LECTURE 15 RECAP

1) A real life example leading up to the problem of computing a high power of a square matrix.
2) Eigenvalue, eigenvector of a matrix.
3) How to find ALL eigenvalues of a matrix - characteristic polynomial and characteristic equation.
4) Eigenvalues of triangular matrices.
5) Eigenspace of a matrix.
6) Several examples on finding eigenvalues and the corresponding dimension of the eigenspaces.

LECTURE 16

Diagonalization

DEFINITION

Given a square matrix $\boldsymbol{A}$, we wanted to know if it is possible to find an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} \text{ (a diagonal matrix)}$$

A square matrix $\boldsymbol{A}$ is called diagonalizable if there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

In here, matrix $\boldsymbol{P}$ is said to diagonalize $\boldsymbol{A}$.

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$.

Then $\boldsymbol{P}$ is invertible (check) and

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0.95 \end{pmatrix}$$

So $\boldsymbol{A}$ is diagonalizable and $\boldsymbol{P}$ diagonalizes $\boldsymbol{A}$.

Recall that 1 and 0.95 are the eigenvalues of $\boldsymbol{A}$ and

$$E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$$

$$E_{0.95} = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

EXAMPLE

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

So $\boldsymbol{B}$ is diagonalizable and $\boldsymbol{P}$ diagonalizes $\boldsymbol{B}$.

Recall that 3 and 0 are the eigenvalues of $\boldsymbol{B}$ and

$$E_3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad E_0 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$.

Then $\boldsymbol{P}$ is invertible (check) and $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

EXAMPLE

$$C = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

So C is diagonalizable and P diagonalizes C.

Recall that $1, \sqrt{2}$ and $-\sqrt{2}$ are the eigenvalues of C and

$$E_1 = \text{span}\left\{\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\right\} \quad E_{\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right\} \quad E_{-\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right\}$$

Let $P = \begin{pmatrix} -2 & -1 & -1 \\ 2 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$.

Then P is invertible (check) and

$$P^{-1}CP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

What about $\boldsymbol{M}$e?

EXAMPLE

$$\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \qquad E_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

We will now show that $\boldsymbol{M}$ is not diagonalizable.

Suppose $\boldsymbol{M}$ is diagonalizable. Then there exists an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

EXAMPLE

What about $\boldsymbol{M}$e?

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We will now show that $\boldsymbol{M}$ is not diagonalizable.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \Leftrightarrow \begin{cases} \boxed{2a = \lambda a \quad (1)} \\ 2b = \mu b \\ \boxed{a + 2c = \lambda c \quad (2)} \\ b + 2d = \mu d \end{cases}$$

If $a \neq 0$, then (1) $\Rightarrow \lambda = 2$

but now (2) $\Rightarrow a = 0$.

$\boxed{\text{So } a = 0, \lambda = 2.}$

$\boxed{\text{Similarly, } b = 0, \mu = 2.}$

EXAMPLE

What about $\boldsymbol{M}$e?

$$\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \qquad E_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

We will now show that $\boldsymbol{M}$ is not diagonalizable.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$ which is singular, a contradiction.

So $\boldsymbol{M}$ is not diagonalizable.

Is there a more efficient way of showing a matrix is not diagonalizable?

THEOREM

Let $\boldsymbol{A}$ be a square matrix of order n. Then $\boldsymbol{A}$ is diagonalizable if and only if $\boldsymbol{A}$ has n linearly independent eigenvectors.

It is important to emphasize the n eigenvectors have to be linearly independent since

E_λ contains ALL the eigenvectors of $\boldsymbol{A}$ associated with λ.

that is, $\boldsymbol{A}$ already has infinitely eigenvectors associated with a particular eigenvalue λ.

AN ALGORITHM

Purpose: Given a square matrix $\boldsymbol{A}$ of order n, we want to determine whether $\boldsymbol{A}$ is diagonalizable.

If $\boldsymbol{A}$ is diagonalizable, find an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

Step 1: Solve $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ to find all eigenvalues of $\boldsymbol{A}$

$\lambda_1, \lambda_2, \ldots, \lambda_k$ (suppose $\boldsymbol{A}$ has k distinct eigenvalues, $k \leq n$)

Step 2: For each λ_i find a basis S_{λ_i} for the eigenspace E_{λ_i}.

AN ALGORITHM

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Step 2: For each λ_i find a basis S_{λ_i} for the eigenspace E_{λ_i}.

Step 3: Let $S = S_{\lambda_1} \cup S_{\lambda_2} \cup \ldots \cup S_{\lambda_k}$ (the union of all bases)

(a) If $|S| < n$, then $\boldsymbol{A}$ is not diagonalizable.

$|S| =$ number of vectors in S

(b) If $|S| = n$, say $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, then let

$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_n \end{pmatrix}$ to be the matrix that diagonalizes $\boldsymbol{A}$.

(c) If $|S| > n$, check your working!

SOME REMARKS

Step 1: Solve $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ to find all eigenvalues of $\boldsymbol{A}$

$\lambda_1, \lambda_2, \ldots, \lambda_k$ (suppose $\boldsymbol{A}$ has k distinct eigenvalues, $k \leq n$)

In general, when solving for the roots of

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$$

it is possible that some λ_i is a complex number.

Complex vector spaces $\mathbb{C}^n$ (instead of $\mathbb{R}^n$) needs to be introduced (but not in this course).

SOME REMARKS

Suppose $\det(\lambda \boldsymbol{I} - \boldsymbol{A})$ is factorised as

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \ldots (\lambda - \lambda_i)^{r_i} \ldots (\lambda - \lambda_k)^{r_k}$$

Note that $\boxed{r_1 + r_2 + \ldots + r_k = n}$ (order of matrix $\boldsymbol{A}$)

Step 2: For each λ_i find a basis S_{λ_i} for the eigenspace $\boxed{E_{\lambda_i}}$.

Then, for each eigenvalue λ_i, $\dim(\boxed{E_{\lambda_i}}) \leq r_i$

Furthermore, $\boldsymbol{A}$ is diagonalizable if and only if for $i = 1, 2, \ldots, n$

$$\left| S_{\lambda_i} \right| = \dim(E_{\lambda_i}) = r_i \quad \Leftrightarrow \left| S \right| = \left| S_{\lambda_1} \cup \ldots \cup S_{\lambda_k} \right| = \boxed{r_1 + \ldots + r_k = n}$$

SOME REMARKS

Step 3: Let $S = S_{\lambda_1} \cup S_{\lambda_2} \cup \ldots \cup S_{\lambda_k}$ (the union of all bases)

(b) If $|S| = n$, say $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, then let

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_n \end{pmatrix}$$

to be the (invertible) matrix that diagonalizes $\boldsymbol{A}$.

How can you be sure that $\boldsymbol{P}$ formed this way will be invertible?

The set S is always linearly independent.

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$$

$$E_{0.95} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$.

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0.95 \end{pmatrix}$$

$$E_3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$E_0 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$.

$$\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

What if

$$\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix}?$$

EXAMPLE

$$\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \qquad \det(\lambda \boldsymbol{I} - \boldsymbol{M}) = (\lambda - 2)^2$$

$\boldsymbol{M}$ is a 2×2 matrix and has one eigenvalue 2.

$$E_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \qquad \dim(E_2) = 1 < 2$$

$\boldsymbol{M}$ has only 1 linearly independent eigenvector and so it is not diagonalizable.

EXAMPLE

Is $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$ diagonalizable? If so, find an invertible matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.

Answer: We know immediately that the eigenvalues of $\boldsymbol{A}$ are 1 and 2.

We need to check if we can find 3 linearly independent eigenvectors of $\boldsymbol{A}$.

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$$

Consider E_1: Solving $(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 3 & -5 & -1 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{-1}{8} & 0 \\ 0 & 1 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x = \frac{t}{8} \\ y = \frac{-t}{8} \\ z = t, \quad t \in \mathbb{R} \end{cases}$$

So $E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 8 \end{pmatrix}\right\}$

and $\dim(E_1) = 1$.

EXAMPLE

$\boldsymbol{A}$ has only two linearly independent eigenvectors

$\boldsymbol{A}$ is not diagonalizable

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$$

$\det(\lambda \boldsymbol{I} - \boldsymbol{A})$

$= (\lambda - 2)^2 (\lambda - 1)$

Consider E_2: Solving $(2\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & -5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & -5 & 0 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x = 0 \\ y = 0 \\ z = t, \quad t \in \mathbb{R} \end{cases}$$

So $E_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

and $\dim(E_2) = 1$.

A SUFFICIENT CONDITION FOR DIAGONALIZABILITY

Let $\boldsymbol{A}$ be a square matrix of order n. If $\boldsymbol{A}$ has n distinct eigenvalues, then $\boldsymbol{A}$ is diagonalizable.

Proof: Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{A}$.

Then $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = (\lambda - \lambda_1)^1 (\lambda - \lambda_2)^1 \ldots (\lambda - \lambda_n)^1$.

Since $\dim(E_{\lambda_i}) \le r_i = 1$ for each $i = 1, 2, \ldots, n$, (by earlier remark)

We must have $\dim(E_{\lambda_i}) = 1$ for all $i = 1, \ldots, n$.

$\boldsymbol{A}$ has n linearly independent eigenvectors and is thus diagonalizable.

Is it possible that $\dim(E_{\lambda_i}) = 0$ for some i?

EXAMPLE

Is $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 3 \end{pmatrix}$ diagonalizable?

Answer: Yes. Since $\boldsymbol{A}$ is a 3×3 matrix that has 3 distinct eigenvalues.

APPLICATION OF DIAGONALIZATION

Once a square matrix $\boldsymbol{A}$ can be diagonalized, we can compute $\boldsymbol{A}^n$ easily.

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} = \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{pmatrix}$$

$$\Leftrightarrow \boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

$$\Leftrightarrow \boldsymbol{A}^k = \boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-1}$$

$$= \boldsymbol{P} \begin{pmatrix} \lambda_1{}^k & & & \mathbf{0} \\ & \lambda_2{}^k & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n{}^k \end{pmatrix} \boldsymbol{P}^{-1}$$

APPLICATION OF DIAGONALIZATION

Suppose $\boldsymbol{A}$ is invertible, by Theorem 6.1.8, $\lambda_i \neq 0$ for all i. Then

$$\boldsymbol{A}^{-1} = \boldsymbol{P}\begin{pmatrix} \lambda_1^{-1} & & & \mathbf{0} \\ & \lambda_2^{-1} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n^{-1} \end{pmatrix}\boldsymbol{P}^{-1}$$

For all positive integers m,

$$\boldsymbol{A}^{-m} = \boldsymbol{P}\begin{pmatrix} \lambda_1^{-m} & & & \mathbf{0} \\ & \lambda_2^{-m} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n^{-m} \end{pmatrix}\boldsymbol{P}^{-1}$$

EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} -4 & 0 & -6 \\ 2 & 1 & 2 \\ 3 & 0 & 5 \end{pmatrix}$. Compute $\boldsymbol{A}^{10}$.

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda+4 & 0 & 6 \\ -2 & \lambda-1 & -2 \\ -3 & 0 & \lambda-5 \end{vmatrix} = (\lambda+4)\begin{vmatrix} \lambda-1 & -2 \\ 0 & \lambda-5 \end{vmatrix} + 6\begin{vmatrix} -2 & \lambda-1 \\ -3 & 0 \end{vmatrix}$$

$$= (\lambda+4)(\lambda-1)(\lambda+5) + 18(\lambda-1)$$

$$= (\lambda-1)[(\lambda^2-\lambda-20+18)]$$

$$= (\lambda-1)(\lambda^2-\lambda-2)$$

$$= (\lambda-1)(\lambda+1)(\lambda-2)$$

So the eigenvalues of $\boldsymbol{A}$ are $-1, 1$ and 2.

EXAMPLE

Using the method described previously, we find:

$$E_{-1} = \text{span}\left\{\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\} \qquad E_1 = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\} \qquad E_2 = \text{span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

(verify it yourself!)

So if we let

$$\boldsymbol{P} = \begin{pmatrix}-2 & 0 & -1\\1 & 1 & 0\\1 & 0 & 1\end{pmatrix}$$

So the eigenvalues of $\boldsymbol{A}$ are $-1,1$ and 2.

EXAMPLE

We will have

$$P^{-1}AP=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&2\end{pmatrix}$$ or equivalently $A=P\begin{pmatrix}-1&0&0\\0&1&0\\0&0&2\end{pmatrix}P^{-1}$

$$\Rightarrow\ A^{10}=P\begin{pmatrix}(-1)^{10}&0&0\\0&1^{10}&0\\0&0&2^{10}\end{pmatrix}P^{-1}$$

$$=\begin{pmatrix}-2&0&-1\\1&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1024\end{pmatrix}\begin{pmatrix}-1&0&-1\\1&1&1\\1&0&2\end{pmatrix}=\begin{pmatrix}-1022&0&-2046\\0&1&0\\1023&0&2047\end{pmatrix}$$

EXAMPLE (FIBONACCI NUMBERS)

Fibonacci numbers: For $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$

$$a_0 = 0 \quad a_1 = 1 \quad a_2 = a_1 + a_0 = 1 \quad a_3 = a_2 + a_1 = 2 \quad \cdots$$

That's easy enough...what is a_{256} ?

I need to find a_{254} and a_{255}..

I need to find a_{253} and a_{254}..

EXAMPLE (FIBONACCI NUMBERS)

Fibonacci numbers: For $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$

$$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = a_1 + a_0 = 1 \qquad a_3 = a_2 + a_1 = 2 \quad \cdots$$

Let $\boldsymbol{x}_n = \begin{pmatrix} \boxed{a_n} \\ \boxed{a_{n+1}} \end{pmatrix}$ and so $\boldsymbol{x}_{n-1} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$.

How do we relate $\boldsymbol{x}_n$ and $\boldsymbol{x}_{n-1}$?

Can we express a_n in terms of a_{n-1} and a_n? $\boxed{a_n = 0a_{n-1} + 1a_n}$

Can we express a_{n+1} in terms of a_{n-1} and a_n? $\boxed{a_{n+1} = 1a_{n-1} + 1a_n}$

EXAMPLE (FIBONACCI NUMBERS)

Fibonacci numbers: For $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$

$$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = a_1 + a_0 = 1 \qquad a_3 = a_2 + a_1 = 2 \quad \cdots$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ and so $\boldsymbol{x}_{n-1} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$.

How do we relate $\boldsymbol{x}_n$ and $\boldsymbol{x}_{n-1}$?

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

$$\boldsymbol{x}_0 = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$a_n = 0a_{n-1} + 1a_n$$

$$a_{n+1} = 1a_{n-1} + 1a_n$$

$$\boldsymbol{x}_n = \boldsymbol{A}\boldsymbol{x}_{n-1} = \boldsymbol{A}^2\boldsymbol{x}_{n-2} = \boldsymbol{A}^3\boldsymbol{x}_{n-3} = \ldots = \boldsymbol{A}^n\boldsymbol{x}_0$$

EXAMPLE (FIBONACCI NUMBERS)

$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$ $\quad$ Let's see if we can diagonalize $\boldsymbol{A}$.

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda - 1$$

Remember $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= (\lambda - (\frac{1+\sqrt{5}}{2}))(\lambda - (\frac{1-\sqrt{5}}{2}))$$

So $\boldsymbol{A}$ has two eigenvalues: $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$.

EXAMPLE (FIBONACCI NUMBERS)

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Let's see if we can diagonalize $\boldsymbol{A}$.

Proceeding as before, we find that

$$E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} \right\} \quad E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \right\}$$

So $\boldsymbol{A}$ has two linearly independent eigenvectors and can be diagonalized.

So $\boldsymbol{A}$ has two eigenvalues: $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$.

EXAMPLE (FIBONACCI NUMBERS)

$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$ $\quad$ Let's see if we can diagonalize $\boldsymbol{A}$.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}$ and we have

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Leftrightarrow \boldsymbol{A} = \boldsymbol{P}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\boldsymbol{P}^{-1}$$

So $\boldsymbol{A}$ has two linearly independent eigenvectors and can be diagonalized.

So $\boldsymbol{A}$ has two eigenvalues: $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$.

EXAMPLE (FIBONACCI NUMBERS)

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Now you can compute a_{256} !

$$\begin{pmatrix} \boxed{a_n} \\ \boxed{a_{n+1}} \end{pmatrix} = \boldsymbol{x}_n = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n} \\ \boxed{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}} \end{pmatrix}$$

FIBONACCI NUMBERS REVISITED

$$\boldsymbol{A}=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} \qquad \boldsymbol{x}_n=\boldsymbol{A}^n\boldsymbol{x}_0$$

By an earlier theorem,
we know that $\boldsymbol{A}$ can
definitely be diagonalized.

$$\det(\lambda \boldsymbol{I}-\boldsymbol{A})=\begin{vmatrix}\lambda & -1\\ -1 & \lambda-1\end{vmatrix}=\lambda^2-\lambda-1$$

We will now solve for a_n
without explicitly finding
the invertible matrix $\boldsymbol{P}$.

$$=(\lambda-(\frac{1+\sqrt{5}}{2}))(\lambda-(\frac{1-\sqrt{5}}{2}))$$

So $\boldsymbol{A}$ has two eigenvalues: $\lambda_1=\dfrac{1+\sqrt{5}}{2}$ and $\lambda_2=\dfrac{1-\sqrt{5}}{2}$.

FIBONACCI NUMBERS REVISITED

$$\boldsymbol{A}=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} \qquad \boldsymbol{x}_n=\boldsymbol{A}^n\boldsymbol{x}_0 \qquad \lambda_1=\frac{1+\sqrt{5}}{2},\ \lambda_2=\frac{1-\sqrt{5}}{2}$$

Let $\boldsymbol{P}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\boldsymbol{P}^{-1}=\begin{pmatrix}e & f\\ g & h\end{pmatrix}$ $\qquad \boxed{\begin{pmatrix}\lambda_1{}^n & 0\\ 0 & \lambda_2{}^n\end{pmatrix}}$

$$\begin{pmatrix}\boxed{a_n}\\ a_{n+1}\end{pmatrix}=\boldsymbol{x}_n=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\boxed{\begin{pmatrix}(\frac{1+\sqrt{5}}{2})^n & 0\\ 0 & (\frac{1-\sqrt{5}}{2})^n\end{pmatrix}}\begin{pmatrix}e & f\\ g & h\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$=\begin{pmatrix}a(\frac{1+\sqrt{5}}{2})^n & b(\frac{1-\sqrt{5}}{2})^n\\ c(\frac{1+\sqrt{5}}{2})^n & d(\frac{1-\sqrt{5}}{2})^n\end{pmatrix}\begin{pmatrix}f\\ h\end{pmatrix}=\begin{pmatrix}\boxed{A(\frac{1+\sqrt{5}}{2})^n+B(\frac{1-\sqrt{5}}{2})^n}\\ C(\frac{1+\sqrt{5}}{2})^n+D(\frac{1-\sqrt{5}}{2})^n\end{pmatrix} \quad A,B,C,D\in\mathbb{R}$$

FIBONACCI NUMBERS REVISITED

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

$a_0 = 0 \qquad a_1 = 1$

(Initial conditions)

$$a_n = A\left(\tfrac{1+\sqrt{5}}{2}\right)^n + B\left(\tfrac{1-\sqrt{5}}{2}\right)^n \qquad a_0 = A\left(\tfrac{1+\sqrt{5}}{2}\right)^0 + B\left(\tfrac{1-\sqrt{5}}{2}\right)^0 = A + B = 0$$

$$a_1 = A\left(\tfrac{1+\sqrt{5}}{2}\right)^1 + B\left(\tfrac{1-\sqrt{5}}{2}\right)^1 = A\left(\tfrac{1+\sqrt{5}}{2}\right) + B\left(\tfrac{1-\sqrt{5}}{2}\right) = 1$$

Solving $\begin{cases} A + B = 0 \\ \left(\frac{1+\sqrt{5}}{2}\right)A + \left(\frac{1-\sqrt{5}}{2}\right)B = 1 \end{cases}$ we have $A = \dfrac{1}{\sqrt{5}}$ and $B = \dfrac{-1}{\sqrt{5}}$,

$$\Rightarrow \boxed{a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

END OF LECTURE 16

Lecture 17:

Inner Products in R^n

Orthogonal and Orthonormal Bases (till Example 5.2.9)