Math 3MB3: practice

1 Fixed points and stability

MS p 38, Exercise 3. For more practice, find the fixed points of the following models. If there are no fixed points, say so. For each fixed point, analyse its stability using the derivative test. If the derivative test is inconclusive, so say.

1.1 Univariate linear (memorization approaches)

Which points are fixed points of the system $x(t+1) = Rx(t)$? (assume that $R \neq 1$) Analyze the stability of any fixed points.
The only fixed point is $x = 0$. It is stable if $|R| < 1$.

1.2 Univariate affine and nonlinear (analytical approaches)

$$x(t+1) = a + x(t) \tag{1}$$

If $x(t+1) = x(t) = x$, then $x = a + x$, which implies $a = 0$. Therefore, if $a = 0$, every x is a fixed point but if $a \neq 0$, there are no fixed points. The derivative of the r.h.s. w.r.t. $x(t)$ is 1. Therefore, when $a = 0$, every fixed point is neutrally stable (i.e. if perturbed, remains at the initial value plus the perturbation).

$$x(t+1) = \frac{3}{16} + x^2(t) \tag{2}$$

$x = \frac{3}{16} + x^2$ implies that $x = \frac{1 \pm \sqrt{1-4(3/16)}}{2}$. Therefore, $x = 1/4$ and $x = 3/4$ are fixed points. Differentiating gives $2x$, which at the first fixed point is $1/2$ (stable) and at the second is $3/2$ (unstable).

$$x(t+1) = a + x(t) + x^2(t), a < 0, x \geq 0 \tag{3}$$

$x = a+x+x^2$ implies $x^2 = -a$. Since by assumption $-a > 0$ and $x \geq 0$, this implies $x = \sqrt{-a}$ is a fixed point. Differentiating gives $1+2x$, which at the fixed point is $1 + 2\sqrt{-a}$. One condition for stability is that $1 + 2\sqrt{-a} < 1$, which implies $2\sqrt{-a} < 0$ and $\sqrt{-a} < 0$ and $-a < 0$ and $a > 0$. But a is restricted to negative numbers and so the fixed point is unstable.

$$x(t+1) = 2 + \sqrt{x(t)} \tag{4}$$

This one's trickier. We need function notation, $f(x) = 2 + \sqrt{x}$. Note that the domain of f is $[0, \inf)$ and the range is $[2, \inf)$. For fixed points $f(x) = x = 2 + \sqrt{x}$, which implies that $(x - 2)^2 = x$. This implies $x^2 - 5x + 4$, which can be factored as $(x - 1)(x - 4) = 0$. Looks like two solutions, but note that x must be in both the domain and range of f for a fixed point (because the definition of a fixed point here is $f(x) = x$). Therefore, $x = 4$ is the only fixed point. Differentiating $f'(x) = \frac{1}{2\sqrt{x}}$ and evaluating at the fixed point is $\frac{1}{4} < 1$, and therefore the fixed point is stable.

$$x(t + 1) = rx(t)(1 - x(t)) \tag{5}$$

You could recognize that this is the logistic model. $x = rx(1 - x)$ implies $x = 1 - \frac{1}{r}$ and $x = 0$ are fixed points. Differentiating gives $r(1 - 2x)$, which at the easy fixed point is r. Therefore the easy fixed point is stable when $|r| < 1$ (i.e. growth/decay rate isn't too large). For the more difficult fixed point the derivative is $r(1 - 2(1 - \frac{1}{r}))$, which simplifies to $2 - r$. The stability condition requires $|2 - r| < 1$, which implies both that $2 - r < 1$ (i.e. $r > 1$) and the $2 - r > -1$ (i.e. $r < 3$). Therefore, the more interesting fixed point is stable when $1 < r < 3$.

$$x(t + 1) = \sin(x(t)) \tag{6}$$

The only point at which $\sin(x) = x$ is when $x = x\star = 0$. The derivative of $\sin(x)$ is $\cos(x)$, which at $x\star = 0$ is 1. *If* this were a linear system we would conclude *neutral stability*. However, it is a *nonlinear* system and so the results of the derivative test are *inconclusive*.

$$x(t + 1) = 1 - e^{-x(t)} \tag{7}$$

where $x(0) > 0$.
$x = 1 - e^{-x}$. By inspectiion we see that if $x = 0$, then $1 - e^0 = 0$. Therefore, $x\star = 0$ is a fixed point. Are there others? No, because the positive initial condition ensures that e^{-x} cannot be lower than 0 or more than 1, and therefore $1 - e^{-x}$ is also between 0 and 1. Since $1 - e^{-x} = 0$ only when $x = 0$, $x\star = 0$ is the only fixed point. For stability, let $f(x) = 1 - e^{-x}$ be the recursion function. The derivative is $f'(x) = e^{-x}$. Therefore the

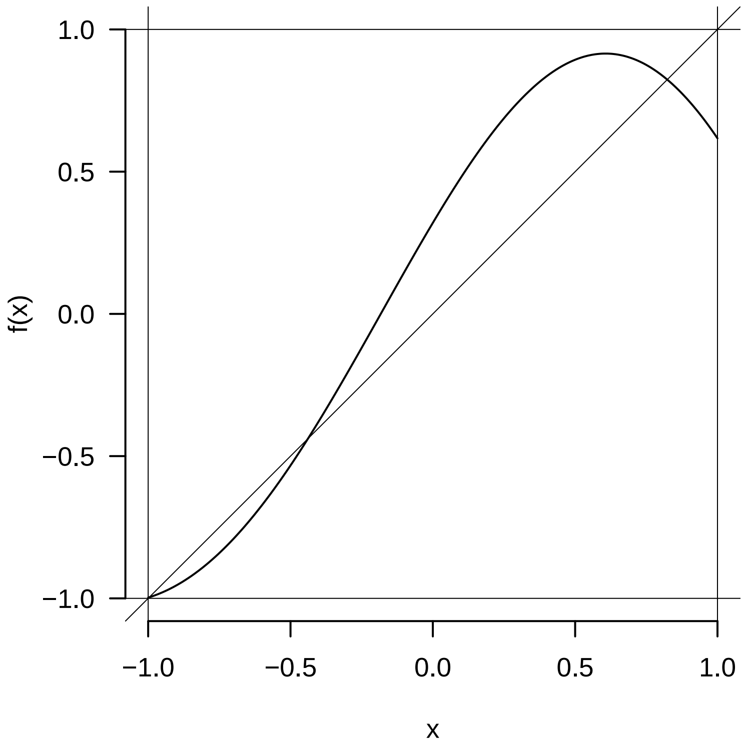
derivative test is inconclusive for the fixed point $x\star = 0$, because $f'(0) = 1$. However, we suspect that it is stable given that *if* there were any other fixed points (which there are not), they would all be stable because $f'(x) < 1$ and positive for all other positive x.

$$x(t+1) = a + x(t)^2, 0 < a < \frac{1}{4} \tag{8}$$

For fixed points we have, $x^2 - x + a = 0$. Therefore, by the quadratic formula we have fixed points $x\star = \frac{1}{2}\left(1 \pm \sqrt{1-4a}\right)$. By the condition on a there is always at least one real solution (usually two). The derivative is $2x$, which at the fixed points is $1 \pm \sqrt{1-4a}$, which is always between 0 and 2. Therefore we don't need to take absolute value with the derivative test. For the larger fixed point the stability criterion is $a > \frac{1}{4}$, which violates the condition. Therefore the larger fixed point is always stable. For the smaller fixed point, the stability criterion is $a < \frac{1}{4}$, which never violates the condition. Therefore the smaller fixed point is stable for all permitted values of a.

1.3 Univariate (graphical approaches)

Assume that $x(t+1) = f(x(t))$ is a discrete-time model with $f(x)$ given by the following graph (thick line, the thin lines are for referrence). Draw on open circle around each unstable fixed point and a closed circle around each stable one. If some points are difficult to classify without algebraic analysis, say so and give a reason why.



The fixed points are where $f(x)$ intersects the 1:1 line (shown). From left to right the fixed points are stable, unstable, and difficult to classify (because it is hard to tell if the slope of f at the last fixed point is greater than or less than -1. If greater than, the point is stable, if less than it is unstable.

1.4 Multivariate linear (eigen approaches)

The multivariate system,

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) \tag{9}$$

always has a fixed point at the origin for any specific $\boldsymbol{A}$. Analyse the stability of this fixed point for the following values for $\boldsymbol{A}$. Recall also that the eigenvalues, d, of a 2-by-2 system like this satisfy $d^2 - Td + \Delta$, where T and Δ are the trace and determinant respectively.

$$\boldsymbol{A} = \begin{bmatrix} 0.25 & 1000 \\ 0 & 0.5 \end{bmatrix} \tag{10}$$

For stability, we need eigenvalues. Well, the trace is $T = 0.75$ and the determinant is $\Delta = 1/8$. The eigenvalues satisfy $d^2 - 3/4d + 1/8 = 0$. By the quadratic formula $d = \frac{6 \pm \sqrt{36-32}}{16} = \frac{6 \pm 2}{16}$, which simplies to two eigenvalues $d = 0.5$ and $d = 0.25$. Therefore the fixed point at the origin is stable because both eigenvalues are positive and less than one.

$$\boldsymbol{A} = \begin{bmatrix} 0.25 & 0.5 \\ 0 & 1 \end{bmatrix} \tag{11}$$

The trace is $T = 1.25$ and the determinant is $\Delta = 0.25$. The eigenvalues satisfy $d^2 - 1.25d + 0.25 = 0$, which factors as $(2d - 0.5)(2d - 2) = 0$. Therefore, the eigenvalues are $d = 1$ and $d = 0.25$. Because the first eigenvalue is one, the origin is not stictly a stable fixed point.

1.5 Multivariate nonlinear (analytical approaches)

$$\begin{aligned} x(t+1) =& x(t) - y(t) + 1 \\ y(t+1) =& y(t) - y(t)^2 + x(t) \end{aligned}$$

$y\star = 1$ by the first equation and $x\star = 1$ by the second. The Jaccobian is $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. Therefore the trace is $\tau = 0$ and determinant is $\delta = 0$. First of all, we have that the determinant δ is less than one and so we can't yet rule out stability. Then we have $|\tau| < \delta + 1$ which in this case is $0 < 1$, which is always true. Therefore, we have stability of $x\star = x\star = 1$.

2 Time dependent solutions

2.1 Time-lagged univariate discrete

Start with MS: pp 27-30. Then try this one for extra practice.

$$x(t+2) - 3x(t+1) + 2x(t) = 0 \tag{12}$$

where $x(0) = 1$ and $x(1) = 0.5$. Hint: begin by guessing that a solution is of the form $x(t) = cd^t$ and then recall that the general solution may be written in the form $x(t) = c_1 d_1^t + c_2 d_2^t$.
Sub in the guess to obtain $d^2 - 3d + 2 = 0$. Factor to obtain $(d-1)(d-2) = 0$. Therefore the eigenvalues are 1 and 2 and the solution is of the form $x(t) = c_1 + c_2(2^t)$. Get the constants from the initial conditions by solving

$1 = c_1 + c_2$ and $0.5 = c_1 + 2c_2$. Therefore $c_1 = 3/2$ and $c_2 = -0.5$ and the final solution is $x(t) = \frac{3}{2} - \frac{1}{2}(2^t)$.

3 R

Assume $x(t+1) = \frac{x(t)}{\sqrt{37-\sin(x(t))}}$ and that $x(0) = 1$. Write no more than five lines of R code (semicolons are not allowed) to calculate $x(10^3)$. What do you think this number will be?

Because the $\sin$ function ranges from -1 to 1, the denominator will range from $\sqrt{36}$ to $\sqrt{38}$. Therefore, x will get cut by a factor of $\frac{1}{\sqrt{37}}$ on average each time step. Therefore, x should be almost zero after 1000 steps. Here is the script in three lines:

```
x <- 1
for(i in 1:1000) x <- x/sqrt(37-sin(x))
print(x)
## [1] 0
```

Assume the continuous model $x(t+1) = x(t) + sin(x(t))$, with initial $x(0) = 0.5$. Write an R script that prints the approximate state of this system at $t = 5$.

```
x <- 0.5
for(i in 1:100) x <- x + sin(x)
print(x)
## [1] 3.142
print(pi)
## [1] 3.142
```

Given the following R commands,

```
set.seed(1)
# 2 vectors of 10 normally distributed random numbers
(x <- rnorm(10))
##  [1] -0.6265  0.1836 -0.8356  1.5953  0.3295
##  [6] -0.8205  0.4874  0.7383  0.5758 -0.3054
(y <- rnorm(10))
##  [1]  1.51178  0.38984 -0.62124 -2.21470  1.12493
##  [6] -0.04493 -0.01619  0.94384  0.82122  0.59390
```

Write one R command to solve each of the following short problems: (a) is the mean of x larger than the mean of y? (b) what are in the first, third, and tenth positions of x? (c) create a vector that contains every element of x such that each element is greater than the corresponding element in y. (d) which elements of y are greater than 0.1? (e) is the inner product of x and y greater than zero? (recall that the inner product is the sum of the element-wise products of two vectors)
Here's the solution:

```
mean(x) > mean(y)
## [1] FALSE
x[c(1,3,10)]
## [1] -0.6265 -0.8356 -0.3054
x[x>y]
## [1] 1.5953 0.4874
which(y>0.1)
## [1]  1  2  5  8  9 10
sum(x*y) > 0
## [1] FALSE
```

Consider the following difference equation,

$$x(t+1) - x(t) = \begin{cases} -0.2 & \text{if } x(t) > 2 \\ 0.5 & \text{if } x(t) \leq 2 \end{cases}$$

where $x(0) = 0$. Write a short R script to calculate the value of x at time 100.
Here's the solution:

```
x <- 0
for(i in 1:20){
  if(x > 2){
    x <- x - 0.2
  } else{
    x <- x + 0.5
  }
}
print(x)
```

```
## [1] 2.3
```

Given a bivariate (2-by-2) system $\boldsymbol{y}(t+1) = \boldsymbol{A}\boldsymbol{y}(t)$ and matrix $\boldsymbol{A}$ entered into R as,

```
(A <- matrix(c(0.1,0.2,0.3,0.4),2,2))
##      [,1] [,2]
## [1,]  0.1  0.3
## [2,]  0.2  0.4
```

Write a short R script to determine whether the system is stable at the origin. Do this even if you can determine stability by inspection. Then print out the first column of $\boldsymbol{A}$.
Here's the solution:

```
all(Mod(eigen(A)$values) < 1)
## [1] TRUE
A[,1]
## [1] 0.1 0.2
```