Announcements

1. Tutorial classes on Tuesday next week.
2. Lab quiz duration: 50 minutes
3. Please bring along your student (matriculation) card.
4. Please be reminded to check your group and arrive on time.
5. Open 'everything' quiz.

MA1101R Lab Quiz

- Once you are inside the lab, find a workstation and get seated.
- If this is NOT your official lab timeslot, you are NOT allowed to take the quiz (unless you have obtained permission).
- Once at your seat, you may take out your belongings and log in to the PC using your NUSNET ID and password.
- Take out your student card and place it on your desk.
- Read the instructions on the cover page of your question paper.
- Once your PC is ready, you may start MATLAB. Remember to change the directory using the command cd c:/ma1101r
- You may flip over (after reading the instructions) and read the question but you **cannot start writing** until you are told to do so.
- This is an open book quiz, but no communication (e.g. email, MSN) is allowed.
- Read the question and enter the matrices CAREFULLY!

Lecture 20 Recap

1) Definition of a linear transformation.

2) The standard matrix and formula for a linear transformation.

3) Abstract definition of a linear transformation.

4) Two properties that linear transformations have.

5) What do we need to know to completely determine a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

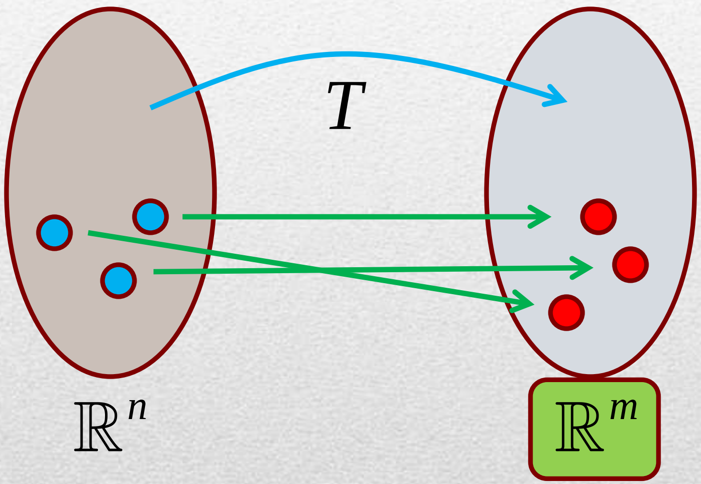
6) Composition of linear transformations.

Lecture 21

Ranges and Kernels

Definition

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.



$$R(T)=\{T(\boldsymbol{u})\,|\,\boldsymbol{u}\in\mathbb{R}^n\}\ \boxed{\subseteq \mathbb{R}^m}$$

$$R(T)=\{\quad\}$$ 'set of all images'

The range, denoted by $R(T)$ of T is the set of images of T.

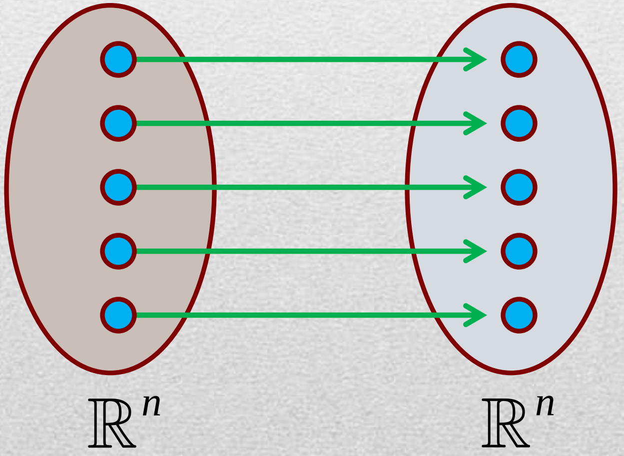
Example

Let $I:\mathbb{R}^n \to \mathbb{R}^n$ be the transformation defined by

$$I(\boldsymbol{u}) = \boldsymbol{u}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'what you put into I,

you get back the same thing'



$$R(I) = \mathbb{R}^n$$

Example

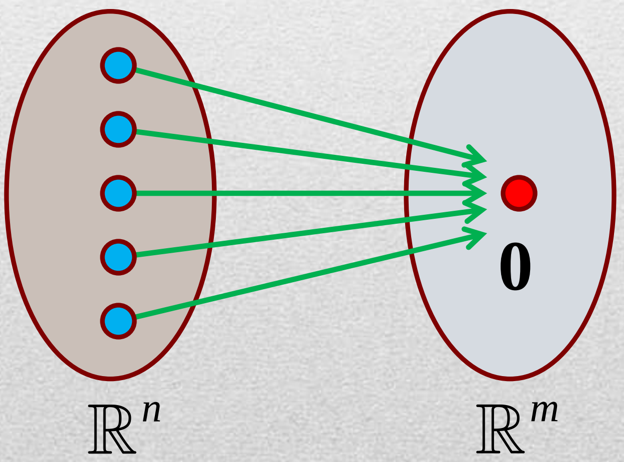
Let $O:\mathbb{R}^n \to \mathbb{R}^m$ be the transformation defined by

$$O(\boldsymbol{u}) = \mathbf{0}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'whatever you put into O,

you get back the zero vector'

(in $\mathbb{R}^m$)



$$R(O) = \{\mathbf{0}\} \boxed{\subseteq \mathbb{R}^m}$$

Example

Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

'all the images are of the form $\begin{pmatrix} x+y \\ y \\ x \end{pmatrix}$ for some real numbers x, y'

$$R(T) = \left\{ \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$$

Example

$$R(T) = \left\{ \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$$

Equation of the plane?

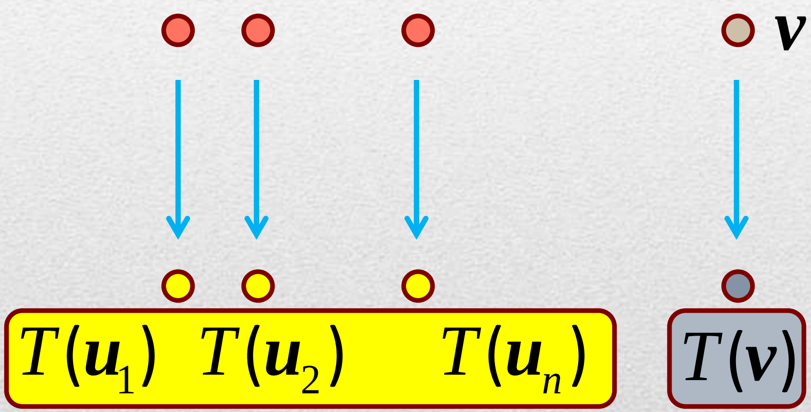
$$= \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ (a plane in } \mathbb{R}^2\text{)}$$

$$ax + by + cz = 0$$

$$\boxed{x - y - z = 0}$$

Discussion

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.



For any $\boldsymbol{v} \in \mathbb{R}^n$, we have already observed that $\boxed{T(\boldsymbol{v})}$ is some linear combination of $\boxed{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n).}$

So $R(T) = \{T(\boldsymbol{v}) \mid \boldsymbol{v} \in \mathbb{R}^n\}$

$\subseteq \text{span}\{T(\boldsymbol{u}_1), \ldots, T(\boldsymbol{u}_n)\}$

Each $T(\boldsymbol{v})$ is a linear combination of $T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \ldots, T(\boldsymbol{u}_n)$.

Discussion

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$. Conversely,

$$\boldsymbol{w} \in \text{span}\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2) \dots, T(\boldsymbol{u}_n)\}$$

since $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ is a basis for $\mathbb{R}^n$

$$\Rightarrow \boldsymbol{w} = a_1 T(\boldsymbol{u}_1), + a_2 T(\boldsymbol{u}_2) + \dots + a_n T(\boldsymbol{u}_n)$$

$$= T(a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + \dots + a_n\boldsymbol{u}_n)$$

$= T(\boldsymbol{v})$ for some $\boldsymbol{v} \in \mathbb{R}^n$

$$\Rightarrow \boldsymbol{w} \in R(T)$$

$$\Rightarrow \text{span}\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \dots, T(\boldsymbol{u}_n)\} \subseteq R(T)$$

So $R(T) = \{T(\boldsymbol{v}) \mid \boldsymbol{v} \in \mathbb{R}^n\}$

$$\subseteq \text{span}\{T(\boldsymbol{u}_1), \dots, T(\boldsymbol{u}_n)\}$$

$R(T)$

$= \text{span}\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \dots, T(\boldsymbol{u}_n)\}$

What does this mean?

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.

$$R(T) = \text{span}\{T(\boldsymbol{u}_1), T(\boldsymbol{u}_2), \dots, T(\boldsymbol{u}_n)\}$$

1) $R(T)$ is always a subspace.

2) $R(T) = \text{span}\{T(\boldsymbol{e}_1), T(\boldsymbol{e}_2) \dots, T(\boldsymbol{e}_n)\}$

$= \text{column space of } \boldsymbol{A}$

$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$ $=$ standard basis for $\mathbb{R}^n$

Remember

standard matrix for $T = \boldsymbol{A} = \begin{bmatrix} T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & \cdots & T(\boldsymbol{e}_n) \end{bmatrix}$

Theorem

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A}$ the standard matrix for T. Then

$$R(T) = \text{column space of } \boldsymbol{A}$$

and is a subspace of $\mathbb{R}^m$.

Definition

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A}$ the standard matrix for T. Then

$$R(T) = \text{column space of } \boldsymbol{A}$$

and is a subspace of $\mathbb{R}^m$.

$\dim(R(T))$ is called the rank of T and is denoted by $\text{rank}(T)$.

By theorem above, $\text{rank}(A) = \text{rank}(T)$ where $\boldsymbol{A}$ is the standard matrix for T.

Example

Let $T:\mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined by

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix} \quad \forall \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$$

Find a basis for $R(T)$ and determine rank(T).

If $\boldsymbol{A}$ is the standard matrix for T...

Find a basis for the column space of $\boldsymbol{A}$ and determine rank($\boldsymbol{A}$).

Example

Let $T:\mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined by

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix} \quad \forall \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$$

We will find a basis for the column space of and rank of

$$= \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{matrix}} \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

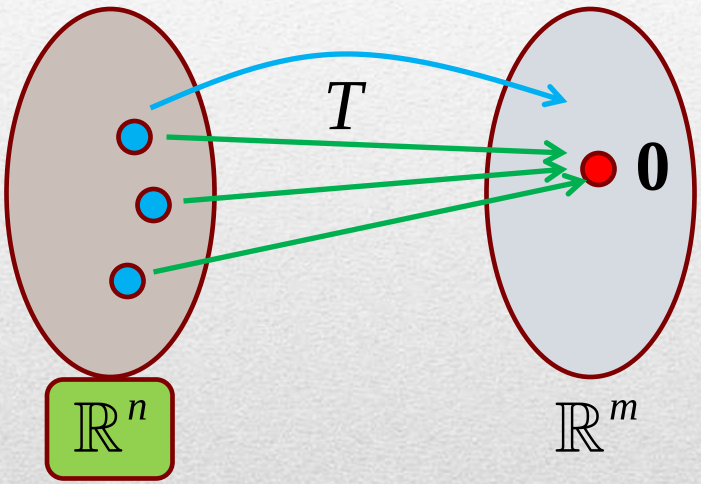
Example

$$\begin{pmatrix} 0 & \boxed{1} & \boxed{2} & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So a basis for $R(T)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \right\}$ and $\operatorname{rank}(T) = 2$.

Definition

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.



$$\mathrm{Ker}(T) = \{\boldsymbol{u} \in \mathbb{R}^n \mid T(\boldsymbol{u}) = \mathbf{0}\}$$

$$\boxed{\subseteq \mathbb{R}^n}$$

The kernel, denoted by Ker(T) of T is the set of vectors in $\mathbb{R}^n$ whose image is the zero vector in $\mathbb{R}^m$.

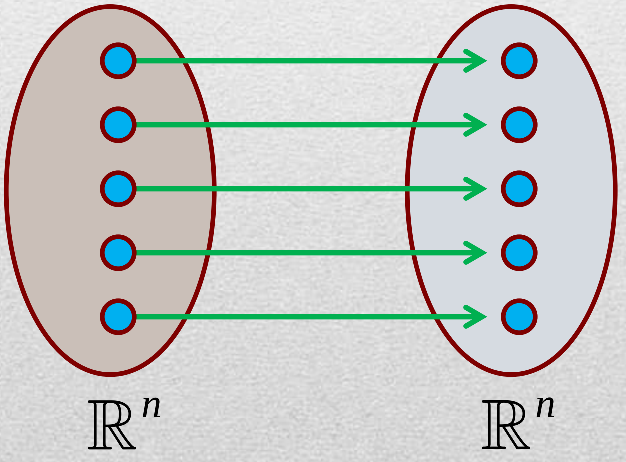
Example

Let $I:\mathbb{R}^n \to \mathbb{R}^n$ be the transformation defined by

$$I(\boldsymbol{u}) = \boldsymbol{u}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'what you put into I,

you get back the same thing'



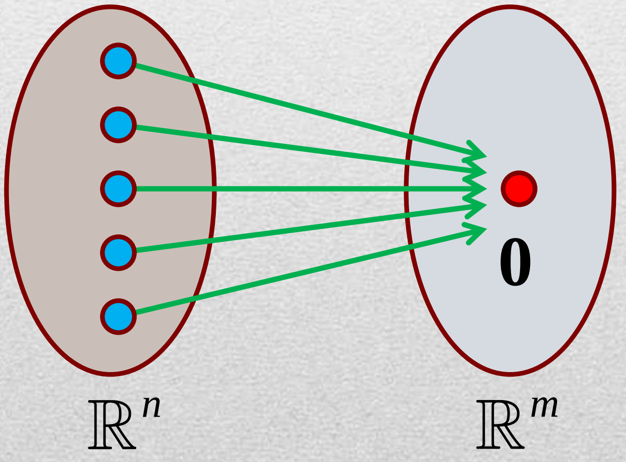
$\mathrm{Ker}(I) = ?$

Example

Let $O:\mathbb{R}^n \to \mathbb{R}^m$ be the transformation defined by

$$O(\boldsymbol{u}) = \mathbf{0}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$$

'whatever you put into O, you get back the zero vector' (in $\mathbb{R}^m$)



$\text{Ker}(O) = ?$

Example

Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

$\text{Ker}(T) = ?$

Example

Let $T:\mathbb{R}^3 \to \mathbb{R}^4$ be the transformation defined by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x-y \\ x-y+3z \\ -5x+y \\ x-z \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

Solve!

To find Ker(T), we need to

find all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Example

$$\begin{cases} 2x - y = 0 \\ x - y + 3z = 0 \\ -5x + y = 0 \\ x - z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x = 0, y = 0, z = 0$

So Ker(T) contains only the zero vector,

that is, $\text{Ker}(T) = \{\mathbf{0}\}$.

Example

Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x-z \\ 0 \\ y \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

To find Ker(T), we need to

find all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Example

Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x-z \\ 0 \\ y \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x-z \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x-z=0 \\ y=0 \end{cases} \quad \Rightarrow \begin{cases} x=z \\ y=0 \end{cases}$$

$$\text{Ker}(T) = \left\{ \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$$

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$$

Theorem

Since Ker(T) is simply the solution space of $\boldsymbol{Ax} = \mathbf{0}$ (where $\boldsymbol{A}$ is the standard matrix for T),

the following theorem is obvious.

$\|$

nullspace of $\boldsymbol{A}$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A}$ the standard matrix for T. Then

$$\text{Ker}(T) = \text{nullspace of } \boldsymbol{A}$$

and is a subspace of $\mathbb{R}^n$.

Definition

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\boldsymbol{A}$ the standard matrix for T. Then

$$\text{Ker}(T) = \text{nullspace of } \boldsymbol{A}$$

and is a subspace of $\mathbb{R}^n$.

dim(Ker(T)) is called the nullity of T and is denoted by nullity(T).

By theorem above, nullity($\boldsymbol{A}$) = nullity(T) where $\boldsymbol{A}$ is the standard matrix for T.

Example

Let $I:\mathbb{R}^n \to \mathbb{R}^n$ be the transformation defined by

$I(\boldsymbol{u}) = \boldsymbol{u}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$ Ker$(I) = \{\mathbf{0}\}$ $\Rightarrow$ nullity$(I) = 0$

Let $O:\mathbb{R}^n \to \mathbb{R}^m$ be the transformation defined by

$O(\boldsymbol{u}) = \mathbf{0}, \quad \forall \boldsymbol{u} \in \mathbb{R}^n$ Ker$(O) = \mathbb{R}^n$ $\Rightarrow$ nullity$(O) = n$

Example

Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x-z \\ 0 \\ y \end{pmatrix} \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

$$\operatorname{Ker}(T) = \left\{ \begin{pmatrix} x \\ 0 \\ x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for Ker(T) and

nullity(T) $= 1$

Example

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the transformation defined by

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix} \quad \forall \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$$

Find a basis for Ker(T) and determine its dimension.

Example

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix} \quad \forall \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$$

The standard matrix for T is $\boldsymbol{A} =$

Let's find a basis for (and determine the dimension of) the nullspace of $\boldsymbol{A}$.

Example

Solving $\boldsymbol{Ax} = \mathbf{0}$:

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} w & = & s \\ x & = & -3t \\ y & = & t \\ z & = & t, \quad s,t \in \mathbb{R}. \end{cases}$$

So $\mathrm{Ker}(T) = \left\{ \begin{pmatrix} s \\ -3t \\ t \\ t \end{pmatrix} \middle| \; s,t \in \mathbb{R} \right\}$

Example

So $\mathrm{Ker}(T) = \left\{ \begin{pmatrix} s \\ -3t \\ t \\ t \end{pmatrix} \middle| \; s,t \in \mathbb{R} \right\} = \left\{ s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \end{pmatrix} \middle| \; s,t \in \mathbb{R} \right\}$

$$= \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathrm{Ker}(T)$

and $\dim(\mathrm{Ker}(T)) = \mathrm{nullity}(T) = 2.$

Theorem

If $T:\mathbb{R}^{n}\to\mathbb{R}^{m}$ is a linear transformation then

$$\operatorname{rank}(T)+\operatorname{nullity}(T)=n.$$

Proof:

Let $\boldsymbol{A}$ be the standard matrix for T. So $\boldsymbol{A}$ is a $m\times n$ matrix.

By dimension theorem for matrices:

$$\begin{array}{ccc} \operatorname{rank}(\boldsymbol{A}) & + & \operatorname{nullity}(\boldsymbol{A})=n \\ \| & & \| \\ \Rightarrow \operatorname{rank}(T) & + & \operatorname{nullity}(T)=n \end{array}$$

by earlier observation

End of Lecture 21

Lecture 22:

Revision