# LECTURE 06 RECAP

Equivalent statements to '$\boldsymbol{A}$ is invertible'.

One way to determine if $\boldsymbol{A}^{-1}$ exists and how to find it if it does.

Proof of the '50% Theorem'.

Definition of the determinant of a square matrix.

Cofactor expansion (along any row or column).

Three theorems about determinants (transposes, triangular matrices, matrices with identical rows or columns).

# LECTURE 07

Determinants (cont'd)

# HOW DO ELEMENTARY ROW OPERATIONS AFFECT DETERMINANTS

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We already know that

$$\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$$

$$\Big\downarrow kR_3$$

Recall that $\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{B}_1$ where

$$\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$\boldsymbol{B}_1 = \begin{pmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{pmatrix}$$

$$\begin{aligned} \det(\boldsymbol{B}_1) &= aeki + bfkg + cdkh - cekg - afkh - bdki \\ &= k(aei + bfg + cdh - ceg - afh - bdi) = k\det(\boldsymbol{A}) \end{aligned}$$

# HOW DO ELEMENTARY ROW OPERATIONS AFFECT DETERMINANTS

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{kR_3} \begin{pmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{pmatrix} = \boldsymbol{B}_1$$

$\boxed{\boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{B}_1}$ where $\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$ $\det(\boldsymbol{E}_1) = k$

$$\boxed{\begin{aligned} \det(\boldsymbol{B}_1) &= k\det(\boldsymbol{A}) \\ &= \det(\boldsymbol{E}_1)\det(\boldsymbol{A}) \end{aligned}}$$

So it seems like elementary row operations of this type changes the determinant of $\boldsymbol{A}$ by a factor of $k$ and we have 'determinant of product equals to product of determinants' (to some extent).

# HOW DO ELEMENTARY ROW OPERATIONS AFFECT DETERMINANTS

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We already know that

$$\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$$

$$R_1 \leftrightarrow R_3$$

$$\boldsymbol{B}_2 = \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}$$

Recall that $\boldsymbol{E}_2 \boldsymbol{A} = \boldsymbol{B}_2$ where

$$\boldsymbol{E}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det(\boldsymbol{B}_2) = gec + hfa + idb - iea - gfb - hdc$$

$$= -(iea + gfb + hdc - gec - hfa - idb) = -\det(\boldsymbol{A})$$

# How do elementary row operations affect determinants

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix} = \boldsymbol{B}_2$$

$\boxed{\boldsymbol{E}_2\boldsymbol{A} = \boldsymbol{B}_2}$ where $\boldsymbol{E}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\det(\boldsymbol{E}_2) = -1$

$$\boxed{\begin{aligned} \det(\boldsymbol{B}_2) &= -\det(\boldsymbol{A}) \\ &= \det(\boldsymbol{E}_2)\det(\boldsymbol{A}) \end{aligned}}$$

So it seems like elementary row operations of this type changes the determinant of $\boldsymbol{A}$ by a factor of $-1$ and we have 'determinant of product equals to product of determinants' (to some extent).

# HOW DO ELEMENTARY ROW OPERATIONS AFFECT DETERMINANTS

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We already know that

$\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$

$$\xrightarrow{R_2 + kR_3}$$

Recall that $\boldsymbol{E}_3\boldsymbol{A} = \boldsymbol{B}_3$ where

$$\boldsymbol{B}_3 = \begin{pmatrix} a & b & c \\ d+kg & e+kh & f+ki \\ g & h & i \end{pmatrix}$$

$$\boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}\det(\boldsymbol{B}_3) &= ai(e+kh) + bg(f+ki) + ch(d+kg) \\ &\quad - cg(e+kh) - ah(f+ki) - bi(d+kg) = \det(\boldsymbol{A})\end{aligned}$$

# HOW DO ELEMENTARY ROW OPERATIONS AFFECT DETERMINANTS

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{R_2 + kR_3} \begin{pmatrix} a & b & c \\ d+kg & e+kh & f+ki \\ g & h & i \end{pmatrix} = \boldsymbol{B}_3$$

$\boxed{\boldsymbol{E}_3 \boldsymbol{A} = \boldsymbol{B}_3}$ where $\boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$ $\det(\boldsymbol{E}_3) = 1$ (why?)

$$\boxed{\begin{aligned} \det(\boldsymbol{B}_3) &= \det(\boldsymbol{A}) \\ &= \det(\boldsymbol{E}_3)\det(\boldsymbol{A}) \end{aligned}}$$

So it seems like elementary row operations of this type does not change the determinant of $\boldsymbol{A}$ and we have 'determinant of product equals to product of determinants' (to some extent).

# THEOREM (E.R.O. AND DETERMINANTS)

Let $\boldsymbol{A}$ be a square matrix. Then

$$\boldsymbol{A} \xrightarrow{kR_i} \boldsymbol{B}_1 \qquad \det(\boldsymbol{B}_1) = k\det(\boldsymbol{A})$$

$$\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}_2 \qquad \det(\boldsymbol{B}_2) = -\det(\boldsymbol{A})$$

$$\boldsymbol{A} \xrightarrow{R_j + kR_i} \boldsymbol{B}_3 \qquad \det(\boldsymbol{B}_3) = \det(\boldsymbol{A})$$

Furthermore, if $\boldsymbol{E}$ is an elementary matrix of the same size as $\boldsymbol{A}$, then $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

# PROOF

We will prove the following:

$$\boldsymbol{A} \xrightarrow{R_j + kR_i} \boldsymbol{B}_3 \qquad \det(\boldsymbol{B}_3) = \det(\boldsymbol{A})$$

Let $\boldsymbol{A}^*$ be the following

$$\begin{array}{r} \\ \\ \text{row } i \\ \\ \\ \text{row } j \\ \\ \\ \end{array} \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\ \vdots & \vdots & & & \vdots \\ a_{j1} & a_{j2} & \cdots & \cdots & a_{jn} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix} \quad \boldsymbol{A}^* = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\ \vdots & \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

# PROOF

Observations:

1) $\boldsymbol{A}$ and $\boldsymbol{A}^*$ differ in only one row (row $j$).

2) $\det(\boldsymbol{A}^*) = 0$ since $\boldsymbol{A}^*$ has two identical rows.

$$
\begin{array}{r}
\\ \\ \text{row } i \\ \boldsymbol{A} = \\ \text{row } j \\ \\ \\
\end{array}
\begin{pmatrix}
a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\
\vdots & \vdots & & & \vdots \\
a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\
\vdots & \vdots & & & \vdots \\
a_{j1} & a_{j2} & \cdots & \cdots & a_{jn} \\
\vdots & \vdots & & & \vdots \\
a_{n1} & a_{n2} & \cdots & \cdots & a_{nn}
\end{pmatrix}
\quad
\boldsymbol{A}^* =
\begin{pmatrix}
a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\
\vdots & \vdots & & & \vdots \\
a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\
\vdots & \vdots & & & \vdots \\
a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\
\vdots & \vdots & & & \vdots \\
a_{n1} & a_{n2} & \cdots & \cdots & a_{nn}
\end{pmatrix}
$$

# PROOF

For $k = 1, 2, \ldots, n,$

cofactors of $\boldsymbol{A}$ $\longleftarrow$ $A_{jk} = A_{jk}{}^{*}$ $\longrightarrow$ cofactors of $\boldsymbol{A}^{*}$

Observations:

3) $(j,1)$-cofactor of $\boldsymbol{A} = (j,1)$-cofactor of $\boldsymbol{A}^{*}$

$(j,2)$-cofactor of $\boldsymbol{A} = (j,2)$-cofactor of $\boldsymbol{A}^{*}$

$$\begin{array}{r} \\ \\ \text{row } i \\ \\ \text{row } j \\ \\ \\ \end{array} \quad \boldsymbol{A} = \begin{pmatrix} & & \cdots & \cdots & a_{1n} \\ & & & & \vdots \\ & & \cdots & \cdots & a_{in} \\ & & & & \vdots \\ & & & & \\ & & & & \vdots \\ & & \cdots & \cdots & a_{nn} \end{pmatrix} \qquad \boldsymbol{A}^{*} = \begin{pmatrix} & & \cdots & \cdots & a_{1n} \\ & & & & \vdots \\ & & \cdots & \cdots & a_{in} \\ & & & & \vdots \\ & & & & \\ & & & & \vdots \\ & & \cdots & \cdots & a_{nn} \end{pmatrix}$$

# PROOF

For $k = 1, 2, \dots, n,$

cofactors of $\boldsymbol{A} \longleftarrow A_{jk} = A_{jk}^{*} \longrightarrow$ cofactors of $\boldsymbol{A}^{*}$

Observations:

3) $(j,1)$-cofactor of $\boldsymbol{A} = (j,1)$-cofactor of $\boldsymbol{A}^{*}$

$(j,2)$-cofactor of $\boldsymbol{A} = (j,2)$-cofactor of $\boldsymbol{A}^{*}$

cofactor expansion along row $j$ of $\boldsymbol{A}^{*}$ :

$$\det(\boldsymbol{A}^{*}) = a_{i1}A_{j1}^{*} + a_{i2}A_{j2}^{*} + \dots + a_{in}A_{jn}^{*}$$

$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{jn}$$

(by observation 3)

$= 0$ (by observation 2)

$$\boldsymbol{A}^{*} = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ \vdots & \vdots & & & \vdots \\ a_{i1} & a_{i2} & \dots & \dots & a_{in} \\ \vdots & \vdots & & & \vdots \\ \boxed{a_{i1}} & a_{i2} & \dots & \dots & a_{in} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{pmatrix}$$

# PROOF

$$\boxed{a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn} = 0}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\ \vdots & \vdots & & & \vdots \\ a_{j1} & a_{j2} & \cdots & \cdots & a_{jn} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & a_{in} \\ \vdots & \vdots & & & \vdots \\ a_{j1}+ka_{i1} & \cdots & \cdots & \cdots & a_{jn}+ka_{in} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

$$\boldsymbol{A} \xrightarrow{R_j + kR_i} \boldsymbol{B}_3$$

# PROOF

$$\boxed{a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn} = 0}$$

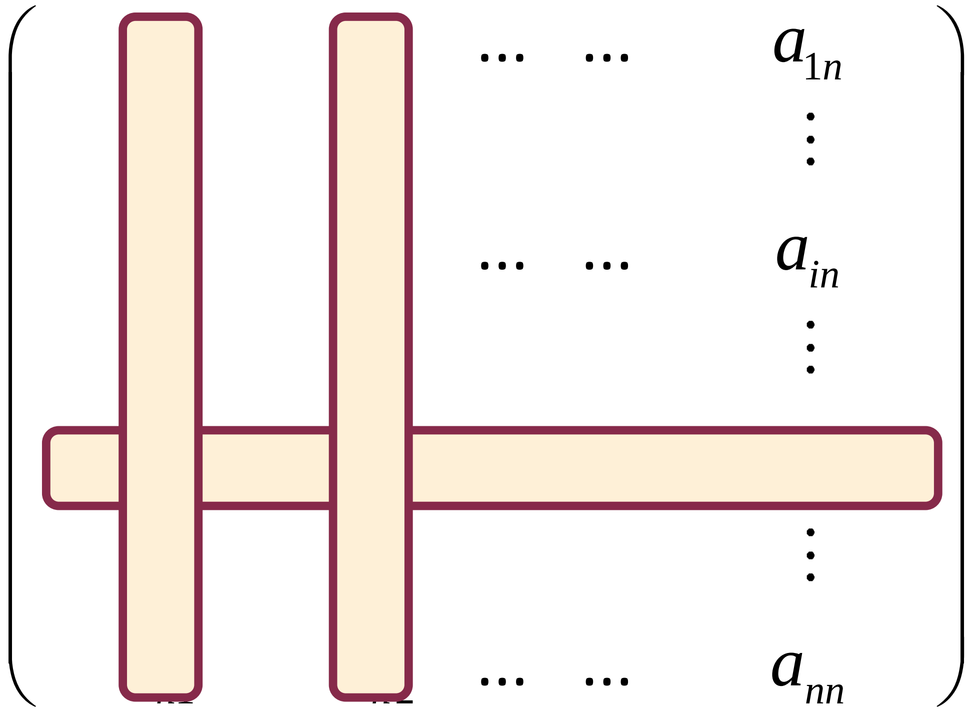
cofactor expansion

along row $j$ of $\boldsymbol{B}_3$ :

$$\det(\boldsymbol{B}_3)$$

$$= (a_{j1} + ka_{i1})A_{j1} + \ldots$$

$$+ (a_{jn} + ka_{in})A_{jn}$$




$$\boldsymbol{A} \xrightarrow{R_j + kR_i} \boldsymbol{B}_3$$

# PROOF

$$\boxed{a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn} = 0}$$

$$\det(\boldsymbol{B}_3) = (a_{j1} + ka_{i1})A_{j1} + \ldots + (a_{jn} + ka_{in})A_{jn}$$

$$= a_{j1}A_{j1} + ka_{i1}A_{j1} + \ldots + a_{jn}A_{jn} + ka_{in}A_{jn}$$

$$= a_{j1}A_{j1} + a_{j2}A_{j2} + \ldots + a_{jn}A_{jn} +$$

$$k\boxed{(a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn}) = 0}$$

$$= a_{j1}A_{j1} + a_{j2}A_{j2} + \ldots + a_{jn}A_{jn}$$

$= \det(\boldsymbol{A})$ (cofactor expansion along row $j$ of $\boldsymbol{A}$)

$$\boldsymbol{A} \xrightarrow{R_j + kR_i} \boldsymbol{B}_3$$

# HOW CAN WE USE THIS THEOREM?

wow, this is quite something, but how can we use such a result?

suppose you want to find the determinant of $\boldsymbol{A}$...

... you first find the determinant of its row-echelon form.

$$\boldsymbol{A} \longrightarrow \boldsymbol{R}$$

# HOW CAN WE USE THIS THEOREM?

oh I know, then we keep track of what e.r.o. have been performed on $\boldsymbol{A}$...

Yes! $\boldsymbol{R}$ is a triangular matrix whose determinant is easy to evaluate... we can now 'backtrack' to find the determinant of $\boldsymbol{A}$.

$$\boldsymbol{A} \longrightarrow \boldsymbol{R}$$

# EXAMPLE (E.R.O. AND DETERMINANTS)

Find the determinant of the following matrix using elementary row operations.

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{pmatrix}$$

# EXAMPLE (E.R.O. AND DETERMINANTS)

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent matrices as shown below:

$$\boldsymbol{A} \xrightarrow{R_1 + \frac{2}{9}R_2} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{4R_2} \begin{pmatrix} 5 & 0 & 8 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} = \boldsymbol{B}$$

Find $\det(\boldsymbol{A})$.

$$\det(\boldsymbol{B}) = 1 \cdot (-1) \cdot (4) \cdot \det(\boldsymbol{A})$$

$$5 \cdot (-2) \cdot 1 \cdot \tfrac{1}{3} = -4\det(\boldsymbol{A})$$

$$\tfrac{5}{6} = \det(\boldsymbol{A})$$

Can you find $\boldsymbol{A}$?

# REMARK (ELEMENTARY COLUMN OPERATIONS)

Instead of performing elementary operations on rows of a matrix, we can also perform these operations on columns. These operations are called elementary column operations.

Corresponding to the previous theorem, there is another one stated in terms of elementary column operations.

However, column operations are not discussed in this module.

# REMEMBER THIS?

If $\boldsymbol{A}$ is a square matrix, then the following statements are equivalent.

1) $\boldsymbol{A}$ is invertible.

2) $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.

3) The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

We can now add one more equivalent statement!

5) $\det(\boldsymbol{A}) \neq 0$

# THEOREM

A square matrix $\boldsymbol{A}$ is invertible if and only if

$\det(\boldsymbol{A}) \neq 0.$

Recall that we have already proven this:

$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if

$$ad - bc = \det(\boldsymbol{A}) \neq 0.$$

# THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of order $n$ and $c$ is a scalar.

1) $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$

Proof:

$$\boldsymbol{A} \xrightarrow{cR_1,\ cR_2, \ldots,\ cR_n} c\boldsymbol{A}$$

Each $cR_i$ changes the determinant by a factor of $c$, so $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$.

# THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of order $n$ and $c$ is a scalar.

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Remark:

This results generalizes one that we had previously:

$$\boxed{\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})}$$

where $\boldsymbol{E}$ is an elementary matrix of the same order as $\boldsymbol{A}$.

# THEOREM

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Proof:

If $\boldsymbol{A}$ is singular, we already know that $\boldsymbol{AB}$ is singular.

In this case, $\det(\boldsymbol{A}) = 0,\ \det(\boldsymbol{AB}) = 0,$ so

$$0 = \det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) = 0$$

Next consider the case when $\boldsymbol{A}$ is invertible.

# THEOREM

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Proof:

Next consider the case when $\boldsymbol{A}$ is invertible.

Since $\boldsymbol{A}$ is invertible, it can be expressed as a product of elementary matrices.

$$\boldsymbol{A} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \dots \boldsymbol{E}_2 \boldsymbol{E}_1$$

So

$$\boldsymbol{AB} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \dots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{B}$$

$$\Rightarrow \det(\boldsymbol{AB}) = \det(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \dots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{B})$$

# THEOREM

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Proof:

We use the result $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$

repeatedly on $\boxed{\phantom{A}}$

$$\Rightarrow \det(\boldsymbol{AB}) = \boxed{\det(\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{B})}$$

$$= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{B})$$

$$\vdots$$

$$= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\ldots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)\det(\boldsymbol{B})$$

# THEOREM

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Proof:

We again use the result $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$ repeatedly on $\boxed{\phantom{x}}$

$$\det(\boldsymbol{AB}) = \boxed{\det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\ldots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)}\det(\boldsymbol{B})$$

$$= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\ldots\det(\boldsymbol{E}_2\boldsymbol{E}_1)\det(\boldsymbol{B})$$

$$\vdots$$

$$= \det(\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_2\boldsymbol{E}_1)\det(\boldsymbol{B})$$

$$= \det(\boldsymbol{A})\det(\boldsymbol{B})$$

# THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of order $n$ and $c$ is a scalar.

3) If $\boldsymbol{A}$ is invertible, then $\det(\boldsymbol{A}^{-1}) = \dfrac{1}{\det(\boldsymbol{A})}$.

Proof:

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \Rightarrow \det(\boldsymbol{A}^{-1}\boldsymbol{A}) = \det(\boldsymbol{I})$$

$$\Rightarrow \det(\boldsymbol{A}^{-1})\det(\boldsymbol{A}) = \det(\boldsymbol{I})$$

by $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

$$\Rightarrow \det(\boldsymbol{A}^{-1})\det(\boldsymbol{A}) = 1$$

$$\Rightarrow \det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}$$

# EXAMPLE

Let $\boldsymbol{A}=\begin{pmatrix}-3 & -2 & 4\\ 4 & 3 & 1\\ 0 & 2 & 4\end{pmatrix}$. We can check that $\det(\boldsymbol{A})=34$.

$\det(4\boldsymbol{A})=4^3\det(\boldsymbol{A})=64\cdot 34=2176.$

If $\boldsymbol{B}=\begin{pmatrix}1 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 2\end{pmatrix}$, $\det(\boldsymbol{B})=$

$\det(\boldsymbol{AB})=$ $\det(\boldsymbol{A}^{-1})=$

# DEFINITION

Let $\boldsymbol{A}$ be a square matrix of order $n$. The adjoint of $\boldsymbol{A}$ is the square matrix of order $n$

$$\operatorname{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

Recall that $A_{ij}$ is the $(i, j)$-cofactor of $\boldsymbol{A}$.

# THEOREM (INVERSE IN TERMS OF ADJOINT)

Let $\boldsymbol{A}$ be a square matrix of order $n$. If $\boldsymbol{A}$ is invertible, then

$$\boxed{\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\operatorname{adj}(\boldsymbol{A}).}$$

Proof:

It suffices to show that

'candidate'

$$\boldsymbol{A}\left[\frac{1}{\det(\boldsymbol{A})}\operatorname{adj}(\boldsymbol{A})\right] = \boldsymbol{I}$$

# THEOREM (INVERSE IN TERMS OF ADJOINT)

Proof:

$$\boldsymbol{A}\left[\frac{1}{\det(\boldsymbol{A})}\operatorname{adj}(\boldsymbol{A})\right]=\boldsymbol{I}$$

$(i, j)$-entry of $\boldsymbol{A}[\operatorname{adj}(\boldsymbol{A})]$

$$=(i,j)\text{-entry of}\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ \vdots & \vdots & & & \vdots \\ \boxed{a_{i1}} & \boxed{a_{i2}} & \boxed{\cdots} & \boxed{\cdots} & \boxed{a_{in}} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} A_{11} & \cdots & \boxed{A_{j1}} & \cdots & A_{n1} \\ A_{12} & \cdots & \boxed{A_{j2}} & \cdots & A_{n2} \\ \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots \\ A_{1n} & \cdots & \boxed{A_{jn}} & \cdots & A_{nn} \end{pmatrix}$$

$$=a_{i1}A_{j1}+a_{i2}A_{j2}+\ldots+a_{in}A_{jn}$$

# THEOREM (INVERSE IN TERMS OF ADJOINT)

Proof:

$(i, j)$-entry of $\boldsymbol{A}[\mathrm{adj}(\boldsymbol{A})] = \boxed{a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn}}$

When $i = j$, $(i, i)$-entry of $\boldsymbol{A}[\mathrm{adj}(\boldsymbol{A})]$ is

$$a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$$

$= \det(\boldsymbol{A})$ (cofactor expansion along row $i$)

When $i \neq j$, $(i, j)$-entry of $\boldsymbol{A}[\mathrm{adj}(\boldsymbol{A})]$ is

$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn}$$

$= 0$ (see slide #13)

# THEOREM (INVERSE IN TERMS OF ADJOINT)

Proof:

$$(i, j)\text{-entry of } \boldsymbol{A}[\mathrm{adj}(\boldsymbol{A})] = \begin{cases} \det(\boldsymbol{A}) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\boldsymbol{A}\left[\frac{1}{\det(\boldsymbol{A})}\mathrm{adj}(\boldsymbol{A})\right] = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(\boldsymbol{A}) & & & \Large 0 \\ & \det(\boldsymbol{A}) & & \\ & \Large 0 & \ddots & \\ & & & \det(\boldsymbol{A}) \end{pmatrix}$$

$$= \boldsymbol{I}$$

So we have shown that $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})}\mathrm{adj}(\boldsymbol{A})$.

# EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find adj($\boldsymbol{A}$).

$$A_{11} = d \qquad A_{12} = -c \qquad A_{21} = -b \qquad A_{22} = a$$

$$\text{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T$$

$$= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

# EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$. We have seen that $\det(\boldsymbol{A}) = 34$.

$$A_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10 \qquad A_{12} = -\begin{vmatrix} 4 & 1 \\ 0 & 4 \end{vmatrix} = -16 \qquad A_{13} = \begin{vmatrix} 4 & 3 \\ 0 & 2 \end{vmatrix} = 8$$

$$A_{21} = -\begin{vmatrix} -2 & 4 \\ 2 & 4 \end{vmatrix} = 16 \quad A_{22} = \begin{vmatrix} -3 & 4 \\ 0 & 4 \end{vmatrix} = -12 \quad A_{23} = -\begin{vmatrix} -3 & -2 \\ 0 & 2 \end{vmatrix} = 6$$

$$A_{31} = \begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix} = -14 \quad A_{32} = -\begin{vmatrix} -3 & 4 \\ 4 & 1 \end{vmatrix} = 19 \quad A_{33} = \begin{vmatrix} -3 & -2 \\ 4 & 3 \end{vmatrix} = -1$$

# EXAMPLE

$$\mathrm{adj}(\boldsymbol{A}) = \begin{pmatrix} 10 & -16 & 8 \\ 16 & -12 & 6 \\ -14 & 19 & -1 \end{pmatrix}^T = \begin{pmatrix} 10 & 16 & -14 \\ -16 & -12 & 19 \\ 8 & 6 & -1 \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{34}\begin{pmatrix} 10 & 16 & -14 \\ -16 & -12 & 19 \\ 8 & 6 & -1 \end{pmatrix}$$

# A SPECIAL METHOD FOR SOLVING SPECIAL KINDS OF LINEAR SYSTEMS

Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is a linear system where $\boldsymbol{A}$ is an invertible square matrix of order $n$. For $i = 1, 2, \ldots, n$, let $\boldsymbol{A}_i$ be the square matrix of order $n$ where the $i$th column of $\boldsymbol{A}$ is replaced by $\boldsymbol{b}$.

Cramer's Rule

Then the unique solution is $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ where

$$x_i = \frac{\det(\boldsymbol{A}_i)}{\det(\boldsymbol{A})},\ i = 1, 2, \ldots, n.$$

# EXAMPLE (CRAMER'S RULE)

Solve the following linear system by Cramer's Rule.

$$\begin{cases} x + y + z = \boxed{-1} \\ 2x - y - z = \boxed{4} \\ x + 2y - 3z = \boxed{7} \end{cases}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -3 \end{pmatrix} \quad \boldsymbol{A}_1 = \begin{pmatrix} -1 & 1 & 1 \\ 4 & -1 & -1 \\ 7 & 2 & -3 \end{pmatrix} \quad \boldsymbol{A}_2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 4 & -1 \\ 1 & 7 & -3 \end{pmatrix}$$

$$\boldsymbol{A}_3 = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{pmatrix}$$

# EXAMPLE (CRAMER'S RULE)

$$\det(\boldsymbol{A}) = 15 \quad \det(\boldsymbol{A}_1) = 15 \quad \det(\boldsymbol{A}_2) = 0 \quad \det(\boldsymbol{A}_3) = -30$$

$$x = \frac{15}{15} = 1 \qquad y = \frac{0}{15} = 0 \qquad z = \frac{-30}{15} = -2$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -3 \end{pmatrix} \quad \boldsymbol{A}_1 = \begin{pmatrix} \boxed{-1} & 1 & 1 \\ \boxed{4} & -1 & -1 \\ \boxed{7} & 2 & -3 \end{pmatrix} \quad \boldsymbol{A}_2 = \begin{pmatrix} 1 & \boxed{-1} & 1 \\ 2 & \boxed{4} & -1 \\ 1 & \boxed{7} & -3 \end{pmatrix}$$

$$\boldsymbol{A}_3 = \begin{pmatrix} 1 & 1 & \boxed{-1} \\ 2 & -1 & \boxed{4} \\ 1 & 2 & \boxed{7} \end{pmatrix}$$

# END OF LECTURE 07

Lecture 08:

Euclidean *n*-spaces

Linear Combinations and Linear Spans (till Example 3.2.8)