boldsymbol{B}$?

A SIMPLE QUESTION

Is $\begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$ the inverse of $\begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix}$?

Check $\begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

Check $\begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix}\begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

CAN YOU SOLVE THIS NOW?

Find $\boldsymbol{X}$ if $\begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{IX} = \begin{pmatrix} -7 & 2 & -3 \\ 8 & -2 & 2 \\ 9 & -3 & 5 \end{pmatrix} \quad \Rightarrow \boldsymbol{X} = \begin{pmatrix} -7 & 2 & -3 \\ 8 & -2 & 2 \\ 9 & -3 & 5 \end{pmatrix}.$$

A SINGULAR MATRIX

Show that $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is singular.

Suppose $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is invertible, then there must be a square matrix of order 2, say $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

(By definition of inverse)

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

A SINGULAR MATRIX

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a+b & 0 \\ c+d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ a contradiction, by looking at the (2,2)-entry

Thus it is impossible for $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ to have an inverse,

and so the matrix has to be singular.

WOW, THIS IS SO HARD!

We knew $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix}$ is invertible because

we were 'given' $\boldsymbol{B} = \begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$ to 'test' whether

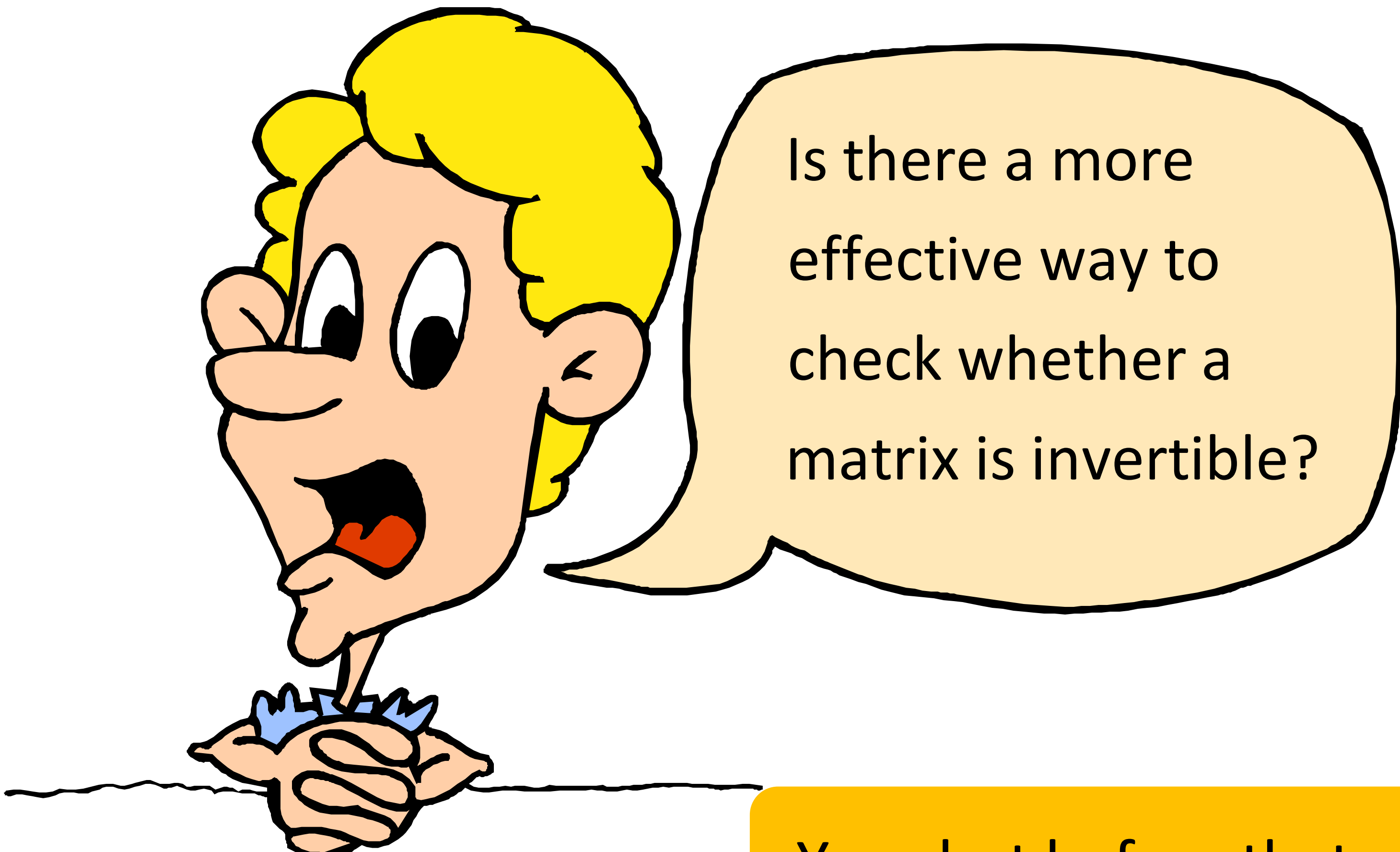
$\boldsymbol{AB}$ and $\boldsymbol{BA}$ are both equal to $\boldsymbol{I}$.

We had to use contradiction to show that $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ has

no inverse. What if the matrix was bigger and more

complicated?

WOW, THIS IS SO HARD!



Yes...but before that...

Cancellation Laws

If $\boldsymbol{A}$ is an invertible square matrix and

$$\boldsymbol{A}\boldsymbol{B}_1 = \boldsymbol{A}\boldsymbol{B}_2$$

then $\boldsymbol{B}_1 = \boldsymbol{B}_2$.

If $\boldsymbol{A}$ is an invertible square matrix and

$$\boldsymbol{C}_1\boldsymbol{A} = \boldsymbol{C}_2\boldsymbol{A}$$

then $\boldsymbol{C}_1 = \boldsymbol{C}_2$.

WARNING! DON'T ANYHOW CANCEL!

If $\boldsymbol{A}$ is not an invertible matrix, the cancellation law may not hold:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$ (known to be singular by previous example)

$$\boldsymbol{B}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Check that $\boldsymbol{AB}_1 = \boldsymbol{AB}_2$, but $\boldsymbol{B}_1 \neq \boldsymbol{B}_2$.

$$\boldsymbol{B}_2 = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

THEOREM (UNIQUENESS OF INVERSE)

If $\boldsymbol{B}$ and $\boldsymbol{C}$ are inverses of a square matrix $\boldsymbol{A}$, then

$$\boldsymbol{B} = \boldsymbol{C}.$$

That is, if $\boldsymbol{A}$ is an invertible square matrix, then it has one and only one inverse.

Inverses are unique!

Since inverses are unique, we will write $\boxed{\boldsymbol{A}^{-1}}$ as the inverse of $\boldsymbol{A}$ if $\boldsymbol{A}$ is invertible.

50% DISCOUNT…

It turns out that to check whether a given square matrix $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$, we only need to check either

$$\boldsymbol{AB} = \boldsymbol{I} \quad \boxed{\text{OR}} \quad \boldsymbol{BA} = \boldsymbol{I}$$

The reason will be explained in Section 2.4.

EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that if $ad - bc \neq 0$, then

$\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$.

Solution:

Check $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$

EXAMPLE

Let $\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that if $ad-bc\neq\ 0$, then

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Solution:

Check $\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

THEOREM (LAWS INVOLVING INVERSES)

1) Let $\boldsymbol{A}$ be an invertible matrix (so $\boldsymbol{A}^{-1}$ exists) and c a non zero scalar. Then $c\boldsymbol{A}$ is invertible and

$$(c\boldsymbol{A})^{-1} = \tfrac{1}{c}\boldsymbol{A}^{-1}.$$

THEOREM (LAWS INVOLVING INVERSES)

2) Let $\boldsymbol{A}$ be an invertible matrix (so $\boldsymbol{A}^{-1}$ exists) then

$\boldsymbol{A}^{T}$ is invertible and

$$(\boldsymbol{A}^{T})^{-1} = (\boldsymbol{A}^{-1})^{T}.$$

Inverse of transpose equal transpose of inverse

THEOREM (LAWS INVOLVING INVERSES)

3) Let $\boldsymbol{A}$ be an invertible matrix (so $\boldsymbol{A}^{-1}$ exists) then $\boldsymbol{A}^{-1}$ is invertible and

$$(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}.$$

THEOREM (LAWS INVOLVING INVERSES)

4) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two invertible matrices of the same size (so $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$ exists) then $\boldsymbol{AB}$ is invertible and

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}.$$

Inverse of product equal product of inverse*

THEOREM (LAWS INVOLVING INVERSES)

4) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two invertible matrices of the same size (so $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$ exists) then $\boldsymbol{AB}$ is invertible and

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}.$$

As an extension to the above, if $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_k$ are invertible matrices of the same size, then $(\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_k)$ is invertible and

$$\boxed{(\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_k)^{-1} = \boldsymbol{A}_k{}^{-1}\ldots\boldsymbol{A}_2{}^{-1}\boldsymbol{A}_1{}^{-1}.}$$

NEGATIVE POWERS

Let $\boldsymbol{A}$ be a square matrix and n be a non negative integer, then

$$\boldsymbol{A}^n = \begin{cases} \boldsymbol{I} & \text{if } n = 0; \\ \underbrace{\boldsymbol{AA}\ldots\boldsymbol{A}}_{n \text{ times}} & \text{if } n \geq 1. \end{cases}$$

If $\boldsymbol{A}$ is an invertible square matrix, then $\boldsymbol{A}^n$ is invertible and we define

$$\boldsymbol{A}^{-n} = (\boldsymbol{A}^{-1})^n = \underbrace{\boldsymbol{A}^{-1}\boldsymbol{A}^{-1}\ldots\boldsymbol{A}^{-1}}_{n \text{ times}} \quad \text{and } (\boldsymbol{A}^n)^{-1} = \boldsymbol{A}^{-n}$$

Example (Powers)

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \boldsymbol{A}^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

$$\boldsymbol{A}^2 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

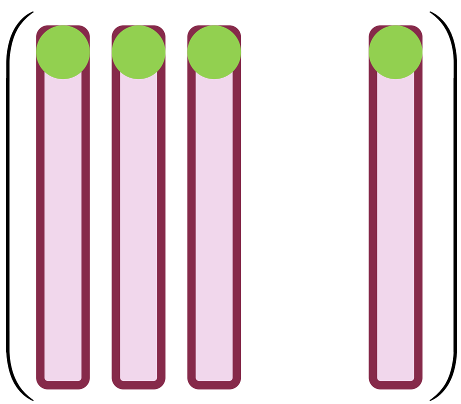
$$\boldsymbol{A}^{-2} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

DISCUSSION

Let $\boldsymbol{B}$ be a $m \times n$ matrix.

$$\boldsymbol{a}_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

$$\boldsymbol{a}_1 \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$ 

$$= \begin{pmatrix} \bullet & \bullet & \bullet & \dots & \bullet \end{pmatrix}$$

$= \text{first row of } \boldsymbol{B}$

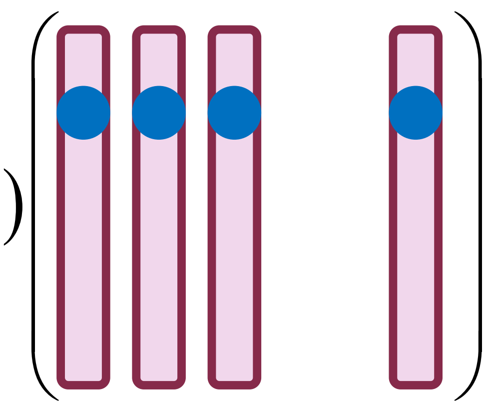
DISCUSSION

Let $\boldsymbol{B}$ be a $m \times n$ matrix.

$$\boldsymbol{a}_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} \quad \end{pmatrix}$$

$$\boldsymbol{a}_2 \boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \end{pmatrix}$$



$$= \begin{pmatrix} \bullet & \bullet & \bullet & \dots & \bullet \end{pmatrix}$$

$=$ second row of $\boldsymbol{B}$

DISCUSSION

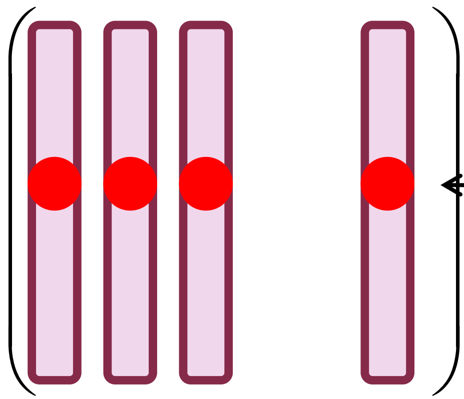
Let $\boldsymbol{B}$ be a $m \times n$ matrix.

$$\boldsymbol{a}_i = \begin{pmatrix} 0 & \dots & c & \dots & 0 \end{pmatrix}$$

ith position

$$\boldsymbol{B} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

$$\boldsymbol{a}_i\boldsymbol{B} = \begin{pmatrix} 0 & \dots & c & \dots & 0 \end{pmatrix}$$



ith row

$$= c\begin{pmatrix} \bullet & \bullet & \bullet & \dots & \bullet \end{pmatrix}$$

$= c$ times ith row of $\boldsymbol{B}$

DISCUSSION

Let $\boldsymbol{B}$ be a $m \times n$ matrix. What is the product equal to?

$$\begin{array}{cc}
 & \overbrace{\hphantom{1 \quad 0 \quad \cdots \quad \cdots \quad \cdots \quad 0}}^{m\text{ columns}} \\
\begin{array}{r} \\ i\text{th row} \\ \\ m \\ \text{rows} \\ \\ \end{array}\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} &
\left(\begin{array}{cccccc}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & \cdots & \cdots & 0 \\
\vdots & & c & & & \vdots \\
\vdots & & & 1 & & \vdots \\
\vdots & & & & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 1
\end{array}\right)
\end{array}
\left(\begin{array}{c} \\ \boldsymbol{B} \\ \\ \end{array}\right)
= \left(\begin{array}{c} \\ \boxed{c \times \text{row } i} \\ \\ \end{array}\right)$$

DISCUSSION

Note that the product can be obtained by performing elementary row operation cR_i to $\boldsymbol{B}$.

$$
\begin{array}{r c}
 & \\
 & \\
i\text{th row} & \\
 & \\
 & \\
 &
\end{array}
\begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & & \cdots & \cdots & 0 \\
\vdots & & c & & & \vdots \\
\vdots & & & 1 & & \vdots \\
\vdots & & & & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
 \\
 \\
\boldsymbol{B} \\
 \\
 \\
\end{pmatrix}
$$

$$
\boldsymbol{B} \xrightarrow{cR_i} = \begin{pmatrix}
 \\
\boxed{c \times \text{row } i} \\
 \\
 \\
\end{pmatrix}
$$

DISCUSSION

Note that the matrix $\boxed{}$ can also be obtained by performing cR_i on $\boldsymbol{I}_m$.

$$\boldsymbol{I}_m \xrightarrow{cR_i}$$

$$\begin{array}{r c} & \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & \cdots & \cdots & 0 \\ \vdots & & c & & & \vdots \\ \vdots & & & 1 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} \\ \\ & \boldsymbol{B} & \\ \\ \\ \end{pmatrix} \\ i\text{th row} & \end{array}$$

$$\boldsymbol{B} \xrightarrow{cR_i} = \begin{pmatrix} \\ \boxed{c \times \text{row } i} \\ \\ \end{pmatrix}$$

DISCUSSION

It seems like we can 'represent' performing the elementary row operation cR_i on $\boldsymbol{B}$ by pre-multiplying a suitable matrix to $\boldsymbol{B}$.

$$\boldsymbol{B} \xrightarrow{\quad cR_i \quad} \boldsymbol{C}$$

$$\boldsymbol{I}_m \xrightarrow{\quad cR_i \quad} \boldsymbol{E}$$

Then we have $\boldsymbol{EB} = \boldsymbol{C}$.

Let's investigate the other two types of elementary row operations!

DISCUSSION

Let $\boldsymbol{B}$ be a $m\times n$ matrix. What is the product equal to?

$$\begin{array}{rc} & \begin{array}{ccccccc} & & i\text{th col} & & j\text{th col} & & \end{array} \\ \begin{array}{r} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array} \begin{pmatrix} \\ \\ \boldsymbol{B} \\ \\ \\ \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{\text{row } j \text{ of } \boldsymbol{B}} \\ \\ \boxed{\text{row } i \text{ of } \boldsymbol{B}} \\ \\ \end{pmatrix}$$

DISCUSSION

Note that the product can be obtained by performing elementary row operation $R_i \leftrightarrow R_j$ to $\boldsymbol{B}$.

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \\ \\ & \boldsymbol{B} & \\ \\ \\ \end{pmatrix}$$

$$\boldsymbol{B} \xrightarrow{R_i \leftrightarrow R_j} = \begin{pmatrix} \boxed{\text{row } j \text{ of } \boldsymbol{B}} \\ \\ \boxed{\text{row } i \text{ of } \boldsymbol{B}} \\ \\ \end{pmatrix}$$

DISCUSSION

Note that the matrix $\boxed{}$ can also be obtained by performing $R_i \leftrightarrow R_j$ on $\boldsymbol{I}_m$.

$$\boldsymbol{I}_m \xrightarrow{R_i \leftrightarrow R_j} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} \\ \\ \boldsymbol{B} \\ \\ \\ \end{pmatrix}$$

$$\boldsymbol{B} \xrightarrow{R_i \leftrightarrow R_j} \; = \begin{pmatrix} \\ \boxed{\text{row } j \text{ of } \boldsymbol{B}} \\ \\ \boxed{\text{row } i \text{ of } \boldsymbol{B}} \\ \\ \end{pmatrix}$$

DISCUSSION

It seems like we can 'represent' performing the elementary row operation $R_i \leftrightarrow R_j$ on $\boldsymbol{B}$ by pre-multiplying a suitable matrix to $\boldsymbol{B}$.

$$\boldsymbol{B} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{C}$$

$$\boldsymbol{I}_m \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$$

Then we have $\boldsymbol{EB} = \boldsymbol{C}$.

Let's investigate the last type of elementary row operation!

DISCUSSION

Let $\boldsymbol{B}$ be a $m\times n$ matrix. What is the product equal to?

$$\begin{array}{cc} & \begin{array}{ccccccc} & & i\text{th col} & & j\text{th col} & & \end{array} \\ \begin{array}{c} \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & k & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array} \begin{pmatrix} & & \\ & \boldsymbol{B} & \\ & & \end{pmatrix}$$

$(i<j)$

$(\text{row } j \text{ of } \boldsymbol{B})+k(\text{row } i \text{ of } \boldsymbol{B})$

$$= \begin{array}{cc} \begin{pmatrix} \\ \\ \boxed{\qquad\qquad\qquad} \\ \\ \end{pmatrix} & \begin{array}{c} \\ \\ j\text{th row} \\ \\ \\ \end{array} \end{array}$$

DISCUSSION

Let $\boldsymbol{B}$ be a $m \times n$ matrix. What is the product equal to?

$$\begin{array}{rc}
 & \begin{array}{ccccccc} & & j\text{th col} & & i\text{th col} & & \end{array} \\
\begin{array}{r} \\ \\ j\text{th row} \\ \\ i\text{th row} \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}
\end{array}
\begin{pmatrix} & & \\ & \boldsymbol{B} & \\ & & \end{pmatrix}
= \begin{pmatrix} \\ \boxed{\qquad\qquad\qquad} \\ \\ \end{pmatrix} \begin{array}{l} \\ j\text{th row} \\ \\ \end{array}$$

$(i > j)$

$(\text{row } j \text{ of } \boldsymbol{B}) + k(\text{row } i \text{ of } \boldsymbol{B})$

DISCUSSION

Note that the product can be obtained by performing elementary row operation $R_j + kR_i$ to $\boldsymbol{B}$.

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} & & \\ & \boldsymbol{B} & \\ & & \end{pmatrix}$$

$(i > j)$

$$\boldsymbol{B} \xrightarrow{R_j + kR_i} = \begin{pmatrix} \boxed{} \\ \\ \\ \end{pmatrix} \begin{array}{l} j\text{th row} \\ \\ \\ \end{array}$$

DISCUSSION

Note that the product can be obtained by performing elementary row operation $R_j + kR_i$ to $\boldsymbol{B}$.

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & k & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} & & \\ & \boldsymbol{B} & \\ & & \end{pmatrix}$$

$(i < j)$

$$\boldsymbol{B} \xrightarrow{R_j + kR_i} = \begin{array}{cl} \begin{pmatrix} \\ \\ \boxed{} \\ \end{pmatrix} & \begin{array}{l} \\ \\ j\text{th row} \\ \\ \end{array} \end{array}$$

DISCUSSION

Note that the matrix $\boxed{}$ can also be obtained by performing $R_j + kR_i$ on $\boldsymbol{I}_m$.

$$\boldsymbol{I}_m \xrightarrow{R_j + kR_i} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} & & \\ & \boldsymbol{B} & \\ & & \end{pmatrix}$$

$$\boldsymbol{B} \xrightarrow{R_j + kR_i} = \begin{pmatrix} \\ \boxed{} \\ \\ \\ \end{pmatrix} \begin{matrix} \\ j\text{th row} \\ \\ \\ \end{matrix}$$

DISCUSSION

Note that the matrix $\boxed{}$ can also be obtained by performing $R_j + kR_i$ on $\boldsymbol{I}_m$.

$$\boldsymbol{I}_m \xrightarrow{R_j + kR_i} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & k & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} & & \\ & \boldsymbol{B} & \\ & & \end{pmatrix}$$

$$\boldsymbol{B} \xrightarrow{R_j + kR_i} \; = \begin{array}{c} \begin{pmatrix} \\ \\ \boxed{} \\ \\ \end{pmatrix} \end{array} \begin{array}{l} \\ \\ j\text{th row} \\ \\ \end{array}$$

DISCUSSION

It seems like we can 'represent' performing the elementary row operation $R_j + kR_i$ on $\boldsymbol{B}$ by pre-multiplying a suitable matrix to $\boldsymbol{B}$.

$$\boldsymbol{B} \xrightarrow{R_j + kR_i} \boldsymbol{C}$$

$$\boldsymbol{I}_m \xrightarrow{R_j + kR_i} \boldsymbol{E}$$

Then we have $\boldsymbol{EB} = \boldsymbol{C}$.

REMARK

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & & \dots & \dots & 0 \\ \vdots & & c & & & \vdots \\ \vdots & & & 1 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 1 \end{pmatrix}$$

$\boldsymbol{E}$ is invertible and $\boldsymbol{E}^{-1}$ is shown below (check it!)

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & & \dots & \dots & 0 \\ \vdots & & \frac{1}{c} & & & \vdots \\ \vdots & & & 1 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 1 \end{pmatrix}$$

REMARK

$$\boldsymbol{E} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

$\boldsymbol{E}$ is invertible and $\boldsymbol{E}^{-1}$ is shown below (check it!)

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

REMARK

$$\boldsymbol{E} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

$\boldsymbol{E}$ is invertible and $\boldsymbol{E}^{-1}$ is shown below (check it!)

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & -k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

REMARK

$$\boldsymbol{E} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & k & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

$\boldsymbol{E}$ is invertible and $\boldsymbol{E}^{-1}$ is shown below (check it!)

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & -k & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

SUMMARY

Let $\boldsymbol{A}$ be a $m \times n$ matrix. For each of the three types of elementary row operations we can perform on $\boldsymbol{A}$:

$$\begin{tikzcd}
 & \boldsymbol{B}_1 \\
\boldsymbol{A} \arrow[ur, "cR_i"] \arrow[r, "R_i \leftrightarrow R_j"] \arrow[dr, "R_j + kR_i"'] & \boldsymbol{B}_2 \\
 & \boldsymbol{B}_3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \boldsymbol{E}_1 \\
\boldsymbol{I}_m \arrow[ur, "cR_i"] \arrow[r, "R_i \leftrightarrow R_j"] \arrow[dr, "R_j + kR_i"'] & \boldsymbol{E}_2 \\
 & \boldsymbol{E}_3
\end{tikzcd}$$

there are matrices $\boldsymbol{E}_i$ $(i = 1,2,3)$ such that

$\boldsymbol{E}_i \boldsymbol{A} = \boldsymbol{B}_i$ for $i = 1,2,3$.

Furthermore, each $\boldsymbol{E}_i$ is an invertible matrix.

DEFINITION

A square matrix is an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation.

REMARK

Note that all elementary matrices are invertible and their inverses are themselves elementary matrices.

$$\begin{tikzcd}
 & \boldsymbol{E}_1 \\
\boldsymbol{I}_m \arrow[ur, "cR_i"] \arrow[r, "R_i \leftrightarrow R_j"] \arrow[dr, "R_j + kR_i"'] & \boldsymbol{E}_2 \\
 & \boldsymbol{E}_3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \boldsymbol{E}_1^{-1} \\
\boldsymbol{I}_m \arrow[ur, "\frac{1}{c}R_i"] \arrow[r, "R_i \leftrightarrow R_j"] \arrow[dr, "R_j - kR_i"'] & \boldsymbol{E}_2^{-1} \\
 & \boldsymbol{E}_3^{-1}
\end{tikzcd}$$

Note that if an elementary matrix $\boldsymbol{E}$ represents a single elementary row operation X, $\boldsymbol{E}^{-1}$ represents the elementary row operation that 'un-do' X.

EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 3 & 1 \\ -1 & 0 & 3 & 4 \end{pmatrix}$. Find a sequence of elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ such that $\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$ is the reduced row-echelon form of $\boldsymbol{A}$.

Write down the inverses of each of the elementary matrices and describe which elementary row operation these inverses represent.

END OF LECTURE 05

Lecture 06:

Elementary matrices (cont’d)

Determinants (till Example 2.5.13)