LECTURE 14 RECAP

1) A theorem leading to the definition of the rank of a matrix.

2) What is a full rank matrix? The rank of a matrix and its transpose. (Invertibility and full rank).

3) Consistency of $\boldsymbol{Ax}=\boldsymbol{b}$ and its relation to rank.

4) The nullspace and nullity of a matrix (dimension Theorem).

5) The solution set of $\boldsymbol{Ax}=\boldsymbol{b}$ in terms of the nullspace of $\boldsymbol{A}$ and a particular solution.

LECTURE 15

Eigenvalues and eigenvectors

A REAL LIFE EXAMPLE

Movement of people between rural and urban district:

1%

4%

rural

urban

Assume: Total population is a constant.

Question: What is going to happen in the long run?

A REAL LIFE EXAMPLE

Movement of people between rural and urban district:

1%

4%

rural

urban

Meaning of a_0 and b_0

Let b_n be the rural population after n years.

Let a_n be the urban population after n years.

A REAL LIFE EXAMPLE

1%

4%

rural (b_n)

urban (a_n)

$$a_n = \boxed{0.96}\, a_{n-1} + \boxed{0.01}\, b_{n-1}$$

$$b_n = \boxed{0.04}\, a_{n-1} + \boxed{0.99}\, b_{n-1}$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ Population distribution after n years

$\boldsymbol{x}_{n-1} = \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$ $(n-1)$

A REAL LIFE EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} \boxed{0.96} & \boxed{0.01} \\ 0.04 & 0.99 \end{pmatrix}$

$$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Initial population distribution

Then $\boldsymbol{x}_n = \boldsymbol{A}\boldsymbol{x}_{n-1} = \boldsymbol{A}^2\boldsymbol{x}_{n-2} = \boldsymbol{A}^3\boldsymbol{x}_{n-3}$

$$\boldsymbol{x}_{n-1} = \boldsymbol{A}\boldsymbol{x}_{n-2}$$

$$\boldsymbol{x}_{n-2} = \boldsymbol{A}\boldsymbol{x}_{n-3}$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$

Population distribution after n years

$$a_n = 0.96\, a_{n-1} + 0.01\, b_{n-1}$$

$$b_n = 0.04\, a_{n-1} + 0.99\, b_{n-1}$$

$$\boldsymbol{x}_{n-1} = \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$$

A REAL LIFE EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$

$$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Initial population distribution

What happens after 1000 years?

First I have to compute $\boldsymbol{A}^{1000}$!

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$

$\boldsymbol{x}_{n-1} = \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$

Population distribution after n years

A REAL LIFE EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$

$$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Suppose somebody tells you that

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.95 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}^{-1}$$

$$= \boldsymbol{PDP}^{-1}$$

($\boldsymbol{P}$ is invertible,

$\boldsymbol{D}$ is diagonal)

$$\boldsymbol{A}^n = (\boldsymbol{PDP}^{-1})^n$$

$$= (\boldsymbol{PDP}^{-1})(\boldsymbol{PDP}^{-1})\ldots(\boldsymbol{PDP}^{-1})$$

$$= (\boldsymbol{PD}\underbrace{\boldsymbol{P}^{-1}\boldsymbol{P}}\boldsymbol{D}\underbrace{\boldsymbol{P}^{-1}\ldots\boldsymbol{P}}\boldsymbol{DP}^{-1})$$

$$= (\boldsymbol{PDD}\ldots\boldsymbol{DP}^{-1})$$

$$= (\boldsymbol{PD}^n\boldsymbol{P}^{-1})$$

What is $\boldsymbol{D}^{1000}$?

A REAL LIFE EXAMPLE

Which was the crucial step?

Let $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$

$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$

So $\lim_{n\to\infty} \boldsymbol{A}^n = \boldsymbol{P}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\boldsymbol{P}^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{pmatrix}$$

$$\lim_{n\to\infty}\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \lim_{n\to\infty} \boldsymbol{x}_n = \lim_{n\to\infty} \boldsymbol{A}^n \boldsymbol{x}_0$$

$$= \begin{pmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{pmatrix}\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

$a_n = 0.2(a_0 + b_0)$ 20% will stay in urban

$b_n = 0.8(a_0 + b_0)$ 80% will stay in rural

A REAL LIFE EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$

$$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Suppose somebody tells you that

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}} \boxed{\begin{pmatrix} 1 & 0 \\ 0 & 0.95 \end{pmatrix}} \boxed{\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}^{-1}}$$

$$= \boldsymbol{PDP}^{-1}$$

($\boldsymbol{P}$ is invertible,
$\boldsymbol{D}$ is diagonal)

$$\begin{aligned} \boldsymbol{A}^n &= (\boldsymbol{PDP}^{-1})^n \\ &= (\boldsymbol{PDP}^{-1})(\boldsymbol{PDP}^{-1})\ldots(\boldsymbol{PDP}^{-1}) \\ &= (\boldsymbol{PD}\underbrace{\boldsymbol{P}^{-1}\boldsymbol{P}}\boldsymbol{D}\underbrace{\boldsymbol{P}^{-1}\ldots\boldsymbol{P}}\boldsymbol{DP}^{-1}) \\ &= (\boldsymbol{PDD}\ldots\boldsymbol{DP}^{-1}) \\ &= (\boldsymbol{PD}^n\boldsymbol{P}^{-1}) \end{aligned}$$

What is $\boldsymbol{D}^{1000}$?

DEFINITION

Let $\boldsymbol{A}$ be a square matrix of order n.

A nonzero column vector $\boldsymbol{u} \in \mathbb{R}^n$ is called an eigenvector of $\boldsymbol{A}$ if

$$\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u} \quad \text{for some scalar } \lambda.$$

'multiplying $\boldsymbol{A}$ to $\boldsymbol{u}$ results in some scalar multiple of $\boldsymbol{u}$.'

The scalar λ is called an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{u}$ is said to be an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ.

A QUESTION BEFORE WE PROCEED

If $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ,

what about vectors like $2\boldsymbol{u}$, $(-1.5)\boldsymbol{u}$, $300\boldsymbol{u}$?

EXAMPLES (EIGENVALUES/EIGENVECTORS)

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\boldsymbol{Ax} =$

1 is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ associated with eigenvalue 1.

$\boldsymbol{Ay} =$

0.95 is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{y}$ is an eigenvector of $\boldsymbol{A}$ associated with eigenvalue 0.95.

EXAMPLES (EIGENVALUES/EIGENVECTORS)

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{B}\boldsymbol{x} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3\boldsymbol{x}$$

3 is an eigenvalue of $\boldsymbol{B}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{B}$ associated with eigenvalue 3.

EXAMPLES (EIGENVALUES/EIGENVECTORS)

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{By} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\boldsymbol{y}$$

0 is an eigenvalue of $\boldsymbol{B}$ and $\boldsymbol{y}$ is an eigenvector of $\boldsymbol{B}$ associated with eigenvalue 0.

$$\boldsymbol{Bz} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\boldsymbol{z}$$

$\boldsymbol{z}$ is an eigenvector of $\boldsymbol{B}$ associated with eigenvalue 0.

EXAMPLES (EIGENVALUES/EIGENVECTORS)

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

associated with eigenvalue 3.

associated with eigenvalue 0.

Observe that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

HOW TO FIND ALL EIGENVALUES OF A MATRIX

Let $\boldsymbol{A}$ be a square matrix of order n.

λ is an eigenvalue of $\boldsymbol{A}$

$\Leftrightarrow \boldsymbol{Au} = \lambda \boldsymbol{u}$ for some non zero column vector $\boldsymbol{u}$ $(\in \mathbb{R}^n)$

$\Leftrightarrow \lambda \boldsymbol{u} - \boldsymbol{Au} = \boldsymbol{0}$ for some non zero column vector $\boldsymbol{u}$ $(\in \mathbb{R}^n)$

$\Leftrightarrow (\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{u} = \boldsymbol{0}$ for some non zero column vector $\boldsymbol{u}$ $(\in \mathbb{R}^n)$

$\Leftrightarrow (\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$ has non trivial solutions

$\Leftrightarrow \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ (that is, $\lambda \boldsymbol{I} - \boldsymbol{A}$ is singular)

HOW TO FIND ALL EIGENVALUES OF A MATRIX

Let $\boldsymbol{A}$ be a square matrix of order $n.$

So the eigenvalues of $\boldsymbol{A}$ are all the numbers λ that makes the matrix $(\lambda \boldsymbol{I} - \boldsymbol{A})$ singular.

λ is an eigenvalue of $\boldsymbol{A}$

$\Leftrightarrow \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ (that is, $\lambda \boldsymbol{I} - \boldsymbol{A}$ is singular)

Note that if expanded (by cofactor expansion), $\det(\lambda \boldsymbol{I} - \boldsymbol{A})$ is a polynomial in λ of degree $n.$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0$$

DEFINITION

Let $\boldsymbol{A}$ be a square matrix of order n.

The polynomial $\det(\lambda \boldsymbol{I} - \boldsymbol{A})$ is called the characteristic polynomial of $\boldsymbol{A}$; and

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0 \longleftarrow \text{characteristic polynomial}$$

the equation $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ is called the characteristic equation of $\boldsymbol{A}$.

$$c_n\lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0 = 0 \longleftarrow \text{characteristic equation}$$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} \qquad \lambda \boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - 0.96 & -0.01 \\ -0.04 & \lambda - 0.99 \end{pmatrix}$$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = (\lambda - 0.96)(\lambda - 0.99) - (-0.01)(-0.04)$$

$$= \lambda^2 - 1.95\lambda + 0.95$$

$$= (\lambda - 1)(\lambda - 0.95)$$

$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \Leftrightarrow \lambda = 1$ or $\lambda = 0.95$.

The eigenvalues of $\boldsymbol{A}$ are 1 and 0.95.

EXAMPLE

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \lambda \boldsymbol{I} - \boldsymbol{B} = \begin{pmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{pmatrix}$$

$\det(\lambda \boldsymbol{I} - \boldsymbol{B}) = \lambda^3 - 3\lambda^2$ (some hardwork required!)

$= \lambda^2(\lambda - 3)$

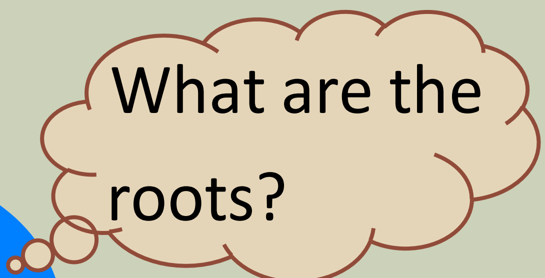
$\det(\lambda \boldsymbol{I} - \boldsymbol{B}) = 0 \Leftrightarrow \lambda = 0$ or λ=3.

The eigenvalues of $\boldsymbol{B}$ are 0 and 3.

EXAMPLE

$$\boldsymbol{C} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \qquad \lambda \boldsymbol{I} - \boldsymbol{C} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & -2 \\ -1 & -1 & \lambda - 1 \end{pmatrix}$$

$\det(\lambda \boldsymbol{I} - \boldsymbol{C}) = \lambda^3 - \lambda^2 - 2\lambda + 2$ (some hardwork required!)



Try $\lambda = -2, -1, 0, 1, 2$

$$\lambda = 1:\ 1^3 - 1^2 - 2(1) + 2 = 0$$

So $\lambda = 1$ is a root of the characteristic equation

EXAMPLE

$$\boldsymbol{C}=\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 2\\ 1 & 1 & 1\end{pmatrix} \qquad \lambda\boldsymbol{I}-\boldsymbol{C}=\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & -2\\ -1 & -1 & \lambda-1\end{pmatrix}$$

$\det(\lambda\boldsymbol{I}-\boldsymbol{C})=\lambda^3-\lambda^2-2\lambda+2$ (some hardwork required!)

$=(\lambda-1)\boxed{(\lambda^2-2)}$ obtained from long division

$=(\lambda-1)(\lambda-\sqrt{2})(\lambda+\sqrt{2})$

$\det(\lambda\boldsymbol{I}-\boldsymbol{C})=0 \Leftrightarrow \lambda=1,\ \sqrt{2}$ or $-\sqrt{2}$.

The eigenvalues of $\boldsymbol{C}$ are 1, $\sqrt{2}$ and $-\sqrt{2}$.

THEOREM

Let $\boldsymbol{A}$ be an $n \times n$ matrix. The following statements are equivalent.

1) $\boldsymbol{A}$ is invertible
2) $\boldsymbol{Ax} = \boldsymbol{0}$ has only trivial solution
3) RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$
4) $\boldsymbol{A}$ can be written as produce of elementary matrices
5) $\det(\boldsymbol{A}) \neq 0$
6) Rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$
7) Columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$
8) $\operatorname{rank}(\boldsymbol{A}) = n$
9) 0 is not an eigenvalue of $\boldsymbol{A}$

THEOREM (EIGENVALUES OF TRIANGULAR MATRICES)

If $\boldsymbol{A}$ is a triangular matrix, the eigenvalues of $\boldsymbol{A}$ are the diagonal entries of $\boldsymbol{A}$.

Proof: Suppose $\boldsymbol{A}$ is a triangular matrix.

Then $(\lambda \boldsymbol{I} - \boldsymbol{A})$ is also a triangular matrix.

THEOREM (EIGENVALUES OF TRIANGULAR MATRICES)

If $\boldsymbol{A}$ is a triangular matrix, the eigenvalues of $\boldsymbol{A}$ are the diagonal entries of $\boldsymbol{A}$.

So the eigenvalues of $\boldsymbol{A}$ are: $a_{11}, a_{22}, \ldots, a_{nn}$

Proof: Suppose $\boldsymbol{A}$ is a triangular matrix.

Then $(\lambda \boldsymbol{I} - \boldsymbol{A})$ is also a triangular matrix.

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{pmatrix} \qquad \lambda \boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ & \lambda - a_{22} & \cdots & -a_{2n} \\ & & \ddots & \vdots \\ 0 & & & \lambda - a_{nn} \end{pmatrix}$$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = (\lambda - a_{11})(\lambda - a_{22})\ldots(\lambda - a_{nn})$$

WHAT MANY OF YOU WILL ASK

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{A}$ are:

$$\boldsymbol{B} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & -3 & 0 & 6 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{B}$ are:

DEFINITION

Let $\boldsymbol{A}$ be a square matrix of order n and λ an eigenvalue of $\boldsymbol{A}$.

$(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$: homogeneous linear system with coefficient matrix $(\lambda \boldsymbol{I} - \boldsymbol{A})$

Note that $\boxed{(\lambda \boldsymbol{I} - \boldsymbol{A}) \text{ is singular}}$ and thus this homogeneous linear system has infinitely many solutions.

The solution space of $(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$ is called the **eigenspace** of $\boldsymbol{A}$ associated with λ and is denoted by E_λ.

REMARK

The solution space of $(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$ is called the eigenspace of $\boldsymbol{A}$ associated with λ and is denoted by E_λ.

Note that a non zero vector $\boldsymbol{v}$ belongs to E_λ

$$\Leftrightarrow (\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v} = \boldsymbol{0} \quad \Leftrightarrow \lambda \boldsymbol{I}\boldsymbol{v} - \boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$$

$$\Leftrightarrow \boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

So E_λ contains ALL the eigenvectors of $\boldsymbol{A}$ associated with λ.

EXAMPLE

The eigenvalues of $\boldsymbol{A}=\begin{pmatrix}0.96 & 0.01\\0.04 & 0.99\end{pmatrix}$ are $\lambda=1$ and $\lambda=0.95$.

$$E_1=\text{span}\left\{\begin{pmatrix}\frac{1}{4}\\1\end{pmatrix}\right\}$$

For $\lambda=1$, we investigate E_1:

$$1\boldsymbol{I}-\boldsymbol{A}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}-\begin{pmatrix}0.96 & 0.01\\0.04 & 0.99\end{pmatrix}=\begin{pmatrix}0.04 & -0.01\\-0.04 & 0.01\end{pmatrix}$$

$$\boxed{\dim(E_1)=1}$$

Solving $(1\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x}=\boldsymbol{0}$,

$$\boldsymbol{x}=\begin{pmatrix}\frac{t}{4}\\t\end{pmatrix}=t\begin{pmatrix}\frac{1}{4}\\1\end{pmatrix},\quad t\in\mathbb{R}$$

$$\left(\begin{array}{cc|c}0.04 & -0.01 & 0\\-0.04 & 0.01 & 0\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}1 & \frac{-1}{4} & 0\\0 & 0 & 0\end{array}\right)$$

$$x_1-\tfrac{1}{4}x_2=0$$

EXAMPLE

$\dim(E_{0.95}) = 1$

The eigenvalues of $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$ are $\lambda = 1$ and $\lambda = 0.95$.

$E_{0.95} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

For $\lambda = 0.95,$ we investigate $E_{0.95}$:

$$0.95\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 0.95 & 0 \\ 0 & 0.95 \end{pmatrix} - \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} = \begin{pmatrix} -0.01 & -0.01 \\ -0.04 & -0.04 \end{pmatrix}$$

Solving $(0.95\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$,

$$\boldsymbol{x} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\left(\begin{array}{cc|c} -0.01 & -0.01 & 0 \\ -0.04 & -0.04 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$x_1 + x_2 = 0$$

EXAMPLE

The eigenvalues of $\boldsymbol{B}=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$ are $\lambda=3$ and $\lambda=0$.

For $\lambda=3$, we investigate E_3 :

$$3\boldsymbol{I}-\boldsymbol{B}=\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$$

Solving $(3\boldsymbol{I}-\boldsymbol{B})\boldsymbol{x}=\boldsymbol{0}$

$$\left(\begin{array}{ccc|c}2&-1&-1&0\\-1&2&-1&0\\-1&-1&2&0\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right)$$

$$\boldsymbol{x}=\begin{pmatrix}t\\t\\t\end{pmatrix}=t\begin{pmatrix}1\\1\\1\end{pmatrix},\ t\in\mathbb{R}$$

$$E_3=\text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

$\boxed{\dim(E_3)=1}$

EXAMPLE

The eigenvalues of $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are $\lambda = 3$ and $\lambda = 0$.

For $\lambda = 0$, we investigate E_0 :

$$0\boldsymbol{I} - \boldsymbol{B} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

Solving $(0\boldsymbol{I} - \boldsymbol{B})\boldsymbol{x} = \boldsymbol{0}$

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

EXAMPLE

For $\lambda = 0$, we investigate E_0 :

$$x_1 + x_2 + x_3 = 0$$

Solving $(0\boldsymbol{I} - \boldsymbol{B})\boldsymbol{x} = \boldsymbol{0}$

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 &= & -s-t \\ x_2 &= & s \\ x_3 &= & t, \;\; s,t \in \mathbb{R} \end{cases}$$

$$\boldsymbol{x} = \left\{ \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} \right\} = \left\{ s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| \, s,t \in \mathbb{R} \right\}$$

$$E_0 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\dim(E_0) = 2$$

EXAMPLE

The eigenvalues of $\boldsymbol{C} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ are $\lambda = 1, \sqrt{2}$ and $-\sqrt{2}$.

Following similar method as before, we have:

$$E_1 = \text{span}\left\{\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\right\} \quad E_{\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right\} \quad E_{-\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right\}$$

EXAMPLE

Let $\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$. 2 is the only eigenvalue of $\boldsymbol{M}$.

Solving $(2\boldsymbol{I} - \boldsymbol{M})\boldsymbol{x} = \boldsymbol{0}$:

$$\begin{pmatrix} 2-2 & 0-0 \\ 0-1 & 2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ s \end{pmatrix}$, $s \in \mathbb{R}$ So $E_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and $\boxed{\dim(E_2) = 1}$.

REMEMBER HOW IT ALL STARTED?

Given a square matrix $\boldsymbol{A}$, we wanted to know if it is possible to find an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} \text{ (a diagonal matrix)}$$

Let's look at a summary of the examples we have seen previously:

WHAT CAN YOU DEDUCE?

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$E_1 = \text{span}\left\{ \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix} \right\}$$

$$\dim(E_1) = 1$$

$$E_{0.95} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

$$\dim(E_{0.95}) = 1$$

$$E_3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\dim(E_3) = 1$$

$$E_0 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\dim(E_0) = 2$$

$$E_1 = \text{span}\left\{ \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\}$$

$$\dim(E_1) = 1$$

$$E_{\sqrt{2}} = \text{span}\left\{ \begin{pmatrix} -1 \\ \sqrt{2} \\ \text{[illegible]} \end{pmatrix} \right\}$$

$$\dim(E_{\sqrt{2}}) = 1$$

$$E_{-\sqrt{2}} = \text{span}\left\{ \begin{pmatrix} -1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \right\}$$

$$\dim(E_{-\sqrt{2}}) = 1$$

WHAT CAN YOU DEDUCE?

$$\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \qquad E_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad \boxed{\dim(E_2) = 1}$$

END OF LECTURE 15

Lecture 16:

Diagonalization (till end of Section 6.2)