Lecture 07 recap

1) How elementary row operations changes the determinant of a matrix.

2) Using elementary row operations to find the determinant of a matrix.

3) $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

4) $\det(c\boldsymbol{A})$, $\det(\boldsymbol{AB})$ and $\det(\boldsymbol{A}^{-1})$.

5) Definition of adjoint of a matrix. Inverse in terms of adjoint.

6) Cramer's rule.

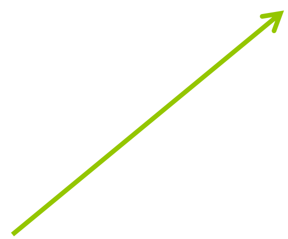
Lecture 08

Euclidean *n*-spaces

Linear combinations and linear spans

How do we represent vectors

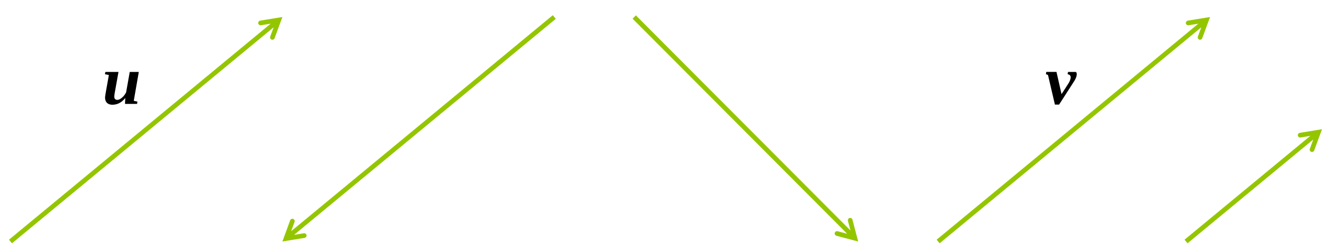
Geometrically, a vector is represented by a directed line segment (or arrow).



direction of the arrow

length of the arrow (magnitude)

Two vectors u and v are said to be equal if they have the same length and direction.

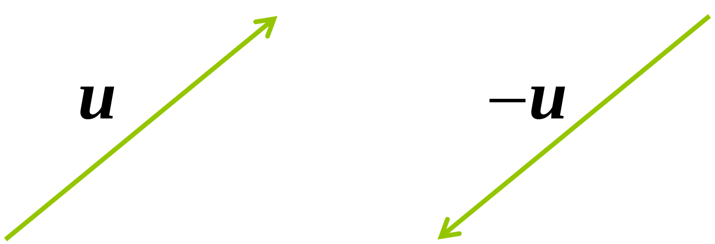


Vector operations

1) Addition: $u + v$

$$\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$$

2) Negative of $\boldsymbol{u}$: $-\boldsymbol{u}$

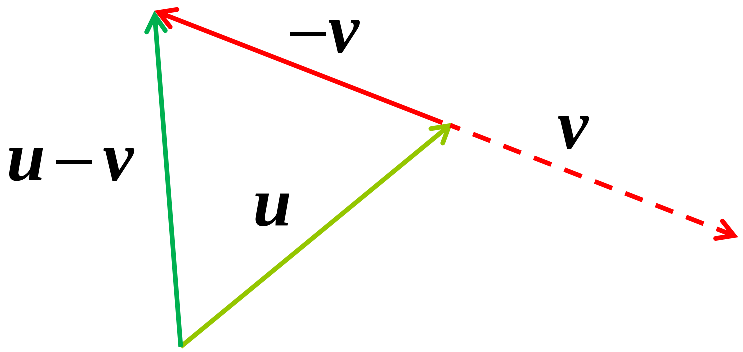


change in direction,

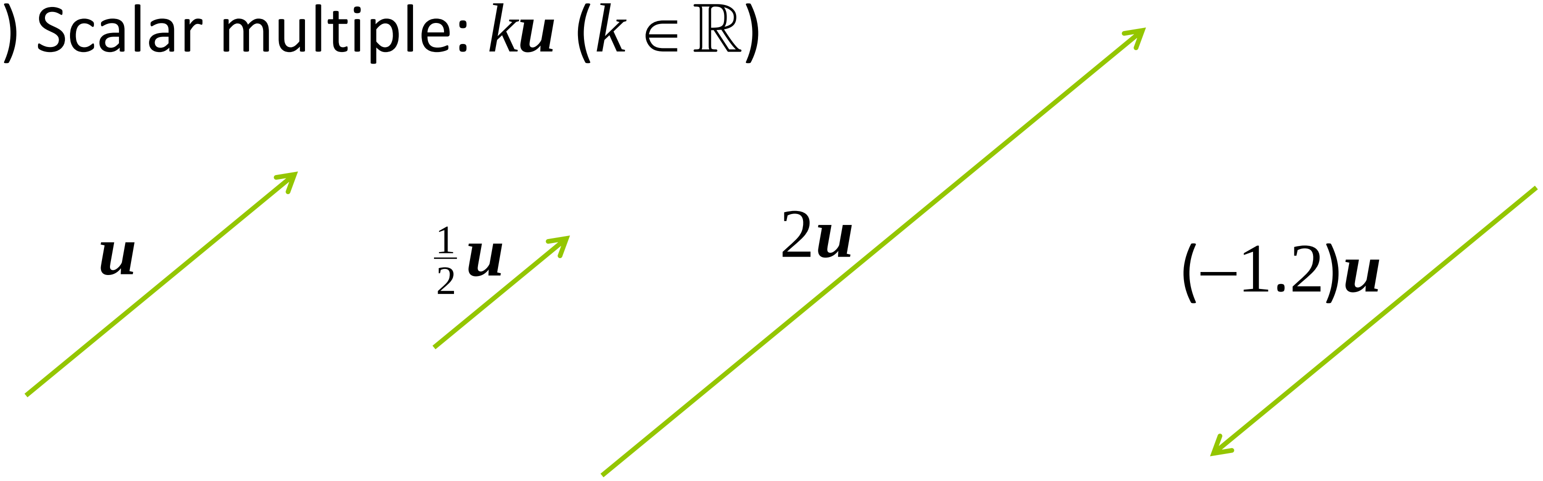
no change in magnitude

Vector operations

3) Difference: $\boldsymbol{u} - \boldsymbol{v}$ (same as $\boldsymbol{u} + (-\boldsymbol{v})$)

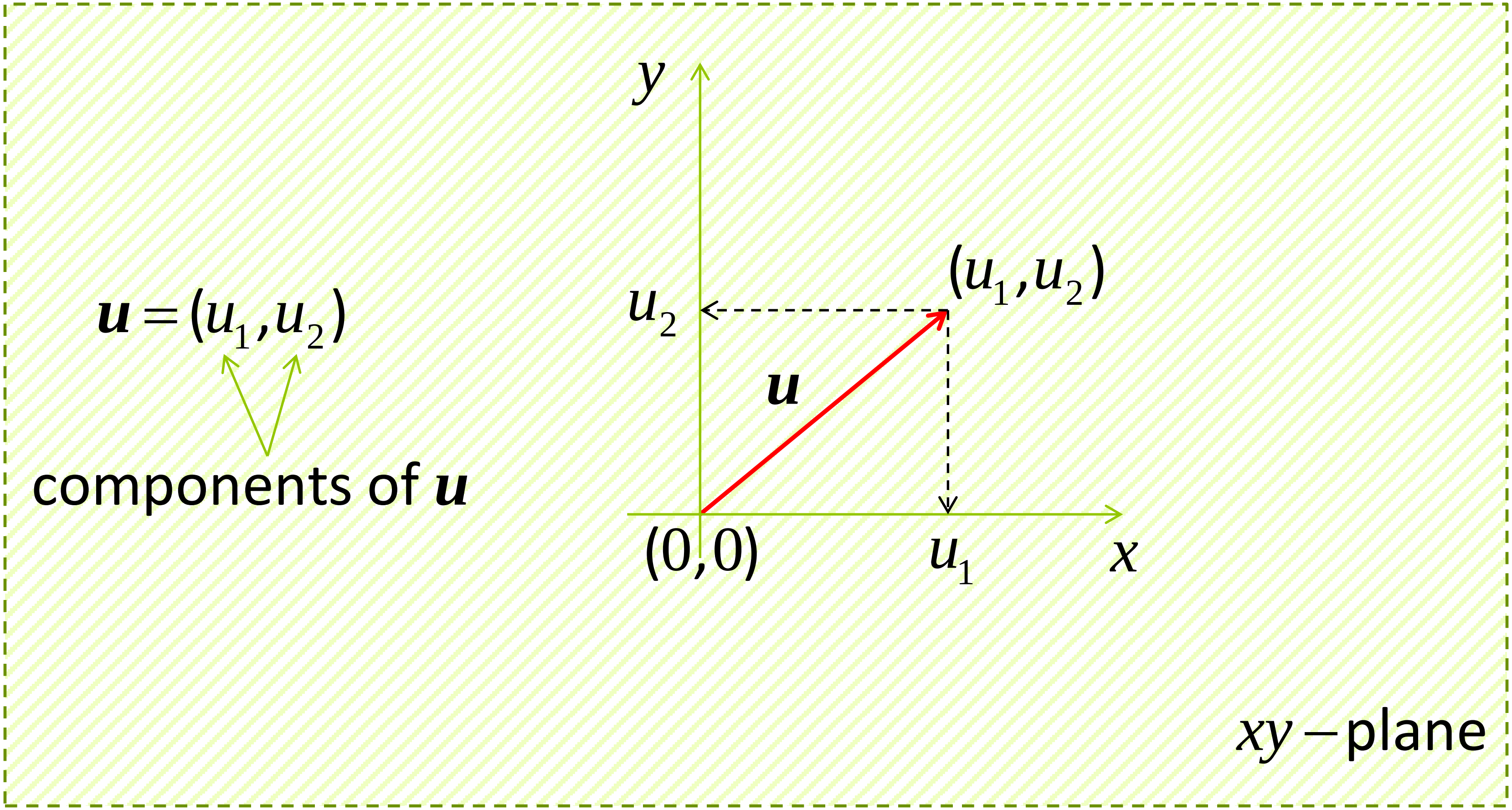


4) Scalar multiple: $k\boldsymbol{u}$ ($k \in \mathbb{R}$)



Coordinate systems (2-space)

Position $\boldsymbol{u}$ with its initial point at $(0,0)$

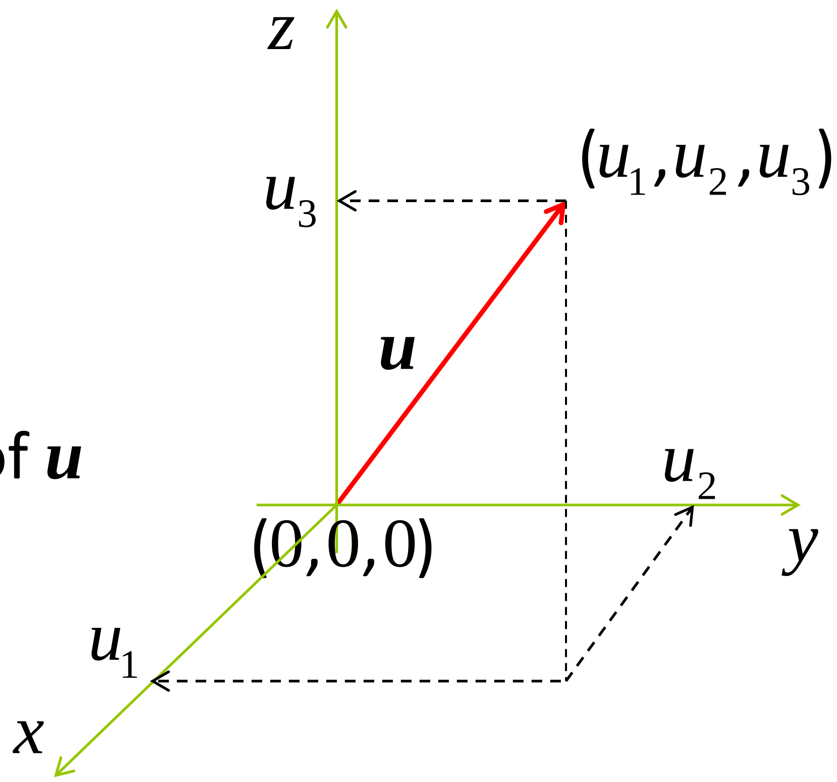


Coordinate systems (3-space)

Position $\boldsymbol{u}$ with its initial point at $(0,0,0)$

$$\boldsymbol{u} = (u_1, u_2, u_3)$$

components of $\boldsymbol{u}$



$xyz -$ space

Addition and scalar multiplication

1) Addition: add component-wise.

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

2) Scalar multiplication: multiply to each component.

$$\boldsymbol{u} = (u_1, u_2), k \in \mathbb{R} \qquad k\boldsymbol{u} = (ku_1, ku_2)$$

$$\boldsymbol{u} = (u_1, u_2, u_3), k \in \mathbb{R} \qquad k\boldsymbol{u} = (ku_1, ku_2, ku_3)$$

Definitions (n-vectors and operations)

n-vector: $(u_1, u_2, \ldots, u_i, \ldots, u_n)$

$u_1, u_2, \ldots, u_n$ are real numbers.

u_1 : 1st component (or 1st coordinate) of the vector

u_i : ith component (or ith coordinate) of the vector

$\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \quad \boldsymbol{v} = (v_1, v_2, \ldots, v_n)$

Equality: $\boldsymbol{u} = \boldsymbol{v}$ if and only if

$u_i = v_i$ for all $i = 1, 2, \ldots, n$.

Definitions (*n*-vectors and operations)

$$\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \qquad \boldsymbol{v} = (v_1, v_2, \ldots, v_n)$$

Addition: $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$

Scalar multiplication: $c \in \mathbb{R},\ c\boldsymbol{u} = (cu_1, cu_2, \ldots, cu_n)$

Negative: $-\boldsymbol{u} = (-u_1, -u_2, \ldots, -u_n)$

Subtraction: $\boldsymbol{u} - \boldsymbol{v} = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n)$

Zero vector: $\boldsymbol{0} = (0, 0, \ldots, 0)$

Example (vector operations)

$$\boldsymbol{u} = (3,4,5,1) \qquad \boldsymbol{v} = (-1,0,1,2)$$

$$\boldsymbol{u} + 2\boldsymbol{v} =$$

$$2\boldsymbol{u} - 3\boldsymbol{v} =$$

If $\boldsymbol{w} = (0,0,0,0,0)$, what is $\boldsymbol{u} - 2\boldsymbol{w}$?

Vectors and matrices

Identifying an n-vector $(u_1, u_2, \ldots, u_n)$ with:

$$1 \times n \text{ matrix } \begin{pmatrix} u_1 & u_2 & \ldots & u_n \end{pmatrix} \text{ (row vector)}$$

or

$$n \times 1 \text{ matrix } \begin{pmatrix} u_1 \\ u_2 \\ : \\ u_n \end{pmatrix} \text{ (column vector)}$$

Properties of vector operations

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be n-vectors and a, b be real numbers.

1) $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$

2) $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}$

3) $\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}=\mathbf{0}+\boldsymbol{u}$

4) $\boldsymbol{u}+(-\boldsymbol{u})=\mathbf{0}$

5) $a(b\mathbf{u})=(ab)\mathbf{u}$

6) $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$

7) $(a+b)\boldsymbol{u}=a\boldsymbol{u}+b\boldsymbol{u}$

8) $1\boldsymbol{u}=\boldsymbol{u}$

Proof: Not discussed here.

Definition (Euclidean *n*-space)

Euclidean n-space, denoted by $\mathbb{R}^n$, is the set of all n-vectors $(u_1, u_2, \ldots, u_n)$ where u_i, $i = 1, \ldots, n$, is a real number.

$$\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n \text{ if and only if } u_1, \ldots, u_n \in \mathbb{R}.$$

Note:

1) For any positive integer n, $\mathbb{R}^n$ is a set.

2) How many vectors does $\mathbb{R}^n$ contain?

Infinitely many!

3) Do $\mathbb{R}^2$ and $\mathbb{R}^3$ have any vector in common?

No!

$(2, -3)$ $(-1.5, 3\pi, 0)$

Example (Subsets of R^n)

$$S = \{(u_1, u_2, u_3) \mid u_1 = 0 \text{ and } u_2 = -u_3\}$$

S contains vectors (u_1, u_2, u_3) from $\mathbb{R}^3$ such that

$$u_1 = 0 \text{ and } u_2 = -u_3$$

S is a subset of $\mathbb{R}^3$ $\quad (0,1,-1) \in S \quad (0,0,0) \in S \quad (1,0,0) \notin S$

We can also write S as

$$S = \{(0, a, -a) \mid a \in \mathbb{R}\}$$

Example (Subsets of R^n)

$$S = \{(w,x,y,z) \mid w - x + y + z = 0, 2w + x - y + 2z = 1\}$$

S contains vectors (w,x,y,z) from $\mathbb{R}^4$ such that w,x,y,z satisfies

$$\begin{cases} w - x + y + z = 0 \\ 2w + x - y + 2z = 1 \end{cases}$$

S is the solution set of the above linear system.

S is a subset of $\mathbb{R}^4$.

Note that the solution set of a linear system involving n variables will be a subset of $\mathbb{R}^n$.

Example (Subsets of R^n)

$$S = \{(w,x,y,z) \mid w - x + y + z = 0, 2w + x - y + 2z = 1\}$$

(Implicit)

Solving the linear system, we have a general solution

$$\begin{cases} w &= \tfrac{1}{3} - t \\ x &= \tfrac{1}{3} + s \\ y &= s \\ z &= t \qquad s,t \in \mathbb{R} \end{cases}$$

We are now able to rewrite the set S in another way:

$$S = \{(\tfrac{1}{3} - t, \tfrac{1}{3} + s, s, t) \mid s,t \in \mathbb{R}\}$$

(Explicit)

Example (Subsets of R^n)

Lines in $\mathbb{R}^2$

Recall that the equation $ax + by = c$ in two variables x, y represents a line in $\mathbb{R}^2$.

Solving

(Implicit) representation:

$$\{(x, y) \mid ax + by = c\},\ a, b \in \mathbb{R} \text{ not both zero.}$$

(Explicit) representation:

$\{(\frac{c - bt}{a}, t) \mid t \in \mathbb{R}\}$, if $a \neq 0$; $\{(t, \frac{c - at}{b}) \mid t \in \mathbb{R}\}$, if $b \neq 0$.

Example (Subsets of R^n)

Planes in $\mathbb{R}^3$

Recall that the equation $ax+by+cz=d$ in three variables x,y,z represents a plane in $\mathbb{R}^3$.

(Implicit) representation:

$$\{(x,y,z)\,|\,ax+by+cz=d\},\ a,b,c\in\mathbb{R} \text{ not all zero.}$$

What about explicit representation?

Solving $ax+by+cz=d$. One equation, 3 unknowns.

Example (Subsets of R^n)

Planes in $\mathbb{R}^3$

Recall that the equation $ax+by+cz=d$ in three variables x, y, z represents a plane in $\mathbb{R}^3$.

Explicit representations: $\{(\frac{d-bs-ct}{a}, s, t) \mid s, t \in \mathbb{R}\}$, if $a \neq 0$;

$\{(s, \frac{d-as-ct}{b}, t) \mid s, t \in \mathbb{R}\}$, if $b \neq 0$;

$\{(s, t, \frac{d-as-bt}{c}) \mid s, t \in \mathbb{R}\}$, if $c \neq 0$;

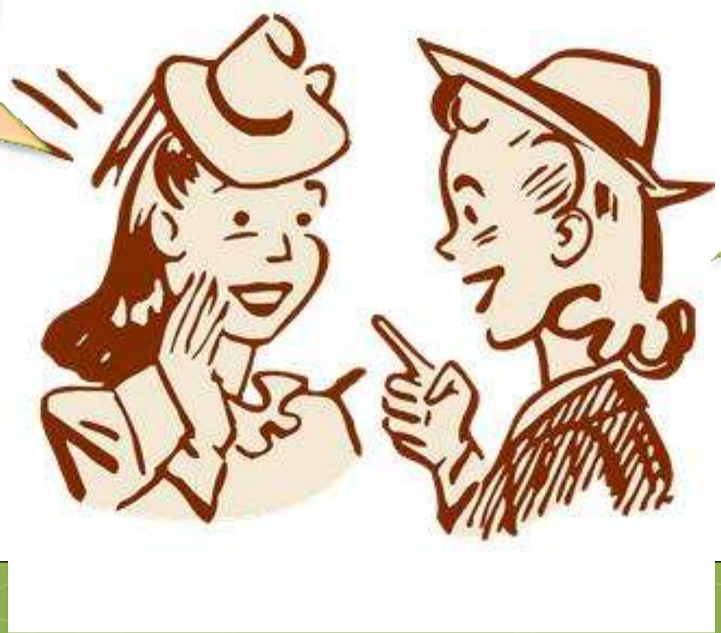
Example (Subsets of R^n)

How do we represent lines in $\mathbb{R}^3$?

A line in $\mathbb{R}^3$ is usually represented explicitly.

So what do we need?

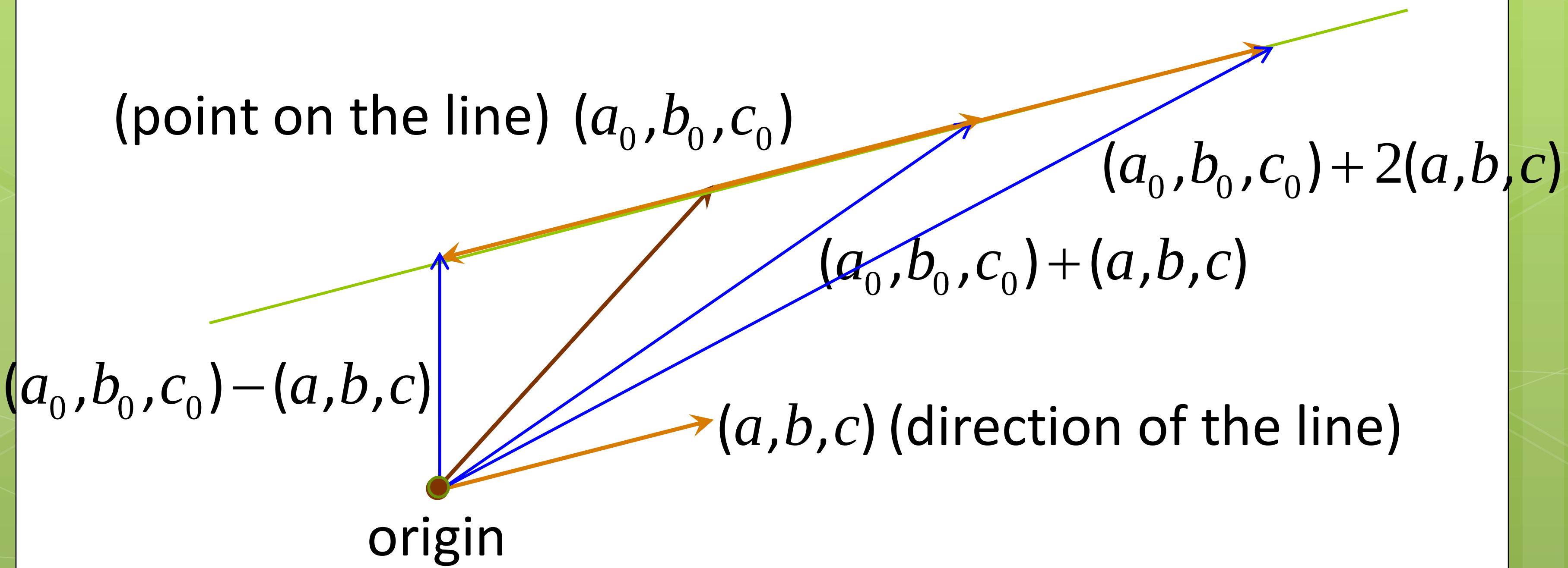
A point and a direction...or two points.



Example (Subsets of R^n)

A line in $\mathbb{R}^3$ is represented by the set

$$\{ \boxed{(a_0, b_0, c_0)} \; \boxed{+t(a,b,c)} \mid t \in \mathbb{R} \}$$



Example (Subsets of R^n)

A line in $\mathbb{R}^3$ is represented by the set

$$\{(a_0, b_0, c_0) + t(a, b, c) \mid t \in \mathbb{R}\} = \{(a_0 + ta, b_0 + tb, c_0 + tc) \mid t \in \mathbb{R}\}$$

Remember:

A line in $\mathbb{R}^3$ cannot be represented by a single linear equation like in $\mathbb{R}^2$.

Notation

S is a finite set. We use $|S|$ to denoted the number of elements in S.

$$S_1 = \{1,2,3\},\ S_2 = \{(1,2,3)\},\ S_3 = \{(1,2,3),(2,3,4)\}$$

$$|S_1| = 3, \quad |S_2| = 1, \quad |S_3| = 2$$

Definition (Linear Combination)

Consider $\boldsymbol{u}=(1,2,-1)$, $\boldsymbol{v}=(0,2,5)$.

$2\boldsymbol{u}+3\boldsymbol{v}=(2,10,13)$ $\qquad$ $\boldsymbol{u}-2\boldsymbol{v}=(1,-2,-11)$

$(2,10,13)$ and $(1,-2,-11)$ are both linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Let $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k$ be vectors in $\mathbb{R}^n$.

For any real numbers $c_1,c_2,\ldots,c_k$, the vector

$$c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k$$

is a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k$.

Example (Linear Combination)

Consider $\boldsymbol{u}=(1,2,-1)$, $\boldsymbol{v}=(0,2,5)$, $\boldsymbol{w}=(1,0,-2)$.

Question: Compute the linear combination $2\boldsymbol{u}+3\boldsymbol{v}-\boldsymbol{w}$

Answer: This is simple.

$$2\boldsymbol{u}+3\boldsymbol{v}-\boldsymbol{w}=$$

Example (Linear Combination)

Consider $\boldsymbol{u}=(1,2,-1)$, $\boldsymbol{v}=(0,2,5)$, $\boldsymbol{w}=(1,0,-2)$.

Question: Is $(0,4,8)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

Answer: We need to check whether there are real numbers a,b,c such that

$$a\boldsymbol{u}+b\boldsymbol{v}+c\boldsymbol{w}=(0,4,8)$$

$$a(1,2,-1)+b(0,2,5)+c(1,0,-2)=(0,4,8)$$

How to check?

Example (Linear Combination)

Consider $\boldsymbol{u}=(2,1,3),\ \boldsymbol{v}=(1,-1,2),\ \boldsymbol{w}=(3,0,5)$.

Question: Is $(3,3,4)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

$$a\mathbf{u}+b\mathbf{v}+c\mathbf{w}=(3,3,4)$$

$$a(2,1,3)+b(1,-1,2)+c(3,0,5)=(3,3,4)$$

$$\begin{cases} 2a & + & b & + & 3c & = & 3 \\ a & - & b & & & = & 3 \\ 3a & + & 2b & + & 5c & = & 4 \end{cases} \qquad \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array}\right)$$

Example (Linear Combination)

Consider $\boldsymbol{u} = (2,1,3),\ \boldsymbol{v} = (1,-1,2),\ \boldsymbol{w} = (3,0,5).$

Question: Is $(3,3,4)$ a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$?

$$a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = (3,3,4) \quad a(2,1,3) + b(1,-1,2) + c(3,0,5) = (3,3,4)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Is the linear system consistent? Are the solutions unique?

Example (Linear Combination)

Consider $\boldsymbol{u} = (2,1,3),\ \boldsymbol{v} = (1,-1,2),\ \boldsymbol{w} = (3,0,5).$

Question: Is $(3,3,4)$ a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$?

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = (3,3,4) \quad a(2,1,3) + b(1,-1,2) + c(3,0,5) = (3,3,4)$$

$$\begin{cases} a &= 2 - t \\ b &= -1 - t \\ c &= t \qquad t \in \mathbb{R} \end{cases}$$

$$(a,b,c) = (2,-1,0)$$

$$2(2,1,3) - (1,-1,2) + 0(3,0,5) = (3,3,4)$$

$$(a,b,c) = (1,-2,1)$$

$$(2,1,3) - 2(1,-1,2) + (3,0,5) = (3,3,4)$$

Example (Linear Combination)

Consider $\boldsymbol{u}=(2,1,3),\ \boldsymbol{v}=(1,-1,2),\ \boldsymbol{w}=(3,0,5)$.

Question: Is $(1,2,4)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

$$a\mathbf{u}+b\mathbf{v}+c\mathbf{w}=(1,2,4)$$

$$a(2,1,3)+b(1,-1,2)+c(3,0,5)=(1,2,4)$$

$$\begin{cases} 2a & + & b & + & 3c & = & 1 \\ a & - & b & & & = & 2 \\ 3a & + & 2b & + & 5c & = & 4 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 2 & 5 & 4 \end{array}\right)$$

Example (Linear Combination)

Consider $\boldsymbol{u} = (2,1,3),\ \boldsymbol{v} = (1,-1,2),\ \boldsymbol{w} = (3,0,5)$.

Question: Is $(1,2,4)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

No!

$$a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = (1,2,4) \quad a(2,1,3) + b(1,-1,2) + c(3,0,5) = (1,2,4)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 3 \end{array}\right)$$

Is the linear system consistent?

Example (Linear Combination)

Consider $\boldsymbol{e}_1 = (1,0,0,0)$, $\boldsymbol{e}_2 = (0,1,0,0)$, $\boldsymbol{e}_3 = (0,0,1,0)$, $\boldsymbol{e}_4 = (0,0,0,1)$

$(1,2,3,4) =$

$(-3, \frac{1}{3}, 0, 2) =$

Any (w, x, y, z) in $\mathbb{R}^4$:

Every vector $\boldsymbol{u} = (w, x, y, z)$ in $\mathbb{R}^4$ is a linear combination of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$.

Example (Linear Combination)

Consider $\boldsymbol{u} = (1,2,-1)$, $\boldsymbol{v} = (0,2,5)$, $\boldsymbol{w} = (1,0,-2)$.

Question: Is every vector in $\mathbb{R}^3$ a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$?

Discussion (Linear Combination)

Consider $\boldsymbol{u} = (1,2,-1)$, $\boldsymbol{v} = (0,2,5)$, $\boldsymbol{w} = (1,0,-2)$.

How many different linear combinations of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ are there?

Quite a bit...

What if I put ALL different linear combinations of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ into a set?

OMG...

Definition (Linear Span)

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

The set of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2 \ldots, \boldsymbol{u}_k$,

$$\{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

is called the linear span of S (or linear span of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$).

This set is denoted by span(S) or span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$.

Example

Consider $\boldsymbol{u}=(2,1,3),\ \boldsymbol{v}=(1,-1,2),\ \boldsymbol{w}=(3,0,5)$.

Question: Is $(3,3,4)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$? Yes!

So $(3,3,4)\in \text{span}\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}$

Question: Is $(1,2,4)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$? No!

So $(1,2,4)\notin \text{span}\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}$

Example

$S = \{(1,1,0),(2,-1,1)\}.$

$\text{span}(S) =$ set of all linear combinations of $(1,1,0)$ and $(2,-1,1)$

Every vector in $\text{span}(S)$ is of the form

$a(1,1,0) + b(2,-1,1)$ where a, b are any real numbers.

So $\text{span}(S) = \{a(1,1,0) + b(2,-1,1) \mid a, b \in \mathbb{R}\}$

Example

$V = \{(2a+b, a, 3b-a) \mid a, b \in \mathbb{R}\}$

V is a subset of $\mathbb{R}^3$. Can V be written as a linear span?

$$(2a+b, a, 3b-a)$$

$$= a(2,1,-1) + b(1,0,3)$$

So $V = \{a(2,1,-1) + b(1,0,3) \mid a, b \in \mathbb{R}\}$

$$= \text{span}\{(2,1,-1),(1,0,3)\}$$

Example

Show that $\text{span}\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.

We need to show that every vector in $\mathbb{R}^3$ can be written as a linear combination of $(1,0,1),(1,1,0),(0,1,1)$.

Show that $\text{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \neq \mathbb{R}^3$.

We need to show that there is some vector in $\mathbb{R}^3$ that cannot be written as a linear combination of $(1,1,1),(1,2,0),(2,1,3),(2,3,1)$.

End of Lecture 08

Lecture 09:

Linear combinations and linear spans (cont’d)

Subspaces (till end of Section 3.3)