Math 3MB3: midterm exam, 09 Oct 2014

Name (1 point):

1 Short answer questions (7 points)

Provide one to three sentence answers to each of the following questions.

1. What is the definition of a fixed point in a univariate model? (2 points)
   A point, $x\star$, of state variable $x(t)$ is a fixed point if $x(t+1) = x(t) = x\star$.

2. Describe a real-life objective or goal in terms of fixed-point/stability analysis. (2 points)
   In epidemiology, a state variable of interest is often the number of infected individuals, I, in a susceptible population. It is usually desired to have a fixed point of I at zero that is stable.

3. When a fixed point in a univariate *linear* model is stable, it is *globally* stable. This means that following *any* perturbation, the state variable will return to the fixed point, given enough time. However, when a fixed point in a univariate *nonlinear* model is stable, it is not necessarily *globally* stable. Why? (3 points)
   In a linear model, the derivative of the recursive function, $f(x)$, is a constant value. Because this derivative determines stability, we know that no matter how far a perturbation takes the state variables from a fixed point, the stability conditions are unchanged. However, the derivative is not constant in a nonlinear model, and so there are no gaurantees about how a general perturbation will respond. Furthermore, in a nonlinear model a perturbation may take the state variable into a region that is attracted to another stable fixed point. In short, in linear models, local stability information is global, but this is not necessarily so in nonlinear models.

2 Fixed points and stability (20 points)

2.1 Univariate non-linear models (10 points)

Consider a general discrete time model:

$$x(t+1) = f(x(t)) \tag{1}$$

for state variable, x. Two special cases of this model are defined below by specifying the function f. For each case, find all of the fixed points and label each as either stable or unstable. Show your work.

MODEL I (5 POINTS)

$$f(x) = 1 + x - \frac{1}{4}x^2$$

$x = 1 + x - \frac{1}{4}x^2$, which implies that $0 = 1 - \frac{1}{4}x^2 = (1 - 0.5x)(1 + 0.5x)$. Therefore, $x = 2$ and $x = -2$ are fixed points. Differentiating $x(t+1)$ w.r.t. $x(t)$ gives $1 - 0.5x$. At the first fixed point this equals $|1 - (0.5)(2)| = |1 - 1| = |0| < 1$, and so this point is stable. At the second fixed point this equals $|1 + (0.5)(2)| = |1 + 1| = |2| > 1$, and so this point is unstable.

MODEL II (5 POINTS)

$$f(x) = 1 - 2|x - 0.5|$$

where $|x - 0.5|$ denotes absolute value of $x - 0.5$. **Hint**: consider $x > 0.5$ and $x < 0.5$ separately.

$$x = f(x) = 1 - 2|x - 0.5| \tag{2}$$

First assume that $x > 0.5$,

$$\begin{aligned} x =& 1 - 2(x - 0.5) \\ x =& 2 - 2x \\ 3x =& 2 \\ x =& \frac{2}{3} \end{aligned} \tag{3}$$

And since $\frac{2}{3} > 0.5$ it is indeed a fixed point. Now assume that $x < 0.5$,

$$\begin{aligned} x =& 1 + 2(x - 0.5) \\ x =& 1 + 2x - 1 \\ -x =& 0 \\ x =& 0 \end{aligned} \tag{4}$$

and again since $0 < 0.5$ it is also indeed a fixed point. Finally, if $x = 0.5$,

$$0.5 \neq 1 - 2|0| = 1 \tag{5}$$

therefore 0.5 is not a fixed point.

Now for stability. If $x > 0.5$,

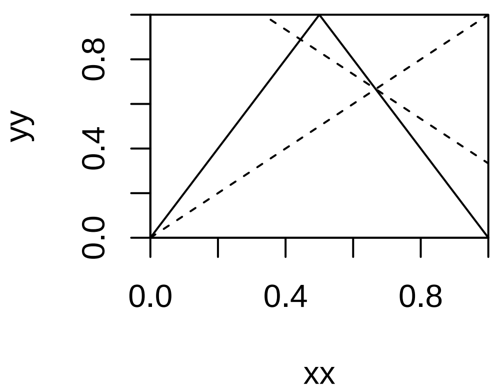
$$\begin{aligned} f(x) &= 1 - 2(x - 0.5) \\ f'(x) &= -2 \end{aligned} \tag{6}$$

and if $x < 0.5$,

$$\begin{aligned} f(x) &= 1 + 2(x - 0.5) \\ f'(x) &= 2 \end{aligned} \tag{7}$$

Therefore neither fixed point is stable because $|f'(x)| = 2 > 1$ in both cases.

```
xx <- seq(0, 1, length = 101)
yy <- 1 - 2*abs(xx - 0.5)
plot(xx, yy, type = "l", xaxs = "i", yaxs = "i")
abline(0, 1, lty = 2)
abline(4/3, -1, lty = 2)
```



2.2 Multivariate linear models (8 points)

Consider the multivariate system,

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) \tag{8}$$

Assume that $\boldsymbol{x}$ has a fixed point at the origin. Further assume that $\boldsymbol{A}$ takes the following form,

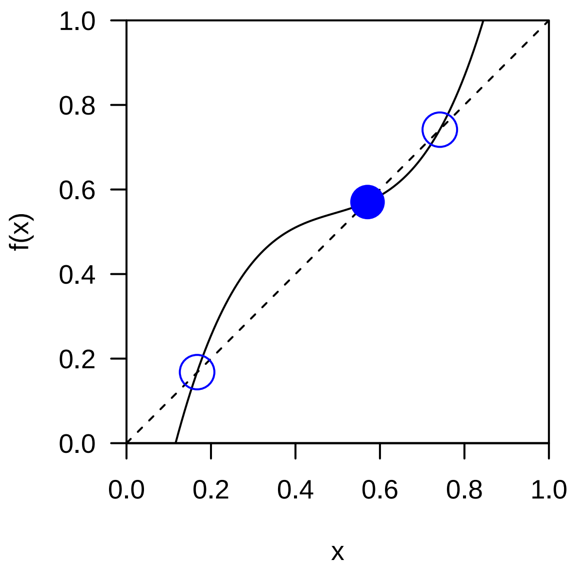
$$\boldsymbol{A} = \begin{bmatrix} 0.5 & -1 \\ a & -0.5 \end{bmatrix} \tag{9}$$

where a is an unknown parameter. For what values of a is this system stable? **Hint**: The absolute value of both eigenvalues is less than one if $\delta < 1$ and $|\tau| < \delta + 1$, where τ is the matrix trace and δ is the matrix determinant.

Find the trace, $\tau = 0$, and determinant, $\delta = a - \frac{1}{4}$. Therefore $a - \frac{1}{4} < 1$ leads to $a < \frac{5}{4}$ and $0 < a - \frac{1}{4} + 1$ leads to $a > -\frac{3}{4}$. Therefore the origin is stable if $-\frac{3}{4} < a < \frac{5}{4}$.

2.3 Graphical analysis (2 points)

Consider another univariate model defined by the function $f(x)$, which is described by the solid curve in the graph below. On the graph, mark any stable fixed points as closed circles and any unstable fixed points as open circles. Point to any fixed points of ambiguous stability with arrows.



3 Time-dependent solutions (10 points)

3.1 Univariate linear models (2 points)

Consider the state variable, $x(t)$, which obeys the following linear recurrence relation,

$$x(t+1) = \frac{x(t)}{2} \tag{10}$$

where $x(0) = 2^{10} = 1024$. What is the numerical value of $x(10)$? That is, what is the value of the state variable at time, $t = 10$? Simplify as far as you can.

This is easy, $x(10) = \frac{2^{10}}{2^{10}} = 1$

3.2 Multivariate linear models (8 points)

Derive a time-dependent solution for the model in section 2.3, but make the following additional assumptions:

- The entry, a, in the lower-left element of the matrix, $\boldsymbol{A}$, is equal to zero.
- The initial conditions are $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boldsymbol{A} = \begin{bmatrix} 0.5 & -1 \\ 0 & -0.5 \end{bmatrix} \tag{11}$$

Let,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{12}$$

The general solution is,

$$\begin{aligned} x_1(t) &= c_{11} 0.5^t + c_{12}(-0.5)^t \\ x_2(t) &= c_{21} 0.5^t + c_{22}(-0.5)^t \end{aligned} \tag{13}$$

For the first variable we have,

$$\begin{aligned} x_1(0) &= c_{11} + c_{12} \\ x_1(1) &= \frac{c_{11}}{2} - \frac{c_{12}}{2} \end{aligned} \tag{14}$$

which after subbing in initial conditions leads to,

$$\begin{aligned} 1 &= c_{11} + c_{12} \\ -\frac{1}{2} &= \frac{c_{11}}{2} - \frac{c_{12}}{2} \end{aligned} \tag{15}$$

and finally,

$$\begin{aligned} c_{12} &= 1 \\ c_{11} &= 0 \end{aligned} \tag{16}$$

Note we have,

$$x(1) = \begin{bmatrix} 0.5 & -1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix} \tag{17}$$

Therefore,

$$x_1(t) = (-0.5)^t \tag{18}$$

and by symmetry,

$$x_2(t) = (-0.5)^5 \tag{19}$$

4 R (8 points)

4.1 For loops (3 points)

Consider the following special case of the discrete-time logistic model,

$$x(t+1) = f(x(t))$$

where $f(x) = \frac{x(1-x)}{2}$ and $x(0) = 1.5$. Write no more than three lines of R code (semicolons are not allowed) to calculate and print out $x(5)$.

Use for loop

```
x <- 1.5
for(i in 1:5) x <- 0.5*x*(1-x)
print(x)
## [1] -0.05155
```

4.2 Subscripting and comparisons (3 points)

Given the following R command,

```
(x <- rnorm(10))
##  [1]  1.05177 -0.75267 -1.43968 -0.28571 -1.03429
##  [6] -0.02815 -0.36632 -1.11525 -0.97556  1.12102
```

Write one R command to solve each of the following short problems: (a) is the first element of x larger than the last element? (b) what is the sum of the elements in x? (c) create a new vector that contains every element of x such that each element is greater than zero.

Here's the solution:

```
x[1] > x[length(x)]
## [1] FALSE
sum(x)
## [1] -3.825
(y <- x[which(x > 0)])
## [1] 1.052 1.121
```

4.3 Matrices (2 points)

Given a bivariate (2-by-2) system $\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t)$ and matrix $\boldsymbol{A}$ entered into R as,

```
(A <- matrix(round(rnorm(4), 1), 2, 2))
##      [,1] [,2]
## [1,]  0.3 -1.0
## [2,] -0.6 -0.5
```

Write a short R script to determine whether the system is stable at the origin. Do this even if you can determine stability by inspection. Then print out the first column of $\boldsymbol{A}$.

Here's the solution:

```
all(Mod(eigen(A)$values) < 1)
## [1] TRUE
A[,1]
## [1]  0.3 -0.6
```

5 Synthesis (4 points)

Let $\mathbf{n}$ be a two-dimensional state vector obeying $\mathbf{n}(t+1) = \mathbf{A}\mathbf{n}(t)$, such that $\mathbf{n}(t) = (1.5)^t c_1 \mathbf{v}_1 + (-1)^t c_2 \mathbf{v}_2$ is the time dependent solution. The $\mathbf{v}$'s are real-valued vectors and the c's are constant real-valued scalars.

1. What are the eigenvalues of $\mathbf{A}$?
   1.5, -1

2. Does this system have a stable fixed point at the origin?
   no

3. How can you tell?
   not all eigenvalues have absolute value less than one

4. Assume $\mathbf{A}$ is stored in an `R` matrix called `A`, and an initial state vector is stored in `n`. Write a for loop that terminates with `n` containing the state at the tenth time step.

```
for(i in 1:10) n <- A%*%n
```

6 Scratch space