Results you need to know how to use (MATH 3MB3)

Univariate linear discrete-time

model $x(t+1) = Rx(t)$

fixed points $x\star = 0$, if $R \neq 1$,
otherwise $x\star$ can be any number

stability $x\star = 0$ stable if $|R| < 1$

solution $x(t) = R^t x(0)$

Univariate affine discrete-time

model $x(t+1) = a + bx(t)$

fixed points $x\star = \frac{a}{1-b}$, if $b \neq 1$,
$x\star$ is any real if $b = 1$, $a = 0$,
otherwise $x\star$ doesn't exist

stability $x\star$ stable if $|b| < 1$

solution $x(t) = b^t x(0) + (1 - b^t)\frac{a}{1-b}$,
or $x(t) = x(0) + at$, if $b = 1$

Univariate nonlinear discrete-time

model $x(t+1) = f(x(t))$

fixed points $x\star = f(x\star)$, no general sol'n

stability $x\star$ stable if $|f'(x\star)| < 1$

solution use a computer

Univariate linear continuous-time

model $\frac{dx}{dt} = rx$

fixed points $x\star = 0$ if $r \neq 0$,
otherwise $x\star$ can be any number

stability $x\star = 0$ stable if $r < 0$

solution $x(t) = x(0)e^{rt}$

Univariate affine continuous-time

model $\frac{dx}{dt} = a + bx$

fixed points $x\star = \frac{-a}{b}$ if $b \neq 0$,
$x\star$ is any real if $a = b = 0$,
otherwise, $x\star$ doesn't exist

stability $x\star$ stable if $b < 0$

solution $x(t) = x(0)e^{bt} + \frac{-a}{b}\left(1 - e^{bt}\right)$,
or $x(t) = at$, if $b = 0$

Univariate nonlinear continuous-time

model $\frac{dx}{dt} = f(x)$

fixed points $0 = f(x\star)$, no general sol'n

stability $x\star$ stable if $f'(x\star) < 0$

solution use a computer unless 1. piecewise linear, 2. partial fractions help, 3. table of integrals after separation, or 4 *maybe* change of variables if you're feeling brave

Bivariate linear discrete-time

model $\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = \begin{bmatrix} a\,b \\ c\,d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

fixed points $\begin{bmatrix} x\star \\ y\star \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, if $\tau \neq \Delta + 1$, otherwise $\begin{bmatrix} x\star \\ y\star \end{bmatrix}$ any vector in a $(2-r)$-dimensional space

stability $\begin{bmatrix} x\star \\ y\star \end{bmatrix}$ is stable if $|\tau| < \Delta + 1$ and $\Delta < 1$

solution See general multivariate affine case below

notation $\tau = a + d$, $\Delta = ad - bc$

Multivariate affine discrete-time

model $\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}$

fixed points $\boldsymbol{x}\star = (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$, if $r = n$, otherwise $\boldsymbol{x}\star$ can be any vector in an $(n-r)$-dimensional space spanned by the $\boldsymbol{v}_i$ associated with $\lambda_i = 1$

stability $\boldsymbol{x}\star$ is stable if $|\lambda_i| < 1\ \forall i$

solution If the $\boldsymbol{v}_i$ are linearly independent, $\boldsymbol{x}(t) = \sum_i \phi_i \lambda_i^t \boldsymbol{v}_i + (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$, otherwise use a computer

notation $\boldsymbol{v}_i$ and λ_i the ith eigenvector and eigenvalue of $\boldsymbol{A}$.
ϕ_i are components of $\boldsymbol{V}^{-1}(\boldsymbol{x}(0) - (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b})$ if it exists, where $\boldsymbol{V}$ has the $\boldsymbol{v}_i$ as columns
n and r are the size and rank of $\boldsymbol{I} - \boldsymbol{A}$, for identity matrix, $\boldsymbol{I}$.

Multivariate nonlinear discrete-time

model $\boldsymbol{x}(t+1) = \boldsymbol{f}(\boldsymbol{x}(t))$

fixed points $\boldsymbol{x}\star = \boldsymbol{f}(\boldsymbol{x}\star)$, no general sol'n

stability $\boldsymbol{x}\star$ stable if $|\lambda_i| < 1\ \forall i$

solution use a computer

notation λ_i is the ith eigenvalue of the Jaccobian matrix of partial derivatives of $\boldsymbol{f}(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ at $\boldsymbol{x}\star$

Bivariate linear continuous-time

model $\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a\,b \\ c\,d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

fixed points $\begin{bmatrix} x\star \\ y\star \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, if $\Delta \neq 0$, otherwise $\begin{bmatrix} x\star \\ y\star \end{bmatrix}$ any vector in a $(2-r)$-dimensional space

stability $\begin{bmatrix} x\star \\ y\star \end{bmatrix}$ stable if $\tau, -\Delta < 0$

solution See general multivariate affine case below

notation $\tau = a + d$, $\Delta = ad - bc$

Multivariate affine continuous-time

model $\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$

fixed points $x\star = -\boldsymbol{A}^{-1}\boldsymbol{b}$ if $r = n$, otherwise $\boldsymbol{x}\star$ can be any vector in an $(n-r)$-dimensional space spanned by the $\boldsymbol{v}_i$ associated with $\lambda_i = 0$

stability $\boldsymbol{x}\star = \boldsymbol{0}$ stable if $\mathrm{Re}(\lambda_i) < 0\ \forall i$

solution If the $\boldsymbol{v}_i$ are linearly independent, $\boldsymbol{x}(t) = \sum_i \phi_i e^{\lambda_i t}\boldsymbol{v}_i - \boldsymbol{A}^{-1}\boldsymbol{b}$, otherwise use a computer

notation $\boldsymbol{v}_i$ and λ_i the ith eigenvector and eigenvalue of $\boldsymbol{A}$
ϕ_i are components of $\boldsymbol{V}^{-1}(\boldsymbol{x}(0) - \boldsymbol{A}^{-1}\boldsymbol{b})$ if it exists, where $\boldsymbol{V}$ has the $\boldsymbol{v}_i$ as columns
n and r are the size and rank of $\boldsymbol{A}$

Multivariate nonlinear continuous-time

model $\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x})$

fixed points $\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}\star)$, no general sol'n

stability $\boldsymbol{x}\star$ stable if $\mathrm{Re}(\lambda_i) < 0\ \forall i$

solution use a computer

notation λ_i is the ith eigenvalue of the Jaccobian matrix of partial derivatives of $\boldsymbol{f}(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ at $\boldsymbol{x}\star$