LECTURE 03 RECAP

What can row-echelon forms tell us about the solution set of linear systems.

Standard notations to use when performing elementary row operations.

Three examples on the 'same' kind of problem.

Homogeneous linear systems and why they are special.

LECTURE 04

Introduction to matrices

Matrix operations

DEFINITION (MATRIX, ENTRIES, SIZE)

A matrix is a rectangular array of numbers.

The numbers in the array are called entries.

The size of a matrix is $m \times n$ if it has m rows and n columns.

$$\boldsymbol{A} = \begin{pmatrix} 3 & 2.4 & 1 & -1 & 0 & 11 \\ -5 & 2 & 2 & 0 & 0 & 1 \\ 4.1 & \pi & 20 & 10 & -2 & 1 \end{pmatrix}$$

$\boldsymbol{A}$ is a 3×6 matrix and the $(1,4)$-entry of $\boldsymbol{A}$ is -1.

What is the $(2,3)$-entry of $\boldsymbol{A}$?

DEFINITION (COLUMN AND ROW MATRICES)

A $\boxed{\text{column matrix}}$ is a matrix with only one column.

$$\begin{pmatrix} 2 \\ 3 \\ -1 \\ 0 \end{pmatrix}$$

A $\boxed{\text{row matrix}}$ is a matrix with only one row.

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$$

(3) is both a row and a column matrix.

A MORE GENERAL WAY TO DENOTE A MATRIX

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$ is a $m \times n$ matrix.

We can also write $\boldsymbol{A} = (a_{ij})_{m \times n}$ where a_{ij} is the (i, j)-entry of $\boldsymbol{A}$.

a_{ij}

i is the 'row index'

j is the 'column index'

We can also write $\boldsymbol{A} = (a_{ij})$.

EXAMPLE (ENTRIES OF A MATRIX)

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad a_{11} = 1,\, a_{12} = 2,\, a_{23} = 6$$

$$\boldsymbol{B} = (b_{ij})_{3\times 4} \text{ where } b_{ij} = \begin{cases} i - j & \text{if } i + j \text{ is even} \\ i + j & \text{if } i + j \text{ is odd} \end{cases}$$

$$\begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

SPECIAL TYPES OF MATRICES

Square matrices are matrices with the same number of rows and columns.

$$(2) \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A square matrix with n rows and n columns is said to be of order n.

SPECIAL TYPES OF MATRICES

Given a square matrix $\boldsymbol{A} = (a_{ij})$ of order n,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

The diagonal of $\boldsymbol{A}$ is the sequence $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$.

$a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the diagonal entries of $\boldsymbol{A}$.

a_{ij}, $i \neq j$, are called the non-diagonal entries of $\boldsymbol{A}$.

SPECIAL TYPES OF MATRICES

$$\begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

A square matrix is a diagonal matrix if all its non-diagonal entries are zero.

$$\boldsymbol{A} = (a_{ij}) \text{ is diagonal } \Leftrightarrow \boxed{a_{ij} = 0 \text{ whenever } i \neq j.}$$

SPECIAL TYPES OF MATRICES

$$\begin{pmatrix} c & 0 & 0 & \dots & 0 \\ 0 & c & 0 & \dots & 0 \\ 0 & 0 & c & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & c \end{pmatrix}$$

A diagonal matrix is a scalar matrix if all its diagonal entries are the same.

$\boldsymbol{A} = (a_{ij})$ is scalar $\Leftrightarrow a_{ij} = 0$ whenever $i \neq j$ and

$a_{ij} = c$ whenever $i = j$

(c is a constant).

SPECIAL TYPES OF MATRICES

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

A diagonal matrix is an identity matrix if all its diagonal entries are equal to 1.

$\boldsymbol{I}_n$ is an identity matrix of order n.

If there is no danger of confusion, we simply write $\boldsymbol{I}$.

SPECIAL TYPES OF MATRICES

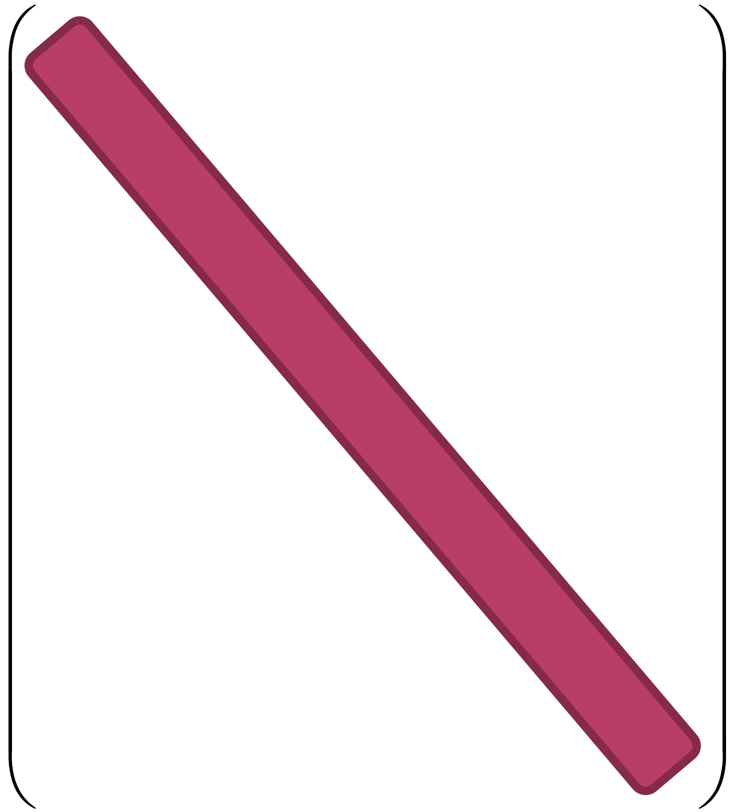
$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

A $\boxed{\text{zero matrix}}$ is a matrix with all entries equal to zero.

$\mathbf{0}_{m\times n}$ is a $m \times n$ zero matrix.

If there is no danger of confusion, we simply write $\mathbf{0}$.

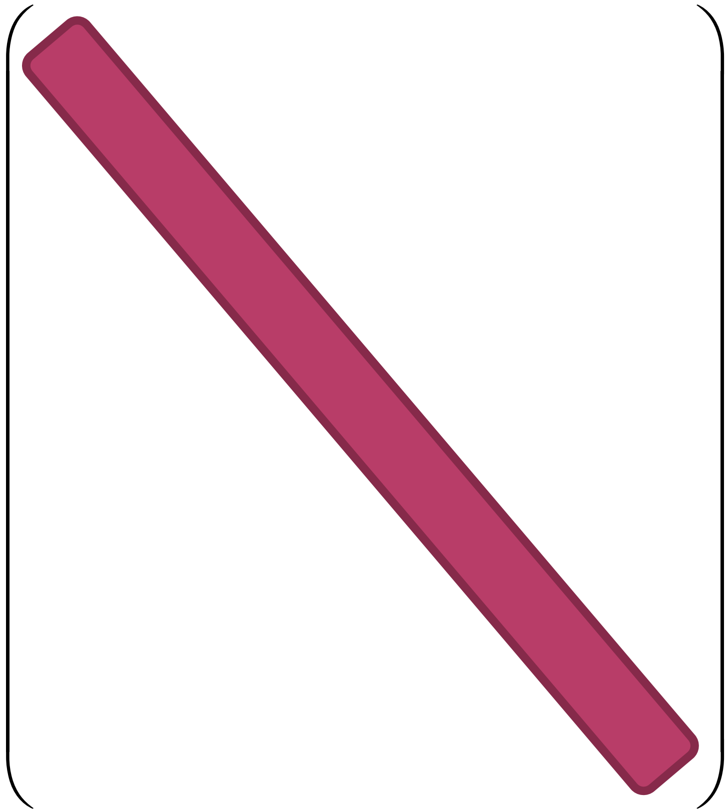
SPECIAL TYPES OF MATRICES



A symmetric matrix (a_{ij}) is a square matrix where

$$\boxed{a_{ij} = a_{ji} \text{ for all } i, j.}$$

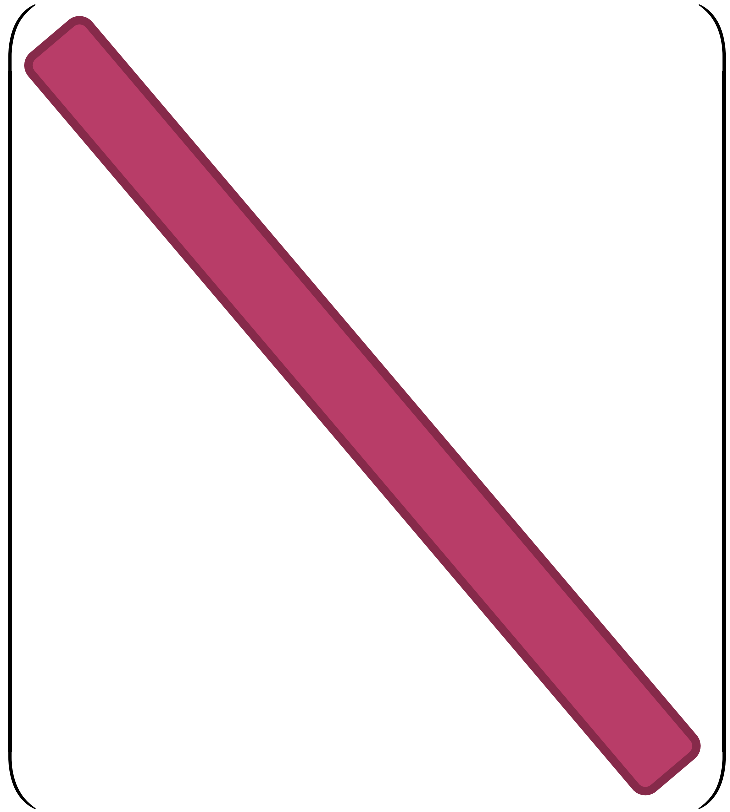
SPECIAL TYPES OF MATRICES



A square matrix (a_{ij}) is an upper triangular matrix if

$$\boxed{a_{ij} = 0 \text{ for all } i > j.}$$

SPECIAL TYPES OF MATRICES



A square matrix (a_{ij}) is an lower triangular matrix if

$$\boxed{a_{ij} = 0 \text{ for all } i < j.}$$

WHAT CAN WE DO WITH MATRICES?

For real numbers, we can add, subtract, multiply and (to some extent) divide. What about for matrices with real number entries?

Before we can write the 'equivalent' of an equation like

$$x = y,$$

we need to define what is meant by $'='$ in terms of matrices

DEFINITION (EQUAL MATRICES)

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be equal, written

$$\boldsymbol{A} = \boldsymbol{B},$$

if they have the same size (rows and columns) and

$$\boxed{a_{ij} = b_{ij} \quad \text{for all } i, j.}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & y \\ x & z \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} 0 & y \\ x & z \end{pmatrix}$$

$$\boldsymbol{D} = \begin{pmatrix} 1 & 1 & x \\ 0 & -1 & 0 \end{pmatrix}$$

Definition (Matrix Addition, Subtraction, Scalar Multiplication)

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $m \times n$ matrices, then

1) $\boldsymbol{A}+\boldsymbol{B}$ is a $m \times n$ matrix such that

$$\boldsymbol{A}+\boldsymbol{B}=(a_{ij}+b_{ij}).$$

2) $\boldsymbol{A}-\boldsymbol{B}$ is a $m \times n$ matrix such that

$$\boldsymbol{A}-\boldsymbol{B}=(a_{ij}-b_{ij}).$$

If $\boldsymbol{A}$ is a $m \times n$ matrix and c is a real number, then

3) $c\boldsymbol{A}$ is a $m \times n$ matrix such that

$$c\boldsymbol{A}=(ca_{ij}).$$

EXAMPLES (MATRIX ADDITION, SUBTRACTION, SCALAR MULTIPLICATION)

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 \\ 1 & 0 \\ 3 & 1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} -1 & 0 \\ 4 & -2 \\ 0 & 3 \end{pmatrix}$$

$$\boldsymbol{A} + \boldsymbol{B} = \qquad \boldsymbol{A} - 3\boldsymbol{B} =$$

SOME REMARKS

1) You cannot add or subtract matrices that are not of the same size.

2) $-\boldsymbol{A}$ is the same as $(-1)\boldsymbol{A}$.

3) $\boldsymbol{A}-\boldsymbol{B}$ is the same as $\boldsymbol{A}+(-1)\boldsymbol{B}$.

4) Just like for real numbers, there are laws for matrix operation which we can use.

THEOREM (MATRIX OPERATION LAWS)

(Commutative Law for Matrix Addition)

For real numbers x, y, we have

$$\boxed{x + y = y + x.}$$

For matrices $\boldsymbol{A}, \boldsymbol{B}$ (of the same size), we have

$$\boxed{\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}.}$$

THEOREM (MATRIX OPERATION LAWS)

(Associative Law for Matrix Addition)

For real numbers x, y, z we have

$$\boxed{x+(y+z)=(x+y)+z.}$$

For matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ (of the same size), we have

$$\boxed{\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}.}$$

THEOREM (MATRIX OPERATION LAWS)

For real numbers x, y, z we have

$$\boxed{x(y+z)=xy+xz} \quad \text{and} \quad \boxed{(xy)z=x(yz).}$$

For matrices $\boldsymbol{A}, \boldsymbol{B}$ (of the same size) and real numbers $a, b,$ we have

$$\boxed{a(\boldsymbol{A}+\boldsymbol{B})=a\boldsymbol{A}+a\boldsymbol{B};} \quad \boxed{(a+b)\boldsymbol{A}=a\boldsymbol{A}+b\boldsymbol{A};}$$

and

$$\boxed{a(b\boldsymbol{A})=(ab)\boldsymbol{A}=b(a\boldsymbol{A}).}$$

CAN THESE LAWS BE PROVEN?

Remember that all the following are just matrix equations.

$$\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}.$$

$$\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}.$$

$$a(\boldsymbol{A}+\boldsymbol{B})=a\boldsymbol{A}+a\boldsymbol{B};$$

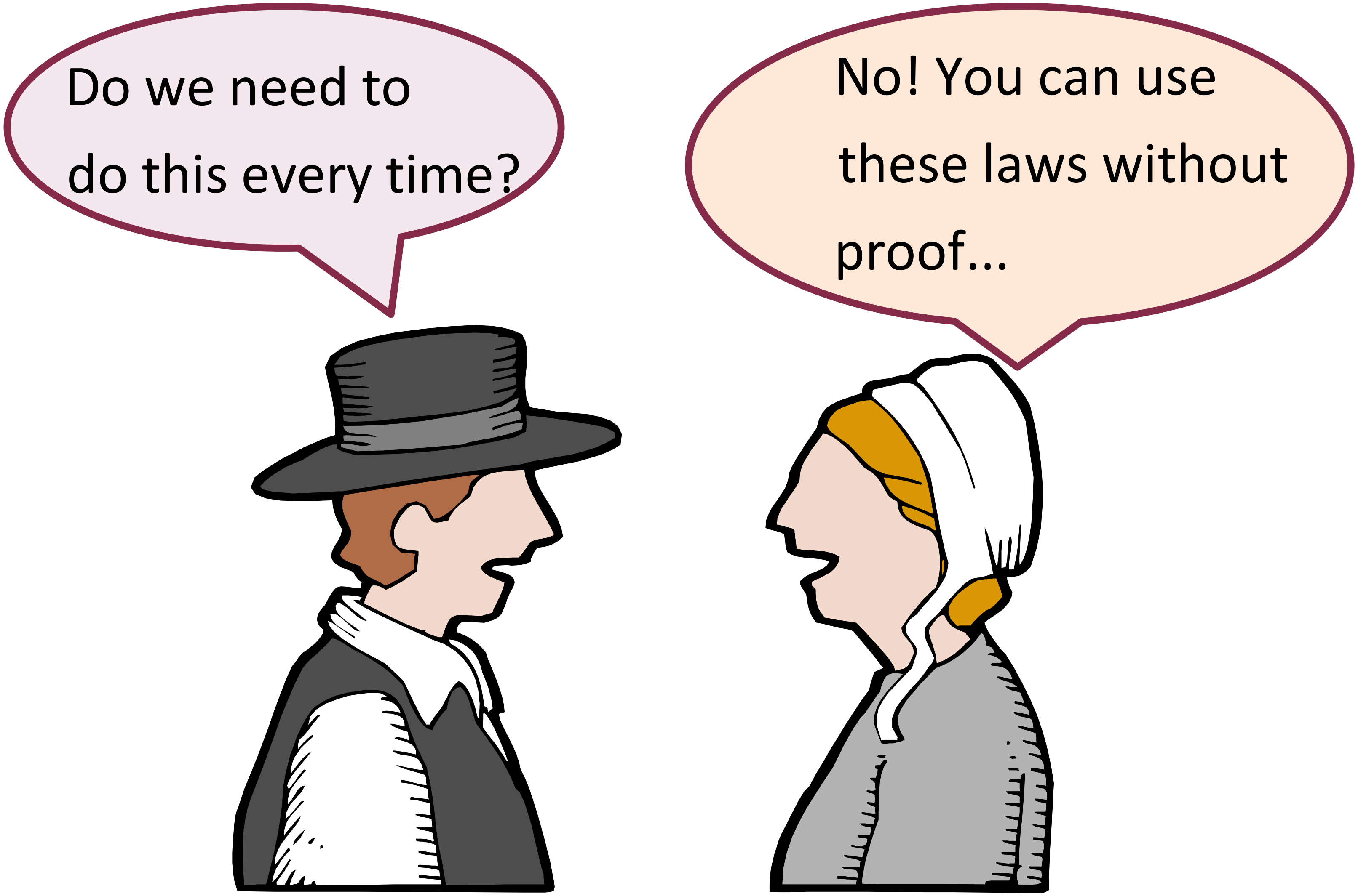
$$(a+b)\boldsymbol{A}=a\boldsymbol{A}+b\boldsymbol{A};$$

$$a(b\boldsymbol{A})=(ab)\boldsymbol{A}=b(a\boldsymbol{A}).$$

Do you remember how to prove that two matrices are equal?

Size and (i, j)-entry consideration.

WOW! ISN'T THIS VERY TEDIOUS?



Working with Matrix Equations

$$(3+2)(\boldsymbol{A}-\boldsymbol{B}+2\boldsymbol{C})$$

$$=(3+2)(\boldsymbol{A}-\boldsymbol{B})+(3+2)(2\boldsymbol{C})$$

$$=(3+2)\boldsymbol{A}-(3+2)\boldsymbol{B}+3(2\boldsymbol{C})+2(2\boldsymbol{C})$$

$$=5\boldsymbol{A}-5\boldsymbol{B}+6\boldsymbol{C}+4\boldsymbol{C}$$

$$=5\boldsymbol{A}-5\boldsymbol{B}+10\boldsymbol{C}.$$

REMARK

Since $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$,

we may simply write

$$\boldsymbol{A}_1+\boldsymbol{A}_2+\ldots+\boldsymbol{A}_k$$

without any parentheses.

A FEW MORE (OBVIOUS) RESULTS

Let $\boldsymbol{A}$ be a matrix.

$$\boldsymbol{A}+\mathbf{0}=\mathbf{0}+\boldsymbol{A}=\boldsymbol{A}.$$

$$\boldsymbol{A}-\boldsymbol{A}=\boldsymbol{A}+(-\boldsymbol{A})=\mathbf{0}.$$

$$0\boldsymbol{A}=\mathbf{0}.$$

CAN WE MULTIPLY MATRICES?

We have already seen that we can add and subtract matrices pretty much like what we do for real numbers. What about multiplication?

For real numbers x, y, $\boxed{xy \text{ is always defined.}}$

What about for matrices $\boldsymbol{A}, \boldsymbol{B}$?

DEFINITION (MATRIX MULTIPLICATION)

Let $\boldsymbol{A}=(a_{ij})_{m\times p}$ and $\boldsymbol{B}=(b_{ij})_{p\times n}$ be two matrices.

$$\boldsymbol{A}=\begin{pmatrix} \\ \boxed{a_{i1} \quad a_{i2} \quad \dots \quad a_{ip}} \\ \\ \end{pmatrix} \qquad \boldsymbol{B}=\begin{pmatrix} & \boxed{\begin{matrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{matrix}} & \end{pmatrix}$$

$i-$th row of $\boldsymbol{A}$

$j-$th column of $\boldsymbol{B}$

Then $\boldsymbol{AB}$ is a $m\times n$ matrix whose (i, j)-entry is given by:

$$\boxed{a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{ip}b_{pj}} \quad \boxed{=\sum_{k=1}^{p} a_{ik}b_{kj}}$$

EXAMPLE (MATRIX MULTIPLICATION)

Computer $\boldsymbol{AB}$ where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 0 & 1 & 3 & 0 \\ 1 & -1 & 0 & 3 \\ 2 & 2 & 0 & -1 \end{pmatrix}$$

$$\boldsymbol{AB} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

SIZE DOES MATTER!

From the way $\boldsymbol{AB}$ is defined, it is clear that the number of columns in $\boldsymbol{A}$ must be EQUAL to the number of rows in $\boldsymbol{B}$.

$$\boldsymbol{A} = \begin{pmatrix} \\ \overbrace{\boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{ip}}}^{\text{number of columns in } \boldsymbol{A}} \\ \\ \end{pmatrix}$$

$$\boldsymbol{B} = \left.\begin{pmatrix} \boxed{\begin{matrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{matrix}} \end{pmatrix}\right\} \text{number of rows in } \boldsymbol{B}$$

DIFFERENCE BETWEEN MATRIX AND REAL NUMBER MULTIPLICATION

We know that for real numbers x, y we have $xy = yx$.

However, for matrices $\boldsymbol{A}, \boldsymbol{B}$, even when both $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are defined, $\boldsymbol{AB}$ does not necessarily equal $\boldsymbol{BA}$.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} -2 & 2 \\ -1 & 3 \end{pmatrix}$$

$$\boldsymbol{AB} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad \boldsymbol{BA} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

DIFFERENCE BETWEEN MATRIX AND REAL NUMBER MULTIPLICATION

We know that for real numbers x, y if $xy = 0$,

then either $x = 0$ or $y = 0$ (or both $= 0$).

However, if $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,

$$\text{then } \boldsymbol{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}.$$

So $\boldsymbol{AB} = \mathbf{0}$ but neither $\boldsymbol{A}$ nor $\boldsymbol{B}$ is the zero matrix.

DEFINITION (PRE- AND POST- MULTIPLICATION)

Pre-multiply of $\boldsymbol{A}$ to $\boldsymbol{B} \to \boldsymbol{AB}$

Post-multiply of $\boldsymbol{A}$ to $\boldsymbol{B} \to \boldsymbol{BA}$

THEOREM (MORE MATRIX OPERATION LAWS)

(Associative Law for Matrix Multiplication)

$$\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$$

(Distributive Law for Matrix Addition and Multiplication)

$$\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{AB}_1 + \boldsymbol{AB}_2$$

$$(\boldsymbol{C}_1 + \boldsymbol{C}_2)\boldsymbol{A} = \boldsymbol{C}_1\boldsymbol{A} + \boldsymbol{C}_2\boldsymbol{A}$$

THEOREM (MORE MATRIX OPERATION LAWS)

If a is a scalar, then

$$a(\boldsymbol{AB}) = (a\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(a\boldsymbol{B})$$

(Distributive Law for Matrix Addition and Multiplication)

$$\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{AB}_1 + \boldsymbol{AB}_2$$

$$(\boldsymbol{C}_1 + \boldsymbol{C}_2)\boldsymbol{A} = \boldsymbol{C}_1\boldsymbol{A} + \boldsymbol{C}_2\boldsymbol{A}$$

TWO MORE (OBVIOUS?) RESULTS

Let $\boldsymbol{A}$ be a $m \times n$ matrix.

$$\boldsymbol{A0} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{0A} = \boldsymbol{0}$$

(What are the sizes of the zero matrices?)

$$\boldsymbol{AI} = \boldsymbol{A} \quad \text{and} \quad \boldsymbol{IA} = \boldsymbol{A}$$

(What are the sizes of the identity matrices?)

DEFINITION (POWER OF MATRICES)

Let $\boldsymbol{A}$ be a square matrix and n a nonnegative integer.

We define

$$\boldsymbol{A}^n = \begin{cases} \boldsymbol{I} & \text{if } n = 0 \\ \underbrace{\boldsymbol{A}\boldsymbol{A}\ldots\boldsymbol{A}}_{n \text{ times}} & \text{if } n \geq 1 \end{cases}$$

1) $\boldsymbol{A}^m \boldsymbol{A}^n = \boldsymbol{A}^{(m+n)}$

2) In general, $(\boldsymbol{A}\boldsymbol{B})^m \neq \boldsymbol{A}^m \boldsymbol{B}^m$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{AB} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$(\boldsymbol{AB})^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\boldsymbol{A}^2\boldsymbol{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

DIFFERENT WAYS TO EXPRESS MATRIX PRODUCT

Let $\boldsymbol{A}$ be a $m \times p$ matrix, $\boldsymbol{B}$ be a $p \times n$ matrix.

Note that $\boldsymbol{AB}$ will be a $m \times n$ matrix.

Write $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{pmatrix} \boxed{\quad \boldsymbol{a}_1 \quad} \\ \boxed{\quad \boldsymbol{a}_2 \quad} \\ \vdots \\ \vdots \\ \boxed{\quad \boldsymbol{a}_m \quad} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} \boxed{\boldsymbol{b}_1} & \boxed{\boldsymbol{b}_2} & \cdots & \cdots & \boxed{\boldsymbol{b}_n} \end{pmatrix}$$

$$\boxed{\boldsymbol{b}_j = j\text{th column of } \boldsymbol{B}} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{pmatrix}$$

$$\boxed{\boldsymbol{a}_i = i\text{th row of } \boldsymbol{A}} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & \cdots & a_{ip} \end{pmatrix}$$

DIFFERENT WAYS TO EXPRESS MATRIX PRODUCT

Let $\boldsymbol{A}$ be a $m \times p$ matrix, $\boldsymbol{B}$ be a $p \times n$ matrix.

First way of computing $\boldsymbol{AB}$: entry by entry

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \boldsymbol{a}_1\boldsymbol{b}_2 & \ldots & \boldsymbol{a}_1\boldsymbol{b}_n \\ \boldsymbol{a}_2\boldsymbol{b}_1 & \boldsymbol{a}_2\boldsymbol{b}_2 & \ldots & \boldsymbol{a}_2\boldsymbol{b}_n \\ \vdots & \vdots & & \vdots \\ \boldsymbol{a}_m\boldsymbol{b}_1 & \boldsymbol{a}_m\boldsymbol{b}_2 & \ldots & \boldsymbol{a}_m\boldsymbol{b}_n \end{pmatrix}$$

$$\boldsymbol{a}_i\boldsymbol{b}_j = (i, j)\text{-entry of } \boldsymbol{AB} = \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{ip} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix}$$

$$= \sum_{k=1}^{p} a_{ik} b_{kj}$$

DIFFERENT WAYS TO EXPRESS MATRIX PRODUCT

Let $\boldsymbol{A}$ be a $m \times p$ matrix, $\boldsymbol{B}$ be a $p \times n$ matrix.

Second way of computing $\boldsymbol{AB}$: row by row

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ : \\ : \\ \boldsymbol{a}_m \end{pmatrix} \boldsymbol{B} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{B} \\ \boldsymbol{a}_2\boldsymbol{B} \\ : \\ : \\ \boldsymbol{a}_m\boldsymbol{B} \end{pmatrix}$$

$$\boldsymbol{a}_i\boldsymbol{B} = i\text{th row of } \boldsymbol{AB} = \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{ip} \end{pmatrix} \boldsymbol{B}$$

DIFFERENT WAYS TO EXPRESS MATRIX PRODUCT

Let $\boldsymbol{A}$ be a $m \times p$ matrix, $\boldsymbol{B}$ be a $p \times n$ matrix.

Third way of computing $\boldsymbol{AB}$: column by column

$$\boldsymbol{AB} = \boldsymbol{A}\begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \dots & \dots & \boldsymbol{b}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{Ab}_1 & \boldsymbol{Ab}_2 & \dots & \dots & \boldsymbol{Ab}_n \end{pmatrix}$$

$$\boldsymbol{Ab}_j = j\text{th column of } \boldsymbol{AB} = \boldsymbol{A}\begin{pmatrix} b_{1j} \\ b_{2j} \\ : \\ : \\ b_{pj} \end{pmatrix}$$

EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{pmatrix}$

$\boldsymbol{Ab}_1 =$

$\boldsymbol{a}_1\boldsymbol{B} =$

$\boldsymbol{Ab}_2 =$

$\boldsymbol{a}_2\boldsymbol{B} =$

$\boldsymbol{AB} =$

$\boldsymbol{a}_3\boldsymbol{B} =$

Representing a Linear System Using Matrix Equation

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases}$$

can be represented by a matrix equation $\boxed{\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}}$ where

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

REPRESENTING A LINEAR SYSTEM USING MATRIX EQUATION

$$\boldsymbol{Ax} = \boldsymbol{b}$$

coefficient matrix

variable matrix

constant matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

REPRESENTING A LINEAR SYSTEM USING MATRIX EQUATION

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$\boldsymbol{Ax} = \boldsymbol{b}$ can also be expressed as

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \ldots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

REPRESENTING A LINEAR SYSTEM USING MATRIX EQUATION

$$\boldsymbol{A}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \boldsymbol{x}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \boldsymbol{b}=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$\boldsymbol{c}_1 \qquad \boldsymbol{c}_2 \qquad \boldsymbol{c}_n$

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}+x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}+\dots+x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$x_1\boldsymbol{c}_1+x_2\boldsymbol{c}_2+\dots+x_n\boldsymbol{c}_n=\boldsymbol{b}$ where $\boldsymbol{c}_i = i$th column of $\boldsymbol{A}$

EXAMPLE

$$\begin{cases} x + 2y - z = 1 \\ 2x - y = 2 \\ x + 2y - 3z = 2 \end{cases}$$

Can you find x, y, z that satisfies

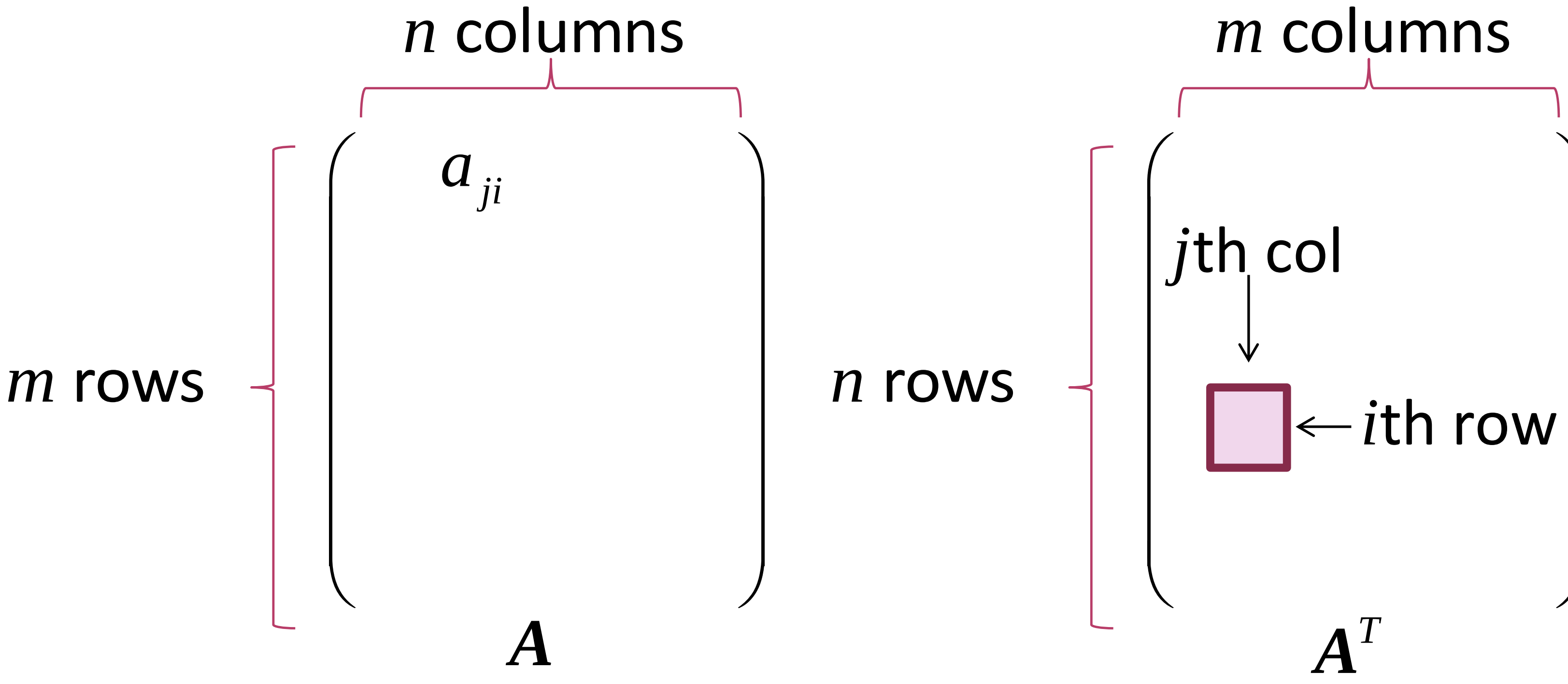
$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

DEFINITION (MATRIX TRANSPOSE)

Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ be a $\boxed{m\times n}$ matrix.

The transpose of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^T$, is a $\boxed{n\times m}$ matrix whose <u>(i, j)-entry is a_{ji}</u>.



EXAMPLE (MATRIX TRANSPOSE)

Write down the transpose of the following matrices:

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 & -1 & 2 & 0 \\ 2 & 1 & 0 & 4 & 6 \\ 1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 1 \\ 3 & 4 & 1 & 0 \end{pmatrix}$$

REMARK

A matrix $\boldsymbol{A}$ is symmetric if and only if

$$\boxed{\boldsymbol{A} = \boldsymbol{A}^T}$$

$$\begin{pmatrix} & & a_{ij} \\ & & \\ a_{ji} & & \end{pmatrix}$$

$a_{ij} = a_{ji}$ for all i, j.

MATRIX PROPERTIES INVOLVING TRANSPOSE

Let $\boldsymbol{A}$ be a $m \times n$ matrix.

1) $(\boldsymbol{A}^T)^T = \boldsymbol{A}$

2) $(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$ (transpose of sum equal sum of transpose).

3) $(a\boldsymbol{A})^T = a\boldsymbol{A}^T$

4) If $\boldsymbol{B}$ is a $n \times p$ matrix, then $(\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T \boldsymbol{A}^T$

(transpose of product equal to product of transpose, but with order reversed).

END OF LECTURE 04

Lecture 05:
Inverses of square matrices
Elementary matrices (till Example 2.4.5)