Test information

Date: 5th October (Friday)
Time: 4.15pm – 5.15pm (60 minutes)
Venue: MPSH1 (Section A + Section B). Seating plan is now available in IVLE.
Scope: Chapter 1, 2, 3 (up to and including Section 3.2)

Format: 3 questions. 40 marks.
Questions 1 and 2 from Chapters 1 and 2.
Question 3 from Chapter 3.

Calculators: Any non-programmable calculator can be used. Graphing calculators cannot be used.

Help sheet: One A4-sized, double sided, handwritten. Do not bring any other paper to the test.

Others: Answer booklet will be provided. You MUST write your answers in pen.

Lecture 11 recap

1) A theorem leading to the definition of coordinate vectors relative to a basis.
2) What are standard basis and why they are useful.
3) Some useful results regarding coordinate vectors.
4) A theorem leading to the definition of dimension.

Lecture 12

Dimensions (continued)

Transition Matrices

Examples (Dimension)

Find a basis for and determine the dimension of the solution space of the homogeneous system

$$\begin{cases} 2v + 2w - x + z = 0 \\ -v - w + 2x - 3y + z = 0 \\ x + y + z = 0 \\ v + w - 2x - z = 0 \end{cases}$$

Consider this…

Question: To find a basis for and determine the dimension of the solution space of a homogeneous linear system.

$$\left(\begin{array}{ccccc|c} \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & & \vdots & & & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdot\cdot & 0 \end{array}\right) \xrightarrow{\text{GJE}} \left(\begin{array}{cccccc|c} * & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & * & \cdots & 0 & \cdots & 0 \\ & & 0 & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & * & \cdots & 0 \end{array}\right)$$

pivot columns

How do we write out a general solution?

Consider this…

$$\begin{array}{ccccccc} & x_i & & x_j & & x_k & \\ \end{array}$$

$$\left(\begin{array}{ccccccc|c} * & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & * & \cdots & 0 & \cdots & 0 \\ & & 0 & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & * & \cdots & 0 \end{array}\right)$$

Assign variables corresponding to non pivot columns arbitrary parameters.

Let's say there are a total of 3 arbitrary parameters s,t,u (it does not matter how many) assigned to variables x_i, x_j, x_k.

How do we write out a general solution?

Consider this…

3 arbitrary parameters s,t,u

(it does not matter how many)

assigned to variables x_i, x_j, x_k.

$$
\begin{array}{c}
\begin{array}{ccccccccc}
 & x_i & & x_j & & x_k & & \\
\end{array} \\
\left(\begin{array}{cccccc|c}
* & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & * & \cdots & 0 & \cdots & 0 \\
 & & 0 & & \vdots & & \vdots \\
0 & \cdots & 0 & \cdots & * & \cdots & 0
\end{array}\right)
\end{array}
$$

$$
\begin{cases}
\vdots & & \vdots \\
x_i & = & s \\
\vdots & & \vdots \\
x_j & = & t \\
\vdots & & \vdots \\
x_k & = & u \\
\vdots & & \vdots
\end{cases}
\qquad
\begin{pmatrix} \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_k \\ \vdots \end{pmatrix}
= s \begin{pmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}
+ t \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}
+ u \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}
$$

Consider this…

$$\begin{pmatrix} \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_k \\ \vdots \end{pmatrix} = s\begin{pmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + t\begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + u\begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

$$\begin{array}{c} \begin{array}{cccccc} & x_i & & x_j & & x_k \end{array} \\ \left(\begin{array}{cccccc|c} * & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & * & \cdots & 0 & \cdots & 0 \\ & & 0 & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & * & \cdots & 0 \end{array}\right) \end{array}$$

Are they always linearly independent?

These 3 vectors will span the solution space.

Consider this…

$$\begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} = c_1 \begin{pmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + c_2 \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + c_3 \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

$$\begin{array}{cccccc|c} & x_i & & x_j & & x_k & \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} * & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & * & \cdots & 0 & \cdots & 0 \\ & & 0 & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & * & \cdots & 0 \end{array}\right)$$

Are they always linearly independent?

$$\Rightarrow c_1 = 0 \;\;, c_2 = 0 \;\;, c_3 = 0$$

YES!

So, this method always produces a basis for the solution space

Consider this…

$$\begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} = c_1 \begin{pmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + c_2 \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + c_3 \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

$$\Rightarrow c_1 = 0 \;\; , c_2 = 0 \;\; , c_3 = 0$$

$$\begin{array}{c} \begin{array}{cccccccc} & x_i & & x_j & & x_k & & \end{array} \\ \left(\begin{array}{cccccc|c} * & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & * & \dots & 0 & \dots & 0 \\ & & 0 & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & * & \dots & 0 \end{array}\right) \end{array}$$

Are they always linearly independent?

YES!

AND

the dimension is always the number of non pivot columns in

Theorem

Let V be a vector space of dimension k and S a subset of V.

The following statements are equivalent:

1) S is a basis for V.

2) S is linearly independent and $|S| = k$.

3) S spans V and $|S| = k$.

What is the significance of this theorem?

Example

Show that $\boldsymbol{u}_1=(2,0,-1), \boldsymbol{u}_2=(4,0,7), \boldsymbol{u}_3=(-1,1,4)$ form a basis for $\mathbb{R}^3$.

If we go by definition, we need to show that the 3 vectors are linearly independent and spans $\mathbb{R}^3$.

But now we know $\dim(\mathbb{R}^3)=3$

So showing either linearly independence or span will do.

We will show that the three vectors are linearly independent.

Example

Show that $\boldsymbol{u}_1 = (2,0,-1), \boldsymbol{u}_2 = (4,0,7), \boldsymbol{u}_3 = (-1,1,4)$ are linearly independent.

Theorem

Let U be a subspace of a vector space V.

Then $\dim(U) \leq \dim(V)$.

Furthermore, if $U \neq V$, then $\dim(U) < \dim(V)$.

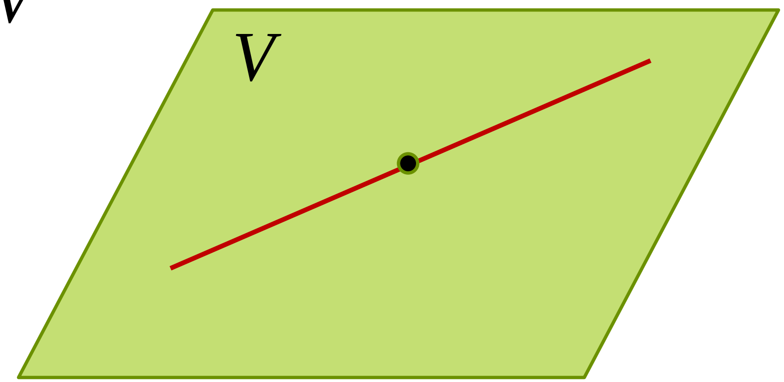
Example

Let V be a plane in $\mathbb{R}^3$ containing the origin.

Suppose U is a subspace of V

and $U \neq V$.

Then $\dim(U) < 2$.



$\dim(V) = 2$

If $\dim(U) = 0$

$\Rightarrow U$ is the zero subspace (that is, just the origin).

If $\dim(U) = 1$

$\Rightarrow U$ is a straight line passing through the origin.

Theorem

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent.

1) $\boldsymbol{A}$ is invertible.

2) $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution.

3) The rref of $\boldsymbol{A}$ is $\boldsymbol{I}$.

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

5) $\det(\boldsymbol{A}) \neq 0$.

Established in Theorem 2.5.19.

6) The rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

7) The columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

Example

Is $\{(1,1,1),(-1,0,2),(3,1,3)\}$ a basis for $\mathbb{R}^3$?

Is $\{(1,1,1,1),(-1,1,-1,1),(0,1,-1,0),(2,1,1,0)\}$ a basis for $\mathbb{R}^4$?

Announcement

MA1101R Lab Classes:

- Starts next week (week 8)
- Worksheets can be downloaded from IVLE (work bin)
- You may choose to download (soft copy) or print (hard copy)
- You do not need to hand in your worksheets.
- Preparations (read worksheet before class)

Attendance taking:

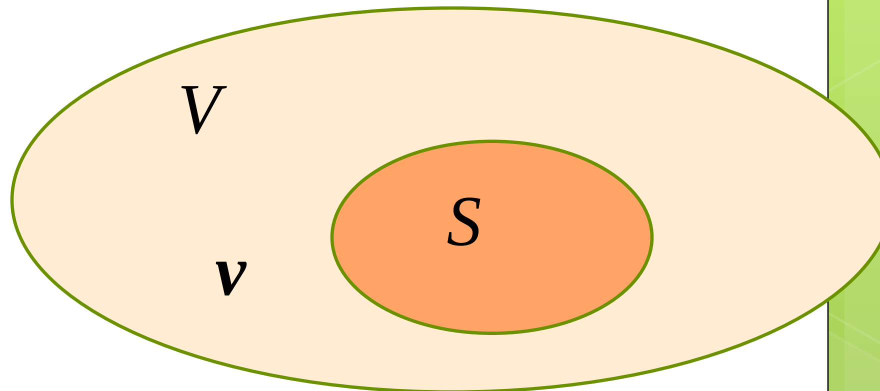
- Automated + manual attendance taking
- You must log in using your own USERID, start MATLAB and use it before your attendance is taken.
- Absentees will be notified by email.
- Absentees without valid reason will be penalized.
- Make up classes – please contact me ASAP.

Recall (coordinate vectors)

V : vector space

$S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$: basis for V

$\boldsymbol{v}$: any vector in V



Suppose $\boldsymbol{v}=c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k$

$(\boldsymbol{v})_S=(c_1,c_2,\ldots,c_k)$ is the coordinate vector of $\boldsymbol{v}$ relative to S.

We can also write $[\boldsymbol{v}]_S=\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ and also call it

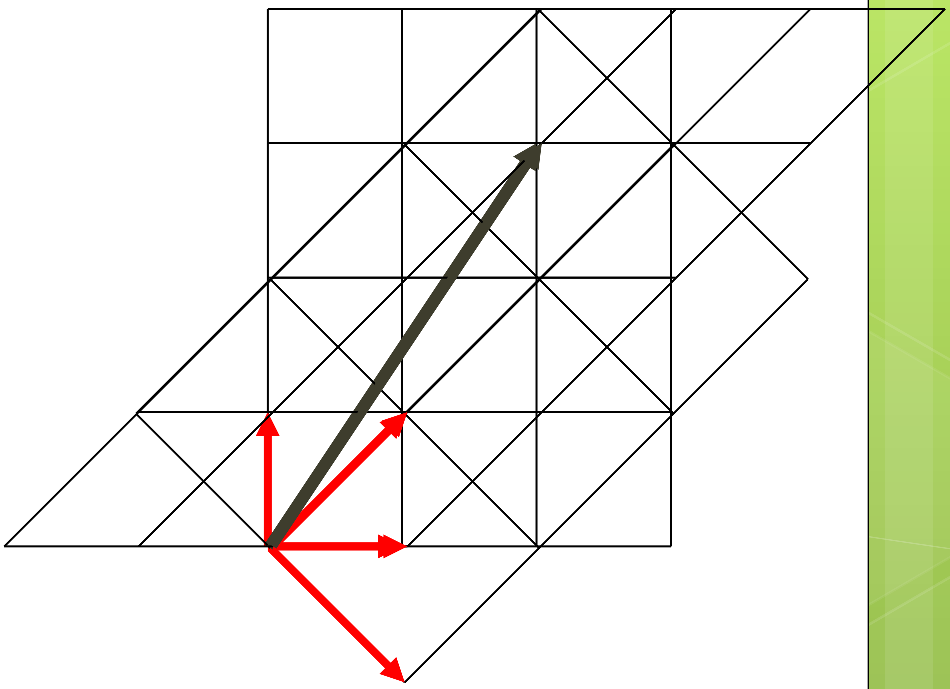
Discussion

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$

$S_1 = \{(1,0),(0,1)\}$ $\quad [\boldsymbol{v}]_{S_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$S_2 = \{(1,-1),(1,1)\}$ $\quad [\boldsymbol{v}]_{S_2} = \begin{pmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$

$S_3 = \{(1,0),(1,1)\}$ $\quad [\boldsymbol{v}]_{S_3} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$



If we know the vectors in S_2 and S_3, is there any relationship between $[\boldsymbol{v}]_{S_2}$ and $[\boldsymbol{v}]_{S_3}$?

Discussion

Generally speaking, given two different bases S and T

for a vector space V, $\quad S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\} \quad T=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_k\}$

for any vector $\boldsymbol{w}\in V$,

can we relate $\boxed{[\boldsymbol{w}]_S}$ and $\boxed{[\boldsymbol{w}]_T}$? $\quad \boxed{[\boldsymbol{w}]_S}=\begin{pmatrix}\vdots\\ \vdots\\ \vdots\end{pmatrix} \quad \boxed{[\boldsymbol{w}]_T}=\begin{pmatrix}\vdots\\ \vdots\\ \vdots\end{pmatrix}$

Suppose

$\boldsymbol{w}=c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k$, so $[\boldsymbol{w}]_S=\begin{pmatrix}c_1\\ c_2\\ \vdots\\ c_k\end{pmatrix}$

Discussion

Generally speaking, given two different bases S and T

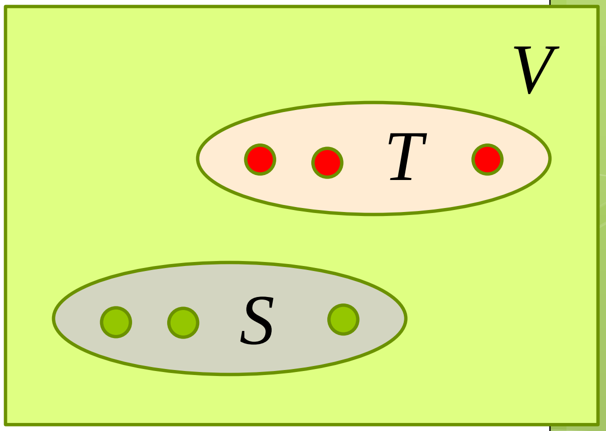
for a vector space V, $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ $T=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_k\}$

Since T is a basis for V

and $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k$ are vectors in V,

each $\boldsymbol{u}_i$ can be written as a linear

combination of $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_k$.



$$\boldsymbol{u}_1 = a_{11}\boldsymbol{v}_1 + a_{21}\boldsymbol{v}_2 + \ldots + a_{k1}\boldsymbol{v}_k$$

$$\boldsymbol{u}_2 = a_{12}\boldsymbol{v}_1 + a_{22}\boldsymbol{v}_2 + \ldots + a_{k2}\boldsymbol{v}_k$$

$$\cdots\cdots \quad \boldsymbol{u}_k = a_{1k}\boldsymbol{v}_1 + a_{2k}\boldsymbol{v}_2 + \ldots + a_{kk}\boldsymbol{v}_k$$

What do these coefficients give us?

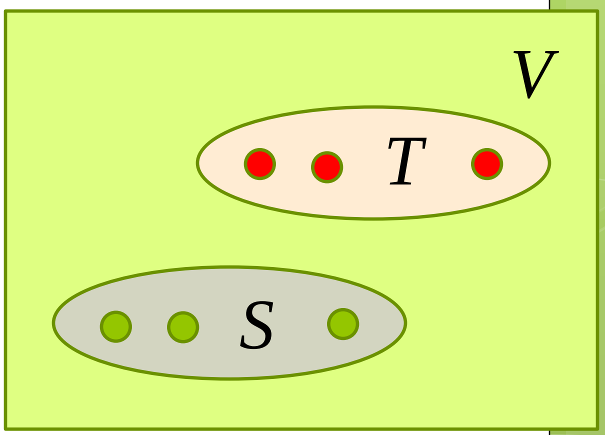
Discussion

Generally speaking, given two different bases S and T for a vector space V, $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ $T=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_k\}$

$$[\boldsymbol{u}_1]_T=\begin{pmatrix} a_{11}\\ a_{21}\\ \vdots\\ a_{k1}\end{pmatrix}$$

$$[\boldsymbol{u}_2]_T=\begin{pmatrix} a_{12}\\ a_{22}\\ \vdots\\ a_{k2}\end{pmatrix}$$

$$\cdots \quad [\boldsymbol{u}_k]_T=\begin{pmatrix} a_{1k}\\ a_{2k}\\ \vdots\\ a_{kk}\end{pmatrix}$$



Coordinate vectors of $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k$ relative to T.

Discussion

Generally speaking, given two different bases S and T

for a vector space V, $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ $T=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_k\}$

for any vector $\boldsymbol{w}\in V$,

Suppose

$$\boldsymbol{w}=c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k,$$

$$\boldsymbol{u}_1=a_{11}\boldsymbol{v}_1+a_{21}\boldsymbol{v}_2+\ldots+a_{k1}\boldsymbol{v}_k$$

$$\boldsymbol{u}_2=a_{12}\boldsymbol{v}_1+a_{22}\boldsymbol{v}_2+\ldots+a_{k2}\boldsymbol{v}_k$$

$$\vdots$$

$$\boldsymbol{u}_k=a_{1k}\boldsymbol{v}_1+a_{2k}\boldsymbol{v}_2+\ldots+a_{kk}\boldsymbol{v}_k$$

$$[\boldsymbol{w}]_S=\begin{pmatrix}c_1\\c_2\\\vdots\\c_k\end{pmatrix}$$

Substitute into

$$\boldsymbol{w}=(\qquad)\boldsymbol{v}_1+(\qquad)\boldsymbol{v}_2+\ldots+(\qquad)\boldsymbol{v}_k$$

Discussion

Generally speaking, given two different bases S and T

for a vector space V, $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ $T=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_k\}$

for any vector $\boldsymbol{w}\in V$,

Suppose

$$\boldsymbol{w}=c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k$$

$$[\boldsymbol{w}]_S=\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

$$=(c_1a_{11}+c_2a_{12}+\ldots+c_ka_{1k})\boldsymbol{v}_1+(c_1a_{21}+c_2a_{22}+\ldots+c_ka_{2k})\boldsymbol{v}_2$$

$$+\ldots+(c_1a_{k1}+c_2a_{k2}+\ldots+c_ka_{kk})\boldsymbol{v}_k$$

What do these coefficients give us?

Coordinate vector of $\boldsymbol{w}$ relative to T.

$$[\boldsymbol{w}]_T=\begin{pmatrix} c_1a_{11}+c_2a_{12}+\ldots+c_ka_{1k} \\ c_1a_{21}+c_2a_{22}+\ldots+c_ka_{2k} \\ \vdots \\ c_1a_{k1}+c_2a_{k2}+\ldots+c_ka_{kk} \end{pmatrix}$$

Discussion

For any vector $\boldsymbol{w} \in V$,

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{pmatrix} \quad [\boldsymbol{u}_2]_T = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{pmatrix} \quad \cdots\cdots \quad [\boldsymbol{u}_k]_T = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \end{pmatrix}$$

$$[\boldsymbol{w}]_T = \begin{pmatrix} c_1 a_{11} + c_2 a_{12} + \ldots + c_k a_{1k} \\ c_1 a_{21} + c_2 a_{22} + \ldots + c_k a_{2k} \\ \vdots \\ c_1 a_{k1} + c_2 a_{k2} + \ldots + c_k a_{kk} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = [\boldsymbol{w}]_S$$

Discussion

For any vector $\boldsymbol{w} \in V$, $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{pmatrix} \quad [\boldsymbol{u}_2]_T = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{pmatrix} \quad \cdots\cdots \quad [\boldsymbol{u}_k]_T = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \end{pmatrix}$$

$$[\boldsymbol{w}]_T = \begin{pmatrix} c_1 a_{11} + c_2 a_{12} + \ldots + c_k a_{1k} \\ c_1 a_{21} + c_2 a_{22} + \ldots + c_k a_{2k} \\ \vdots \\ c_1 a_{k1} + c_2 a_{k2} + \ldots + c_k a_{kk} \end{pmatrix} = \left([\boldsymbol{u}_1]_T \quad [\boldsymbol{u}_2]_T \quad \cdots \quad [\boldsymbol{u}_k]_T \right) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = [\boldsymbol{w}]_S$$

Discussion

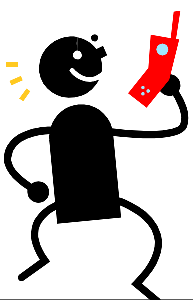
For any vector $\boldsymbol{w} \in V$,

$$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \quad T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$$

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{pmatrix} \quad [\boldsymbol{u}_2]_T = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{pmatrix} \quad \cdots\cdots \quad [\boldsymbol{u}_k]_T = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \end{pmatrix}$$

Question:
How to get this matrix?

$$[\boldsymbol{w}]_T = \underbrace{\left([\boldsymbol{u}_1]_T \quad [\boldsymbol{u}_2]_T \quad \cdots \quad [\boldsymbol{u}_k]_T\right)}[\boldsymbol{w}]_S$$



Pre-multiplying this matrix to $[\boldsymbol{w}]_S$ gives us $[\boldsymbol{w}]_T$.

from $\longrightarrow$ to

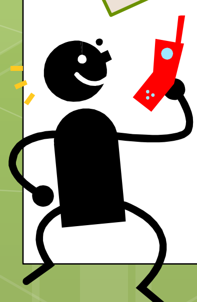
Discussion

For any vector $\boldsymbol{w} \in V$, $\quad S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \quad T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{pmatrix} \quad [\boldsymbol{u}_2]_T = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{pmatrix} \quad \cdots\cdots \quad [\boldsymbol{u}_k]_T = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \end{pmatrix}$$

$$[\boldsymbol{w}]_T = \underbrace{\left([\boldsymbol{u}_1]_T \quad [\boldsymbol{u}_2]_T \quad \cdots \quad [\boldsymbol{u}_k]_T\right)}[\boldsymbol{w}]_S$$

Answer: Write each vector from S in terms of vectors from T.



Pre-multiplying this matrix to $[\boldsymbol{w}]_S$ gives us $[\boldsymbol{w}]_T$.

from $\longrightarrow$ to

Discussion

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{pmatrix} \quad [\boldsymbol{u}_2]_T = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{pmatrix} \quad \cdots\cdots \quad [\boldsymbol{u}_k]_T = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \end{pmatrix}$$

$\boldsymbol{P} = \begin{pmatrix} [\boldsymbol{u}_1]_T & [\boldsymbol{u}_2]_T & \cdots & [\boldsymbol{u}_k]_T \end{pmatrix}$ is called the transition matrix from S to T.

Note that for any $\boldsymbol{w} \in V$, $\boldsymbol{P}[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$.

Example (transition matrix)

$\boldsymbol{u}_1 = (1,0,-1),\ \boldsymbol{u}_2 = (0,-1,0),\ \boldsymbol{u}_3 = (1,0,2)$

$\boldsymbol{v}_1 = (1,1,1),\ \boldsymbol{v}_2 = (1,1,0),\ \boldsymbol{v}_3 = (-1,0,0)$

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are both bases for $\mathbb{R}^3$.

To find the transition matrix from S to T:

Write each vector from S in terms of vectors in T.

That is, find $[\boldsymbol{u}_1]_T, [\boldsymbol{u}_2]_T$ and $[\boldsymbol{u}_3]_T$.

Example (transition matrix)

$$\boldsymbol{u}_1 = (1,0,-1),\ \boldsymbol{u}_2 = (0,-1,0),\ \boldsymbol{u}_3 = (1,0,2)$$

$$\boldsymbol{v}_1 = (1,1,1),\ \boldsymbol{v}_2 = (1,1,0),\ \boldsymbol{v}_3 = (-1,0,0)$$

Find $[\boldsymbol{u}_1]_T, [\boldsymbol{u}_2]_T$ and $[\boldsymbol{u}_3]_T$.

$$\boldsymbol{u}_1 = a_{11}\boldsymbol{v}_1 + a_{21}\boldsymbol{v}_2 + a_{31}\boldsymbol{v}_3$$

$$\boldsymbol{u}_2 = a_{12}\boldsymbol{v}_1 + a_{22}\boldsymbol{v}_2 + a_{32}\boldsymbol{v}_3$$

$$\boldsymbol{u}_3 = a_{13}\boldsymbol{v}_1 + a_{23}\boldsymbol{v}_2 + a_{33}\boldsymbol{v}_3$$

Example (transition matrix)

$$\boldsymbol{u}_1 = (1,0,-1),\ \boldsymbol{u}_2 = (0,-1,0),\ \boldsymbol{u}_3 = (1,0,2)$$

$$\boldsymbol{v}_1 = (1,1,1),\ \boldsymbol{v}_2 = (1,1,0),\ \boldsymbol{v}_3 = (-1,0,0)$$

Find $[\boldsymbol{u}_1]_T, [\boldsymbol{u}_2]_T$ and $[\boldsymbol{u}_3]_T$.

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right)$$

$$\boldsymbol{u}_1 = a_{11}\boldsymbol{v}_1 + a_{21}\boldsymbol{v}_2 + a_{31}\boldsymbol{v}_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$a_{11} = -1,\quad a_{21} = 1,\quad a_{31} = -1$$

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

Example (transition matrix)

$\boldsymbol{u}_1 = (1,0,-1),\ \boldsymbol{u}_2 = (0,-1,0),\ \boldsymbol{u}_3 = (1,0,2)$

$\boldsymbol{v}_1 = (1,1,1),\ \boldsymbol{v}_2 = (1,1,0),\ \boldsymbol{v}_3 = (-1,0,0)$

Find $[\boldsymbol{u}_1]_T, [\boldsymbol{u}_2]_T$ and $[\boldsymbol{u}_3]_T$.

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right)$$

$$\boldsymbol{u}_2 = a_{12}\boldsymbol{v}_1 + a_{22}\boldsymbol{v}_2 + a_{32}\boldsymbol{v}_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$a_{12} = 0,$
$a_{22} = -1,$
$a_{32} = -1$

$$[\boldsymbol{u}_2]_T = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

Example (transition matrix)

$$\boldsymbol{u}_1=(1,0,-1),\ \boldsymbol{u}_2=(0,-1,0),\ \boldsymbol{u}_3=(1,0,2)$$

$$\boldsymbol{v}_1=(1,1,1),\ \boldsymbol{v}_2=(1,1,0),\ \boldsymbol{v}_3=(-1,0,0)$$

Find $[\boldsymbol{u}_1]_T, [\boldsymbol{u}_2]_T$ and $[\boldsymbol{u}_3]_T$.

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right)$$

$$\boldsymbol{u}_3=a_{13}\boldsymbol{v}_1+a_{23}\boldsymbol{v}_2+a_{33}\boldsymbol{v}_3$$

$$[\boldsymbol{u}_3]_T=\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$a_{13}=2,\quad a_{23}=-2,\quad a_{33}=-1$$

Example (transition matrix)

$\boldsymbol{u}_1 = (1,0,-1),\ \boldsymbol{u}_2 = (0,-1,0),\ \boldsymbol{u}_3 = (1,0,2)$

$\boldsymbol{v}_1 = (1,1,1),\ \boldsymbol{v}_2 = (1,1,0),\ \boldsymbol{v}_3 = (-1,0,0)$

Find $[\boldsymbol{u}_1]_T, [\boldsymbol{u}_2]_T$ and $[\boldsymbol{u}_3]_T$.

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \qquad [\boldsymbol{u}_2]_T = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \qquad [\boldsymbol{u}_3]_T = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$$

So the transition matrix from S to T is

$$\boldsymbol{P} = \left([\boldsymbol{u}_1]_T \quad [\boldsymbol{u}_2]_T \quad [\boldsymbol{u}_3]_T \right) = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}$$

Example (transition matrix)

So the transition matrix from S to T is

$$\boldsymbol{P} = \begin{pmatrix} [\boldsymbol{u}_1]_T & [\boldsymbol{u}_2]_T & [\boldsymbol{u}_3]_T \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}$$

If $\boldsymbol{w} \in \mathbb{R}^3$ is such that $(\boldsymbol{w})_S = (1, -1, 4)$, find $(\boldsymbol{w})_T$.

Example (transition matrix)

$\boldsymbol{u}_1 = (1,1), \boldsymbol{u}_2 = (1,-1), \boldsymbol{v}_1 = (1,0), \boldsymbol{v}_2 = (1,1)$

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ are both bases for $\mathbb{R}^2$.

Transition matrix from S to T is:

$$\begin{aligned} \boldsymbol{u}_1 &= \boldsymbol{v}_2 \\ \boldsymbol{u}_2 &= 2\boldsymbol{v}_1 - \boldsymbol{v}_2 \end{aligned}$$

Transition matrix from T to S is:

$$\begin{aligned} \boldsymbol{v}_1 &= \tfrac{1}{2}\boldsymbol{u}_1 + \tfrac{1}{2}\boldsymbol{u}_2 \\ \boldsymbol{v}_2 &= \boldsymbol{u}_1 \end{aligned}$$

Example (transition matrix)

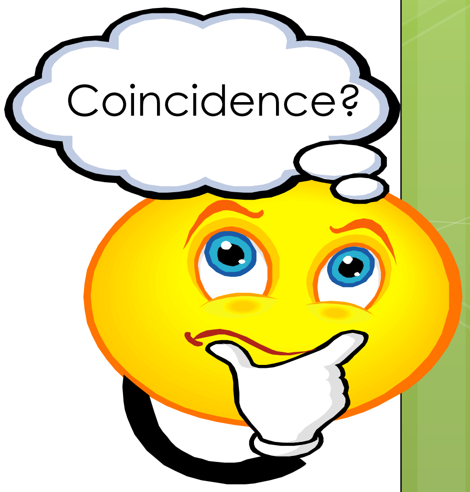
$$\boldsymbol{u}_1 = (1,1), \boldsymbol{u}_2 = (1,-1), \boldsymbol{v}_1 = (1,0), \boldsymbol{v}_2 = (1,1)$$

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ are both bases for $\mathbb{R}^2$.

Transition matrix from S to T is: $\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$

Inverses of each other

Transition matrix from T to S is: $\begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$



Theorem

Let S and T be two bases of a vector space and let $\boldsymbol{P}$ be the transition matrix from S to T. Then

1) $\boldsymbol{P}$ is invertible; and

2) $\boldsymbol{P}^{-1}$ is the transition matrix from T to S.

Idea:

1) Let $\boldsymbol{Q}$ be the transition matrix from T to S;
2) Show that $\boldsymbol{QP} = \boldsymbol{I}$ (that is, $\boldsymbol{Q}$ and $\boldsymbol{P}$ are inverses of each other).

Example (transition matrix)

$\boldsymbol{u}_1 = (1,0,-1),\ \boldsymbol{u}_2 = (0,-1,0),\ \boldsymbol{u}_3 = (1,0,2)$

$\boldsymbol{v}_1 = (1,1,1),\ \boldsymbol{v}_2 = (1,1,0),\ \boldsymbol{v}_3 = (-1,0,0)$

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are both bases for $\mathbb{R}^3$.

The transition matrix from S to T is $\begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}$

The transition matrix from T to S is $\begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}^{-1}$

Example (transition matrix)

$\boldsymbol{u}_1=(1,0,-1),\ \boldsymbol{u}_2=(0,-1,0),\ \boldsymbol{u}_3=(1,0,2)$

$\boldsymbol{v}_1=(1,1,1),\ \boldsymbol{v}_2=(1,1,0),\ \boldsymbol{v}_3=(-1,0,0)$

$S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$ and $T=\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$ are both bases for $\mathbb{R}^3$.

The transition matrix from T to S is $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ -1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \end{pmatrix}$

For any $\boldsymbol{w}\in\mathbb{R}^3$, $\boxed{\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ -1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & \frac{-1}{3} \end{pmatrix}[\boldsymbol{w}]_T=[\boldsymbol{w}]_S}$

End of Lecture 12

Lecture 13:

Row spaces and Column spaces (till end of Section 4.1)