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CS161, Spring 2021

Homework 7

1. Prove:
   $Pr(\alpha_1,\ldots,\alpha_n \mid \beta) = Pr(\alpha_1 \mid \alpha_2,\ldots, \alpha_n, \beta)Pr(\alpha_2 \mid \alpha_3,\ldots, \alpha_n, \beta) \ldots Pr(\alpha_n \mid \beta)$

   Base Case:
   - *n = 1*
     $Pr(\alpha_1 \mid \beta) = Pr(\alpha_1 \mid \beta)$ ✓
   - *n = 2*
     $Pr(\alpha_1, \alpha_2 \mid \beta) = Pr(\alpha_1 \mid \alpha_2 \mid \beta)Pr(\alpha_2 \mid \beta)$ Bayes Conditioning Rules
     $= Pr(\alpha_1 \mid \alpha_2, \beta)Pr(\alpha_2 \mid \beta)$ ✓

   Inductive Step:
   Assume that $Pr(\alpha_1,\ldots,\alpha_n \mid \beta) = Pr((\alpha_1,\ldots,\alpha_{n-1}), \alpha_n \mid \beta)$ is true for all n between 1 and n-1.
   We have:
   $Pr(\alpha_1,\ldots,\alpha_n \mid \beta) = Pr(\alpha_1,\ldots,\alpha_{n-1} \mid \alpha_n, \beta)Pr(\alpha_n \mid \beta)$ Bayes Conditioning Rules

   Since the assumption is true for up to n-1, we can apply it as follows:
   $Pr(\alpha_1,\ldots,\alpha_{n-1} \mid \alpha_n, \beta) = Pr(\alpha_1 \mid \alpha_2,\ldots, \alpha_n, \beta)Pr(\alpha_2 \mid \alpha_3,\ldots, \alpha_n, \beta) \ldots Pr(\alpha_{n-1} \mid \alpha_n, \beta)$
   $Pr(\alpha_1,\ldots,\alpha_n \mid \beta) = Pr(\alpha_1 \mid \alpha_2,\ldots, \alpha_n, \beta)Pr(\alpha_2 \mid \alpha_3,\ldots, \alpha_n, \beta) \ldots Pr(\alpha_n \mid \beta)$ ■

2. Let the notation be as follows:
   Oil = O
   Gas = G
   Positive Test = T

$Pr(O) = 0.5$
$Pr(\neg O) = 1 - 0.5 = 0.5$
$Pr(G) = 0.2$
$Pr(\neg G) = 1 - 0.2 = 0.8$
$Pr(\neg O \land \neg G) = 0.3$
$Pr(O \lor G) = 1 - 0.3 = 0.7$
$Pr(T \mid O) = 0.9$
$Pr(\neg T \mid O) = 1 - 0.9 = 0.1$
$Pr(T \mid G) = 0.3$
$Pr(\neg T \mid G) = 1 - 0.3 = 0.7$
$Pr(T \mid \neg O \land \neg G) = 0.1$
$Pr(\neg T \mid \neg O \land \neg G) = 1 - 0.1 = 0.9$

Using Bayes Rules, we can use the following formula to find the probability that oil is present

$$Pr(O \mid T) = \frac{Pr(T|O)Pr(O)}{Pr(T)}$$

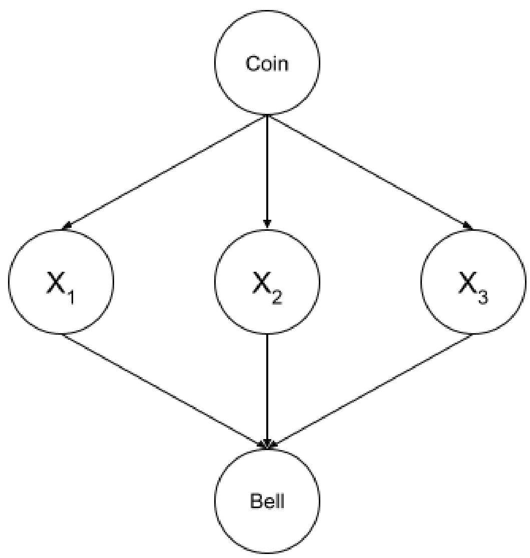
To find Pr(T), we consider a case analysis since O, G, and $\neg O \wedge \neg G$ are mutually exclusive and exhaustive as the three conditions cannot be present at the same instance.

$$Pr(T) = Pr(T|O)Pr(O) + Pr(T|G)Pr(G) + Pr(T \mid \neg O \wedge \neg G)Pr(\neg O \wedge \neg G)$$
$$Pr(T) = (0.9)(0.5) + (0.3)(0.2) + (0.1)(0.3) = 0.54$$

$$Pr(O \mid T) = \frac{(0.9)(0.5)}{(0.54)} = 0.8\overline{33}$$

3.

Bayesian Network:



Coin = { a, b, c }
$X_1 = X_2 = X_3$ = { H, T } where H = Heads and T = Tails
Bell = { T, F } where T = On and F = Off

Defining CPTs:

Coin	Θ_{Coin}
a	0.33
b	0.33
c	0.33

Coin	X_1	$\theta_{X1 \mid Coin}$
a	H	0.2
a	T	0.8
b	H	0.4
b	T	0.6
c	H	0.8
c	T	0.2

Coin	X_2	$\theta_{X2 \mid Coin}$
a	H	0.2
a	T	0.8
b	H	0.4
b	T	0.6
c	H	0.8
c	T	0.2

Coin	X_3	$\theta_{X3 \mid Coin}$
a	H	0.2
a	T	0.8
b	H	0.4
b	T	0.6
c	H	0.8
c	T	0.2

X_1	X_2	X_3	Bell	$\theta_{Bell \mid X1,X2,X3}$
H	H	H	T	1
H	H	H	F	0
H	H	T	T	0
H	H	T	F	1
H	T	H	T	0
H	T	H	F	1
H	T	T	T	0
H	T	T	F	1
T	H	H	T	0
T	H	H	F	1
T	H	T	T	0
T	H	T	F	1
T	T	H	T	0
T	T	H	F	1
T	T	T	T	1
T	T	T	F	0

4.

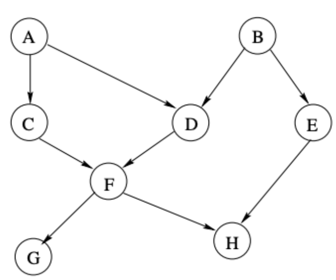


Figure 1: The DAG of a Bayesian network.

a) Markovian assumptions:

1. I(A, ∅, BE)
2. I(B, ∅, AC)
3. I(C, A, BDE)
4. I(D, AB, CE)
5. I(E, B, ACDFG)
6. I(F, CD, ABE)
7. I(G, F, ABCDEH)
8. I(H, EF, ABCDG)

b)

i) d_separated(A,F,E) is false.
Path ADBE is not blocked by any valves.
A and E are not d-separated given F.

ii) d_separated(G,B,E) is true.
Valves B and H are closed when B is known
All paths from G to E are blocked and G and E are d-separated given B.

iii) d_separated(AB, CDE, GH) is true.
C, D, or E are sequential valves that all paths from AB to GH pass through.
Since CDE is known, these valves are blocked.
All paths between AB and GH are blocked and AB and GH are d-separated given CDE.

c) $Pr(a,b,c,d,e,f,g,h) = Pr(a)Pr(b)Pr(c \mid a)Pr(d \mid a,b)Pr(e \mid b)Pr(f \mid c,d)Pr(f \mid c,d)Pr(g \mid f)Pr(h \mid e,f)$

d)

$Pr(A = 1,\ B = 1)$

A and B are independent from the Markovian assumption (1) I(A, ∅, BE).
We can calculate Pr(A=1, B = 1) = Pr(A=1)Pr(B=1) = (0.2)(0.7) = **0.14**

$Pr(E = 0 \mid A = 0)$

Using Bayes Conditioning Rules, we have

$Pr(E = 0 \mid A = 0) = \frac{Pr(E=0, A=0)}{Pr(A=0)}$

A and E are independent from the Markovian assumption (1) I(A, ∅, BE).
Pr(E=0, A=0) = Pr(E=0)Pr(A=0)

$$Pr(E = 0 \mid A = 0) = \frac{Pr(E=0, A=0)}{Pr(A=0)} = \frac{Pr(E=0)Pr(A=0)}{Pr(A=0)} = Pr(E = 0)$$

Using Case Analysis, we can derive

$$Pr(E = 0) = Pr(E = 0 \mid B = 0)Pr(B = 0) + Pr(E = 0 \mid B = 1)Pr(B = 1)$$

$$Pr(E = 0) = (0.1)(0.3) + (0.9)(0.7) = 0.66$$

5. $\alpha : A \Rightarrow B$

	A	B	Pr(A,B)	$A \Rightarrow B$
w_0	T	T	0.3	✓
w_1	T	F	0.2	x
w_2	F	T	0.1	✓
w_3	F	F	0.4	✓

a) $M(\alpha) = \{ w_0, w_2, w_3 \}$

b) $Pr(\alpha) = Pr(w_0) + Pr(w_2) + Pr(w_3) = 0.3 + 0.1 + 0.4 =$ **0.8**

c) Compute $Pr(A,B \mid \alpha)$

	A	B	$Pr(A,B \mid \alpha)$
w_0	T	T	0.3/0.8 = 0.375
w_1	T	F	0
w_2	F	T	0.1/0.8 = 0.125
w_3	F	F	0.4/0.8 = 0.5

d) Compute $Pr(A \Rightarrow \neg B \mid \alpha)$
$M(A \Rightarrow \neg B) = \{w_1, w_2, w_3\}$
$Pr(A \Rightarrow \neg B \mid \alpha) = Pr(w_1) + Pr(w_2) + Pr(w_3) = 0 + 0.125 + 0.5 =$ **0.625**