Lecture 10 recap

1) Discussion of redundancy.

2) Definition of linear independence (in terms of solutions of a vector equation).

3) Linear indepenence - one or two vectors.

4) Linear independence in terms of redundancy.

5) Geometrical examples - linear independence/dependence of two or three vectors.

6) A theorem of guaranteed dependence.

Lecture 11

Linear independence (cont'd)

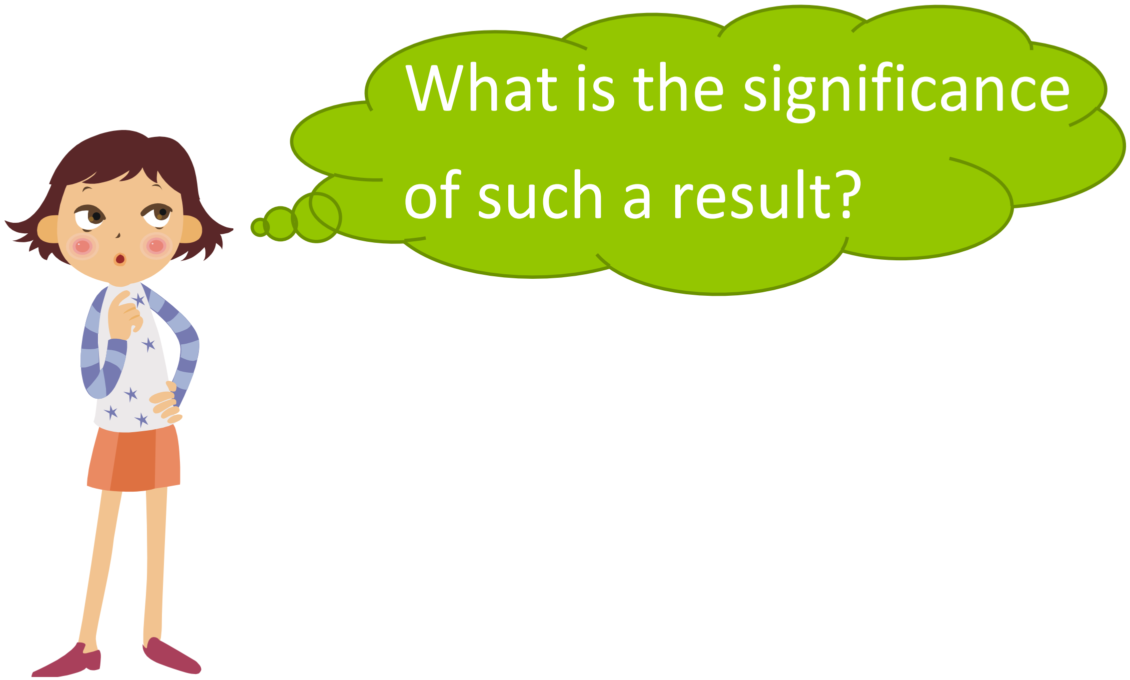
Bases

Dimensions

Theorem

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be linearly independent vectors in $\mathbb{R}^n$.

Suppose $\boldsymbol{u}_{k+1} \in \mathbb{R}^n$ is NOT a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$, then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}$ are linearly independent.



Discussion

1) A set V is called a vector space if either $V = \mathbb{R}^n$ or V is a subspace of $\mathbb{R}^n$ for some positive integer n.

2) Let W be a vector space. A set V is called a subspace of W if V is a vector space contained in W.

Example

$U = \text{span}\{(1,1,1)\}$ $V = \text{span}\{(1,1,-1)\}$

$W = \text{span}\{(1,0,0),(0,1,1)\}$

U, V, W are all subspaces of $\mathbb{R}^3$. So they are vector spaces.

$(1,1,1) = (1,0,0) + (0,1,1) \Rightarrow U \subseteq W$

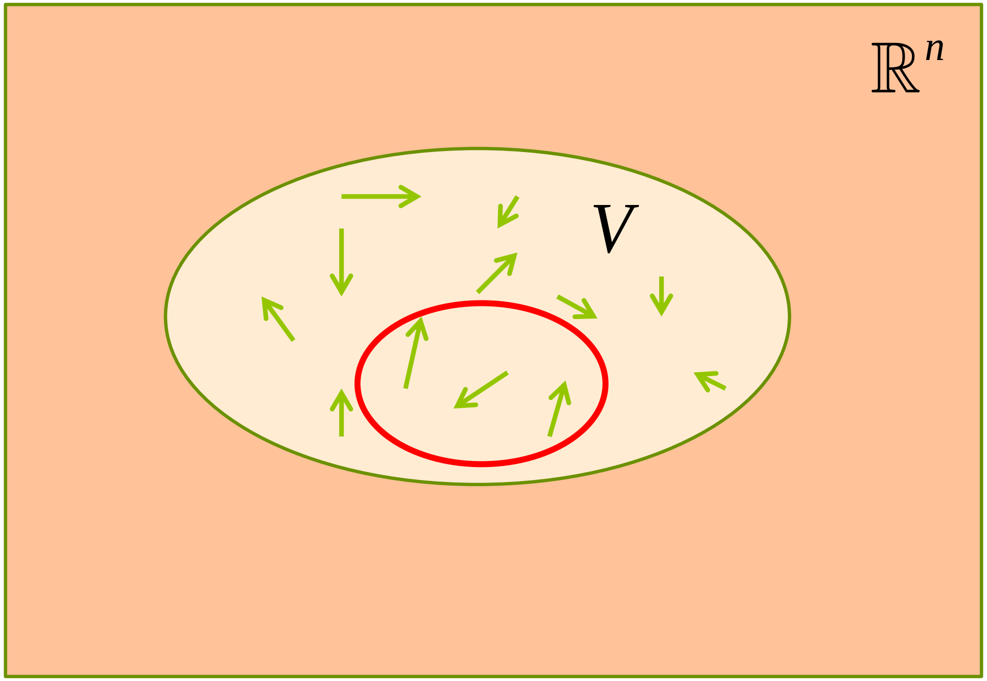
So U is a subspace of W

$(1,1,-1)$ is not a linear combination of $(1,0,0),(0,1,1)$

$\Rightarrow V \not\subset W$ and so V is not a subspace of W

Discussion

Consider a vector space V.



Such a set can then be used to build a 'coordinate system' for V.

Question:

Find a subset S of V, containing as few vectors as possible, so that every vector in V is a linear combination of the vectors in S (that is, $\text{span}(S) = V$).

Definition (Basis)

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a subset of a vector space V.

Then S is called a basis (plural bases) for V if

1. S is linearly independent and
2. S spans V.

Examples (Basis)

Show that $S = \{(2,4),(1,0)\}$ is a basis for $\mathbb{R}^2$.

Let $S = \{(1,4,1),(2,0,-1)\}$. Show that S is a basis for span(S).

Show that $S = \{(1,2,1),(2,9,0),(3,3,4)\}$ is a basis for $\mathbb{R}^3$.

Examples (Basis)

Is $S = \{(1,1,0,1),(2,1,0,3),(3,-1,0,3)\}$ a basis for $\mathbb{R}^4$?

Is $S = \{(1,1,1,1),(0,0,1,2),(-1,0,0,1)\}$ a basis for $\mathbb{R}^4$?

No. $(1,0,0,0)$ is not a linear combination of vectors in S.

Examples (Basis)

Is $S = \{(1,1,1),(0,0,1),(1,1,0)\}$ a basis for $\mathbb{R}^3$?

Some remarks

1) A basis for a vector space V contains the smallest possible number of vectors that can span V.

2) For convenience, we say that the empty set $\varnothing$ is the basis for the zero space.

3) Except the zero space, any vector space has infinitely many different bases.

Theorem

If $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is a basis for a vector space V, then every vector $\boldsymbol{v} \in V$ can be expressed in the form (as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$)

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k$$

in exactly one way, where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Definition

Let $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ be a basis for a vector space V and $\boldsymbol{v}$ be a vector in V. If

$$\boldsymbol{v}=\boxed{c_1}\boldsymbol{u}_1+\boxed{c_2}\boldsymbol{u}_2+\ldots+\boxed{c_k}\boldsymbol{u}_k$$

then the coefficients $\boxed{c_1,c_2,\ldots,c_k}$ are called the coordinates of $\boldsymbol{v}$ relative to the basis S.

The vector

$$(\boldsymbol{v})_S=(c_1,c_2,\ldots,c_k)\ \text{(belonging to } \mathbb{R}^k\text{)}$$

is called the coordinate vector of $\boldsymbol{v}$ relative to the basis S.

Some remarks

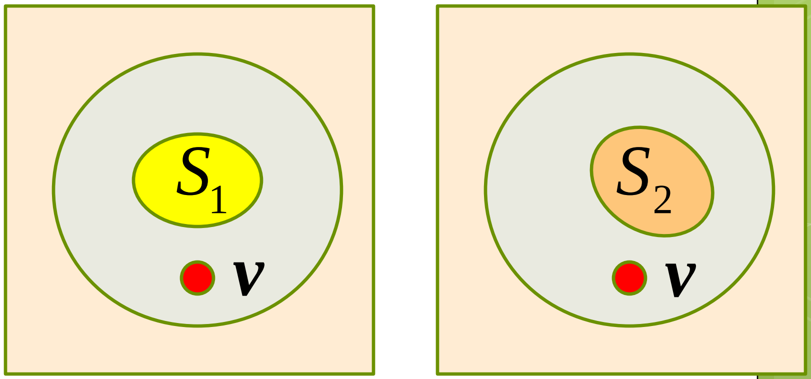
1) In order to discuss coordinate vectors meaningfully, the vectors in $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ must be ordered.

$$\boldsymbol{v}=c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k \qquad (\boldsymbol{v})_S=(c_1,c_2,\ldots,c_k)$$

2) Once $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ is fixed, $(\boldsymbol{v})_S$ is unique and well-defined for each $\boldsymbol{v}\in V$.

3) Different basis, different coordinate vectors.

$$\boxed{(\boldsymbol{v})_{S_1}\neq(\boldsymbol{v})_{S_2}}$$



Example (coordinate vectors)

$S = \{(1,2,1),(2,9,0),(3,3,4)\}$

1) Prove that S is a basis for $\mathbb{R}^3$.

2) Find the coordinate vector of $\boldsymbol{v} = (5,-1,9)$ relative to S.

Solution: Solve for the coefficients a,b,c in the equation

$$\boldsymbol{v} = (5,-1,9) = a(1,2,1) + b(2,9,0) + c(3,3,4)$$

$$\begin{cases} a + 2b + 3c = 5 \\ 2a + 9b + 3c = -1 \\ a + 4c = 9 \end{cases}$$

Example (coordinate vectors)

$$S=\{(1,2,1),(2,9,0),(3,3,4)\}$$

$$\boldsymbol{v}=(5,-1,9)=a(1,2,1)+b(2,9,0)+c(3,3,4)$$

$$\begin{cases} a + 2b + 3c = 5 \\ 2a + 9b + 3c = -1 \\ a + 4c = 9 \end{cases}$$

$$a=1, b=-1, c=2$$

So $(\boldsymbol{v})_S=(1,-1,2)$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 9 & 3 & -1 \\ 1 & 0 & 4 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

unique solution

Example (coordinate vectors)

$S = \{(1,2,1),(2,9,0),(3,3,4)\}$

1) Prove that S is a basis for $\mathbb{R}^3$.

2) Find the coordinate vector of $\boldsymbol{v} = (5,-1,9)$ relative to S.

3) Find a vector $\boldsymbol{w}$ in $\mathbb{R}^3$ such that $(\boldsymbol{w})_S = (1,3,3)$.

Test information

Date: 5th October (Friday)
Time: 4.15pm – 5.15pm (60 minutes)
Venue: MPSH1 (Section A + Section B). Seating plan will be made available in IVLE next week. Seating will be according to your tutorial group.

Scope: Chapter 1, 2, 3 (up to and including Section 3.2)

Format: 3 questions. 40 marks.
Questions 1 and 2 from Chapters 1 and 2.
Question 3 from Chapter 3.

Calculators: Any non-programmable calculator can be used. Graphing calculators cannot be used.

Help sheet: One A4-sized, double sided, handwritten. Do not bring any other paper to the test.

Others: Answer booklet will be provided. You MUST write your answers in pen.

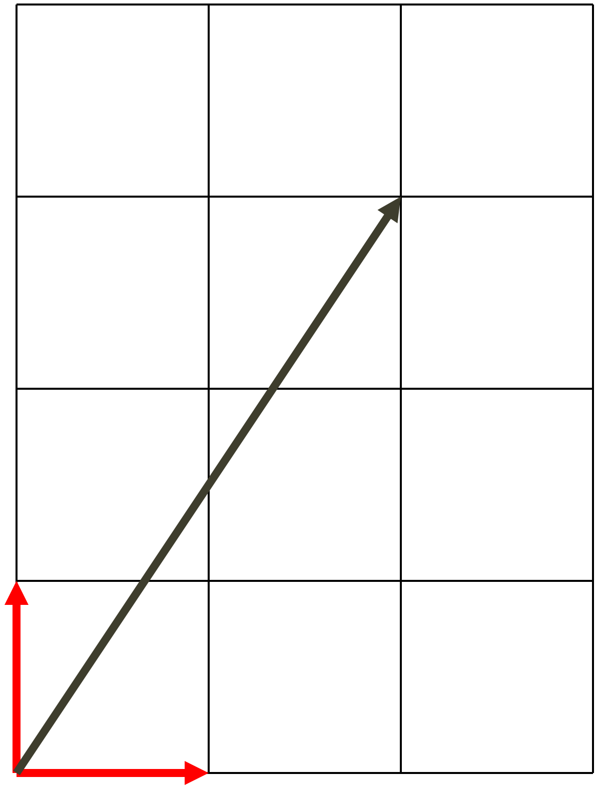
Example (different bases)

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$

$S_1 = \{(1,0),(0,1)\}$

$S_2 = \{(1,-1),(1,1)\}$

$S_3 = \{(1,0),(1,1)\}$



$(2,3) = 2(1,0) + 3(0,1)$

$\Rightarrow (\boldsymbol{v})_{S_1} = (2,3)$

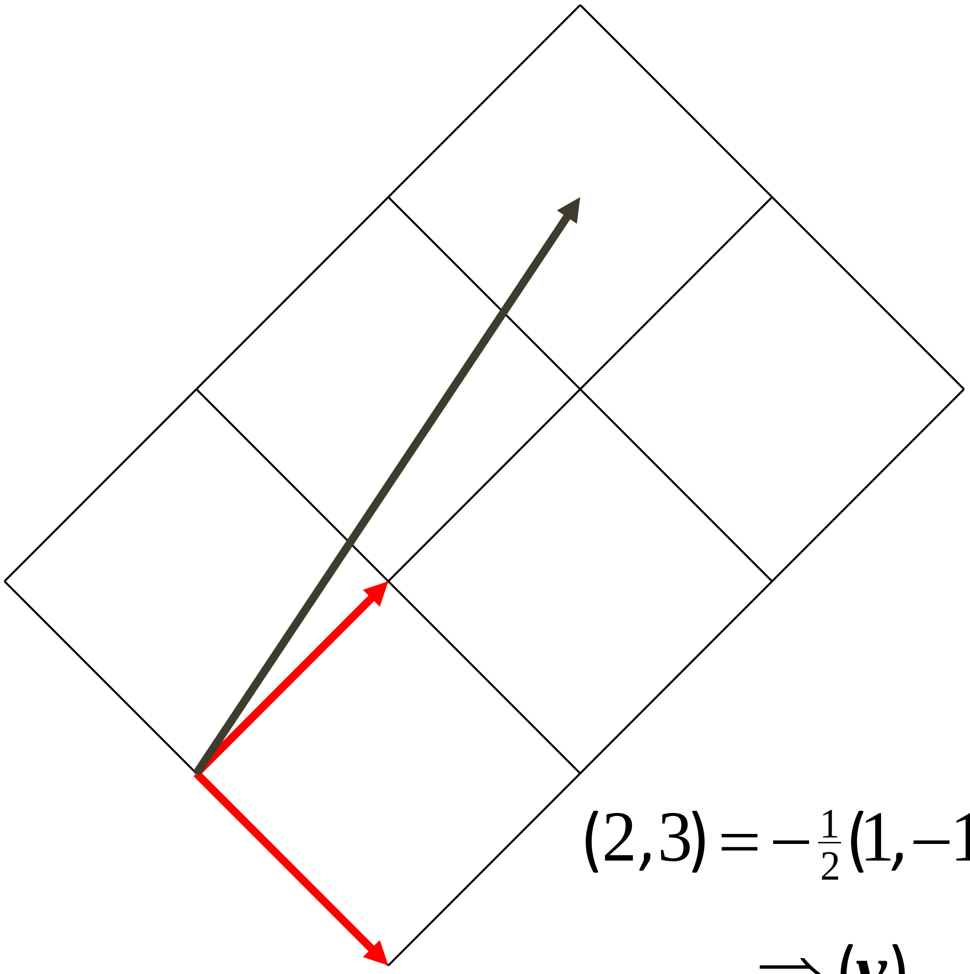
Example (different bases)

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$

$S_1 = \{(1,0),(0,1)\}$

$S_2 = \{(1,-1),(1,1)\}$

$S_3 = \{(1,0),(1,1)\}$



$(2,3) = -\tfrac{1}{2}(1,-1) + \tfrac{5}{2}(1,1)$

$\Rightarrow (\boldsymbol{v})_{S_2} = (-\tfrac{1}{2}, \tfrac{5}{2})$

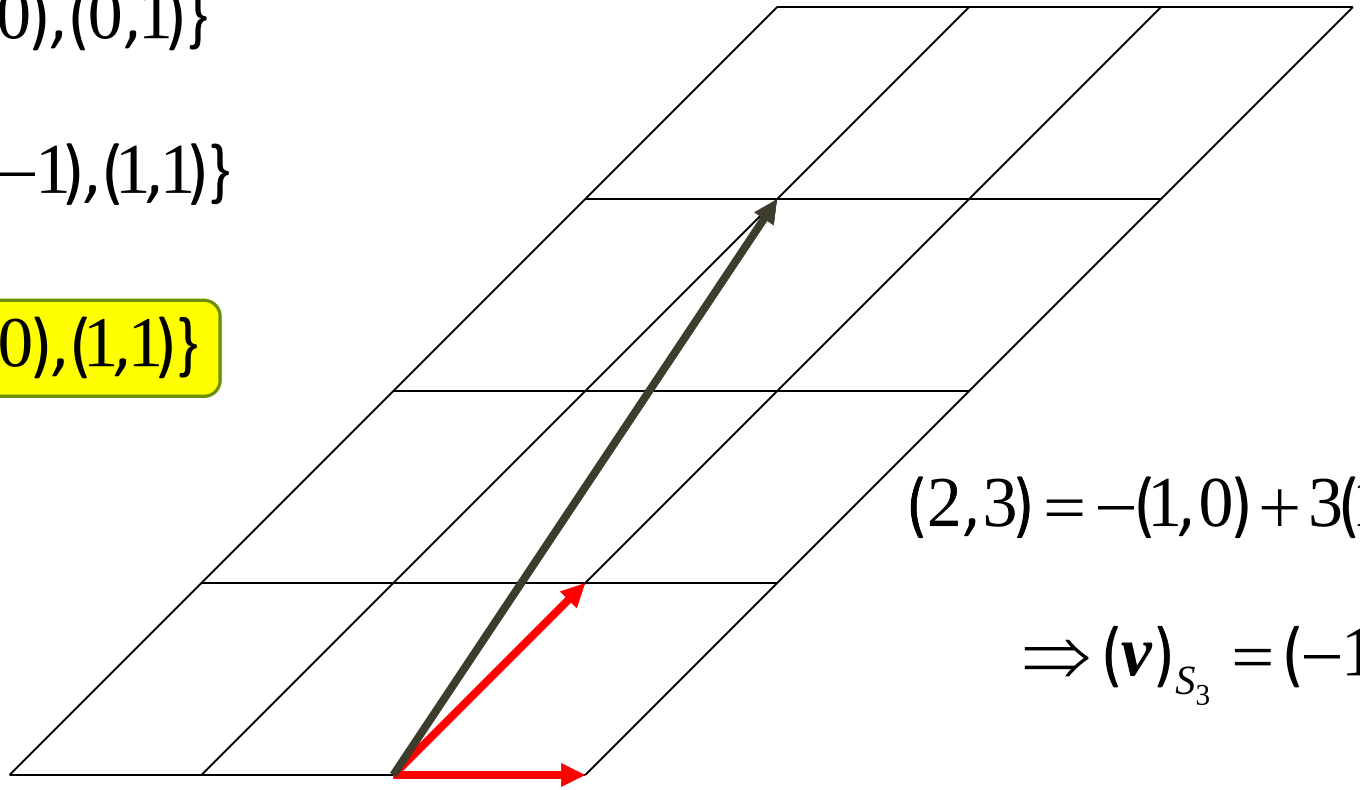
Example (different bases)

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$

$S_1 = \{(1,0),(0,1)\}$

$S_2 = \{(1,-1),(1,1)\}$

$\boxed{S_3 = \{(1,0),(1,1)\}}$



$(2,3) = -(1,0) + 3(1,1)$

$\Rightarrow (\boldsymbol{v})_{S_3} = (-1,3)$

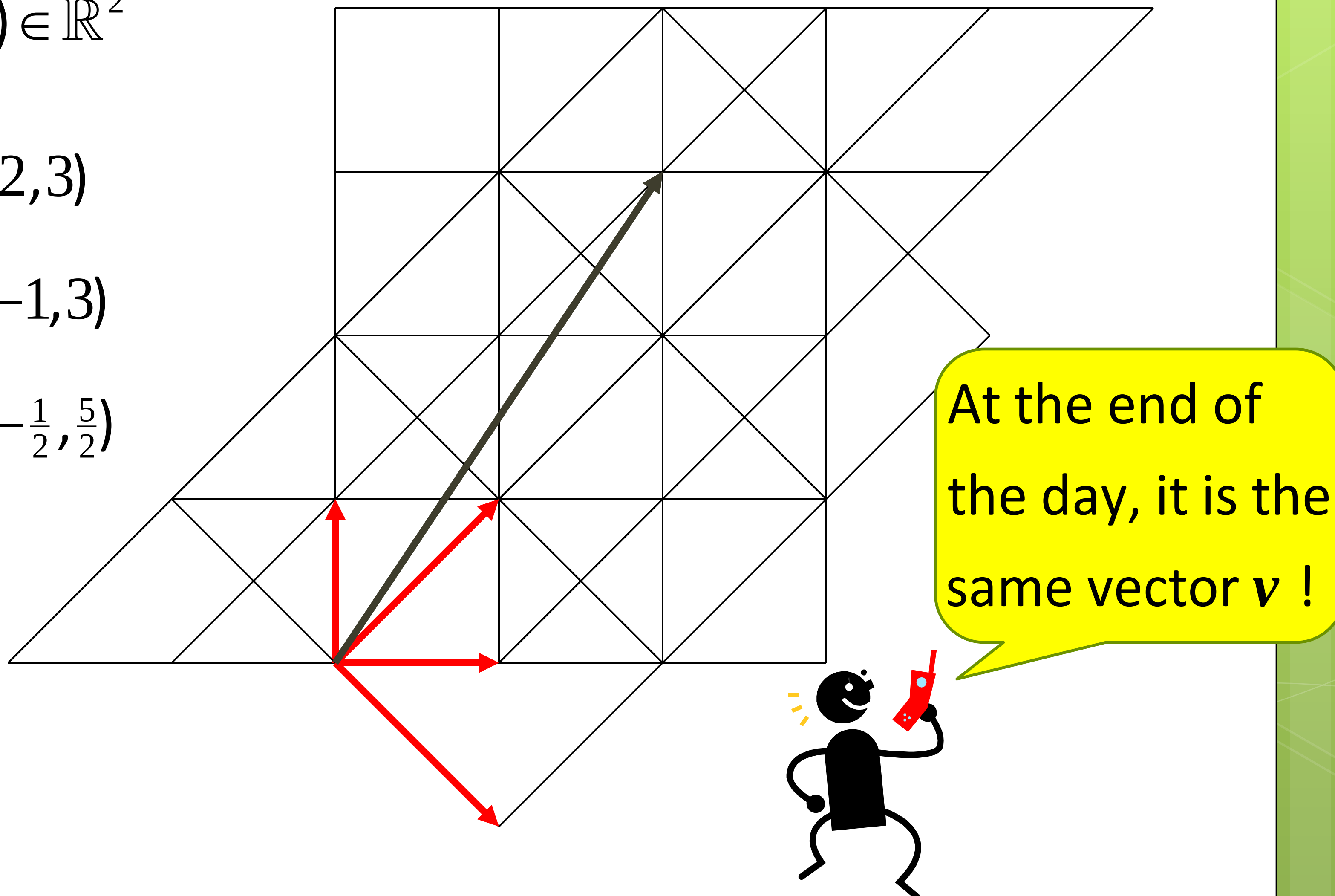
What is the significance?

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$

$(\boldsymbol{v})_{S_1} = (2,3)$

$(\boldsymbol{v})_{S_3} = (-1,3)$

$(\boldsymbol{v})_{S_2} = (-\frac{1}{2}, \frac{5}{2})$

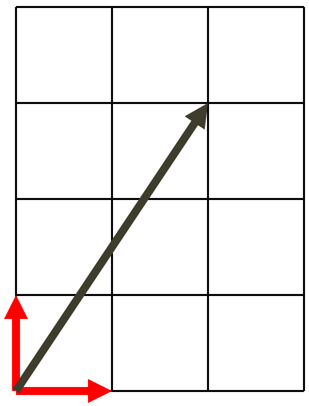


Example (standard basis)

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$ $\qquad S_1 = \{(1,0),(0,1)\}$ $\qquad (\boldsymbol{v})_{S_1} = (2,3) = \boldsymbol{v}$

In fact, for any $\boldsymbol{v} = (x, y) \in \mathbb{R}^2$,

$$(\boldsymbol{v})_{S_1} = (x, y) = \boldsymbol{v}$$



Such a basis (like S_1) is convenient to use.

Let $E = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ where

For any $\boldsymbol{v} \in \mathbb{R}^n$, $(\boldsymbol{v})_E = \boldsymbol{v}$

$$\boldsymbol{e}_1 = (1,0,\ldots,0) \quad \boldsymbol{e}_2 = (0,1,\ldots,0) \quad \cdots\cdots\cdots \quad \boldsymbol{e}_n = (0,0,\ldots,1)$$

E is called the standard basis for $\mathbb{R}^n$.

Remark (standard basis)

Remember the standard basis for $\mathbb{R}^3$ and the standrd basis for $\mathbb{R}^4$ contains entirely different vectors.

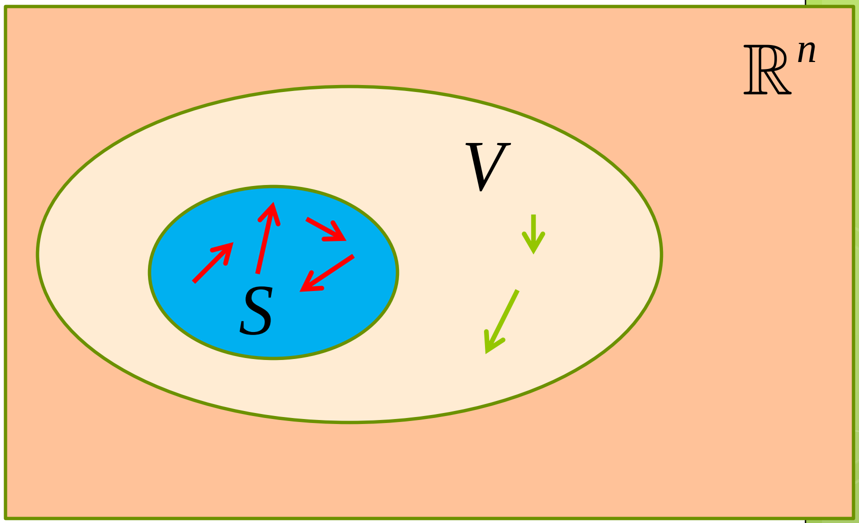
Do not be confused!

Standard basis for $\mathbb{R}^3$

Standard basis for $\mathbb{R}^4$

Some useful rules

Let S be a basis for a vector space V.



1) For any $\boldsymbol{u},\boldsymbol{v}$ in V, $\boldsymbol{u}=\boldsymbol{v}$ if and only if $(\boldsymbol{u})_S=(\boldsymbol{v})_S$.

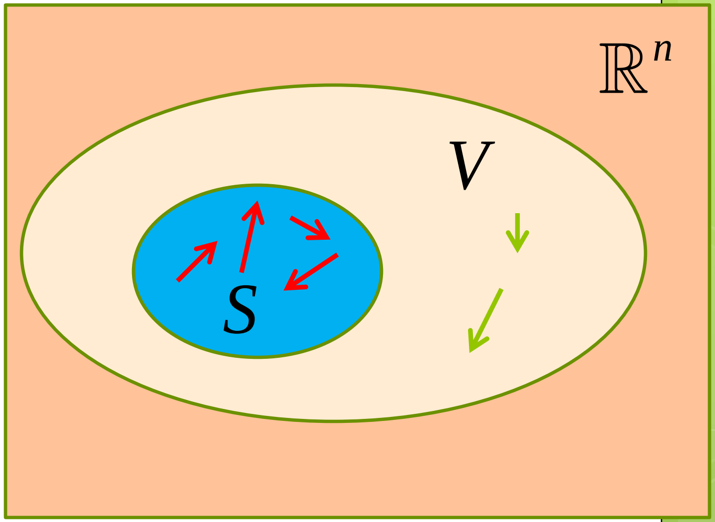
2) For any $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_r\in V$, $c_1,c_2,\ldots,c_r\in\mathbb{R}$,

$$(c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\ldots+c_r\boldsymbol{v}_r)_S=c_1(\boldsymbol{v}_1)_S+c_2(\boldsymbol{v}_2)_S+\ldots+c_r(\boldsymbol{v}_r)_S$$

Theorem

Let S be a basis for a vector space V where $|S| = k$.

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in V$.



Note that $(\boldsymbol{v}_i)_S \in \mathbb{R}^k$ for each i.

1) $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$ are linearly dependent (resp. independent) vectors in V if and only if $(\boldsymbol{v}_1)_S, (\boldsymbol{v}_2)_S, \ldots, (\boldsymbol{v}_r)_S$ are linearly dependent (resp. independent) vectors in $\mathbb{R}^k$.

2) $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\} = V$ if and only if $\operatorname{span}\{(\boldsymbol{v}_1)_S, (\boldsymbol{v}_2)_S, \ldots, (\boldsymbol{v}_r)_S\} = \mathbb{R}^k$.

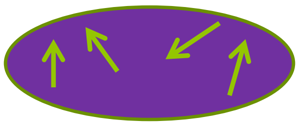
Proof: Omitted.

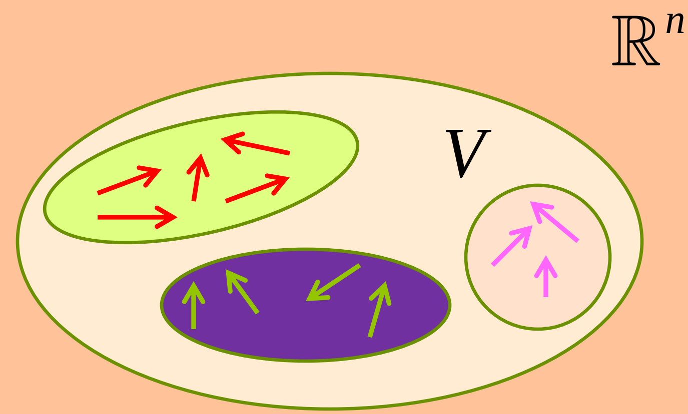
A question to ponder

For a vector space V, we know that V can have many different bases. But do all these bases have the same number of vectors?

Theorem

Let V be a vector space which has a basis with k vectors.

 $=$ basis for V



"You must have the right number"

Then

(1) any subset of V with more than k vectors is always linearly dependent (so cannot be a basis);

(2) any subset of V with less than k vectors cannot span V (so also cannot be a basis);

Definition (Dimension)

The dimension of a vector space V denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V.

The dimension of the zero space is defined to be zero.

Examples (Dimension)

1) $\dim(\mathbb{R}^n) = n$

(recall a basis for $\mathbb{R}^n$ can be $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$).

2) Subspaces of $\mathbb{R}^2$: $\longrightarrow$ $\{\mathbf{0}\}$: dimension $\boxed{0}$

lines through the origin: dimension $\boxed{1}$

$\mathbb{R}^2$: dimension $\boxed{2}$

3) Subspaces of $\mathbb{R}^3$: $\longrightarrow$ $\{\mathbf{0}\}$: dimension $\boxed{0}$

lines through the origin: dimension $\boxed{1}$

$\mathbb{R}^3$: dimension $\boxed{3}$

planes containing the origin: dimension $\boxed{2}$

Examples (Dimension)

Find a basis for and determine the dimension of the subspace $W = \{(x, y, z) \mid y = 2z\}$.

$$= \{(x, 2z, z) \mid x, z \in \mathbb{R}\}$$

$$= \{x(1,0,0) + z(0,2,1) \mid x, z \in \mathbb{R}\}$$

$$= \text{span}\{(1,0,0), (0,2,1)\}$$

$\{(1,0,0), (0,2,1)\}$ spans W

$\{(1,0,0), (0,2,1)\}$ is linearly independent (why?)

$\{(1,0,0), (0,2,1)\}$ is a basis for W and $\dim(W) = 2$.

End of Lecture 11

Lecture 12:

Dimensions (cont'd)

Transition matrices (till end of Chapter 3)