LECTURE 05 RECAP

Definition of invertible matrices.

Uniqueness of inverse.

Some laws involving inverses.

Representing an elementary row operation by a matrix.

Definition of an elementary matrix.

Elementary matrices are invertible and their inverses are also elementary matrices.

LECTURE 06

Elementary matrices (cont'd)

Determinants

FROM CHAPTER 1

$$
\begin{tikzcd}
\boxed{\text{Linear system } X} \arrow[rr, leftrightarrow, "\text{same}", "\text{solution set}"'] \arrow[d] & & \boxed{\text{Linear system } Y} \arrow[d] \\
\boxed{\begin{array}{l}\text{Augmented matrix}\\ \text{of linear system } X\end{array}} \arrow[rr, leftrightarrow, "\text{row}", "\text{equivalent}"'] & & \boxed{\begin{array}{l}\text{Augmented matrix}\\ \text{of linear system } Y\end{array}}
\end{tikzcd}
$$

$$\left(\boldsymbol{A} \mid \boldsymbol{b}\right) \xrightarrow[\text{operations}]{\text{elementary row}} \left(\boldsymbol{C} \mid \boldsymbol{d}\right)$$

We want to show that $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Cx} = \boldsymbol{d}$ have the same solution set.

FROM CHAPTER 1

$$\begin{tikzcd} \text{Linear system } X \arrow[r, leftrightarrow, "\text{same}", "\text{solution set}"'] \arrow[d] & \text{Linear system } Y \arrow[d] \\ \text{Augmented matrix of linear system } X \arrow[r, leftrightarrow, "\text{row}", "\text{equivalent}"'] & \text{Augmented matrix of linear system } Y \end{tikzcd}$$

$$(\boldsymbol{A} \mid \boldsymbol{b}) \xrightarrow[\text{row operation}]{\text{one elementary}} (\boldsymbol{C} \mid \boldsymbol{d})$$

We want to show that $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Cx} = \boldsymbol{d}$ have the same solution set.

FROM CHAPTER 1

$$(\boldsymbol{A} \mid \boldsymbol{b}) \xrightarrow[\text{row operation}]{\text{one elementary}} (\boldsymbol{C} \mid \boldsymbol{d})$$

We want to show that $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Cx} = \boldsymbol{d}$ have the same solution set.

$\subseteq$

$\supseteq$

There exists an elementary matrix $\boldsymbol{E}$ such that

$$(\boldsymbol{C} \mid \boldsymbol{d}) = \boldsymbol{E}(\boldsymbol{A} \mid \boldsymbol{b}) \Rightarrow (\boxed{\boldsymbol{C}} \mid \boxed{\boldsymbol{d}}) = (\boxed{\boldsymbol{EA}} \mid \boxed{\boldsymbol{Eb}})$$

$\boldsymbol{C} = \boldsymbol{EA}$

and $\boldsymbol{d} = \boldsymbol{Eb}$

Let $\boldsymbol{u}$ be a solution to $\boldsymbol{Ax} = \boldsymbol{b}$

$$\boldsymbol{Au} = \boldsymbol{b} \Rightarrow \boldsymbol{EAu} = \boldsymbol{Eb} \Rightarrow \boldsymbol{Cu} = \boldsymbol{d}$$

$\Rightarrow \boldsymbol{u}$ is a solution to $\boldsymbol{Cx} = \boldsymbol{d}$

FROM CHAPTER 1

$$(A \mid b) \xrightarrow[\text{row operation}]{\text{one elementary}} (C \mid d)$$

We want to show that $Ax = b$ and $Cx = d$ have the same solution set.

$\subseteq$

$\supseteq$

There exists an elementary matrix E such that

$$(C \mid d) = E(A \mid b) \Rightarrow (\boxed{C} \mid \boxed{d}) = (\boxed{EA} \mid \boxed{Eb})$$

$C = EA$

and $d = Eb$

Let v be a solution to $Cx = d$

$$Cv = d \Rightarrow E^{-1}Cv = E^{-1}d \Rightarrow Av = b$$

$$\Rightarrow v \text{ is a solution to } Ax = b$$

$E^{-1}C = A$

and $E^{-1}d = b$

A PART OF A BIGGER THEOREM

If A is a square matrix, then the following statements are equivalent.

(If one of them is true, so are the rest. If one of them is false, so are the rest.)

1) A is invertible.

2) $Ax = \mathbf{0}$ has only the trivial solution.

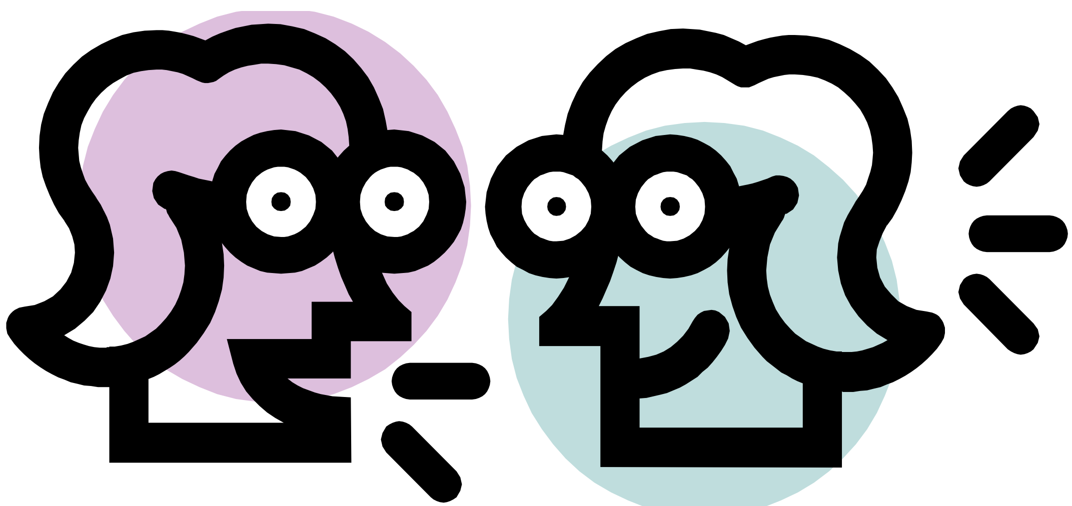
3) The reduced row-echelon form of A is I.

4) A can be expressed as a product of elementary matrices.

A VERY IMPORTANT QUESTION

So what if we know A is invertible? How can we find A^{-1}?

The proof of the previous theorem actually tells us how...



REMEMBER WHAT ELEMENTARY MATRICES CAN DO

Recall that for any matrix $\boldsymbol{A}$, there exists elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ such that

$$\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A}$$

is the reduced row-echelon form of $\boldsymbol{A}$.

If $\boldsymbol{A}$ is invertible, we know that

$$\boxed{\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}_n}$$

REMEMBER WHAT ELEMENTARY MATRICES CAN DO

$$\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}_n$$

By the '50%' remark in Lecture 05, since $(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1)$ and $\boldsymbol{A}$ are both square matrices of the same size, we can conclude that

$$\boxed{(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1) = \boldsymbol{A}^{-1}}$$

If $\boldsymbol{A}$ is a square matrix of order n, consider the following $n \times 2n$ matrix:

$$\left(\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{I}_n \end{array}\right)$$

REMEMBER WHAT ELEMENTARY MATRICES CAN DO

$\boxed{\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}_n}$ $\boxed{(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1) = \boldsymbol{A}^{-1}}$

What if we premultiply $(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1)$ to this $n \times 2n$ matrix?

$$\left(\boxed{\boldsymbol{A}} \;\middle|\; \boxed{\boldsymbol{I}_n} \right)$$

$$\begin{aligned}
&(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1)\left(\boldsymbol{A} \mid \boldsymbol{I}_n \right) \\
&= \left(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{I}_n \right) \\
&= \left(\boldsymbol{I}_n \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \right) \\
&= \left(\boldsymbol{I}_n \mid \boldsymbol{A}^{-1} \right)
\end{aligned}$$

REMEMBER WHAT ELEMENTARY MATRICES CAN DO

$$
\begin{aligned}
&(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1)\left(\boldsymbol{A} \mid \boldsymbol{I}_n\right) \\
&= \left(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{I}_n\right) \\
&= \left(\boldsymbol{I}_n \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1\right) \\
&= \left(\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}\right)
\end{aligned}
$$

This provides us with a way to find the inverse of an invertible matrix $\boldsymbol{A}$.

Question:

What happens if the matrix $\boldsymbol{A}$ is not invertible?

REMEMBER WHAT ELEMENTARY MATRICES CAN DO

$$(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1)(\boldsymbol{A} \mid \boldsymbol{I}_n)$$
$$= (\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{I}_n)$$

when $\boldsymbol{A}$ is singular

$$= (\boldsymbol{I}_n \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1)$$
$$= (\boldsymbol{I}_n \mid \boldsymbol{A}^{-1})$$

$$= (\boldsymbol{R} \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1)$$

Question:

What happens if the matrix $\boldsymbol{A}$ is not invertible?

Answer:

If $\boldsymbol{A}$ is singular, then its reduced row-echelon form will not be the identity matrix.

EXAMPLE (FINDING INVERSE)

Determine if the following matrix is invertible and if so, find its inverse.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

EXAMPLE (FINDING INVERSE)

Determine if the following matrix is invertible and if so, find its inverse.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 2 & 6 & 3 \\ 1 & -2 & -6 & -4 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

EXAMPLE (FINDING INVERSE)

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 6 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -6 & -4 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_3 - R_1 \\ R_4 - R_1}]{R_2 + R_1} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 7 & 4 & 1 & 1 & 0 & 0 \\ 0 & -3 & -7 & -5 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

EXAMPLE (FINDING INVERSE)

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 7 & 4 & 1 & 1 & 0 & 0 \\ 0 & -3 & -7 & -5 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + R_2} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 7 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

EXAMPLE (FINDING INVERSE)

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 7 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$$

row-echelon form has only 3 leading entries

$$\xrightarrow{R_4 - R_3} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 7 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right)$$

The reduced row-echelon form of $\boldsymbol{A}$ can never be $\boldsymbol{I}_4$, so $\boldsymbol{A}$ is singular.

GENERAL 2X2 MATRICES

We have already seen (from Lecture 5) that:

If $ad - bc \neq 0$, then $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible

and $\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$.

We will now show:

If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

GENERAL 2X2 MATRICES

If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

Case 1: If $a = 0$, and $c = 0$.

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

This matrix will not have $\boldsymbol{I}_2$ as its reduced row-echelon form and so is not invertible.

So we do not need to consider this case, since the hypothesis "If $\boldsymbol{A}$ is invertible" is not satisfied.

GENERAL 2X2 MATRICES

If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

Case 2: $a \neq 0$ or $c \neq 0$. First suppose $a \neq 0$.

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2 - \frac{c}{a}R_1} \begin{pmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{pmatrix}$$

So if $\boldsymbol{A}$ is invertible, we must have two leading entries and thus $\frac{ad-bc}{a} \neq 0$ (that is, $ad - bc \neq 0$).

GENERAL 2X2 MATRICES

If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

Case 2: $a \neq 0$ or $c \neq 0$. Now suppose $a = 0, c \neq 0.$

$$\boldsymbol{A} = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$

So if $\boldsymbol{A}$ is invertible, we must have two leading entries and thus $b \neq 0.$

For this case, this implies $ad - bc \neq 0$.

GENERAL 2X2 MATRICES

If $ad - bc \neq 0$, then $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible

and $\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$.

If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.

THE '50% THEOREM'

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same size.

If $\boldsymbol{AB} = \boldsymbol{I}$, then

$$\boldsymbol{BA} = \boldsymbol{I} \qquad \boldsymbol{A} = \boldsymbol{B}^{-1} \qquad \boldsymbol{B} = \boldsymbol{A}^{-1}$$

Proof:

Consider the homogeneous linear system $\boldsymbol{Bx} = \mathbf{0}$.

Strategy: If we can show $\boldsymbol{Bx} = \mathbf{0}$ has only the trivial solution, then $\boldsymbol{B}$ is invertible.

THE '50% THEOREM'

If $\boldsymbol{AB} = \boldsymbol{I}$, then

$$\boldsymbol{BA} = \boldsymbol{I} \qquad \boldsymbol{A} = \boldsymbol{B}^{-1} \qquad \boldsymbol{B} = \boldsymbol{A}^{-1}$$

Proof:

Consider the homogeneous linear system $\boldsymbol{Bx} = \boldsymbol{0}.$

Let $\boldsymbol{u}$ be a solution to $\boldsymbol{Bx} = \boldsymbol{0}.$

$$\boldsymbol{Bu} = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{ABu} = \boldsymbol{A0}$$

$$\Rightarrow \boldsymbol{Iu} = \boldsymbol{0} \Rightarrow \boldsymbol{u} = \boldsymbol{0}$$

Strategy: If we can show $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution, then $\boldsymbol{B}$ is invertible.

So $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{u} = \boldsymbol{0}$ and thus $\boldsymbol{B}$ is invertible (that is, $\boldsymbol{B}^{-1}$ exists).

THE '50% THEOREM'

If $\boldsymbol{AB} = \boldsymbol{I}$, then

$$\boxed{\boldsymbol{BA} = \boldsymbol{I}} \qquad \boxed{\boldsymbol{A} = \boldsymbol{B}^{-1}} \qquad \boxed{\boldsymbol{B} = \boldsymbol{A}^{-1}}$$

Proof:

So $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{u} = \boldsymbol{0}$ and

thus $\boldsymbol{B}$ is invertible (that is, $\boldsymbol{B}^{-1}$ exists).

$$\boldsymbol{AB} = \boldsymbol{I} \Rightarrow \boldsymbol{ABB}^{-1} = \boldsymbol{IB}^{-1} \Rightarrow \boldsymbol{AI} = \boldsymbol{B}^{-1} \Rightarrow \boxed{\boldsymbol{A} = \boldsymbol{B}^{-1}}$$

Since $\boldsymbol{A} = \boldsymbol{B}^{-1}$, $\boldsymbol{A}$ is invertible and

$$\boxed{\boldsymbol{A}^{-1} = (\boldsymbol{B}^{-1})^{-1} = \boldsymbol{B}}$$

Finally,

$$\boxed{\boldsymbol{BA} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}}$$

EXAMPLE

If $\boldsymbol{A}$ is a square matrix such that

$$\boldsymbol{A}^2 - 6\boldsymbol{A} + 8\boldsymbol{I} = \mathbf{0},$$

prove that $\boldsymbol{A}$ is invertible.

$$\boldsymbol{A}^2 - 6\boldsymbol{A} + 8\boldsymbol{I} = \mathbf{0} \Rightarrow \boldsymbol{A}^2 - 6\boldsymbol{A} = -8\boldsymbol{I}$$

$$\Rightarrow \boldsymbol{A}(\boldsymbol{A} - 6) = -8\boldsymbol{I}$$

something wrong??

$$\Rightarrow \boldsymbol{A}(\boldsymbol{A} - 6\boldsymbol{I}) = -8\boldsymbol{I}$$

$$\boldsymbol{A}^{-1} = \boxed{[-\tfrac{1}{8}(\boldsymbol{A} - 6\boldsymbol{I})]} \qquad \Rightarrow \boldsymbol{A}\boxed{[-\tfrac{1}{8}(\boldsymbol{A} - 6\boldsymbol{I})]} = \boldsymbol{I}$$

WHAT IF ONE MATRIX IS SINGULAR?

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both square matrices of the same size and $\boldsymbol{B}$ is singular, then

$$\boldsymbol{AB} \text{ and } \boldsymbol{BA} \text{ are both singular.}$$

POST MULTIPLICATION OF ELEMENTARY MATRICES

$$\begin{tikzcd} & & E_1 \\ I_m \arrow[urr, "cR_i"] \arrow[rr, "R_i \leftrightarrow R_j"] \arrow[drr, "\boxed{R_j + kR_i}"'] & & E_2 \\ & & E_3 \end{tikzcd}$$

$\boldsymbol{A}: p \times m$ matrix

$\boldsymbol{E}_i : m \times m$ elementary matrix

$\boldsymbol{AE}_1$: matrix resulting from multiplying

column i of $\boldsymbol{A}$ by c

$\boldsymbol{AE}_2$: matrix resulting from interchanging

columns i and j of $\boldsymbol{A}$

$\boldsymbol{AE}_3$: matrix resulting from adding k times of

column j of $\boldsymbol{A}$ to column i

Note that 'reversal' of role of i and j

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -1 & 2 \\ 0 & 3 & -2 & 3 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

2 times column 3

$$\boldsymbol{AE}_1 = \begin{pmatrix} 1 & 4 & \boxed{-2} & 2 \\ 0 & 3 & \boxed{-4} & 3 \\ 1 & 1 & \boxed{0} & 0 \end{pmatrix}$$

$$\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (2R_3)$$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -1 & 2 \\ 0 & 3 & -2 & 3 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

interchange

columns 1 and 4

$$\boldsymbol{AE}_2 = \begin{pmatrix} \boxed{2} & 4 & -1 & \boxed{1} \\ \boxed{3} & 3 & -2 & \boxed{0} \\ \boxed{0} & 1 & 0 & \boxed{1} \end{pmatrix}$$

$$\boldsymbol{E}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} (R_1 \leftrightarrow R_4)$$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -1 & 2 \\ 0 & 3 & -2 & 3 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

column 3 minus 2

times column 2

$$\boldsymbol{AE}_3 = \begin{pmatrix} 1 & 4 & \boxed{-9} & 2 \\ 0 & 3 & \boxed{-8} & 3 \\ 1 & 1 & \boxed{-2} & 0 \end{pmatrix}$$

$$\boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (R_2 - 2R_3)$$

Remember for 2x2 matrices…

Recall that we have have shown

$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$.

The quantity $ad - bc$ is known as the <u>determinant</u> of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

DEFINITION (DETERMINANTS)

Let $\boldsymbol{A} = (a_{ij})$ be a square matrix of order n.

Let $\boldsymbol{M}_{ij}$ be a square matrix of order $n-1$ obtained by removing the ith row and jth column of $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{pmatrix} & & & \\ & & 3 & 1 \\ & & & \\ & & 1 & -1 \end{pmatrix}$$

$$\boldsymbol{M}_{11} = \begin{pmatrix} -1 & 3 & 1 \\ -4 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\boldsymbol{M}_{32} = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

DEFINITION (DETERMINANTS)

Let $\boldsymbol{A} = (a_{ij})$ be a square matrix of order n.

Let $\boldsymbol{M}_{ij}$ be a square matrix of order $n-1$ obtained by removing the ith row and jth column of $\boldsymbol{A}$.

The determinant of $\boldsymbol{A}$ is defined as

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} & \text{if } n \geq 2 \end{cases}$$

where $A_{ij} = (-1)^{i+j} \det(\boldsymbol{M}_{ij})$.

$A_{ij} = (-1)^{i+j} \det(\boldsymbol{M}_{ij})$ is called the (i, j)-cofactor of $\boldsymbol{A}$.

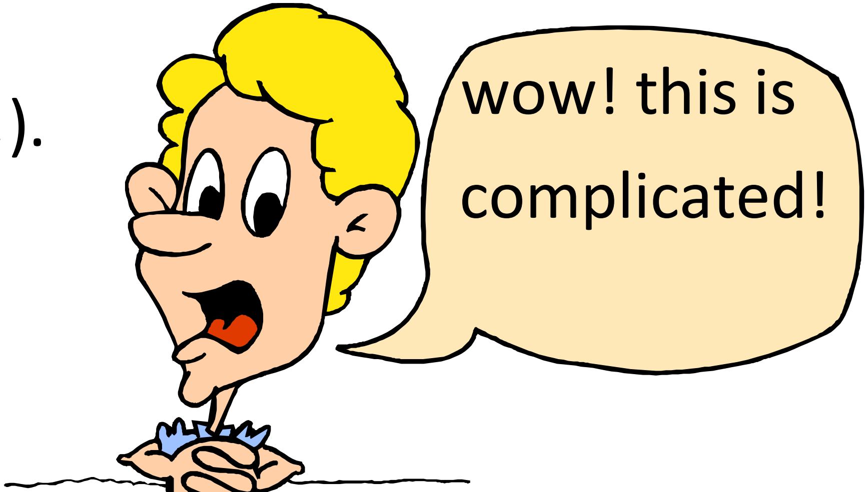
DEFINITION (DETERMINANTS)

To know the determinant of a $n \times n$ matrix, we need to know the determinants of $(n-1)\times(n-1)$ matrices...

To know the determinant of a $(n-1)\times(n-1)$ matrix, we need to know the determinants of $(n-2)\times(n-2)$ matrices...

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} & \text{if } n \geq 2 \end{cases}$$

where $A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij})$.



DEFINITION (DETERMINANTS)

The determinant of $\boldsymbol{A}$ is defined as

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} & \text{if } n \geq 2 \end{cases}$$

where $A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij})$.

$A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij})$ is called the (i, j)-cofactor of $\boldsymbol{A}$.

This is known as cofactor expansion.

NOTATION (DETERMINANTS)

The determinant of $\boldsymbol{A} = (a_{ij})$ is usually written as

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

DETERMINANT OF A 2X2 MATRIX

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\boldsymbol{M}_{11} = (d) \qquad A_{11} = (-1)^{1+1}\det(d) = d$$

$$\boldsymbol{M}_{12} = (c) \qquad A_{12} = (-1)^{1+2}\det(c) = -c$$

$$\det(\boldsymbol{A}) = aA_{11} + bA_{12} = \boxed{ad - bc}$$

We have seen this expression before!

If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

EXAMPLE (COFACTOR EXPANSION)

Evaluate $\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix}$.

By cofactor expansion,

$$\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = (-1)\cdot(-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 2 & -9 \end{vmatrix} + (3)\cdot(-1)^{1+2}\begin{vmatrix} 2 & 1 \\ -4 & -9 \end{vmatrix}$$

$$+(-4)\cdot(-1)^{1+3}\begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix}$$

EXAMPLE (COFACTOR EXPANSION)

By cofactor expansion,

$$\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = (-1)\cdot(-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 2 & -9 \end{vmatrix} + (3)\cdot(-1)^{1+2}\begin{vmatrix} 2 & 1 \\ -4 & -9 \end{vmatrix}$$

$$+(-4)\cdot(-1)^{1+3}\begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix}$$

$$= -(4\times -9 - 1\times 2) - 3(2\times -9 - 1\times -4)$$

$$-4(2\times 2 - 4\times -4)$$

$$= 0.$$

DETERMINANT OF A 3X3 MATRIX

What is the determinant of the following matrix?

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Answer:

$$\begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

$$\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$$

Verify this expression using cofactor expansion!

DETERMINANT OF A 4X4 MATRIX

What is the determinant of the following matrix?

$$\boldsymbol{A} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

Answer:

No 'special formula'! Use cofactor expansion!

A VERY USEFUL FACT

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} & \text{if } n \geq 2 \end{cases}$$

This is actually performing cofactor expansion along the first row of $\boldsymbol{A}$.

A VERY USEFUL FACT

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

It turns out that we can compute det($\boldsymbol{A}$) by performing cofactor expansion along any row or any column of $\boldsymbol{A}$.

cofactor expansion

$$\det(\boldsymbol{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} \quad \text{along } i\text{th row}$$

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj} \quad \text{along } j\text{th column}$$

EXAMPLE (COFACTOR EXPANSION)

Check that $\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = 0$ by cofactor expansion along second row.

$$\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = 2\cdot(-1)^{2+1}\begin{vmatrix} 3 & -4 \\ 2 & -9 \end{vmatrix} + 4\cdot(-1)^{2+2}\begin{vmatrix} -1 & -4 \\ -4 & -9 \end{vmatrix}$$

$$+1\cdot(-1)^{2+3}\begin{vmatrix} -1 & 3 \\ -4 & 2 \end{vmatrix}$$

EXAMPLE (COFACTOR EXPANSION)

$$\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = 2\cdot(-1)^{2+1}\begin{vmatrix} 3 & -4 \\ 2 & -9 \end{vmatrix} + 4\cdot(-1)^{2+2}\begin{vmatrix} -1 & -4 \\ -4 & -9 \end{vmatrix}$$

$$+1\cdot(-1)^{2+3}\begin{vmatrix} -1 & 3 \\ -4 & 2 \end{vmatrix}$$

$$= -2(-27+8)+4(9-16)-1(-2+12)$$

$$= 38-28-10$$

$$= 0$$

Try cofactor expansion along another row or column!

THEOREM (DETERMINANT OF TRIANGULAR MATRICES)

If $\boldsymbol{A}$ is a triangular matrix, then $\det(\boldsymbol{A})$ is the product of its diagonal entries.

$$\boldsymbol{A} = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 3 & & \\ & \text{\huge 0} & & 0 & \\ & & & & 3 \end{pmatrix} \qquad \boldsymbol{A} = \begin{pmatrix} 2 & & & \text{\huge 0} & \\ & 3 & & & \\ & & \frac{1}{2} & & \\ & & & -1 & \\ & & & & 4 \end{pmatrix}$$

$$\det(\boldsymbol{A}) = 0 \qquad \det(\boldsymbol{A}) = 2 \cdot 3 \cdot \tfrac{1}{2} \cdot -1 \cdot 4 = -12$$

Proof is omitted.

THEOREM (DETERMINANT OF TRANSPOSE)

If $\boldsymbol{A}$ is a square matrix, then

$$\boxed{\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T).}$$

Proof is omitted.

$$\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = 0 \qquad \begin{vmatrix} -1 & 2 & -4 \\ 3 & 4 & 2 \\ -4 & 1 & -9 \end{vmatrix} = 0$$

Theorem (Matrices with Identical Rows or Columns)

The determinant of a square matrix with two identical rows is 0.

The determinant of a square matrix with two identical columns is 0.

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 0 & 1 & -1 \\ 2 & -3 & 2 & 3 \\ 4 & 1 & 4 & 3 \\ -1 & 2 & -1 & 0 \end{vmatrix} = 0$$

Proof is omitted.

WHAT ABOUT THE PROOFS?

The three theorems we have stated without proofs can all be proven using a technique known as Mathematical Induction.

If you are interested, you are encouraged to figure out the proofs.

END OF LECTURE 06

Lecture 07:

Determinants (till end of Chapter 2)