Univariate linear (and affine) discrete-time deterministic models

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Basic model: $N(t+1) = f(N(t))$, where f is a linear function and t is on the integers. Typically, state variable is continuous. What are the units of time? When does discrete time make sense?

We will use this extremely simple model to illustrate the general approach to dynamical modelling, which is,

1. solve for equilibria
2. evaluate stability of equilibria
3. possibly evaluate for small N (near system boundaries)
4. solve for time-dependent solution or simulate various special cases through time

Geometric growth (decay)

Simplest possible discrete-time deterministic dynamical model. $f(N) = RN$ (sometimes stated as $f(N) = (1+r)N$). Solve recursion analytically (can you do this?). Now we know everything about the dynamics. Suppose $N(0) > 0$, $R > 0$ (note: the R language indexes vectors starting from 1). What happens if $N(0) < 0$? model of debt?

What happens if $R < 0$?

(Even this ridiculously simple rule — or generalizations of it — is the basis of serious modeling in conservation biology.)

If $R < 1$, $N \to 0$ but $N = 0$ only in the limit, unless it starts there. A value of N such that $f(N^*) = N^*$ is called an *equilibrium* (or a *fixed point*).

Stability: what happens for perturbations in the neighborhood of the fixed point? Consider displacing the population away from N^* by δ, where $\delta \ll 1$; what happens? If $|R| < 1$ then then N will return to $N*$.

$N = 0$ is always an equilibrium, stable iff $|R| < 1$.

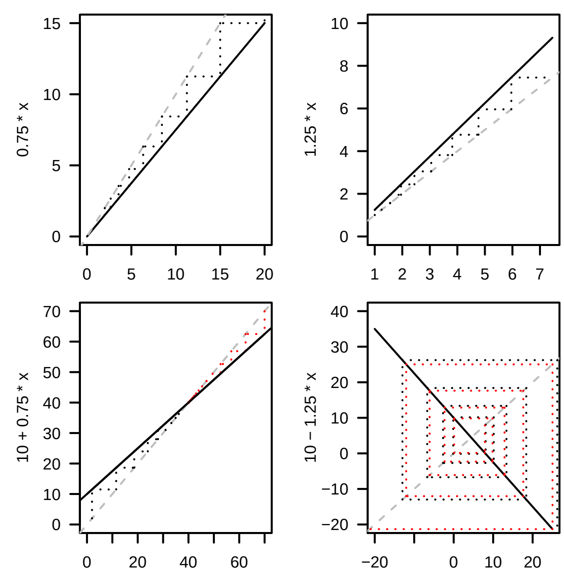
Affine models

Now suppose (as in the example in the book) we are adding or subtracting a fixed amount per time step: $N(t+1) = a + bN(t)$. As before we can work out the recursion.

Summing the series for t steps gives $a(1-b^t)/(1-b) + b^t N$; the limit is $a/(1-b) + \lim_{t\to\infty} b^t N(0)$. If $|b| > 1$ this is a bit boring. If $|b| < 1$ we get a stable equilibrium at $a/(1-b)$. (For $b < 0$ ("bucket model"): a is the supply rate, $1/(1-b)$ is the average *residence time*.)

Useful component for larger models. (Autoregressive model in time series analysis; sometimes used as the bottom level in food chain modeling.)

Graphical approaches: cobwebbing



Multiple lags

What if $N(t+1)$ depends on previous time steps $N(t-1)$ etc. as well as $N(t)$? Homogeneous linear equations: $\sum_{i=0}^{m} a_i N(t-i) = 0$. Plug in $N(t) = C\lambda^t$. Solve characteristic equation … get a linear combination of geometric growth/decay, $\sum C_i \lambda_i^n$: largest *eigenvalue* dominates long-term behavior.