Lecture 08 recap

1) Geometric vectors and Euclidean $n-$space.

2) Identifying a vector with a matrix.

3) Subsets of $\mathbb{R}^n$. Implicit and explicit representations.

4) Linear combination

5) Linear span (set of all linear combinations).

6) Determining whether a set spans $\mathbb{R}^n$.

Lecture 09

Linear combinations and linear spans (cont’d)

Subspaces

Discussion

Suppose $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}\subseteq\mathbb{R}^n$. We want to determine if $\operatorname{span}(S)=\mathbb{R}^n$.

$$\boldsymbol{u}_1=(a_{11},a_{12},\ldots,a_{1n})\quad \boldsymbol{u}_2=(a_{21},a_{22},\ldots,a_{2n})\quad \ldots\ \boldsymbol{u}_k=(a_{k1},a_{k2},\ldots,a_{kn})$$

For any $\boldsymbol{v}=(v_1,v_2,\ldots,v_n)\in\mathbb{R}^n$, consider the equation:

$$c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k=\boldsymbol{v}$$

$$c_1(a_{11},a_{12},\ldots,a_{1n})+c_2(a_{21},a_{22},\ldots,a_{2n})+\ldots+c_k(a_{k1},a_{k2},\ldots,a_{kn})$$
$$=(v_1,v_2,\ldots,v_n)$$

Discussion

Suppose $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}\subseteq\mathbb{R}^n$. We want to determine if $\text{span}(S)=\mathbb{R}^n$.

$$c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k=\boldsymbol{v}$$

$$c_1(a_{11},a_{12},\ldots,a_{1n})+c_2(a_{21},a_{22},\ldots,a_{2n})+\ldots+c_k(a_{k1},a_{k2},\ldots,a_{kn})$$
$$=(v_1,v_2,\ldots,v_n)$$

$$\begin{cases} a_{11}c_1 & + & a_{21}c_2 & + & \ldots & + & a_{k1}c_k & = & v \\ a_{12}c_1 & + & a_{22}c_2 & + & \ldots & + & a_{k2}c_k & = & v_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{1n}c_1 & + & a_{2n}c_2 & + & \ldots & + & a_{kn}c_k & = & v_n \end{cases}$$

Discussion

Suppose $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\dots,\boldsymbol{u}_k\}\subseteq\mathbb{R}^n$. We want to determine if $\text{span}(S)=\mathbb{R}^n$.

$$c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\dots+c_k\boldsymbol{u}_k=\boldsymbol{v} \quad \text{(*)}$$

$$c_1(a_{11},a_{12},\dots,a_{1n})+c_2(a_{21},a_{22},\dots,a_{2n})+\dots+c_k(a_{k1},a_{k2},\dots,a_{kn})$$
$$=(v_1,v_2,\dots,v_n)$$

$$\begin{array}{cc} & k \text{ columns} \\ n \text{ rows} & \left(\begin{array}{c|c} \boldsymbol{A} & \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} \end{array}\right) \end{array}$$

If a row-echelon form of $\boldsymbol{A}$ does not have a zero row, (*) is always consistent regardless of $\boldsymbol{v}$

$\Rightarrow \text{span}(S)=\mathbb{R}^n$

Discussion

Suppose $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}\subseteq\mathbb{R}^n$. We want to determine if $\mathrm{span}(S)=\mathbb{R}^n$.

$$c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k=\boldsymbol{v} \quad \boxed{(*)}$$

$$c_1(a_{11},a_{12},\ldots,a_{1n})+c_2(a_{21},a_{22},\ldots,a_{2n})+\ldots+c_k(a_{k1},a_{k2},\ldots,a_{kn})$$
$$=(v_1,v_2,\ldots,v_n)$$

$$\begin{array}{cc} & k \text{ columns} \\ \begin{array}{c} n \\ \text{rows} \end{array} & \left(\begin{array}{ccccc|c} & & & & & v_1 \\ & & \boldsymbol{A} & & & v_2 \\ & & & & & \vdots \\ 0 & 0 & \ldots & 0 & 0 & v_n \end{array}\right) \end{array}$$

If a row-echelon form of $\boldsymbol{A}$ has at least one zero row, $\boxed{(*)}$ is not always consistent

$$\boxed{\Rightarrow \mathrm{span}(S)\neq\mathbb{R}^n}$$

Example

From earlier example:

Show that $\text{span}\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

No zero row

Show that $\text{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \neq \mathbb{R}^3$.

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 0 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Has zero row

Theorem

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

If $k < n$, then S cannot span $\mathbb{R}^n$.

> If $k < n$, then a row-echelon form of $\boldsymbol{A}$ has at least one zero row $\Rightarrow S$ cannot span $\mathbb{R}^n$.

Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. We want to determine if span$(S) = \mathbb{R}^n$.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \boldsymbol{v} \quad (*)$$

$$\begin{array}{cc} & k \text{ columns} \\ n \text{ rows} & \left(\begin{array}{ccccc|c} & & \boldsymbol{A} & & & \begin{matrix} v_1 \\ v_2 \\ \vdots \end{matrix} \\ 0 & 0 & \ldots & 0 & 0 & v_n \end{array}\right) \end{array}$$

If a row-echelon form of $\boldsymbol{A}$ has at least one zero row, (*) is not always consistent

Example

1) One vector cannot span $\mathbb{R}^2$.

2) One or two vectors cannot span $\mathbb{R}^3$.

Theorem

Let $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}\subseteq\mathbb{R}^n$.

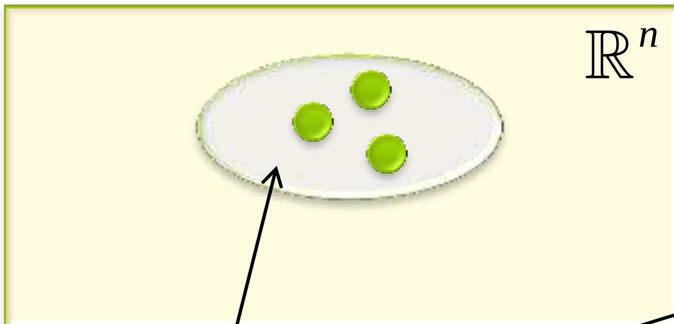
1) $\mathbf{0}\in\text{span}(S)$

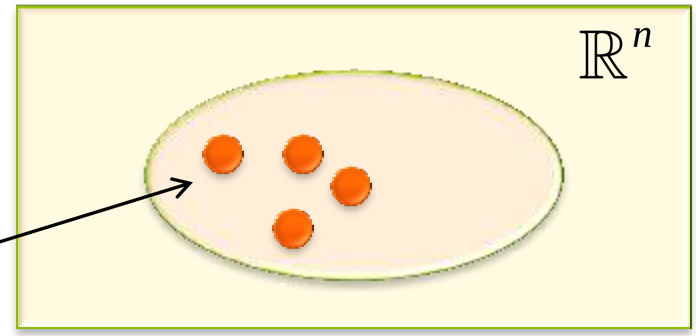
2) For any $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_r\in\text{span}(S)$ and $c_1,c_2,\ldots,c_r\in\mathbb{R}$,

$$c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\ldots+c_r\boldsymbol{v}_r\in\text{span}(S).$$

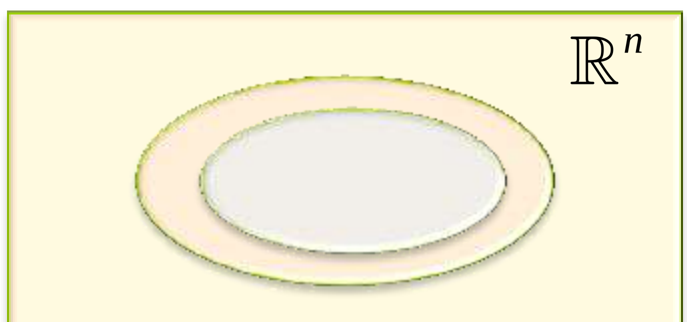
Theorem

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.





Then $\text{span}(S_1) \subseteq \text{span}(S_2) \Leftrightarrow$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.



Example

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Idea: The previous theorem gives us a necessary and sufficient condition for

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}.$$

Each of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$.

Example

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Each of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$.

$$= a(1,2,3) + b(2,-1,1)$$

Example

$$\left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cc|c|c|c} 1 & 0 & \frac{1}{5} & \frac{3}{5} & \frac{3}{5} \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} & \frac{-4}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$(1,0,1) = \quad (1,2,3) + \quad (2,-1,1)$$

$$(1,1,2) = \quad (1,2,3) + \quad (2,-1,1)$$

$$(-1,2,1) = \quad (1,2,3) + \quad (2,-1,1)$$

Since each of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$

is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$,

$\boxed{\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}.}$

Example

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Can we show

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \supseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$?

Shown:

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Each of $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

$$= a(1,0,1) + b(1,1,2) + c(-1,2,1)$$

Example

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

$$\left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c} 1 & 0 & -3 & -1 & 3 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\boldsymbol{v}_1 = a(1,0,1) + b(1,1,2) + c(-1,2,1)$$

Example

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1 (1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

$$\left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c} 1 & 0 & -3 & -1 & 3 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since each of $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$,

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \supseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}.$$

Together with $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, we have shown

$$\boxed{\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}.}$$

Example

Let $\boldsymbol{u}_1=(1,1,0,2),\boldsymbol{u}_2=(1,0,0,1),\boldsymbol{u}_3=(0,1,0,1),$
$\boldsymbol{v}_1(1,1,1,1),\boldsymbol{v}_2=(-1,1,-1,1),\boldsymbol{v}_3=(-1,1,1,-1).$

Show that $\text{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}\subseteq\text{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$ but

$$\text{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}\neq\text{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}.$$

To show $\text{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}\subseteq\text{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$,

$$\left(\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & 1 & 1 \end{array}\right)$$

How did this matrix come about?

Example

Let $\boldsymbol{u}_1 = (1,1,0,2), \boldsymbol{u}_2 = (1,0,0,1), \boldsymbol{u}_3 = (0,1,0,1)$,

$\boldsymbol{v}_1(1,1,1,1), \boldsymbol{v}_2 = (-1,1,-1,1), \boldsymbol{v}_3 = (-1,1,1,-1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ but

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \neq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

To show $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$,

$$\left(\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c|c|c} 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Example

Let $\boldsymbol{u}_1 = (1,1,0,2), \boldsymbol{u}_2 = (1,0,0,1), \boldsymbol{u}_3 = (0,1,0,1)$,

$\boldsymbol{v}_1(1,1,1,1), \boldsymbol{v}_2 = (-1,1,-1,1), \boldsymbol{v}_3 = (-1,1,1,-1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ but

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \neq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}.$$

To show $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \neq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$,

$$\boldsymbol{v}_1 = (1,1,1,1) = a(1,1,0,2) + b(1,0,0,1) + c(0,1,0,1)$$

has no solution.

So $\boldsymbol{v}_1$ is not a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

Example

Let $\boldsymbol{u}_1 = (1,0,0,1), \boldsymbol{u}_2 = (0,1,-1,2), \boldsymbol{u}_3 = (2,1,-1,4),$

$\boldsymbol{v}_1(1,1,1,1), \boldsymbol{v}_2 = (-1,1,-1,1), \boldsymbol{v}_3 = (-1,1,1,-1).$

Show that $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \not\subset \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

We try to write each $\boldsymbol{v}_i$ as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Which $\boldsymbol{v}_i$ is NOT a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$?

Theorem ('useless vector')

Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are vectors taken from $\mathbb{R}^n$.

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

set of ALL linear combinations of

=

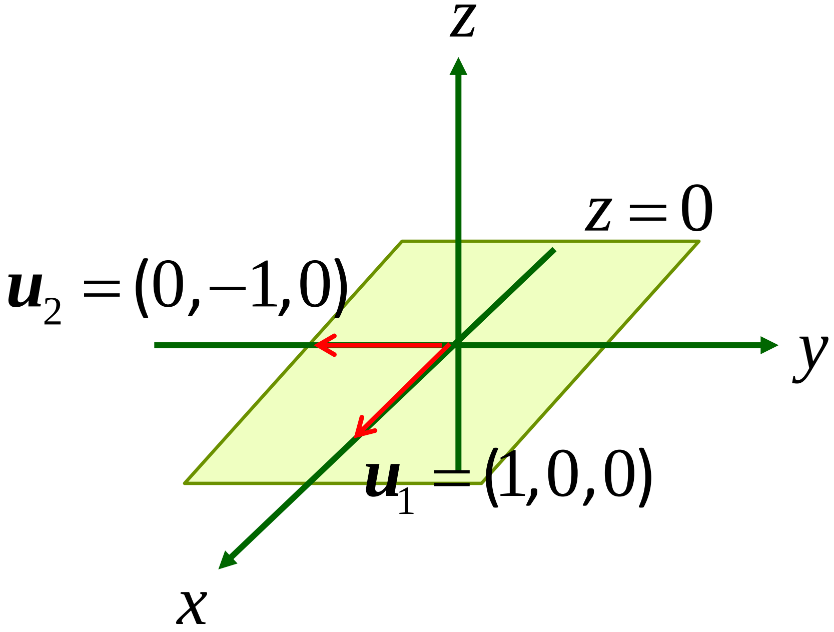
set of ALL linear combinations of

Example

Let $\boldsymbol{u}_1 = (1,0,0), \boldsymbol{u}_2 = (0,-1,0), \boldsymbol{u}_3 = (2,3,0)$.

Clearly, $\boldsymbol{u}_3 = 2\boldsymbol{u}_1 - 3\boldsymbol{u}_2$. So $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

Can you describe $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ geometrically?



Definition (Subspaces)

Let V be a subset of $\mathbb{R}^n$.

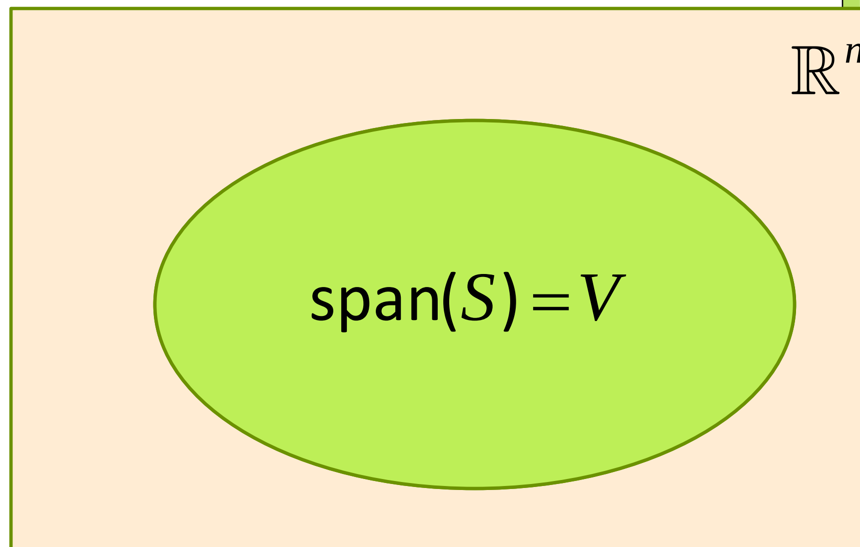
If there exists a set of vectors

$$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \text{ in } \mathbb{R}^n$$

such that $\text{span}(S) = V$,

then V is said to be

a subspace of $\mathbb{R}^n$.



Remember: $\text{span}(S) =$ set of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$.

Definition (Subspaces)

Let V be a subset of $\mathbb{R}^n$.

If there exists a set of vectors

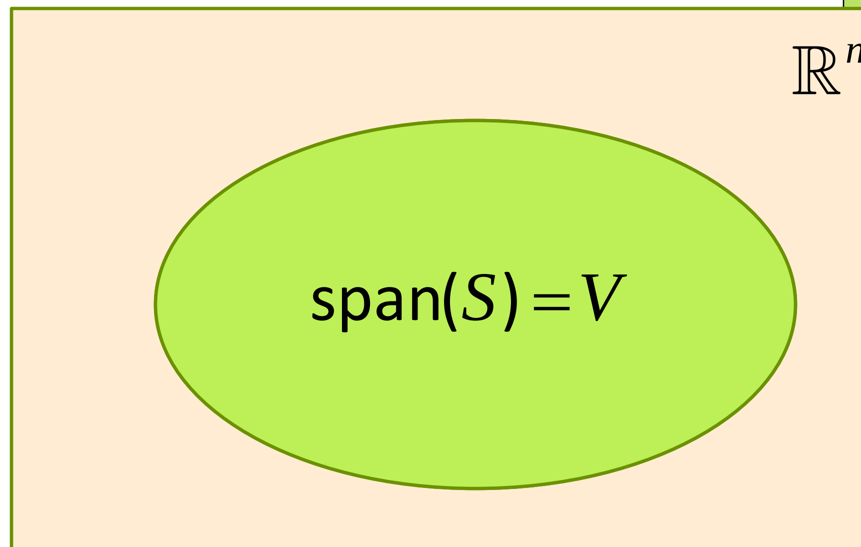
$$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \text{ in } \mathbb{R}^n$$

$$\{\bullet \;\; \bullet \qquad \bullet\}$$

such that $\text{span}(S) = V$,

then V is said to be

a subspace of $\mathbb{R}^n$.



More precisely, we say:

1) V is the subspace spanned by S.

2) V is the subspace spanned by $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$.

3) S spans V.

Remark

$\{\mathbf{0}\} = \text{span}\{\mathbf{0}\}$ is the zero subspace of $\mathbb{R}^n$.

(Here, $\mathbf{0}$ is the zero vector of $\mathbb{R}^n$.)

It is also called the zero space of $\mathbb{R}^n$.

It is also the only subspace of $\mathbb{R}^n$ with a finite number (in this case, one), and thus the least, of vectors.

Remark

Consider the vectors in $\mathbb{R}^n$:

$$\boldsymbol{e}_1 = (1,0,....,0) \qquad \boldsymbol{e}_2 = (0,1,....,0) \qquad \cdots\cdot \qquad \boldsymbol{e}_n = (0,0,....,1)$$

$$\begin{aligned}\mathbb{R}^n &= \{(u_1,u_2,...,u_n) \,|\, u_1,u_2,...,u_n \in \mathbb{R}\} \\ &= \{u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + ... + u_n\boldsymbol{e}_n \,|\, u_1,u_2,...,u_n \in \mathbb{R}\} \\ &= \text{the set of all linear combinations of } \boldsymbol{e}_1,\boldsymbol{e}_2,...,\boldsymbol{e}_n \\ &= \text{span}\{\boldsymbol{e}_1,\boldsymbol{e}_2,...,\boldsymbol{e}_n\} = \text{a subspace of } \mathbb{R}^n\end{aligned}$$

So $\mathbb{R}^n$ is a subspace of itself and to some extent, it can be thought of as the subspace of $\mathbb{R}^n$ with the 'largest' number of vectors.

Remark

In more advance textbooks on linear algebra, 'subspace' is defined differently. We will discuss this briefly later in today's lecture.

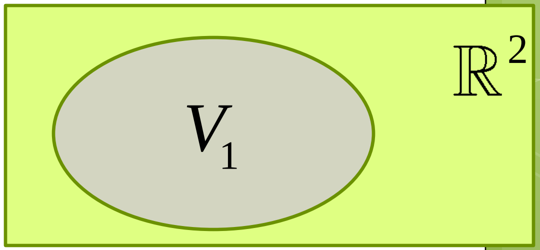
Example (subspaces)

$V_1 = \{(a-2b, 3b) \mid a, b \in \mathbb{R}\}$. V_1 is a subset of $\mathbb{R}^2$.

Is V_1 a subspace of $\mathbb{R}^2$?

Idea: Can you try to write V_1 as a linear span?

Is $V_1 = \mathbb{R}^2$?



So V_1 is a subspace of $\mathbb{R}^2$.

Is there a vector in $\mathbb{R}^2$ that is not in V_1?

Example (subspaces)

$V_1 = \{(a-2b, 3b) \mid a, b \in \mathbb{R}\}.$

Is $V_1 = \mathbb{R}^2$?

Idea: Is every vector in $\mathbb{R}^2$ a linear combination of $(1,0)$ and $(-2,3)$?

Example (subspaces)

$V_2 = \{(x, y, z) \mid x - 3y + 2z = 0\}$. V_2 is a subset of $\mathbb{R}^3$.

Describe V_2 geometrically. Is V_2 a subspace of $\mathbb{R}^3$?

Idea: V_2 is given in implicit form. Can you find its explicit form?

Example (subspaces)

$V_3 = \{(x, y, z) \mid x - 3y + 2z = 1\}$. V_3 is a subset of $\mathbb{R}^3$.

Describe V_3 geometrically. Is V_3 a subspace of $\mathbb{R}^3$?

$(0,0,0) \notin V_3$ since $0 - 3(0) + 2(0) \neq 1$.

If V_3 is a subspace, then it can be expressed as a linear span, that is, $V_3 = \text{span}(S)$ for some finite set S.

By (first) theorem in today's lecture, $(0,0,0) \in \text{span}(S) = V_3$.

Contradiction! So V_3 is not a subspace of $\mathbb{R}^3$.

Example (subspaces)

$V_4 = \{(x, y, z) \mid x \le y \le z\}$. V_4 is a subset of $\mathbb{R}^3$.

Is V_4 a subspace of $\mathbb{R}^3$? No!

If V_4 is a subspace, then it can be expressed as a linear span, that is, $V_4 = \text{span}(T)$ for some finite set T.

By (first) theorem in today's lecture,

For any $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in \text{span}(T) = V_4$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_r\boldsymbol{v}_r \in \text{span}(T) = V_4.$$

$(1,1,2), (0,2,4) \in V_4$ but $(1,1,2) - 2(0,2,4) = (1,-3,-6) \notin V_4$.

End of Lecture 09

Lecture 10:

Subspaces (cont'd)

Linear independence

Bases (till Example 3.5.5.1)