Lecture 02 Recap

A strategy to solve linear systems using elementary row operations.

Definition of row-echelon form and reduced row-echelon form of a matrix.

Gaussian Elimination and Gauss-Jordan Elimination.

Lecture 03

Gaussian Elimination (continued)

Homogeneous Linear Systems

What can row-echelon form tell us?

Remember that any linear system has either

(i) no solution (that is, inconsistent); or

(ii) exactly one solution (that is, an unique solution); or

(iii) infinitely many solutions.

By looking at a row-echelon form of the augmented matrix of the linear system, we can determine which of the above holds for the linear system.

$$\left(\begin{array}{ccc|c} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{array}\right)$$

How can we tell? (consistent vs. inconsistent)

If the augmented matrix of a linear system has a row-echelon form whose last column is a pivot column, then the linear system is inconsistent.

$$\left(\begin{array}{cccc|c} \otimes & & & & \\ & \otimes & & & \\ & & \otimes & & \\ 0 & 0 & \cdots & 0 & \otimes \end{array}\right)$$

last column

$\otimes$: leading entry

In other words, there is a row where every entry is zero except the last entry, which is non zero.

How can we tell? (consistent vs. inconsistent)

If the augmented matrix of a linear system has a row-echelon form whose last column is a pivot column, then the linear system is inconsistent.

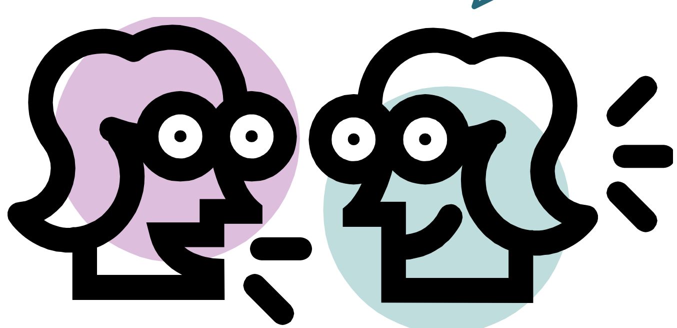
If the augmented matrix of a linear system has a row-echelon form whose last column is NOT a pivot column, then the linear system is consistent.

$$\left(\begin{array}{cccc|c} \otimes & & & & \\ & \otimes & & & \\ & & & \otimes & \\ 0 & 0 & \dots & 0 & 0 \end{array}\right)$$

Can we say more?

But a consistent linear system could still behave in two different ways right?

Yes you are right, row-echelon forms can also tell us how many solutions a linear system have!



How can we tell? (unique vs. infinitely many solutions)

If the augmented matrix of a consistent linear system has a row-echelon form where every column (except the last) is a pivot column, then the linear system has a unique (that is, exactly one) solution.

$$\left(\begin{array}{ccccc|c} \otimes & & & & & \\ & \otimes & & & & \\ & & \otimes & & & \\ & & & \ddots & & \\ & & & & \otimes & \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{array}\right)$$

Every column here is a pivot column

How can we tell? (unique vs. infinitely many solutions)

If the augmented matrix of a consistent linear system has a row-echelon form where some column (other than the last) is NOT a pivot column, then the linear system has infinitely many solutions.

$$
\left(\begin{array}{ccccc|c}
\otimes & * & & & & \\
 & * & & & & \\
 & 0 & \otimes & & & \\
 & \vdots & & \ddots & & \\
 & 0 & & & \otimes & \\
0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right)
$$

Some column here

is NOT a pivot column

Some examples

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & \boxed{0} \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Linear system is inconsistent

(last column is a pivot column)

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Linear system is consistent

and has infinitely many

solutions

(some column other than the

last is a non pivot column)

Some examples

$$\left(\begin{array}{cccc|c} -1 & 0 & \boxed{1} & 4 & 0 \\ 0 & 1 & \boxed{-1} & 3 & 1 \\ 0 & 0 & \boxed{0} & 2 & -1 \\ 0 & 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & \boxed{0} & 0 & 0 \end{array}\right)$$

Linear system is consistent and has <u>infinitely many solutions</u>

(some column other than the last is a non pivot column)

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 3 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Linear system is consistent and has a <u>unique solution</u>

(every column other than the last is a pivot column)

Notations to use

To ensure that we (your examiners) understand what you are doing, please use the following notations when performing elementary row operations:

1) When you want to multiply row i by a non zero constant c, write cR_i.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & \tfrac{1}{2} & 0 \end{array}\right) \xrightarrow[2R_3]{-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\tfrac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Notations to use

2) When you want interchange rows i and j,

write $R_i \leftrightarrow R_j$.

$$\left(\begin{array}{ccc|c} 0 & 0 & -1 & 2 \\ 1 & -2 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right)$$

Notations to use

3) When you want add k times of row i to row j,

write $R_j + kR_i$.

Remember that in this case,

row j changes but row i does not.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & -2 & 0 & 1 \\ -2 & 0 & \frac{1}{2} & 0 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 1 & -1 \\ 0 & 2 & -\frac{3}{2} & 4 \end{array}\right)$$

Can I 'combine' operations 'under one arrow'?

Answer: Only if there is no confusion or ambiguity.

$$\begin{pmatrix} & \bigg| \end{pmatrix} \xrightarrow[R_3+2R_1]{R_2-R_1}$$

No ambiguity

$$\begin{pmatrix} & \bigg| \end{pmatrix} \xrightarrow[R_3+2R_2]{R_2-R_1}$$

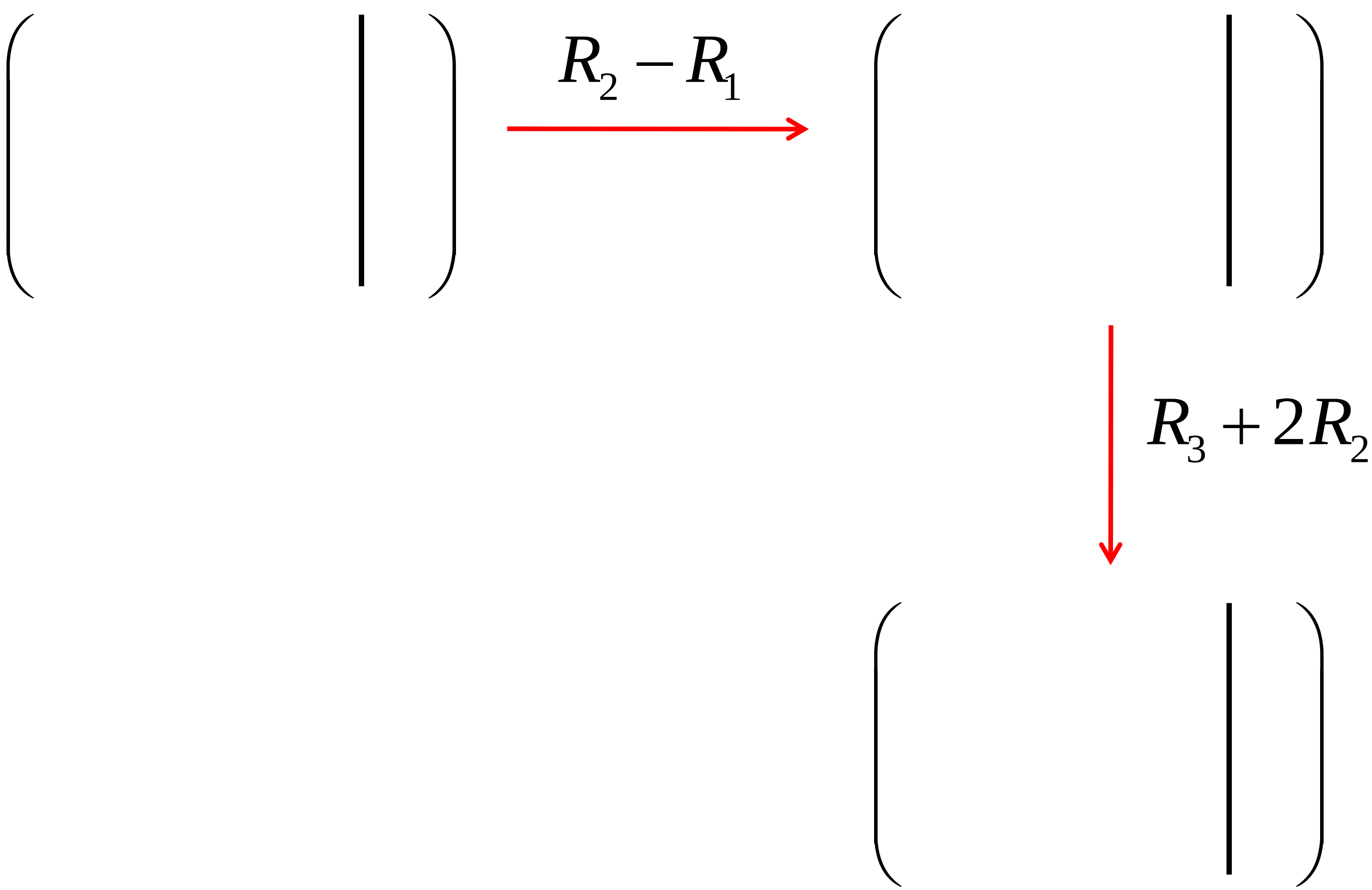
$R_2 - R_1$ first then $R_3 + 2R_2$

OR

$R_3 + 2R_2$ first then $R_2 - R_1$

Can I ‘combine’ operations ‘under one arrow’?

$R_2 - R_1$ first then $R_3 + 2R_2$



Example

What condition(s) must be satisfied by a, b, c such that the linear system

$$\begin{cases} x + 2y - 3z = a \\ 2x + 6y - 11z = b \\ x - 2y + 7z = c \end{cases}$$

has at least one solution?

Example

For what values of a will the following linear system

$$\begin{cases} x + 2y - 3z = 4 \\ 3x - y + 5z = 2 \\ 4x + y + (a^2 - 14)z = a + 2 \end{cases}$$

have no solution? Exactly one solution?

Infinitely many solutions?

Example

For what values of a and b will the following linear system

$$\begin{cases} ax + y = a \\ x + y + z = 1 \\ y + az = b \end{cases}$$

have no solution? Exactly one solution? Infinitely many solutions?

Example (quadric surface)

Find a formula for the quadric surface

$$ax^2 + by^2 + cz^2 = d$$

that passes through the points $(1,1,-1)$, $(1,3,3)$ and $(-2,0,2)$.

Solution (quadric surface)

Since the quadric surface

$$ax^2 + by^2 + cz^2 = d \qquad (*)$$

passes through $(1,1,-1)$, $(1,3,3)$ and $(-2,0,2)$,

$$x = 1, y = 1, z = -1,$$

$$x = 1, y = 3, z = 3 \text{ and}$$

$$x = -2, y = 0, z = 2$$

must satisfy equation (*).

Solution (quadric surface)

$$ax^2 + by^2 + cz^2 = d \qquad (*)$$

$x = 1, y = 1, z = -1,$
$x = 1, y = 3, z = 3$ and
$x = -2, y = 0, z = 2$

$$\begin{cases} a + b + c = d \\ a + 9b + 9c = d \\ 4a + 4c = d \end{cases}$$

$$\Leftrightarrow \begin{cases} a + b + c - d = 0 \\ a + 9b + 9c - d = 0 \\ 4a + 4c - d = 0 \end{cases}$$

Solution (quadric surface)

$$\begin{cases} a + b + c - d = 0 \\ a + 9b + 9c - d = 0 \\ 4a + 4c - d = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 9 & 9 & -1 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right) \xrightarrow[R_3 - 4R_1]{R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 8 & 8 & 0 & 0 \\ 0 & -4 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 + \frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 8 & 8 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right)$$

Solution (quadric surface)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 8 & 8 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right) \xrightarrow[\frac{1}{4}R_3]{\frac{1}{8}R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 \end{array}\right)$$

$$\xrightarrow[R_2 - R_3]{R_1 - R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{7}{4} & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 \end{array}\right)$$

$$\xrightarrow{R_1 - R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 \end{array}\right)$$

Solution (quadric surface)

$$ax^2 + by^2 + cz^2 = d$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 \end{array}\right) \qquad \begin{cases} a = t \\ b = \frac{3}{4}t \\ c = -\frac{3}{4}t \\ d = t, \quad t \in \mathbb{R}. \end{cases}$$

Let $d = 4$, then $\boxed{a = 4, b = 3, c = -3, d = 4}$ is a solution.

An equation of the quadric surface is

$$\underline{4x^2 + 3y^2 - 3z^2 = 4.}$$

Think for a while…

$$ax^2 + by^2 + cz^2 = d$$

Let $d = 4$, then $\boxed{a = 4, b = 3, c = -3, d = 4}$ is a solution.

An equation of the quadric surface is

$$\underline{4x^2 + 3y^2 - 3z^2 = 4.}$$

If I choose another value for d, does that mean

I get another equation of another quadric surface?

Let $d = 8$, then $a = 8, b = 6, c = -6, d = 8$ is a solution.

$$8x^2 + 6y^2 - 6z^2 = 8 \quad \Leftrightarrow \quad 4x^2 + 3y^2 - 3z^2 = 4.$$

Discussion

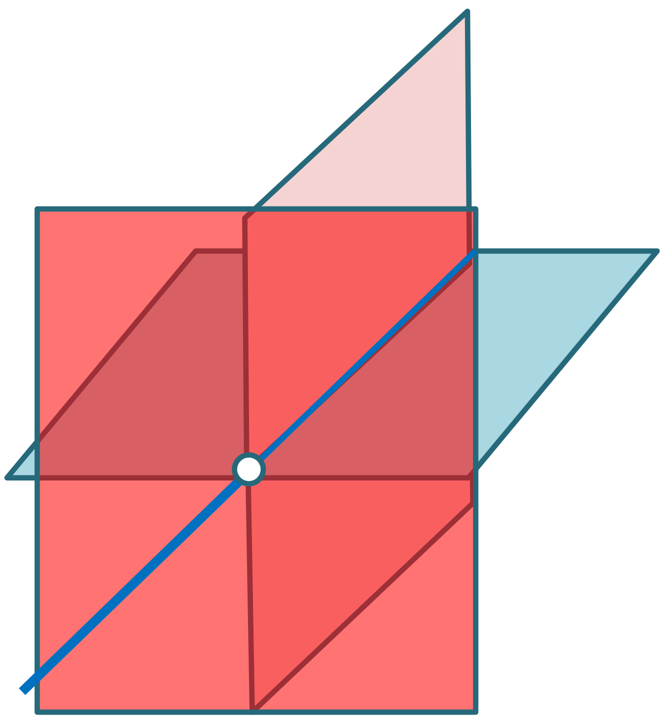
Suppose we have a consistent linear system involving three unknowns x, y, z

at most 3 leading entries

row-echelon form

$$\boldsymbol{R} = \left(\begin{array}{ccc|c} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \end{array}\right) \qquad \left(\begin{array}{ccc|c} \bullet & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \end{array}\right)$$

inconsistent



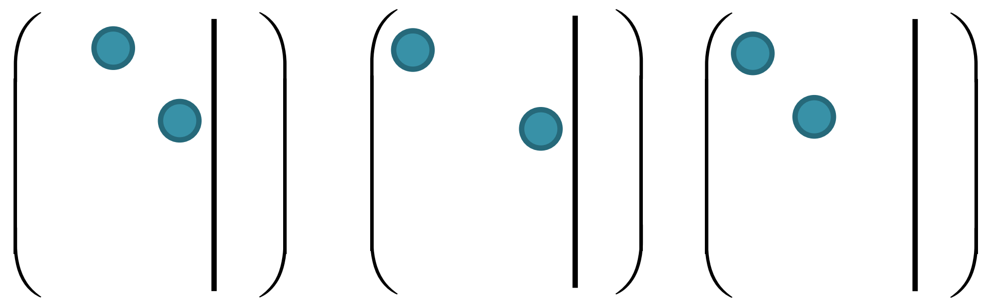
1) if 3 non zero rows (that is, 3 leading entries)

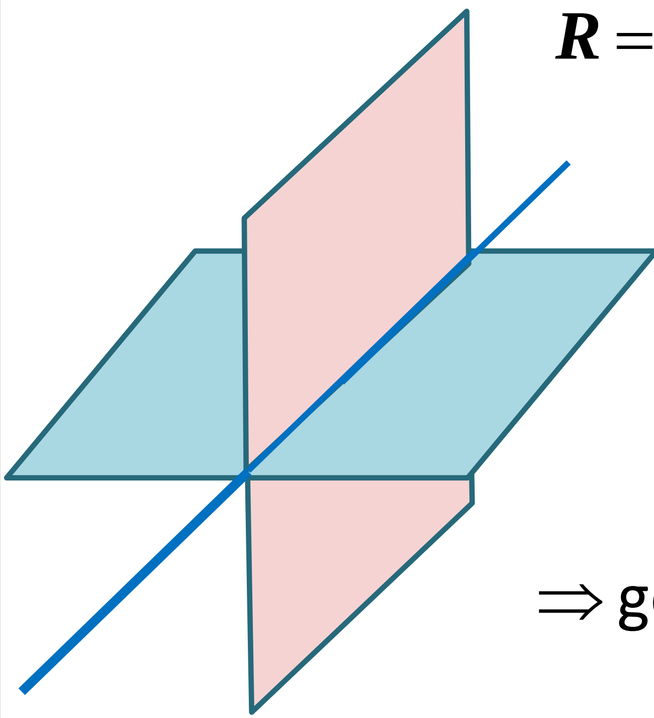
$\Rightarrow$ unique solution

$\Rightarrow$ solution set is a point

Discussion

Suppose we have a consistent linear system involving three unknowns x, y, z

$\boldsymbol{R} =$ 



2) if 2 non zero rows (that is, 2 leading entries)

$\Rightarrow$ infinitely many solutions

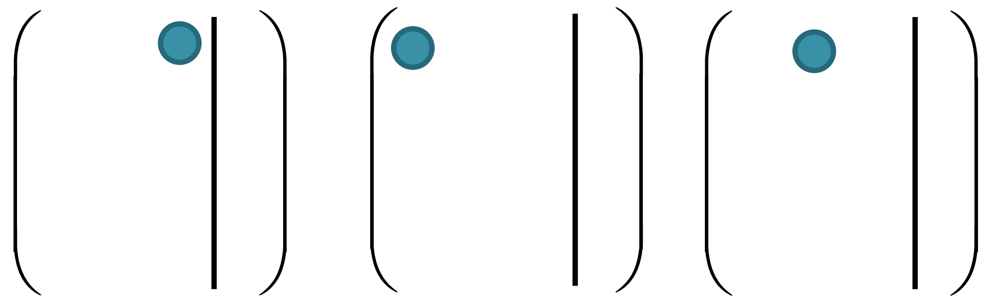
$\Rightarrow$ general solution has one parameter

$\Rightarrow$ solution set is a line

Discussion

Suppose we have a $\boxed{\text{consistent}}$ linear system involving three unknowns x, y, z

$$\boldsymbol{R} =$$



3) if 1 non zero row (that is, 1 leading entry)

$\Rightarrow$ infinitely many solutions

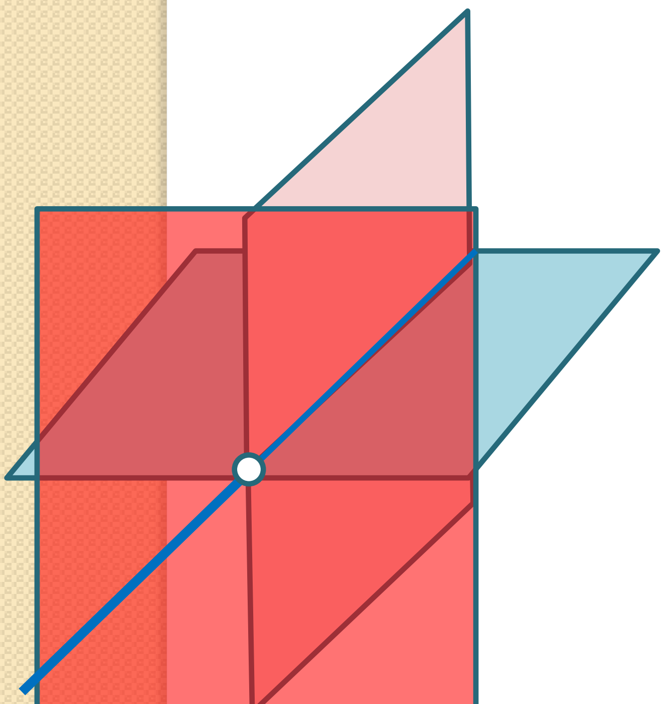
$\Rightarrow$ general solution has $\boxed{\text{two parameters}}$

$\Rightarrow$ solution set is a $\boxed{\text{plane}}$

Example

$$\begin{cases} x + y + 2z = 1 \\ x - y - z = 0 \\ x + y - z = 2 \end{cases}$$

Three planes in $xyz-$space



$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right)$$

3 leading entries

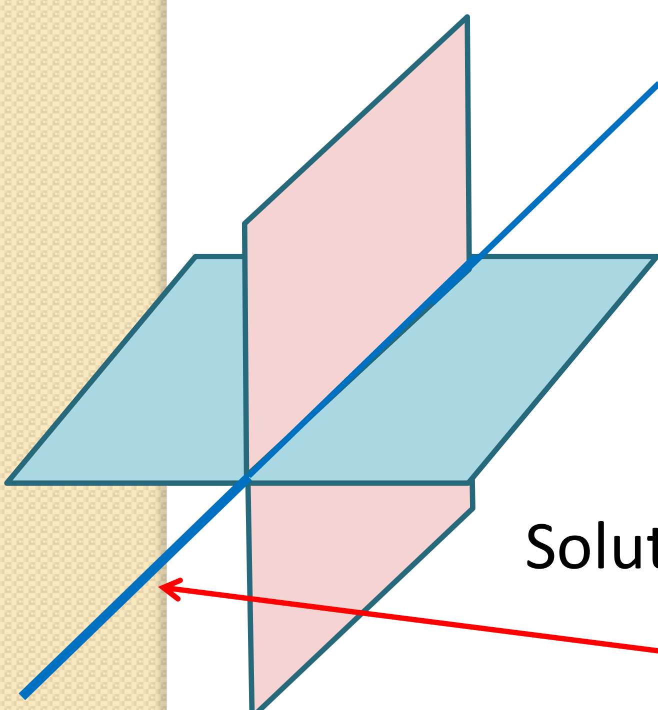
$x = \frac{2}{3}, y = 1, z = -\frac{1}{3}$ is the unique solution

Example

$$\begin{cases} x + y + 2z = 1 \\ x - y - z = 0 \\ 2x + z = 1 \\ 3x - y = 1 \end{cases}$$

Four planes in xyz – space

$$\begin{cases} x = \frac{1}{2} - \frac{1}{2}t \\ y = \frac{1}{2} - \frac{3}{2}t \\ z = t \qquad t \in \mathbb{R} \end{cases}$$



$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & -1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

2 leading entries

Solution is a straight line consisting of points:

$(\frac{1}{2} - \frac{1}{2}t, \frac{1}{2} - \frac{3}{2}t, t)$ for all $t \in \mathbb{R}$

Example

$$\begin{cases} x + y + 2z = 1 \\ 3x + 3y + 6z = 3 \end{cases}$$

Two planes in $xyz-$space

$$\begin{cases} x = 1 - s - 2t \\ y = s \\ z = t \end{cases} \qquad s,t \in \mathbb{R}$$

$x + y + 2z = 1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & 3 & 6 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

1 leading entry

Solution is a plane consisting of points:

$(1 - s - 2t, s, t)$ for all $s,t \in \mathbb{R}$

Definition (Homogeneous Linear Systems)

A linear system is said to be homogeneous if it has the following form:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & \boxed{0} \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{cases}$$

$a_{11}, a_{12}, \dots, a_{mn}$ are real constants.

Something special

Recall that any linear system behaves in exactly one of the following three ways:

Linear system

no solutions (inconsistent)	exactly one solution	infinitely many solutions

(consistent)

Something special

no solutions (inconsistent)

exactly one solution

infinitely many solutions

(consistent)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases}$$

What if we let $x_1 = 0, x_2 = 0, \dots, x_n = 0$?

Something special

~~no solutions (inconsistent)~~

exactly one solution

infinitely many solutions

(consistent)

$$\begin{cases} a_{11}0 & + & a_{12}0 & + & \dots & + & a_{1n}0 & = & 0 \\ a_{21}0 & + & a_{22}0 & + & \dots & + & a_{2n}0 & = & 0 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}0 & + & a_{m2}0 & + & \dots & + & a_{mn}0 & = & 0 \end{cases}$$

$x_1 = 0, x_2 = 0, \dots, x_n = 0$ is ALWAYS a solution!

Definition (Trivial and non-trivial solution)

A homogeneous linear system is always consistent.

$x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is called the **trivial solution** of the homogeneous linear system.

Any other solution (if there exists) is called a **non-trivial solution.**

~~no solutions (inconsistent)~~

exactly one solution

$\downarrow$

only trivial solution

infinitely many solutions

$\downarrow$

trivial + non-trivial solutions

Example

How many solutions does the following homogeneous linear system have?

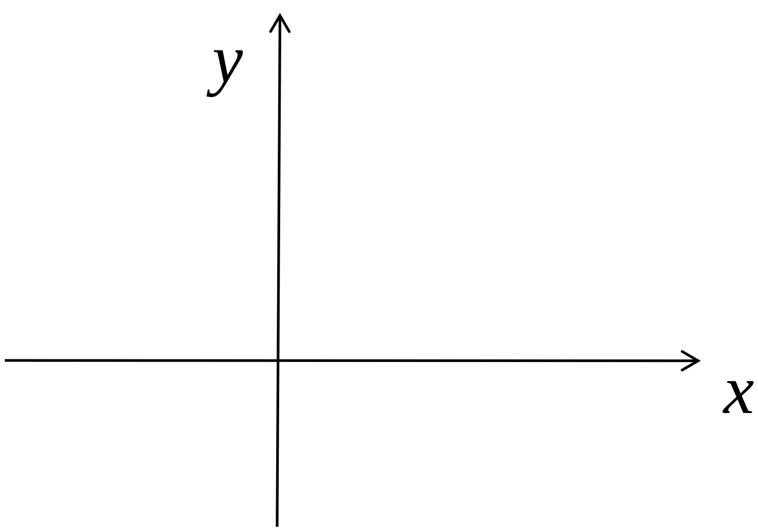
$$\begin{cases} 2x & - & y & - & 3z & = & 0 \\ -x & + & 2y & - & 3z & = & 0 \\ x & + & y & + & 4z & = & 0 \end{cases}$$

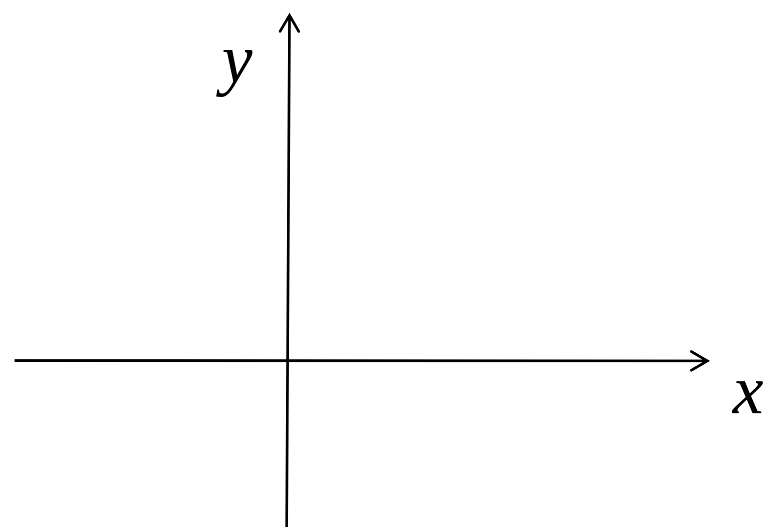
Note that the above linear system represents 3 planes in the three dimensional space, each plane contains the origin (that is, contains the point $x = 0, y = 0, z = 0$).

Example (2 variables)

l_1 and l_2 are two lines in the xy plane passing through the origin.

$$\begin{cases} a_1 x + b_1 y = 0 & (l_1) \\ a_2 x + b_2 y = 0 & (l_2) \end{cases}$$





Example (2 variables)

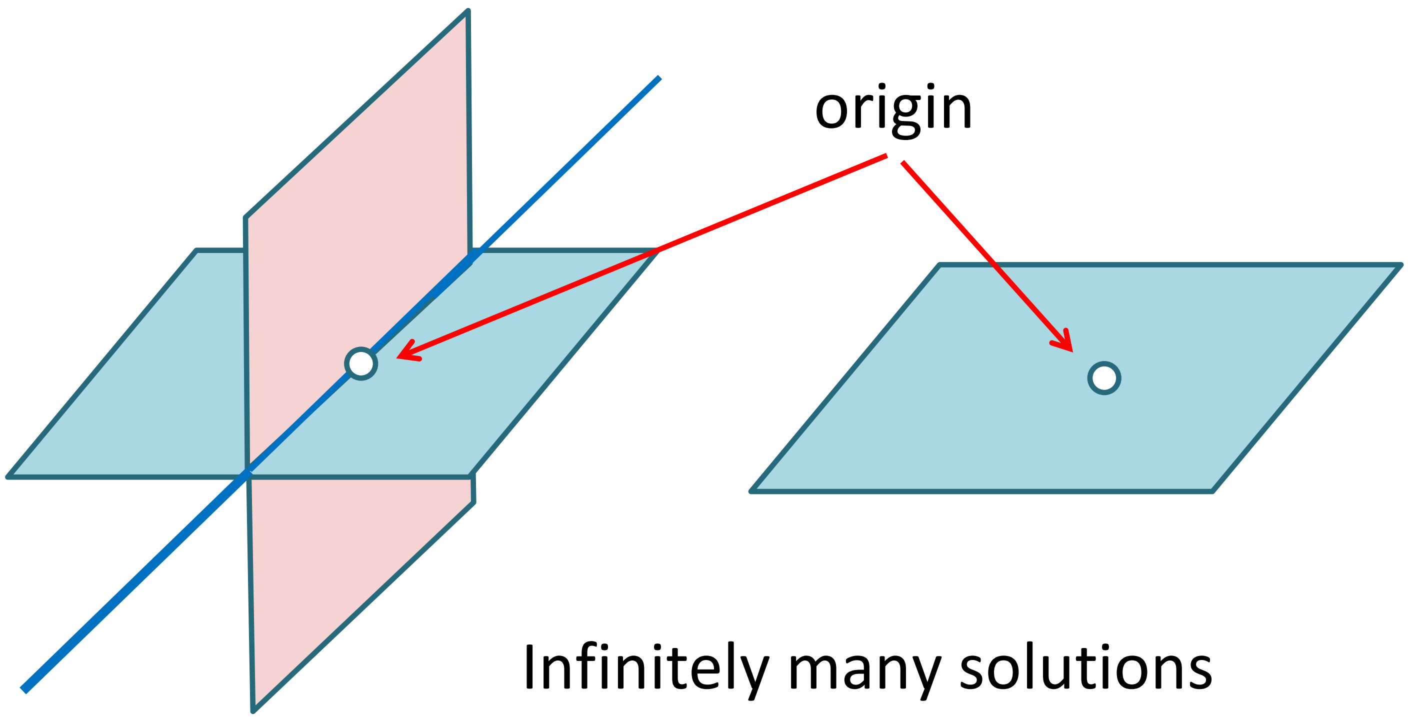
$$\begin{cases} a_1 x & + & b_1 y & = & 0 & & (l_1) \\ a_2 x & + & b_2 y & = & 0 & & (l_2) \end{cases}$$

The linear system has only the trivial solution if and only if the two lines are not the same.

The linear system has non-trivial solutions if and only if the two lines are identical.

3 variables

$$\begin{cases} a_1x + b_1y + c_1z = 0 & (p1) \\ a_2x + b_2y + c_2z = 0 & (p2) \end{cases}$$



Infinitely many solutions

(trivial + non-trivial)

3 variables and 3 planes

p_1, p_2 and p_3 are three planes in the three dimensional space, each containing the origin ($x=0, y=0, z=0$).

$$\begin{cases} a_1x + b_1y + c_1z = 0 & (p_1) \\ a_2x + b_2y + c_2z = 0 & (p_2) \\ a_3x + b_3y + c_3z = 0 & (p_3) \end{cases}$$

Discuss the relative positions of the planes such that the linear system has (i) only the trivial solution; (ii) non-trivial solutions.

What similarities can you find?

$$\begin{cases} x + y - 2z = 0 \\ 2x - 3y + 9z = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -3 & 9 & 0 \end{array}\right)$$

$$\begin{cases} x_1 - x_2 + 2x_3 - 4x_4 = 0 \\ 2x_1 + x_2 - 3x_3 + 5x_5 = 0 \\ 3x_1 - 2x_2 + x_3 - 3x_4 + x_5 = 0 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & -4 & 0 & 0 \\ 2 & 1 & -3 & 0 & 5 & 0 \\ 3 & -2 & 1 & -3 & 1 & 0 \end{array}\right)$$

What similarities can you find?

What is the **maximum number of leading entries** in a row-echelon form of each of the following augmented matrices?

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -3 & 9 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & -4 & 0 & 0 \\ 2 & 1 & -3 & 0 & 5 & 0 \\ 3 & -2 & 1 & -3 & 1 & 0 \end{array}\right)$$

What similarities can you find?

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & -4 & 0 & 0 \\ 2 & 1 & -3 & 0 & 5 & 0 \\ 3 & -2 & 1 & -3 & 1 & 0 \end{array}\right)$$

If a homogeneous linear system has more unknowns than equations...

... there will ALWAYS be non-pivot columns in a row-echelon form of the augmented matrix (other than the last column).

$\Rightarrow$ infinitely many solutions $\Rightarrow$ non-trivial solutions

What similarities can you find?

$$\begin{cases} x + y - 2z = 0 \\ 2x - 3y + 9z = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + 2x_3 - 4x_4 = 0 \\ 2x_1 + x_2 - 3x_3 + 5x_5 = 0 \\ 3x_1 - 2x_2 + x_3 - 3x_4 + x_5 = 0 \end{cases}$$

Homogeneous linear systems with more unknowns than equations always has infinitely many solutions.

End of Lecture 03

Lecture 04

Introduction to matrices

Matrix operations (till Theorem 2.2.22)