Lecture 01 Recap

Linear equations (systems), their solutions, solution sets and general solutions.

Geometrical interpretation of linear equations (systems) and their solutions.

Consistent and inconsistent linear systems. Three possibilities for the solution set of a linear system.

Augmented matrix and elementary row operations.

Row equivalent augmented matrices and Theorem 1.2.7.

Lecture 02

Row-echelon forms

Gaussian Elimination

Which is easier to solve?

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \quad\quad - 15z = 9 & (6) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

Our strategy

Since row equivalent augmented matrices corresponds to linear systems having the same solution set...

... our strategy is to perform elementary row operations on the (starting) augemented matrix until it changes into a 'nice' form.

Given linear system we want to solve

augmented matrix in 'nice' form

$$\left(\begin{array}{ccc|c} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right) \xrightarrow[\text{operations}]{\text{elementary row}} \left(\begin{array}{ccc|c} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right)$$

Definition (Row-echelon form)

An augmented matrix is said to be in row-echelon form if it has the following two properties.

1) If there are any rows consisting entirely of zeros, then they are grouped at the bottom of the matrix.

$$\left(\begin{array}{ccccc} * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * \\ \hline 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right)$$

non zero rows

zero rows

(if any)

Definition (Row-echelon form)

2) In any two successive rows that are not entirely zeros, the first non zero number in the lower row occurs further to the right than the first non zero number in the higher row.

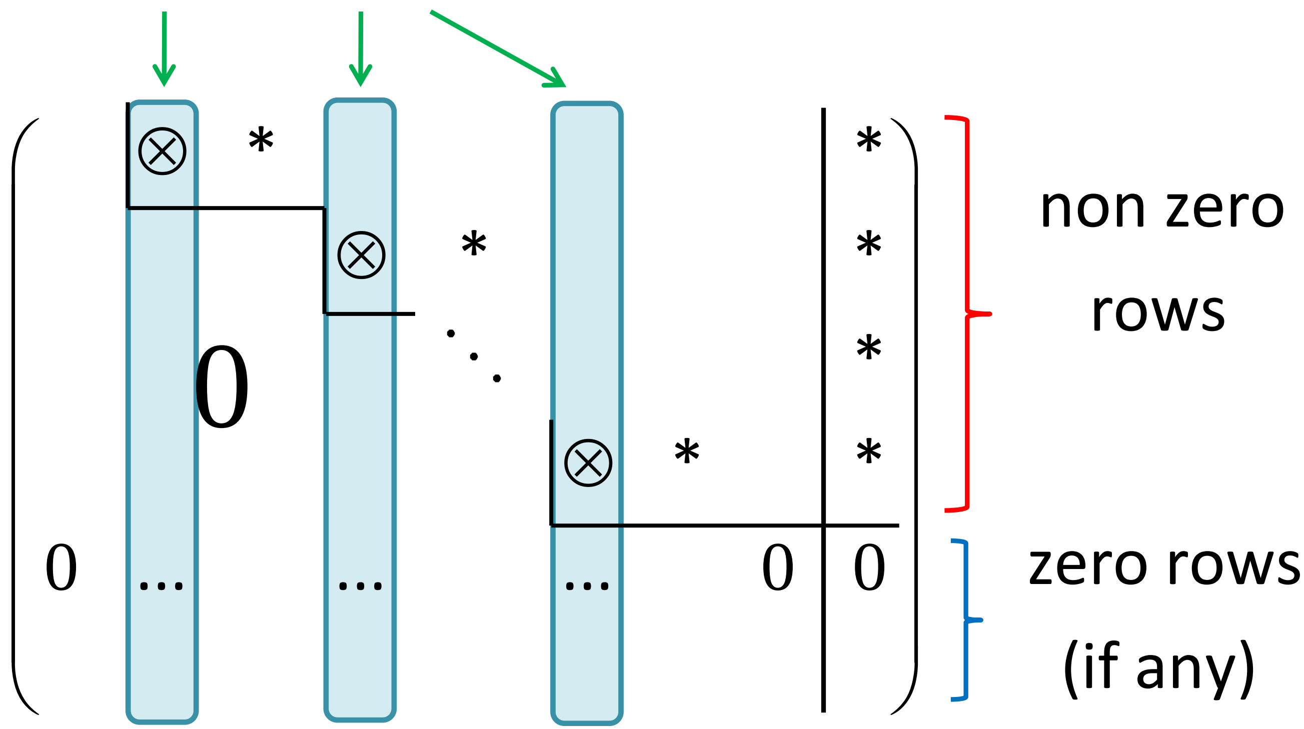
$$\text{non zero rows}\left\{\begin{pmatrix} 0 & 0 & \otimes & * & \cdots & \cdots & * \\ 0 & 0 & 0 & 0 & \otimes & * & * \end{pmatrix}\right\}\text{two successive rows}$$

The first non zero number in every row is called the leading entry of that row. $\otimes$: leading entry

Definition (Pivot column, pivot point)

A leading entry is also called a **pivot point.**

A column containing a leading entry / pivot point is called a **pivot column.**

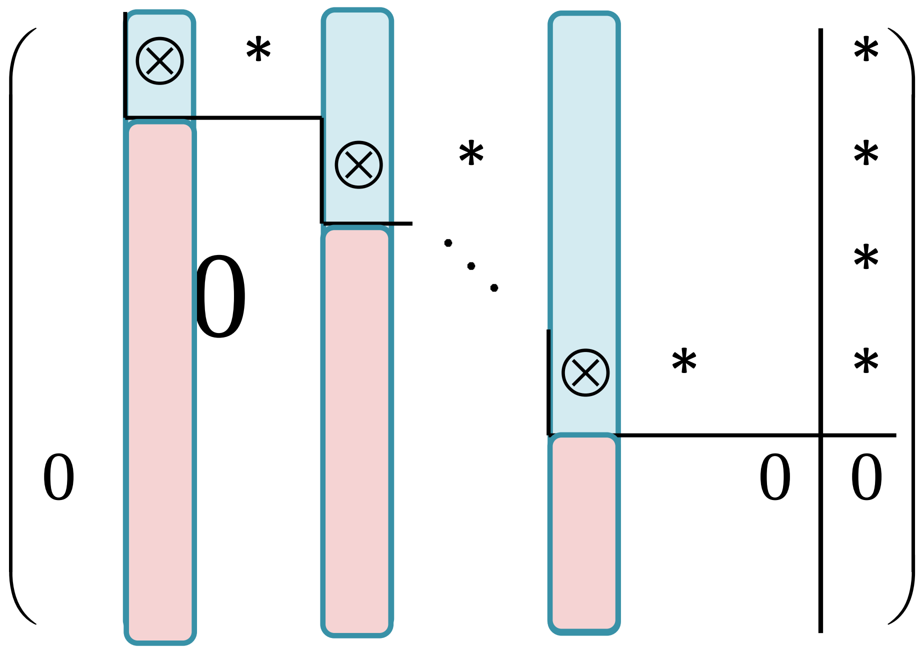


Remark

The concept of row-echelon forms can be applied to matrices in general, not just for augmented matrices.

Some observations

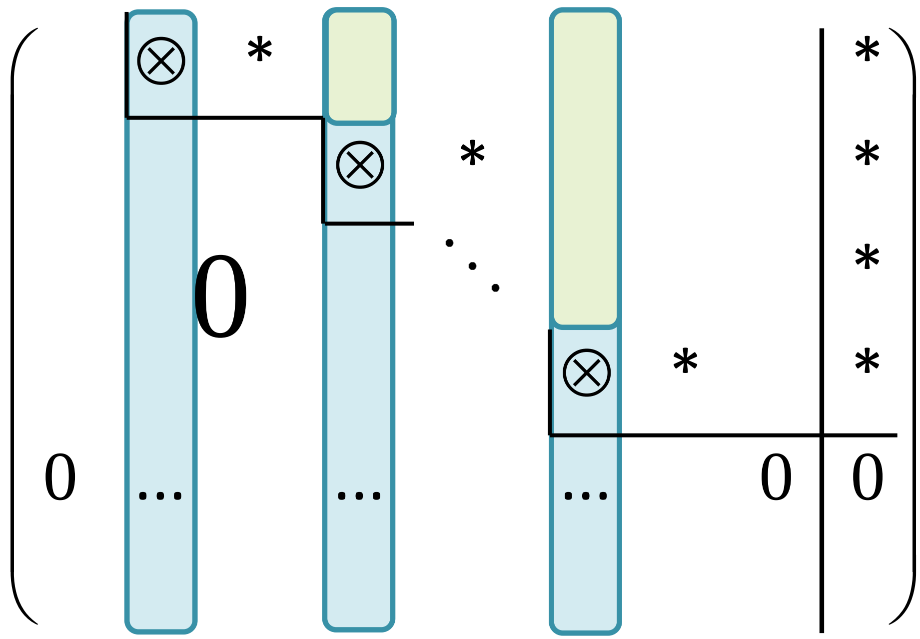
If an augmented matrix is in row-echelon form,



1) all entries (in the same column) below each leading entry must be 0.

Some observations

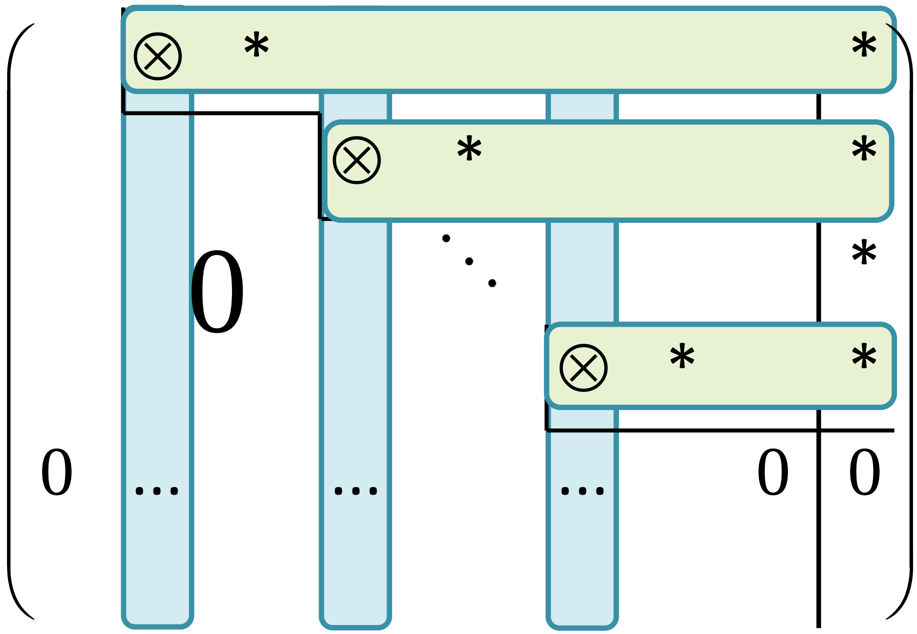
If an augmented matrix is in row-echelon form,



2) entries (in the same column) above each leading entry do not have to be 0.

Some observations

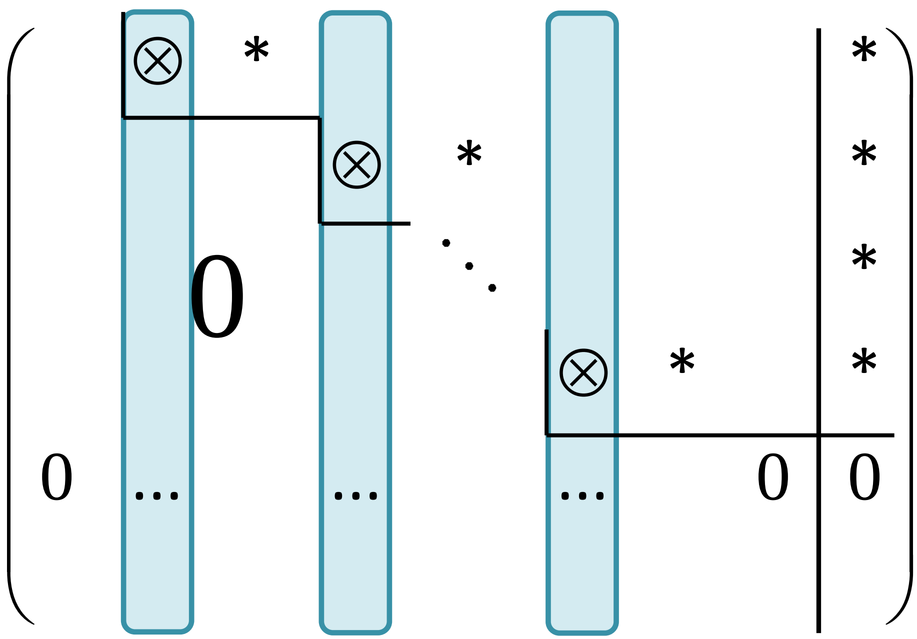
If an augmented matrix is in row-echelon form,



3) every non zero row has one and exactly one leading entry.

Some observations

If an augmented matrix is in row-echelon form,



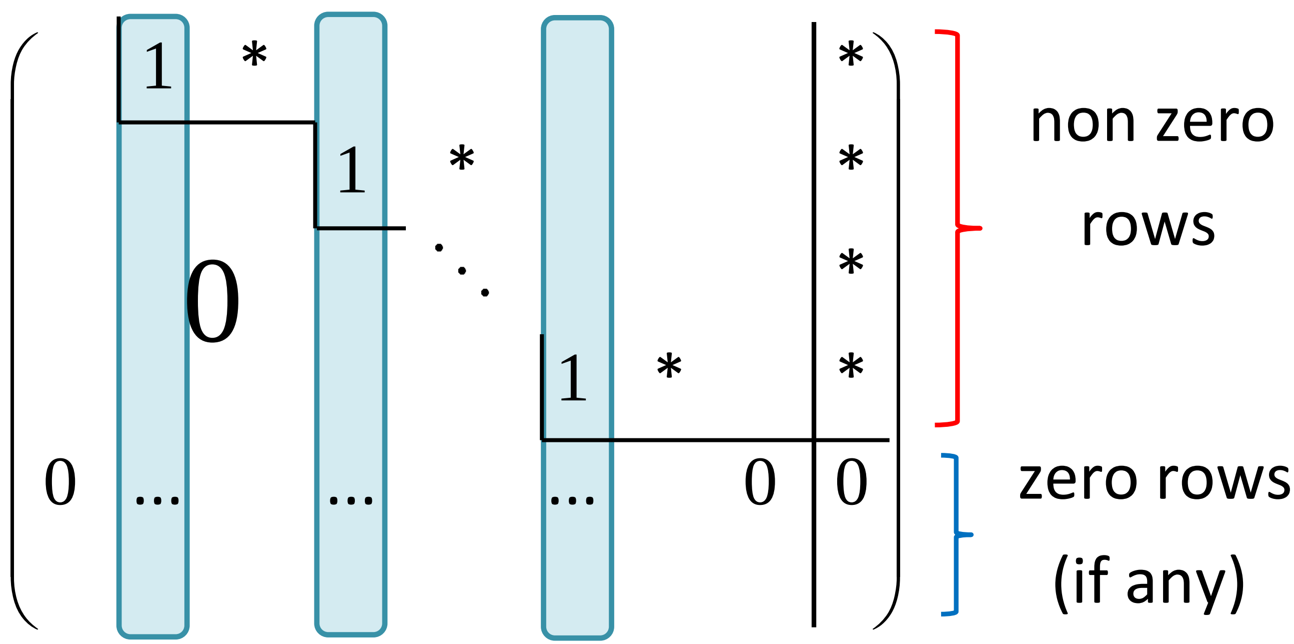
4) every pivot column has one and exactly one pivot point;

a column without a pivot point is a non-pivot column.

Definition (Reduced row-echelon form)

An augmented matrix is said to be in reduced row-echelon form if it is in row-echelon form and has the following two additional properties.

3) all leading entries must be 1.



Definition (Reduced row-echelon form)

4) in each pivot column, other than the pivot point, all other entries are zero.

$$\left(\begin{array}{cccccccc|c} & 1 & * & 0 & & 0 & & & * \\ & & & 1 & * & 0 & & & * \\ & & \mathbf{0} & & \ddots & 0 & & & * \\ & & & & & 1 & * & & * \\ 0 & \cdots & & \cdots & & \cdots & & 0 & 0 \end{array}\right)$$

Examples (row-echelon form)

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

Examples (reduced row-echelon form)

$$\begin{cases} v + w = 0 \\ x + y = -2 \\ z = 4 \end{cases}$$

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

reduced

row-echelon form?

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

How easy is it?

3 variables: x, y, z

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

reduced row-echelon form?

Yes

Writing out the linear system:

$$\begin{cases} x \qquad\qquad = 2 \\ \qquad y \qquad = 0 \\ \qquad\qquad z = 4 \end{cases}$$

Linear system has a unique solution.

How easy is it?

5 variables: x_1, x_2, x_3, x_4, x_5

$$\left(\begin{array}{ccccc|c} 0 & -1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 & 6 \end{array}\right)$$

reduced row-echelon form?

No. Row-echelon form.

Writing out the linear system:

$$\begin{cases} -x_2 + 4x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 3 \\ 3x_4 + 3x_5 = 6 \end{cases}$$

What happened to x_1?

How easy is it?

For any 'variable' column (left side of vertical line) that is a non pivot column, we will assign an arbitrary parameter to that variable.

$$\begin{array}{cccccc} x_1 & & & & x_5 & \\ \end{array}$$

$$\left(\begin{array}{ccccc|c} \boxed{0} & -1 & 4 & 0 & \boxed{-1} & 0 \\ \boxed{0} & 0 & 1 & 1 & \boxed{1} & 3 \\ \boxed{0} & 0 & 0 & 3 & \boxed{3} & 6 \end{array}\right)$$

In this case, since x_1 does not appear in any of the equations, none of the variables will 'depend' on the value of x_1.

$$\begin{cases} -x_2 + 4x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 3 \\ 3x_4 + 3x_5 = 6 \end{cases}$$

How easy is it?

Let $x_1 = s, x_5 = t$, where $s,t \in \mathbb{R}$.

$$\begin{cases} \quad - x_2 + 4x_3 \qquad\qquad - x_5 = 0 \\ \qquad\qquad\quad x_3 + x_4 + x_5 = 3 \\ \qquad\qquad\qquad\quad 3x_4 + 3x_5 = 6 \end{cases}$$

Starting from the last equation:

$3x_4 + 3t = 6 \Leftrightarrow 3x_4 = 6 - 3t \Leftrightarrow x_4 = 2 - t.$

Substitute into next 'higher' equation:

$x_3 + (2 - t) + t = 3 \Leftrightarrow x_3 = 1.$

Substitute into next 'higher' equation:

$-x_2 + 4(1) - t = 0 \Leftrightarrow x_2 = 4 - t.$

How easy is it?

$$\left.\begin{array}{l}\text{Let } \underline{x_1 = s, x_5 = t}, \text{ where } s,t \in \mathbb{R}. \\[1em] 3x_4 + 3t = 6 \Leftrightarrow 3x_4 = 6 - 3t \Leftrightarrow \underline{x_4 = 2 - t}. \\[1em] x_3 + (2-t) + t = 3 \Leftrightarrow \underline{x_3 = 1}. \\[1em] -x_2 + 4(1) - t = 0 \Leftrightarrow \underline{x_2 = 4 - t}.\end{array}\right\} \text{Back substitution}$$

A general solution is:

Linear system has infinitely many solutions.

$$\begin{cases} x_1 &= s \\ x_2 &= 4 - t \\ x_3 &= 1 \\ x_4 &= 2 - t \\ x_5 &= t, \qquad s,t \in \mathbb{R}. \end{cases}$$

How easy is it?

5 variables: v,w,x,y,z

$$\begin{array}{cccccc} & w & & y & & \\ \multicolumn{6}{c}{\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right)} \end{array}$$

reduced row-echelon form?

Yes

Writing out the linear system:

$$\begin{cases} v + w = 0 \\ x + y = -2 \\ z = 4 \end{cases}$$

Let $w = s, y = t,\ s,t \in \mathbb{R}$.

How easy is it?

$$\begin{cases} v + w & & & = 0 \\ & x + y & & = -2 \\ & & z & = 4 \end{cases}$$

No back substitution!

Let $w = s, y = t,\ s,t \in \mathbb{R}$.

Starting from the last equation:

$z = 4$.

Next equation:

$x + t = -2 \Leftrightarrow x = -2 - t$.

Next equation:

$v + s = 0 \Leftrightarrow v = -s$.

A general solution is:

$$\begin{cases} v &= -s \\ w &= s \\ x &= -2 - t \\ y &= t \\ z &= 4, \quad s,t \in \mathbb{R}. \end{cases}$$

How easy is it?

3 variables: x, y, z

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

reduced row-echelon form?

No. Row-echelon form.

Writing out the linear system:

$$\begin{cases} x - y = 2 \\ y = 0 \\ \boxed{0x + 0y + 0z = 2} \end{cases}$$

Inconsistent!

How useful are row-echelon forms?

Recall that any linear system either has (i) no solution; (ii) a unique solution; (iii) infinitely many solutions.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

Linear system has a unique solution.

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

Linear system is inconsistent.

$$\left(\begin{array}{ccccc|c} 0 & -1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 & 6 \end{array}\right)$$

Linear system has infinitely many solutions.

So how to get them?

1) Row-echelon forms allows us to determine how many (if any) solutions a linear system has. (Trust me...you will learn this later).

2) Row-echelon forms helps us to write down a general solution for a linear system that is consistent.

3) We now need to develop a systematic way of finding an augmented matrix in row-echelon form that is row equivalent to the augmented matrix of the original linear system.

Definition

Suppose $\boldsymbol{A}$ and $\boldsymbol{R}$ are row equivalent augmented matrices.

$$\left(\begin{array}{c|c} \boldsymbol{A} & \end{array}\right) \xrightarrow[\text{operations}]{\text{Elementary row}} \left(\begin{array}{c|c} \boldsymbol{R} & \end{array}\right)$$

If $\boldsymbol{R}$ is in (reduced) row-echelon form, we say

1) $\boldsymbol{R}$ is a (reduced) row-echelon form of $\boldsymbol{A}$

2) $\boldsymbol{A}$ has a (reduced) row-echelon form $\boldsymbol{R}$

Gaussian Elimination

Gaussian Elimination is an algorithm (systematic way of doing things) to reduce an augmented matrix to a row-echelon form using elementary row operations.

Again, this is not restricted to augmented matrices.

Step 1

Locate the leftmost column that is not entirely zero.

$$\begin{pmatrix} \boxed{0} & 2 & \cdots & \cdots \\ \boxed{1} & -1 & \cdots & \cdots \\ \boxed{2} & 3 & \cdots & \cdots \end{pmatrix} \qquad \begin{pmatrix} \boxed{2} & 1 & \cdots & \cdots \\ \boxed{-1} & 1 & \cdots & \cdots \\ \boxed{3} & 2 & \cdots & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 0 & \boxed{0} & \cdots & \cdots \\ 0 & \boxed{1} & \cdots & \cdots \\ 0 & \boxed{2} & \cdots & \cdots \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \boxed{1} & \cdots \\ 0 & 0 & \boxed{2} & \cdots \\ 0 & 0 & \boxed{-1} & \cdots \end{pmatrix}$$

Step 2

Look at the topmost entry in the column identified in Step 1.

(i) if the entry is non zero, do nothing. This entry is your leading entry.

$$\begin{pmatrix} 0 & 2 & \ldots & \ldots \\ 1 & -1 & \ldots & \ldots \\ 2 & 3 & \ldots & \ldots \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & \ldots & \ldots \\ -1 & 1 & \ldots & \ldots \\ 3 & 2 & \ldots & \ldots \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \ldots & \ldots \\ 0 & 1 & \ldots & \ldots \\ 0 & 2 & \ldots & \ldots \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & \ldots \\ 0 & 0 & 2 & \ldots \\ 0 & 0 & -1 & \ldots \end{pmatrix}$$

Step 2

Look at the topmost entry in the column identified in Step 1.

(ii) if the entry is zero, interchange the top row with another row to bring a non zero entry to the top.

$$\begin{pmatrix} 0 & 2 & \cdots & \cdots \\ 1 & -1 & \cdots & \cdots \\ 2 & 3 & \cdots & \cdots \end{pmatrix} \xrightarrow[\text{rows 1 and 2}]{\text{interchange}} \begin{pmatrix} 1 & -1 & \cdots & \cdots \\ 0 & 2 & \cdots & \cdots \\ 2 & 3 & \cdots & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \cdots & \cdots \\ 0 & 1 & \cdots & \cdots \\ 0 & 2 & \cdots & \cdots \end{pmatrix} \xrightarrow[\text{rows 1 and 2}]{\text{interchange}} \begin{pmatrix} 0 & 1 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ 0 & 2 & \cdots & \cdots \end{pmatrix}$$

Wait a minute…

(ii) if the entry is zero, interchange the top row with another row to bring a non zero entry to the top.

$$\begin{pmatrix} 0 & 2 & \dots & \dots \\ 1 & -1 & \dots & \dots \\ 2 & 3 & \dots & \dots \end{pmatrix} \xrightarrow[\text{rows 1 and 2}]{\text{interchange}} \begin{pmatrix} 1 & -1 & \dots & \dots \\ 0 & 2 & \dots & \dots \\ 2 & 3 & \dots & \dots \end{pmatrix}$$

Can we interchange rows 1 and 3 instead?

$$\begin{pmatrix} 0 & 2 & \dots & \dots \\ 1 & -1 & \dots & \dots \\ 2 & 3 & \dots & \dots \end{pmatrix} \xrightarrow[\text{rows 1 and 3}]{\text{interchange}} \begin{pmatrix} 2 & 3 & \dots & \dots \\ 1 & -1 & \dots & \dots \\ 0 & 2 & \dots & \dots \end{pmatrix}$$

Remember…

At the end of Step 2, you would have identified a pivot column and a leading entry in that column.

$$\begin{pmatrix} \boxed{2} & 1 & \dots & \dots \\ -1 & 1 & \dots & \dots \\ 3 & 2 & \dots & \dots \end{pmatrix} \qquad \begin{pmatrix} \boxed{1} & -1 & \dots & \dots \\ 0 & 2 & \dots & \dots \\ 2 & 3 & \dots & \dots \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \boxed{1} & \dots \\ 0 & 0 & 2 & \dots \\ 0 & 0 & -1 & \dots \end{pmatrix} \qquad \begin{pmatrix} 0 & \boxed{1} & \dots & \dots \\ 0 & 0 & \dots & \dots \\ 0 & 2 & \dots & \dots \end{pmatrix}$$

Step 3

For each row below the row with the leading entry, add a suitable multiple of the row with the leading entry so that all the entries below the leading entry (in the same column) becomes zero.

$$-2 = -\left(\frac{2}{1}\right) \qquad \begin{pmatrix} 0 & 1 & \dots & \dots \\ 0 & 0 & \dots & \dots \\ 0 & 2 & \dots & \dots \end{pmatrix}$$

Add -2 times of row 1 to row 3 so that 2 becomes 0.

Step 3

For each row below the row with the leading entry, add a suitable multiple of the row with the leading entry so that all the entries below the leading entry (in the same column) becomes zero.

$$\frac{1}{2} = -\left(\frac{-1}{2}\right)$$

$$\begin{pmatrix} 2 & 1 & \dots & \dots \\ -1 & 1 & \dots & \dots \\ 3 & 2 & \dots & \dots \end{pmatrix}$$

$$-\frac{3}{2} = -\left(\frac{3}{2}\right)$$

Add $\frac{1}{2}$ times of row 1 to row 2 so that (-1) becomes 0.

Add $-\frac{3}{2}$ times of row 1 to row 3 so that 3 becomes 0.

Step 4

Cover the row containing the leading entry and apply Step 1 again to the remaining submatrix.

Add $\frac{1}{2}$ times of row 1 to row 2 so that (-1) becomes 0.

$$\begin{pmatrix} 2 & 1 & \cdots & \cdots \\ -1 & 1 & \cdots & \cdots \\ 3 & 2 & \cdots & \cdots \end{pmatrix} \longrightarrow \begin{pmatrix} & & & \\ 0 & \frac{3}{2} & \cdots & \cdots \\ 0 & \frac{1}{2} & \cdots & \cdots \end{pmatrix}$$

remaining submatrix

Add $-\frac{3}{2}$ times of row 1 to row 3 so that 3 becomes 0.

Continue this way until the entire matrix is in row-echelon form.

Example (Gaussian Elimination)

Using Gaussian Elimination, find a row-echelon form of the following augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{array}\right)$$

Gauss-Jordan Elimination

Gauss-Jordan Elimination is an algorithm to reduce an augmented matrix to the reduced row-echelon form using elementary row operations.

Again, this is not restricted to augmented matrices.

Step 5

First, apply Gaussian Elimination to obtain a row-echelon form for the augmented matrix.

row-echelon forms

$$\begin{pmatrix} 2 & 1 & \ldots & \ldots \\ 0 & \frac{3}{2} & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & \ldots \\ 0 & 0 & -\frac{11}{3} & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}$$

Multiply each (non zero) row by a suitable constant so that each leading entry becomes 1.

$$\begin{pmatrix} 2 & 1 & \ldots & \ldots \\ 0 & \frac{3}{2} & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \xrightarrow[\text{Multiply row 2 by } \frac{2}{3}]{\text{Multiply row 1 by } \frac{1}{2}} \begin{pmatrix} 1 & \frac{1}{2} & \ldots & \ldots \\ 0 & 1 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix}$$

Step 5

First, apply Gaussian Elimination to obtain a row-echelon form for the augmented matrix.

row-echelon forms

$$\begin{pmatrix} 2 & 1 & \ldots & \ldots \\ 0 & \frac{3}{2} & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & \ldots \\ 0 & 0 & -\frac{11}{3} & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}$$

Multiply each (non zero) row by a suitable constant so that each leading entry becomes 1.

$$\begin{pmatrix} 0 & 1 & 1 & \ldots \\ 0 & 0 & -\frac{11}{3} & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix} \xrightarrow{\text{Multiply row 2 by } -\frac{3}{11}} \begin{pmatrix} 0 & 1 & 1 & \ldots \\ 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}$$

Step 6

Starting with the leading entry in the 'lowest' row, add suitable multiples of this row to rows above so that entries (in the same column) above the leading entry becomes 0.

$$\begin{pmatrix} 1 & -2 & \dots & \dots & -4 & \dots \\ & 1 & \dots & \dots & 3 & \dots \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 2 & \dots \\ & & & & 1 & \dots \end{pmatrix}$$

$$-2 = -\left(\frac{2}{1}\right)$$

Add (–2) times row $\boxed{}$ to this row so that 2 becomes 0.

Step 6

Starting with the leading entry in the 'lowest' row, add suitable multiples of this row to rows above so that entries (in the same column) above the leading entry becomes 0.

$$\begin{pmatrix} 1 & -2 & \cdots & \cdots & -4 & \cdots \\ & 1 & \cdots & \cdots & 3 & \cdots \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 2 & \cdots \\ & & & & 1 & \cdots \end{pmatrix}$$

$$-3 = -\left(\frac{3}{1}\right)$$

Add (-3) times row $\square$ to this row so that 3 becomes 0.

Step 6

Starting with the leading entry in the 'lowest' row, add suitable multiples of this row to rows above so that entries (in the same column) above the leading entry becomes 0.

$$\begin{pmatrix} 1 & -2 & \ldots & \ldots & -4 & \ldots \\ & 1 & \ldots & \ldots & 3 & \ldots \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 2 & \ldots \\ & & & & 1 & \ldots \end{pmatrix}$$

Add (4) times row $\boxed{}$ to this row so that -4 becomes 0.

$$4 = -\left(\frac{-4}{1}\right)$$

Step 6

Repeat the same procedure with the 'next lowest' leading entry.

$$\begin{pmatrix} 1 & -2 & \dots & \dots & 0 & \dots \\ & 1 & \dots & \dots & 0 & \dots \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 0 & \dots \\ & & & & 1 & \dots \end{pmatrix}$$

Continue this way until reduced row-echelon form is attained.

Example (Gauss-Jordan Elimination)

Using Gauss-Jordan Elimination, find the reduced row-echelon form of the following augmented matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & -1 & 1 \\ 5 & 9 & 1 & 6 \end{array}\right)$$

Remark

Every matrix has a unique reduced row-echelon form but can have many different row-echelon forms.

Example (solving linear systems)

Solve the following linear system using Gaussian Elimination.

$$\begin{cases} \phantom{-2x_1 - 4x_2 -{}} 2x_3 + 4x_4 + 2x_5 = 8 \\ x_1 + 2x_2 + 4x_3 + 5x_4 + 3x_5 = -9 \\ -2x_1 - 4x_2 - 5x_3 - 4x_4 + 3x_5 = 6 \end{cases}$$

Exercise (solving linear systems)

Solve the following linear system using Gauss-Jordan Elimination.

$$\begin{cases} & & & & 2x_3 & + & 4x_4 & + & 2x_5 & = & 8 \\ x_1 & + & 2x_2 & + & 4x_3 & + & 5x_4 & + & 3x_5 & = & -9 \\ -2x_1 & - & 4x_2 & - & 5x_3 & - & 4x_4 & + & 3x_5 & = & 6 \end{cases}$$

End of Lecture 02

Lecture 03:

Gaussian Elimination (cont'd)

Homogeneous linear systems (end of Chapter 1)