LECTURE 13 RECAP

1) Definition of the row space and column space of a $m \times n$ matrix.

2) How to find bases for the row space or column space of a matrix systematically.

3) Row equivalent matrices have the same row space (generally they do not have the same column space).

4) How to obtain a basis for the row (or column) space of $\boldsymbol{A}$ by using information from $\boldsymbol{R}$, a row-echelon form of $\boldsymbol{A}$.

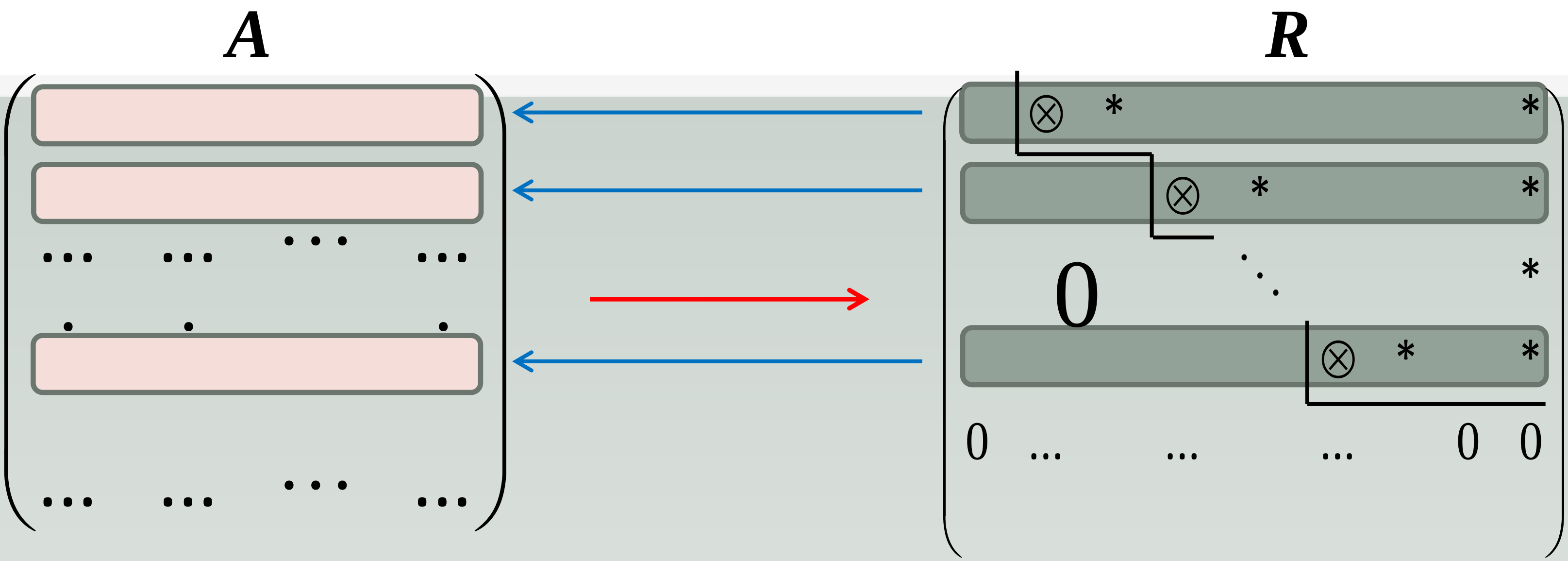
5) Column space and linear systems.

LECTURE 14

RANKS

NULLSPACES AND NULLITIES

A COMMON QUESTION



A basis for the row space of A can be obtained by taking the non zero rows of R.

Can I take the corresponding rows from A instead?

A COMMON QUESTION

A basis for the row space of $\boldsymbol{A}$ can be obtained by taking the non zero rows of $\boldsymbol{R}$.

Can I take the corresponding rows from $\boldsymbol{A}$ instead?

NO!

$$\boldsymbol{A}=\begin{pmatrix} \boxed{1 \quad 1 \quad 1 \quad 1} \\ \boxed{0 \quad 1 \quad 1 \quad 1} \\ \boxed{0 \quad 1 \quad 1 \quad 1} \\ 0 \quad 0 \quad 1 \quad 1 \end{pmatrix} \xrightarrow{\text{G.E.}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{R}$$

THEOREM

The row space and the column space of a matrix have the same dimension.

$$\boldsymbol{A} = \overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}^{n} \left.\vphantom{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}}\right\} m$$

Note that the row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$ while

the column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^m$.

These two subspaces may have nothing to do with each other (if $m \neq n$) but the theorem states that they have the same dimension.

PROOF

Strategy: We will use what we learnt in the previous section to find bases for both subspaces.

Let $\boldsymbol{R}$ be a row echelon form of $\boldsymbol{A}$.

$$\overset{\boldsymbol{A}}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \overset{\boldsymbol{R}}{\begin{pmatrix} \otimes & * & & & & * \\ & & \otimes & * & & * \\ & 0 & & \ddots & & * \\ & & & & \otimes \quad * & * \\ 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix}}$$

PROOF

Question: What is a basis for the row space of $\boldsymbol{A}$?

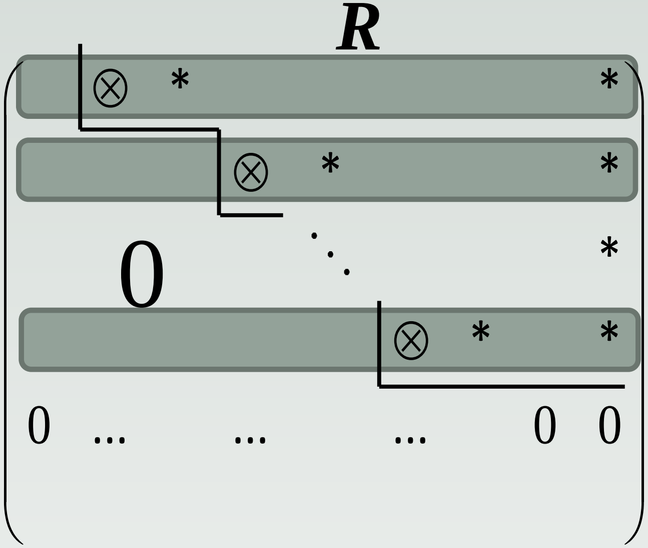
Answer: The non-zero rows of $\boldsymbol{R}$.

Question: What is a dimension of the row space of $\boldsymbol{A}$?

Answer: The number of non-zero rows of $\boldsymbol{R}$.

$=$ number of leading entries in $\boldsymbol{R}$.

$$\begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}}$$

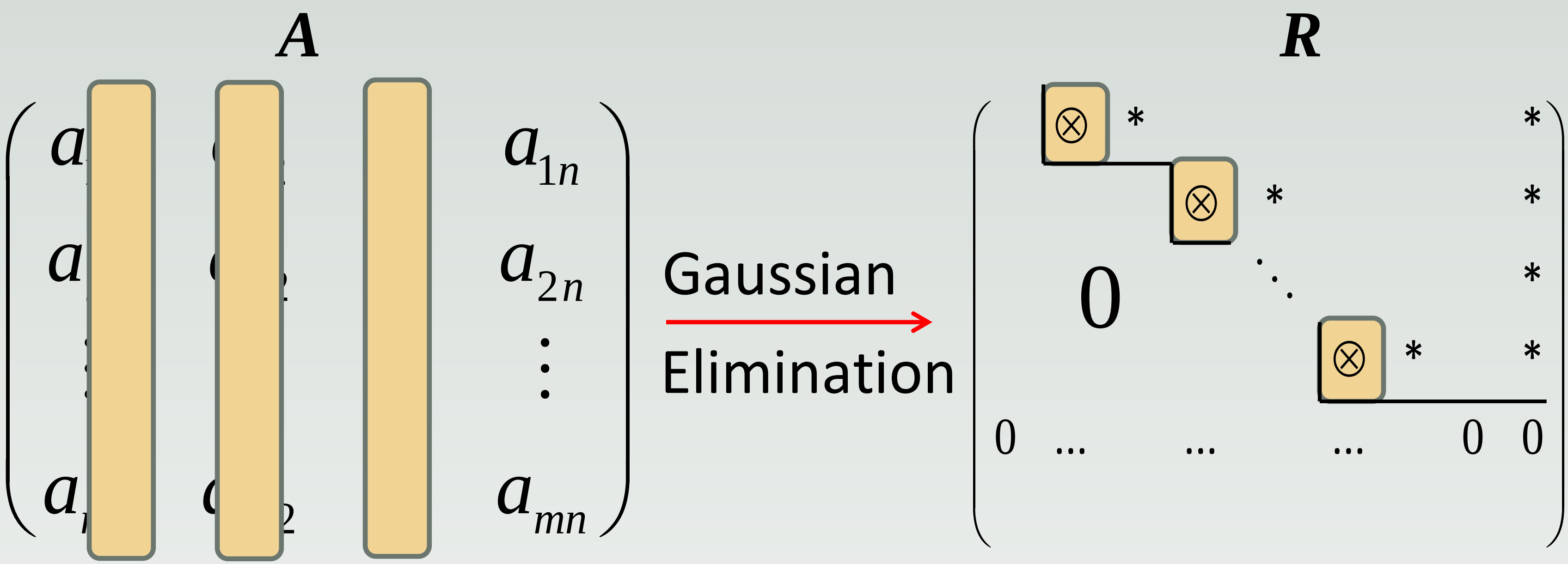


PROOF

Question: What is a basis for the column space of $\boldsymbol{A}$?

Answer: The columns of $\boldsymbol{A}$ corresponding to the pivot columns of $\boldsymbol{R}$.

Question: What is a dimension of the column space of $\boldsymbol{A}$?



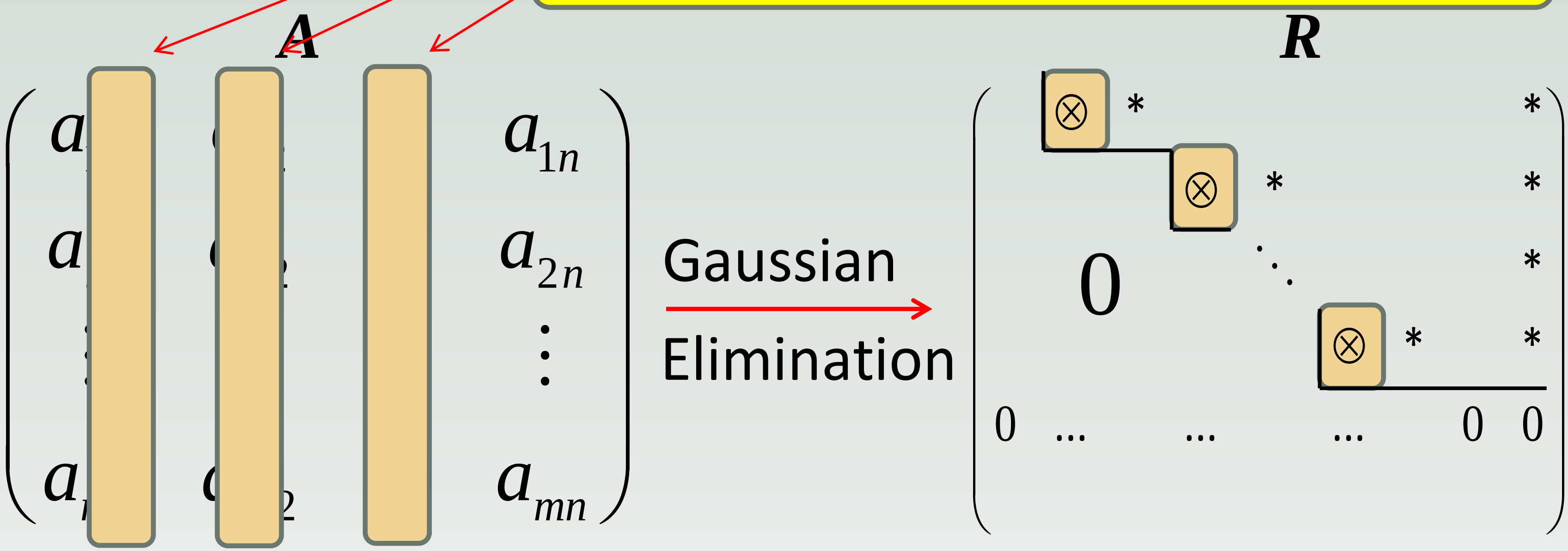
PROOF

Question: What is a dimension of the column space of $\boldsymbol{A}$?

Answer: The number of columns here.

$=$ The number of pivot columns in $\boldsymbol{R}$.

$=$ The number of leading entries in $\boldsymbol{R}$.



PROOF

Question: What is a dimension of the row space of $\boldsymbol{A}$?

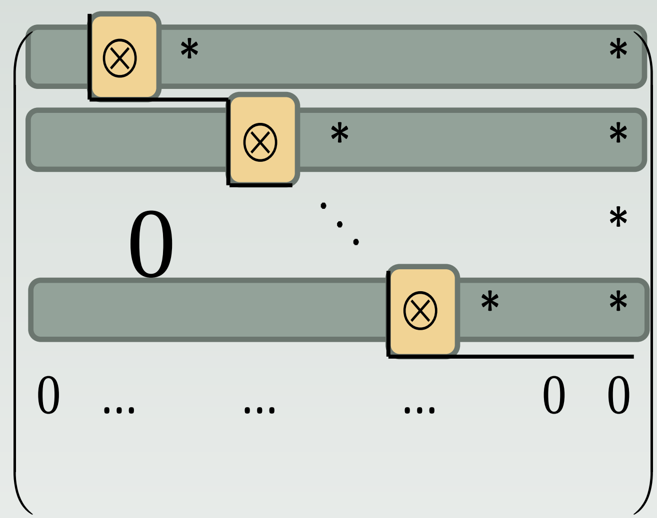
Answer: The number of leading entries in $\boldsymbol{R}$.

Question: What is a dimension of the column space of $\boldsymbol{A}$?

Answer: The number of leading entries in $\boldsymbol{R}$.

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}}$$

$\boldsymbol{R}$



EXAMPLE

$$C = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 2 & 1 & 1 & -2 & 5 \\ -4 & -3 & 0 & 5 & -7 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the row space of C is:

$$\{(2,0,3,-1,8),(0,1,-2,-1,-3)\}$$

A basis for the column space of C is:

$$\left\{\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}\right\}$$

The dimension of both the row space and column space of C is 2.

DEFINITION

The rank of a matrix is the dimension of its row space (or column space).

The rank of $\boldsymbol{A}$ is denoted by rank($\boldsymbol{A}$).

By our earlier discussion,

rank($\boldsymbol{A}$) $=$ number of non zero rows in a row echelon form of $\boldsymbol{A}$

$=$ number of pivot columns in a row echelon form of $\boldsymbol{A}$

EXAMPLES (RANK)

1) $\text{rank}(\mathbf{0}) = 0$

2) $\text{rank}(\boldsymbol{I}_n) = n$

3) $\text{rank}(\boldsymbol{C}) = 2$

$$\boldsymbol{C} = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 2 & 1 & 1 & -2 & 5 \\ -4 & -3 & 0 & 5 & -7 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

EXAMPLES (RANK)

$$\boldsymbol{A} = \left.\overbrace{\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdot\cdot & \cdots & \cdots \end{pmatrix}}^{m}\right\} n$$

$$m \geq n$$

What is the largest possible value for rank($\boldsymbol{A}$)?

$$\boxed{n}$$

$$\boldsymbol{A} = \left.\overbrace{\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots \end{pmatrix}}^{m}\right\} n$$

$$n \geq m$$

What is the largest possible value for rank($\boldsymbol{A}$)?

$$\boxed{m}$$

REMARKS

1) For a $m \times n$ matrix $\boldsymbol{A}$, $\text{rank}(\boldsymbol{A}) \le \min\{m, n\}$.

2) For a $m \times n$ matrix $\boldsymbol{A}$, if $\text{rank}(\boldsymbol{A}) = \min\{m, n\}$,

we say that $\boldsymbol{A}$ is of full rank.

3) $\text{rank}(\boldsymbol{A}) =$ dimension of row space of $\boldsymbol{A}$

$=$ dimension of column space of $\boldsymbol{A}^T$

$= \text{rank}(\boldsymbol{A}^T)$

4) A square matrix $\boldsymbol{A}$ is of full rank if and only if $\det(\boldsymbol{A}) \neq 0$.

EXAMPLE

Consider the following linear system

$$\begin{cases} 2x - y = 1 \\ x - y + 3z = 0 \\ -5x + y = 0 \\ x + z = 0 \end{cases}$$

Rewritting as $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

EXAMPLE

Solving $\boldsymbol{Ax} = \boldsymbol{b}$ using Gaussian Elimination,

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The linear system is inconsistent since the last column of a row echelon form of $(\boldsymbol{A} \mid \boldsymbol{b})$ is a pivot column.

is a row echelon form of $\boldsymbol{A}$, so $\operatorname{rank}(\boldsymbol{A}) = 3$.

EXAMPLE

Solving $\boldsymbol{Ax} = \boldsymbol{b}$ using Gaussian Elimination,

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The linear system is inconsistent since the last column of a row echelon form of $(\boldsymbol{A}|\boldsymbol{b})$ is a pivot column.

Since the last column is a pivot column, $\text{rank}(\boldsymbol{A}|\boldsymbol{b}) = \text{rank}(\boldsymbol{A}) + 1 = 4$

REMARK

A linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{A}$ and the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$ have the same rank.

THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ and $n \times p$ matrices respectively.

Then

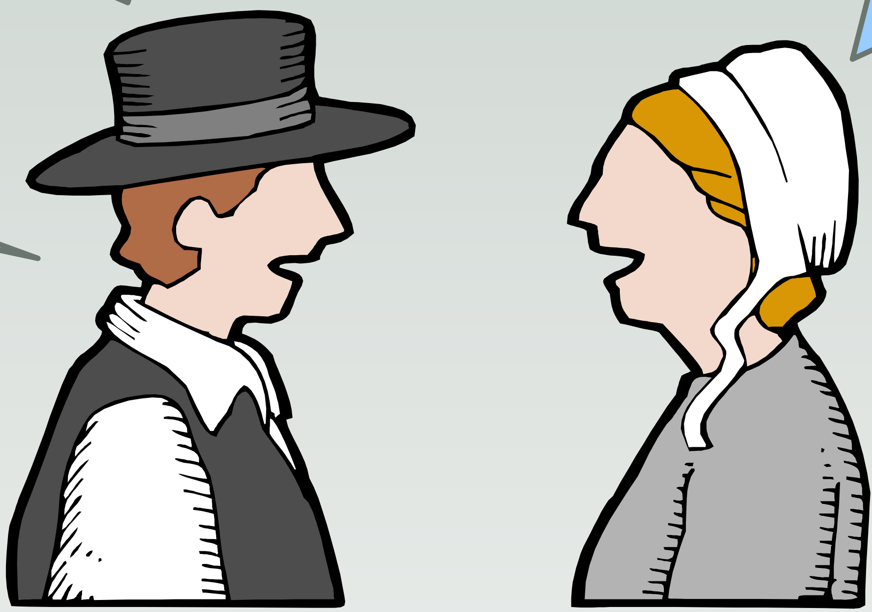
$$\operatorname{rank}(\boldsymbol{AB}) \leq \min\{\operatorname{rank}(\boldsymbol{A}), \operatorname{rank}(\boldsymbol{B})\}$$

ANOTHER SUBSPACE

We have learnt the row space and column space of a matrix $\boldsymbol{A}$.

Are there any other?

Yes, there is. But it is not something new...



DEFINITION

Let $\boldsymbol{A}$ be a $m \times \boxed{n}$ matrix.

$\boldsymbol{Ax} = \boldsymbol{0}$ is a homogeneous linear system with $\boxed{n}$ variables.

The solution set of $\boldsymbol{Ax} = \boldsymbol{0}$ is a subspace of $\mathbb{R}^n$

and is also called the <u>solution space of $\boldsymbol{Ax} = \boldsymbol{0}$.</u>

This is also called the **nullspace of the matrix $\boldsymbol{A}$.**

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

DEFINITION

Let $\boldsymbol{A}$ be a $m \times \boxed{n}$ matrix.

$\boldsymbol{Ax} = \boldsymbol{0}$ is a homogeneous linear system with $\boxed{n}$ variables.

The solution set of $\boldsymbol{Ax} = \boldsymbol{0}$ is a subspace of $\mathbb{R}^n$

and is also called the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$.

This is also called the nullspace of the matrix $\boldsymbol{A}$.

Since the nullspace of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$,

its dimension is $\boxed{\leq n.}$

The dimension of the nullspace of $\boldsymbol{A}$ is called the nullity of $\boldsymbol{A}$

and denoted by nullity($\boldsymbol{A}$).

EXAMPLE

Find a basis for the nullspace and determine the nullity of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 0 \\ -1 & 3 & 2 & -4 \\ 2 & 1 & 0 & 8 \\ 3 & 1 & -1 & 12 \end{pmatrix}$$

EXAMPLE

The reduced row echelon form of the augmented matrix

$(\boldsymbol{A} \,|\, \mathbf{0})$ is
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4s \\ 0 \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}.$$

A general solution for $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ is
$$\begin{cases} x_1 & = & -4s \\ x_2 & = & 0 \\ x_3 & = & 0 \\ x_4 & = & s, \quad s \in \mathbb{R} \end{cases}$$

EXAMPLE

The reduced row echelon form of the augmented matrix

$(\boldsymbol{A} \mid \boldsymbol{0})$ is $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4s \\ 0 \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}.$$

So a basis for the nullspace of $\boldsymbol{A}$ is $\left\{\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

$\text{nullity}(\boldsymbol{A}) = 1$

1 non pivot column

What is the rank of $\boldsymbol{A}$?

$\text{rank}(\boldsymbol{A}) = 3$

3 pivot columns

EXAMPLE

Find a basis for the nullspace and determine the nullity of the following matrix:

$$\boldsymbol{B} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$$

EXAMPLE

The reduced row echelon form of the augmented matrix $(\boldsymbol{B}\,|\,\boldsymbol{0})$ is

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A general solution for $\boldsymbol{Bx} = \boldsymbol{0}$ is

$$\begin{cases} x_1 &= -s - t \\ x_2 &= s \\ x_3 &= -t \\ x_4 &= 0 \\ x_5 &= t, \qquad s, t \in \mathbb{R} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

EXAMPLE

The reduced row echelon form of the augmented matrix

$(\boldsymbol{B}\,|\,\boldsymbol{0})$ is $\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

So a basis for the nullspace of $\boldsymbol{B}$ is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\text{nullity}(\boldsymbol{B}) = 2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

What is the rank of $\boldsymbol{B}$?

$\text{rank}(\boldsymbol{B}) = 3$

2 non pivot columns

3 pivot columns

THEOREM

(Dimension Theorem for matrices)

Let $\boldsymbol{A}$ be a matrix with n columns. Then

$$\boxed{\operatorname{rank}(\boldsymbol{A}) + \operatorname{nullity}(\boldsymbol{A}) = n}$$

PROBLEM

In each of the following cases, find rank($\boldsymbol{A}$), nullity($\boldsymbol{A}$) and nullity($\boldsymbol{A}^T$).

1) $\boldsymbol{A}$ is a 3×4 matrix and rank($\boldsymbol{A}$) $= 3$.

2) $\boldsymbol{A}$ is a 7×5 matrix and nullity($\boldsymbol{A}$) $= 3$.

3) $\boldsymbol{A}$ is a 3×2 matrix and nullity($\boldsymbol{A}^T$) $= 3$.

ANOTHER RELATION

We have already seen how the consistency of a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is related to the column space of $\boldsymbol{A}$.

It turns out that the entire solution set for $\boldsymbol{Ax} = \boldsymbol{b}$ can be obtained if we have the nullspace of $\boldsymbol{A}$ and a 'particular' solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Nullspace of $\boldsymbol{A}$
(solution set/space for $\boldsymbol{Ax} = \boldsymbol{0}$)

A particular solution (say $\boldsymbol{v}$)
to $\boldsymbol{Ax} = \boldsymbol{b}$

Solution set for
$\boldsymbol{Ax} = \boldsymbol{b}$

THEOREM

Suppose the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $\boldsymbol{v}$. Then the solution set of the system is given by

$$M = \{\boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \text{ is an element of the nullspace of } \boldsymbol{A}\}$$

That is, $\boldsymbol{Ax} = \boldsymbol{b}$ has a general solution

$$\boldsymbol{x} = (\text{a general solution for } \boldsymbol{Ax} = \boldsymbol{0})$$

$$+(\text{a particular solution to } \boldsymbol{Ax} = \boldsymbol{b})$$

PROOF

$M = \{\boldsymbol{u}+\boldsymbol{v} \mid \boldsymbol{u}$ is an element of the nullspace of $\boldsymbol{A}\}$

Since $\boldsymbol{v}$ is a solution to $\boldsymbol{Ax}=\boldsymbol{b}$, we have $\boxed{\boldsymbol{Av}=\boldsymbol{b}.}$

Let T be the solution set of $\boldsymbol{Ax}=\boldsymbol{b}$.

We will show $T \subseteq M$ and $M \subseteq T$.

$\boxed{\text{To show } T \subseteq M:}$ Done!

Let $\boldsymbol{w} \in T$. So $\boldsymbol{w}$ is a solution to $\boldsymbol{Ax}=\boldsymbol{b}$ and thus $\boxed{\boldsymbol{Aw}=\boldsymbol{b}.}$

We want to show that $\boldsymbol{w}$ can be written in the form $\underline{\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{v}}$ for some $\boldsymbol{u}$ belonging to the nullspace of $\boldsymbol{A}$.

Define $\boldsymbol{u}=\boldsymbol{w}-\boldsymbol{v}$ (so that $\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{v}$)

$$\boldsymbol{Au}=\boldsymbol{A}(\boldsymbol{w}-\boldsymbol{v})=\boldsymbol{Aw}-\boldsymbol{Av}=\boldsymbol{b}-\boldsymbol{b}=\boldsymbol{0}$$

So $\boldsymbol{u} \in$ nullspace of $\boldsymbol{A}$.

PROOF

$M = \{\boldsymbol{u}+\boldsymbol{v} \mid \boldsymbol{u}$ is an element of the nullspace of $\boldsymbol{A}\}$

Since $\boldsymbol{v}$ is a solution to $\boldsymbol{Ax}=\boldsymbol{b}$, we have $\boxed{\boldsymbol{Av}=\boldsymbol{b}.}$

Let T be the solution set of $\boldsymbol{Ax}=\boldsymbol{b}$.

We will show $T \subseteq M$ and $M \subseteq T$.

$\boxed{\text{To show } M \subseteq T:}$ Done!

Let $\boldsymbol{w} \in M$. So $\boldsymbol{w}$ can be expressed as the $\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{v}$ for some $\boldsymbol{u}$ in the nullspace of $\boldsymbol{A}$ (and so $\boxed{\boldsymbol{Au}=\boldsymbol{0}}$).

$\boldsymbol{Aw}=\boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{Au}+\boldsymbol{Av} = \boldsymbol{0}+\boldsymbol{b}=\boldsymbol{b}$

So $\boldsymbol{w}$ is a solution to $\boldsymbol{Ax}=\boldsymbol{b}$.

We want to show that $\boldsymbol{w} \in T$, that is, $\boldsymbol{w}$ is a solution to $\boldsymbol{Ax}=\boldsymbol{b}$, that is $\boldsymbol{Aw}=\boldsymbol{b}$.

EXAMPLE

Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}$$

From an earlier example today, we found that the nullspace of $\boldsymbol{A}$ is

$$\left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

EXAMPLE

Check that

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} \boxed{1} \\ \boxed{-1} \\ \boxed{1} \\ \boxed{1} \\ \boxed{1} \end{pmatrix} = \begin{pmatrix} 2-2-1+1 \\ -1+1+2-3+1 \\ 1+1+1 \\ 1-1-2-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix} = \boldsymbol{b}$$

So $\boxed{}$ is a particular solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

So the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ can be written as:

$$\left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

EXAMPLE

Check that

$$\boldsymbol{A}=\begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} \boxed{1} \\ \boxed{-1} \\ \boxed{1} \\ \boxed{1} \\ \boxed{1} \end{pmatrix}=\begin{pmatrix} 2-2-1+1 \\ -1+1+2-3+1 \\ 1+1+1 \\ 1-1-2-1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}=\boldsymbol{b}$$

So $\boxed{}$ is a particular solution to $\boldsymbol{Ax}=\boldsymbol{b}$.

So the solution set of $\boldsymbol{Ax}=\boldsymbol{b}$ can be written as:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s,t\in\mathbb{R} \right\}$$

EXAMPLE

Check that

$$A=\begin{pmatrix}2&2&-1&0&1\\-1&-1&2&-3&1\\0&0&1&1&1\\1&1&-2&0&-1\end{pmatrix}\begin{pmatrix}\boxed{\begin{matrix}1\\-1\\1\\1\\1\end{matrix}}\end{pmatrix}=\begin{pmatrix}2-2-1+1\\-1+1+2-3+1\\1+1+1\\1-1-2-1\end{pmatrix}=\begin{pmatrix}0\\0\\3\\-3\end{pmatrix}=b$$

A general solution to $Ax=b$ is

$$x=\begin{pmatrix}\boxed{\begin{matrix}1\\-1\\1\\1\\1\end{matrix}}\end{pmatrix}+s\begin{pmatrix}\boxed{\begin{matrix}-1\\1\\0\\0\\0\end{matrix}}\end{pmatrix}+t\begin{pmatrix}\boxed{\begin{matrix}-1\\0\\-1\\0\\1\end{matrix}}\end{pmatrix}\quad s,t\in\mathbb{R}$$

END OF LECTURE 14

LECTURE 15: EIGENVALUES AND EIGENVECTORS (TILL END OF SECTION 6.1)