LECTURE 12 RECAP

1) A theorem on the usefulness of knowing the dimension of a vector space.

2) Two more equivalent statements to "$\boldsymbol{A}$ is an invertible square matrix of order n."

3) Transition matrices: how to find and how to use.

4) Transition matrices are invertible and their inverses are also transition matrices.

LECTURE 13

ROW SPACES AND COLUMN SPACES

DEFINITION

Given any $m \times n$ matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \begin{pmatrix} \overbrace{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}}^{n} \\ a_{21} \quad a_{22} \quad \cdots \quad a_{2n} \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \\ a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn} \end{pmatrix}$$

$$\boldsymbol{r}_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$$
$$\boldsymbol{r}_2 = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$
$$\vdots$$
$$\boldsymbol{r}_m = \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The rows of $\boldsymbol{A}$ can be considered as vectors in $\mathbb{R}^n$.

$\Rightarrow$ span$\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m\}$ is a subspace of $\mathbb{R}^n$,

This subspace is called the row space of $\boldsymbol{A}$.

DEFINITION

Given any $m \times n$ matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Bigg\} m$$

The columns of $\boldsymbol{A}$ can be considered as vectors in $\mathbb{R}^m$.

$\Rightarrow$ span$\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n\}$ is a subspace of $\mathbb{R}^m$,

$$\boldsymbol{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \boldsymbol{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \quad \boldsymbol{c}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

This subspace is called the column space of $\boldsymbol{A}$.

REMARK

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & -2 \\ 1 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix}$$

The row space of $\boldsymbol{A}$ is the column space of $\boldsymbol{A}^T$

The column space of $\boldsymbol{A}$ is the row space of $\boldsymbol{A}^T$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$.

The column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^4$.

Note that if $\boldsymbol{A}$ is not a square matrix, then the row space and column space of $\boldsymbol{A}$ contains totally 'different type' of vectors.

EXAMPLE

$$\boldsymbol{A} = \begin{array}{cc} \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} & \begin{array}{c} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_3 \\ \boldsymbol{r}_4 \end{array} \end{array}$$

We write $\boldsymbol{r}_1 = (2,-1,0)$ (as a vector) rather than a row matrix $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$.

The row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$.

$$= \text{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4\}$$

$$= \{a(2,1,0) + b(1,-1,3) + c(-5,1,0) + d(1,0,1) \mid a,b,c,d \in \mathbb{R}\}$$

$$= \{(2a + b - 5c + d, -a - b + c, 3b + d) \mid a,b,c,d \in \mathbb{R}\}$$

EXAMPLE

$$\boldsymbol{A} = \begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ -5 \\ 1 \end{matrix}} & \boxed{\begin{matrix} -1 \\ -1 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 3 \\ 0 \\ 1 \end{matrix}} \end{pmatrix} \\ \begin{matrix} \quad \boldsymbol{c}_1 & \quad \boldsymbol{c}_2 & \quad \boldsymbol{c}_3 \end{matrix} \end{array}$$

The column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^4$.

$= \text{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$

$$= \left\{ a\begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \middle| \, a,b,c \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 2a-b \\ a-b+3c \\ -5a+b \\ a+c \end{pmatrix} \middle| \, a,b,c \in \mathbb{R} \right\}$$

NOTATION

We have observed that a vector in $\mathbb{R}^n$ can be identified as a row or matrix.

Henceforth, when we write

Identified with:

$(u_1, u_2, \ldots, u_n)$ → row vector $\begin{pmatrix} u_1 & u_2 & \ldots & u_n \end{pmatrix}$

$(u_1, u_2, \ldots, u_n)^{\boxed{T}}$ → column vector $\begin{pmatrix} u_1 \\ u_2 \\ : \\ u_n \end{pmatrix}$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space or column space of $\boldsymbol{A}$?

The row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^5$.

The column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$.

EXAMPLE

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space or column space of A?

row space of $A = \text{span}\{(1,0,-1,1,4),(0,1,4,2,1),(0,0,-2,0,1)\}$

If $(1,0,-1,1,4),(0,1,4,2,1),(0,0,-2,0,1)$ (that is, the rows of A) are linearly independent, then obviously they will form a basis for the row space of A.

EXAMPLE

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space or column space of A?

row space of $A = \text{span}\{(1,0,-1,1,4),(0,1,4,2,1),(0,0,-2,0,1)\}$

$$(0,0,0,0,0) = a(1,0,-1,1,4) + b(0,1,4,2,1) + c(0,0,-2,0,1)$$

$$\Rightarrow a = 0, b = 0, c = 0$$

So the three rows of A are linearly independent and thus form a basis for the row space of A.

EXAMPLE

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

Note that $\boldsymbol{A}$ is in row echelon form.

What if we want to find a basis for the row space of a matrix $\boldsymbol{R}$ that is in row echelon form?

$$\boldsymbol{R} = \begin{pmatrix} \otimes & * & & & & * \\ & & \otimes & * & & * \\ & \mathbf{0} & & \ddots & & * \\ & & & & \otimes \quad * & * \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \\ & & & & & \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{non zero rows} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{zero rows (if any)} \end{matrix}$$

EXAMPLE

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

Note that $\boldsymbol{A}$ is in row echelon form.

The non zero rows of $\boldsymbol{R}$ are always linearly independent and thus forms a basis for the row space of $\boldsymbol{R}$.

$$\boldsymbol{R} = \left(\begin{array}{cccccc} \otimes & * & & & & * \\ & & \otimes & * & & * \\ & 0 & & \ddots & & * \\ & & & & \otimes \quad * & * \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \\ & & & & & \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \text{non zero rows} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \text{zero rows (if any)} \end{array}$$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space or column space of $\boldsymbol{A}$?

$$\text{column space of } \boldsymbol{A} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \right\}$$

column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$,

$\Rightarrow$ the dimension of this subspace is at most 3

So if we can identify 3 linearly independent vectors (out of the 5) from the set above...

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space or column space of $\boldsymbol{A}$?

$$\text{column space of } \boldsymbol{A} = \text{span}\{\underbrace{\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \boxed{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \boxed{\begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}}}_{\text{linearly independent}}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}\}$$

linearly independent

column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$,

$\Rightarrow$ the dimension of this subspace is at most 3

Column space of $\boldsymbol{A}$ is the entire $\mathbb{R}^3$.

DISCUSSION

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space or column space of $\boldsymbol{A}$?

That was based on observations...

Yes, you are right. A more systematic approach is needed.

DISCUSSION

Let S be the set of all matrices of the same size (say $m \times n$).

Recall the definition of row equivalent.

$$\boldsymbol{A} \xleftrightarrow{\text{Series of e.r.o}} \boldsymbol{B}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

Row equivalence is an equivalence relation on S.

(Reflexive) $\boldsymbol{A}$ is row equivalent to $\boldsymbol{A}$

(Symmetric) If $\boldsymbol{A} \longleftrightarrow \boldsymbol{B}$, then $\boldsymbol{B} \longleftrightarrow \boldsymbol{A}$

(Transitive) If $\boldsymbol{A} \longleftrightarrow \boldsymbol{B}$ and $\boldsymbol{B} \longleftrightarrow \boldsymbol{C}$ then $\boldsymbol{A} \longleftrightarrow \boldsymbol{C}$

DISCUSSION

Let S be the set of all matrices of the same size (say $m \times n$).

Recall the definition of row equivalent.

$$\boldsymbol{A} \xleftrightarrow{\text{Series of e.r.o}} \boldsymbol{B}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ (of the same size) are row equivalent if and only if they have a similar row-echelon form (or they have the same unique reduced row-echelon form).

THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Then the row space of $\boldsymbol{A}$ and the row space of $\boldsymbol{B}$ are identical.

$$\boldsymbol{A} \xleftrightarrow{\text{Series of e.r.o}} \boldsymbol{B}$$

same row space

That is to say, performing elementary row operations on $\boldsymbol{A}$ does not change its row space.

EXAMPLE

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

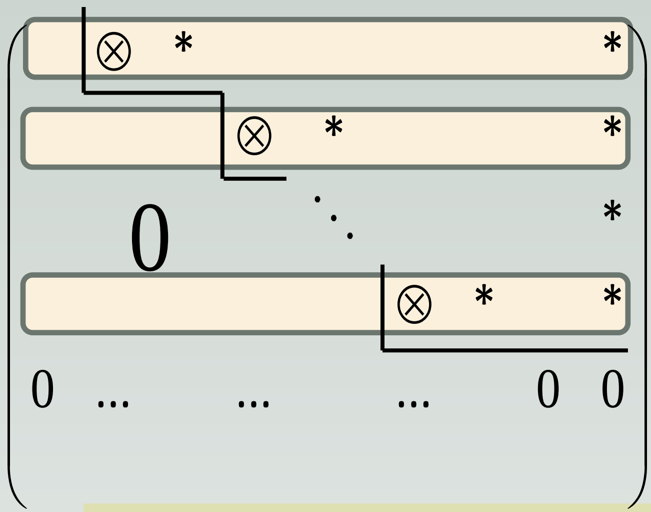
$$\begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} &\operatorname{span}\{(0,0,1),(0,2,4),(\tfrac{1}{2},1,2)\} \\ =\ &\operatorname{span}\{(\tfrac{1}{2},1,2),(0,2,4),(0,0,1)\} \\ =\ &\operatorname{span}\{(1,2,4),(0,2,4),(0,0,1)\} \\ =\ &\operatorname{span}\{(1,0,0),(0,2,4),(0,0,1)\} \end{aligned}$$

BACK TO THIS QUESTION

Question : How to find a basis for the row space of a matrix $\boldsymbol{A}$?

$\boldsymbol{R} =$



The non zero rows of $\boldsymbol{R}$ are always linearly independent and thus forms a basis for the row space of $\boldsymbol{R}$.

Answer :

Find a row-echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$.

A basis for the row space of $\boldsymbol{R}$ is also a basis for the row space of $\boldsymbol{A}$.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Then the row space of $\boldsymbol{A}$ and the row space of $\boldsymbol{B}$ are identical.

EXAMPLE

Find a basis for the row space of the following matrix.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} \boxed{2 \quad 2 \quad -1 \quad 0 \quad 1} \\ \boxed{0 \quad 0 \quad \tfrac{3}{2} \quad -3 \quad \tfrac{3}{2}} \\ \boxed{0 \quad 0 \quad 0 \quad 3 \quad 0} \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \end{pmatrix}$$

Performing Gaussian Elimination on $\boldsymbol{A}$:

A basis for the row space of $\boldsymbol{A}$ is

$$\{(2,2,-1,0,1),(0,0,\tfrac{3}{2},-3,\tfrac{3}{2}),(0,0,0,3,0)\}$$

FINDING BASIS FOR COLUMN SPACES

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

column space of A

$=$ row space of A^T

$$A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & -2 \\ 1 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix}$$

Ok, what about column spaces?

Note the relationship between column space of A and row space of A^T ...

FINDING BASIS FOR COLUMN SPACES

$$A = \begin{pmatrix} \boxed{1} & \boxed{0} & \boxed{-1} & \boxed{1} & \boxed{4} \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

column space of A $=$ row space of A^T

$$A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & -2 \\ 1 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix}$$

So to find a basis for the column space of A, we can use the previous method to find a basis for the row space of A^T.

In what follows, we will discuss another method.

IMPORTANT TO NOTE

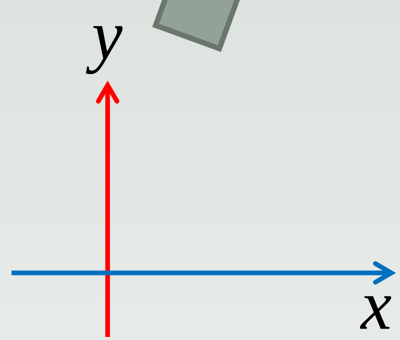
Elementary row operations preserve the row space of a matrix but NOT the column space.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \boldsymbol{B}$$

column space of $\boldsymbol{A}$ $\neq$ column space of $\boldsymbol{B}$

$= \text{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ $\qquad$ $= \text{span}\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$

$= \boxed{x - \text{axis}}$ $\qquad$ $= \boxed{y - \text{axis}}$



THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Then the following statements hold:

$$\boldsymbol{A} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & * \end{pmatrix} \xleftrightarrow[\text{e.r.o}]{\text{Series of}} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & * \end{pmatrix} = \boldsymbol{B}$$

A given set of columns of $\boldsymbol{A}$ is linearly independent if and only if the corresponding columns of $\boldsymbol{B}$ is linearly independent.

THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Then the following statements hold:

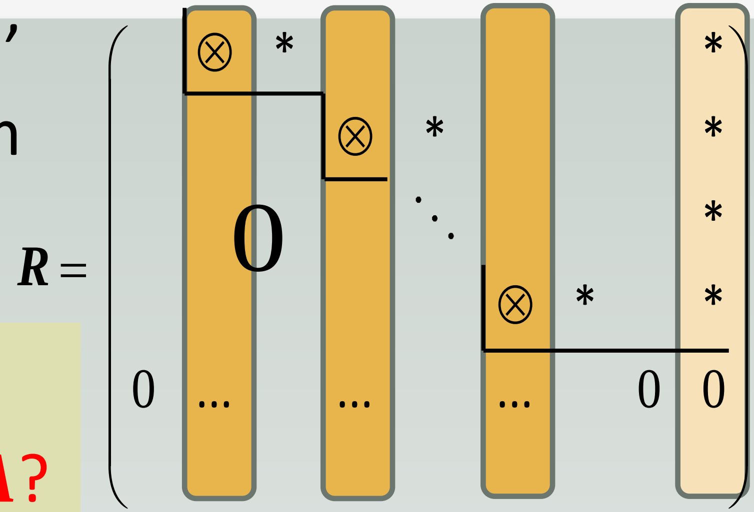
$$\boldsymbol{A} = \begin{pmatrix} \boxed{*} & * & \boxed{*} & * & \boxed{*} \\ \boxed{*} & * & \boxed{*} & * & \boxed{*} \\ \boxed{:} & : & \boxed{:} & : & \boxed{:} \\ \boxed{*} & * & \boxed{*} & * & \boxed{*} \end{pmatrix} \xrightarrow{\text{e.r.o}} \begin{pmatrix} \boxed{*} & * & \boxed{*} & * & \boxed{*} \\ \boxed{*} & * & \boxed{*} & * & \boxed{*} \\ \boxed{:} & : & \boxed{:} & : & \boxed{:} \\ \boxed{*} & * & \boxed{*} & * & \boxed{*} \end{pmatrix} = \boldsymbol{B}$$

Remember: Column space of $\boldsymbol{A} \neq$ Column space of $\boldsymbol{B}$

A given set of columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$ if and only if the corresponding columns of $\boldsymbol{B}$ forms a basis for the column space of $\boldsymbol{B}$.

OBSERVATION

If $\boldsymbol{R}$ is a matrix in row echelon form, the pivot columns of $\boldsymbol{R}$ always form a basis for the column space of $\boldsymbol{R}$.



Question: How to find a basis for the column space of a matrix $\boldsymbol{A}$?

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices.
A given set of columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$ if and only if the corresponding columns of $\boldsymbol{B}$ forms a basis for the column space of $\boldsymbol{B}$.

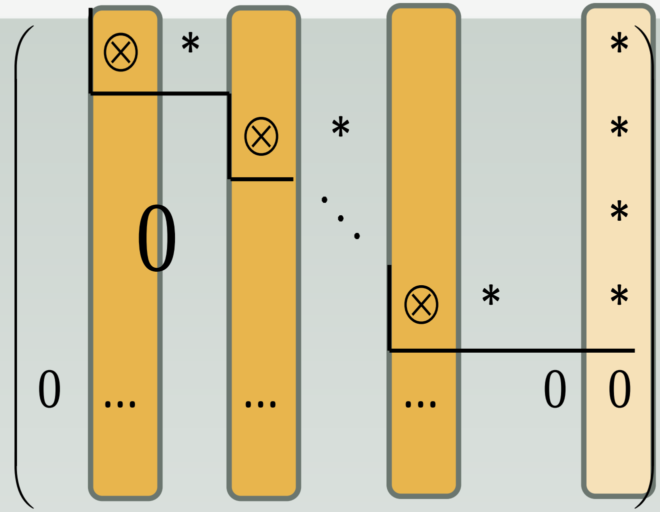
FINDING A BASIS FOR COLUMN SPACE

If $\boldsymbol{R}$ is a matrix in row echelon form,

the pivot columns of $\boldsymbol{R}$ always form

a basis for the column space of $\boldsymbol{R}$.

$\boldsymbol{R} =$



Question: How to find a basis for the column space of a matrix $\boldsymbol{A}$?

Answer

Let $\boldsymbol{R}$ be a row echelon form of $\boldsymbol{A}$.

Remember NOT to take the columns of $\boldsymbol{R}$ as your answer!

A basis for the column space of $\boldsymbol{A}$ can be obtained by taking the columns of $\boldsymbol{A}$ that correspond to the pivot columns in $\boldsymbol{R}$.

EXAMPLE

Find a basis for the column space of the following matrix.

$$\boldsymbol{A} = \begin{pmatrix} \boxed{2} & 2 & \boxed{-1} & \boxed{0} & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

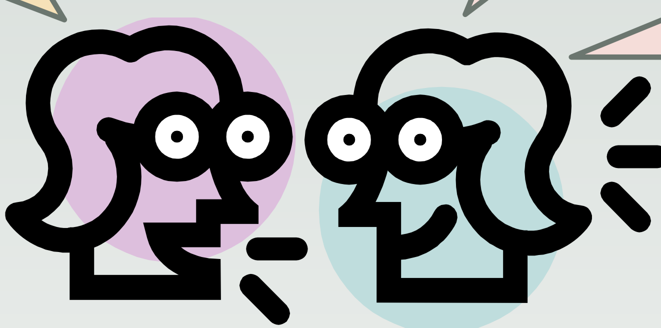
Performing Gaussian Elimination on $\boldsymbol{A}$:

A basis for the column space of $\boldsymbol{A}$ is $\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}$

APPLYING THIS KNOWLEDGE

Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$, $\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

Find a basis for V.



METHOD 1

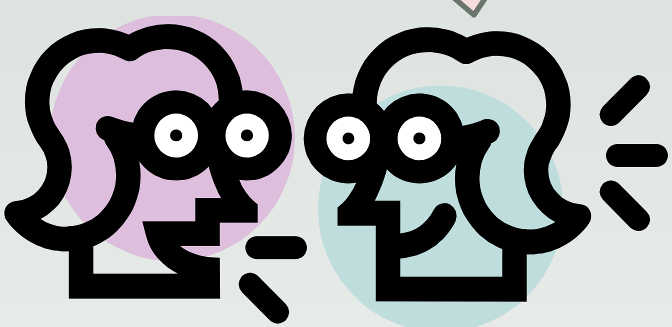
Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$, $\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

Find a basis for V.

same as "Find a basis for the row space of $\boldsymbol{A}$".

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 0 \\ -1 & 3 & 2 & -4 \\ 2 & 1 & 0 & 8 \\ 3 & 1 & -1 & 12 \end{pmatrix}$.

Put the vectors as rows. So V is the row space of the matrix $\boldsymbol{A}$.



METHOD 1

Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$,

$\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

Find a basis for V.

same as "Find a basis for the row space of $\boldsymbol{A}$".

$\{(1,2,0,4),(0,1,5,0),(0,0,-23,0)\}$ is a basis for V.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 0 \\ -1 & 3 & 2 & -4 \\ 2 & 1 & 0 & 8 \\ 3 & 1 & -1 & 12 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -23 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{R}$$

METHOD 2

Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$, $\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

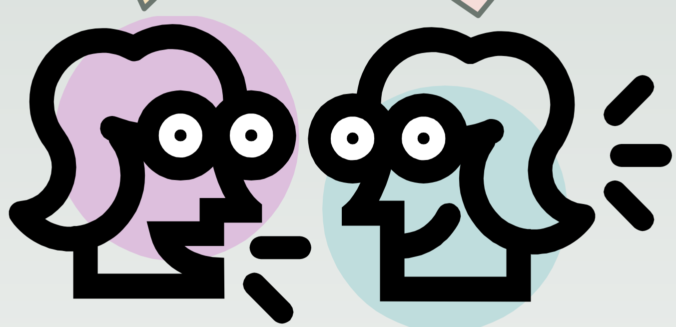
Find a basis for V.

same as "Find a basis for the column space of $\boldsymbol{B}$".

Note that $\boldsymbol{B} = \boldsymbol{A}^T$ ($\boldsymbol{A}$ from Method 1)

Put the vectors as columns. So V is the columns space of the matrix $\boldsymbol{B}$.

Let $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 2 & 1 & 3 & 1 & 1 \\ 0 & 5 & 2 & 0 & -1 \\ 4 & 0 & -4 & 8 & 12 \end{pmatrix}$.



METHOD 2

Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$,

$\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

Find a basis for V.

same as "Find a basis for the column space of $\boldsymbol{B}$".

$\{(1,2,0,4),(0,1,5,0),(-1,3,2,-4)\}$ is a basis for V.

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 2 & 1 & 3 & 1 & 1 \\ 0 & 5 & 2 & 0 & -1 \\ 4 & 0 & -4 & 8 & 12 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 5 & -3 & -5 \\ 0 & 0 & -23 & 15 & 24 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{R}$$

METHOD 1 VS METHOD 2

Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$,

$\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

Find a basis for V.

Method 1

$\{(1,2,0,4),(0,1,5,0),(0,0,-23,0)\}$

is a basis for V.

Method 2

$\{(1,2,0,4),(0,1,5,0),(-1,3,2,-4)\}$

is a basis for V.

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for V.

The basis found using Method 2 will always be a subset of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

EXAMPLE

$$S = \{(1,4,-2,5,1),(2,9,-1,8,2),(2,9,-1,9,3)\}.$$

1) Show that S is a linearly independent set.

2) Extend S to a basis for $\mathbb{R}^5$.

Here we mean add vectors to the set S so that the resulting set becomes a basis for $\mathbb{R}^5$.

How many vectors do we need to add?

COLUMN SPACE AND LINEAR SYSTEMS

$$\begin{cases} 2x - y = -1 \\ x - y + 3z = 4 \\ -5x + y = -2 \\ x + z = 3 \end{cases}$$

$$\Leftrightarrow \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \text{ where}$$

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow x\begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}$$

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}$$

COLUMN SPACE AND LINEAR SYSTEMS

$$\begin{cases} 2x - y = -1 \\ x - y + 3z = 4 \\ -5x + y = -2 \\ x + z = 3 \end{cases} \Leftrightarrow \boldsymbol{Ax} = \boldsymbol{b}$$

$$\Leftrightarrow x\begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}$$

(*)

$\boldsymbol{Ax} = \boldsymbol{b}$ is consistent means x, y, z can be found to satisfy (*)

$\Rightarrow$ $\boldsymbol{b}$ is a linear combination of the columns of $\boldsymbol{A}$.

That is, $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$.

COLUMN SPACE AND LINEAR SYSTEMS

$$\begin{cases} 2x - y = -1 \\ x - y + 3z = 4 \\ -5x + y = -2 \\ x + z = 3 \end{cases} \Leftrightarrow \boldsymbol{Ax} = \boldsymbol{b}$$

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(*)

$\Rightarrow \boldsymbol{Ax} = \boldsymbol{b}$ is consistent

$\Rightarrow x, y, z$ can be found to satisfy (*)

$\Rightarrow \boldsymbol{b}$ is a linear combination of the columns of $\boldsymbol{A}$.

$\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$.

THEOREM

Let $\boldsymbol{A}$ be a $m \times n$ matrix. Then the column space of $\boldsymbol{A}$ is

$$\left\{ \boldsymbol{A}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \middle| \, u_1, u_2, \ldots, u_n \in \mathbb{R} \right\} = \left\{ \boldsymbol{A}\boldsymbol{u} \,|\, \boldsymbol{u} \in \mathbb{R}^n \right\}$$

A system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ lies in the column space of $\boldsymbol{A}$.

END OF LECTURE 14

LECTURE 15: RANKS
NULLSPACE AND NULLITY(END OF CHAPTER 4)