Chapter 4. Probability models

- A **probability model** is an assignment of probabilities to possible outcomes.
- We don't observe these probabilities. We observe a particular dataset.
- If we treat the dataset as an outcome of a probability model, we can answer questions such as,

"*If there really is no association between unemployment and life expectancy, what is the probability we would see a least squares linear model coefficient as large as the one we actually observed, due to random fluctuations in the data?*"

- Here, we are particularly interested in developing a probability model for the linear model.
- First, we need some basic tools for probability models: random variables, the normal distribution, mean, variance and standard deviation.

Random variables and events

- A **random variable** X is a random number with probabilities assigned to outcomes.

  Example: Let X be a roll of a fair die. A natural probability model is to assign probability of $1/6$ to each of the possible outcomes $1, 2, 3, 4, 5, 6$.

- An **event** is a set of possible outcomes.

  Example: For a die, $E = \{X \geq 4\} = \{4, 5, 6\}$ is the event that the die shows 4 or more.

- We can assign probabilities to events just like to outcomes.

  Example: For a die, $\mathrm{P}(E) = \mathrm{P}(X \geq 4) = 3/6 = 1/2$.

Question 4.1. If an experiment can be repeated many times (like rolling a die) how can you check whether the probability model is correct?

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Notation for combining events

- $\{E \text{ or } F\}$ is the event that either E or F or both happens.
- Since E and F are sets, we can write this as a union, $\{E \text{ or } F\} = E \cup F$
- $\{E \text{ and } F\}$ is the event that both E and F happen.
- We can write this as an intersection,

$$\{E \text{ and } F\} = E \cap F$$

- Usually, we prefer "and/or" to "intersection/union".

Question 4.2. When does this formal use of "and" and "or" agree with usual English usage? When does it disagree?

The basic rules of probability

1. Probabilities are numbers between 0 (impossible) and 1 (certain).
2. Let $\mathcal{S}$ be the set of all possible outcomes. Then, $\mathrm{P}(\mathcal{S}) = 1$.
   Example: For a die, $\mathrm{P}(X \in \{1, 2, 3, 4, 5, 6\}) = 1$.
3. Events E and F are called **mutually exclusive** if they cannot happen at the same time. In other words, their intersection is the empty set. In this case,

$$\mathrm{P}(E \text{ or } F) = \mathrm{P}(E) + \mathrm{P}(F).$$

Question 4.3. You roll a red die and a blue die. Let
$E = \{\text{red die shows 1}\}$, $F = \{\text{blue die shows 1}\}$, $G = \{\text{red die shows 6}\}$.
(a) Are E and F mutually exclusive? (b) How about E and G? (c) How about F and G?

Discrete random variables

- A **discrete random variable** is one where we can list all possible outcomes. Let's call them $x_1, x_2, \ldots$.
- A discrete random variable is specified by probability that the random variable takes each possible outcome,

$$p_i = \mathrm{P}[X = x_i], \text{for } i = 1, 2, 3, \ldots$$

- It can be helpful to plot a graph of p_i against x_i.
- This graph is called the **probability mass function**.

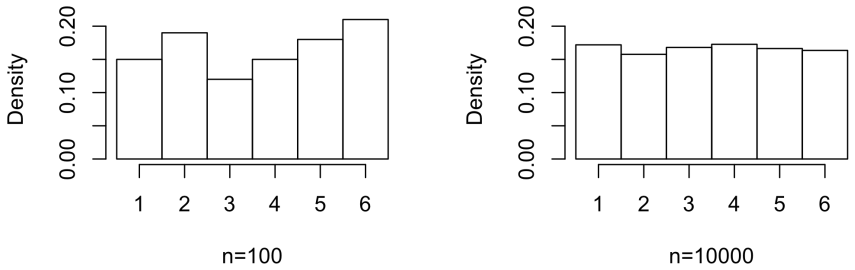
Question 4.4. Sketch the probability mass function for a fair die.

Simulating the law of large numbers

- The "law of large numbers" says that the proportion of each outcome i in a large number of draws of a discrete random variable approaches p_i.
- We can test this by simulation, using the `replicate()` command.

Worked example 4.1. In R, a random draw with replacement from $\{1, 2, 3, 4, 5, 6\}$ can be obtained by `sample(1:6,size=1)` This is equivalent to one roll of a fair die.

```
hist(replicate(n=100,sample(1:6,size=1) ),
  main="",prob=TRUE,breaks=0.5:6.5,xlab="n=100",ylim=c(0,0.21))
```



Continuous random variables: the normal distribution

- A **continuous random variable** is one which can take any value in an interval of the real numbers.

  Example: physical quantities such as time and speed are not limited to a discrete set of possible values.
- We will often see the **normal distribution**.
- Let's look at **normal random variables** simulated by R using `rnorm()`.

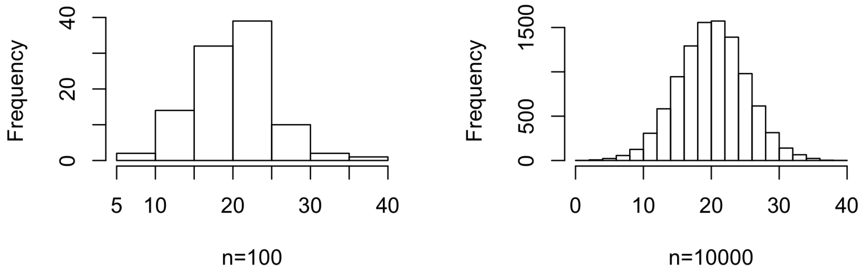
```
rnorm(n=10,mean=20,sd=5)

##  [1] 14.01141 26.18597 17.18972 21.12226 15.20211 30.34660
##  [7] 19.94277 22.05179 27.73804 25.94906
```

- The arguments `mean=20,sd=5` of `rnorm()` are the **parameters** of the normal distribution.
- A normal random variable can take any numeric value: it is continuous.
- Values are centered on the mean and are usually less than twice the standard deviation (sd) from the mean.

A histogram of normal distribution simulations

```r
hist(rnorm(n=100,mean=20,sd=5),
  main="",xlab="n=100")
```



- Large samples from the normal distribution follow a **bell curve** histogram.
- From smaller samples, this is harder to see.

Finding probabilities for a continuous random variable

- A continuous random variable X has a **probability density function** $f(x)$ with

$$\mathrm{P}(a < X < b) = \int_a^b f(x)\,dx$$

- If X is $\mathrm{normal}(\mu, \sigma)$, i.e., X follows the normal distribution with mean μ and sd σ, the probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

and so

$$\mathrm{P}(a < X < b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

- This integral has no closed form solution.
- Fortunately, R provides `pnorm()` and `qnorm()` that let us work with probabilities for the normal distribution numerically.

Calculating probabilities for the normal distribution

- `pnorm()` finds the **left tail** of the normal distribution.
- `pnorm(25,mean=20,sd=5)` computes the shaded area: