

STA 578 - Statistical Computing Notes

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Preface

This is a set of lecture notes that I will use for Northern Arizona University's STA 578 course titled "Statistical Computing". For this course we will primarily be using the textbook "An Introduction to Statistical Learning" by James, Witten, Hastie, and Tibshirani. As such, the notes will primarily be to supplant the text book and these shouldn't be considered comprehensive, stand-alone notes.

Chapter 1

Data Manipulation

Most of the time, our data is in the form of a data frame and we are interested in exploring the relationships. This chapter explores how to manipulate data frames and methods.

1.1 Classic R functions for summarizing rows and columns

1.1.1 `summary()`

The first method is to calculate some basic summary statistics (minimum, 25th, 50th, 75th percentiles, maximum and mean) of each column. If a column is categorical, the summary function will return the number of observations in each category.

```
# use the iris data set which has both numerical and categorical variables
```

```
data( iris )
```

```
str(iris) # recall what columns we have
```

```
## 'data.frame':   150 obs. of  5 variables:
## $ Sepal.Length: num  5.1 4.9 4.7 4.6 5 5.4 4.6 5 4.4 4.9 ...
## $ Sepal.Width : num  3.5 3 3.2 3.1 3.6 3.9 3.4 3.4 2.9 3.1 ...
## $ Petal.Length: num  1.4 1.4 1.3 1.5 1.4 1.7 1.4 1.5 1.4 1.5 ...
## $ Petal.Width : num  0.2 0.2 0.2 0.2 0.2 0.4 0.3 0.2 0.2 0.1 ...
## $ Species      : Factor w/ 3 levels "setosa","versicolor",...: 1 1 1 1 1 1 1 1 1 1 ...
```

```
# display the summary for each column
```

```
summary( iris )
```

```
##   Sepal.Length   Sepal.Width   Petal.Length   Petal.Width
## Min.   :4.300   Min.   :2.000   Min.   :1.000   Min.   :0.100
## 1st Qu.:5.100   1st Qu.:2.800   1st Qu.:1.600   1st Qu.:0.300
## Median :5.800   Median :3.000   Median :4.350   Median :1.300
## Mean   :5.843   Mean   :3.057   Mean   :3.758   Mean   :1.199
## 3rd Qu.:6.400   3rd Qu.:3.300   3rd Qu.:5.100   3rd Qu.:1.800
## Max.   :7.900   Max.   :4.400   Max.   :6.900   Max.   :2.500
##      Species
## setosa   :50
## versicolor:50
## virginica :50
##
```

```
##
```

1.1.2 apply()

The summary function is convenient, but we want the ability to pick another function to apply to each column and possibly to each row. To demonstrate this, suppose we have data frame that contains students grades over the semester.

```
# make up some data
grades <- data.frame(
  l.name = c('Cox', 'Dorian', 'Kelso', 'Turk'),
  Exam1 = c(93, 89, 80, 70),
  Exam2 = c(98, 70, 82, 85),
  Final = c(96, 85, 81, 92) )
```

The `apply()` function will apply an arbitrary function to each row (or column) of a matrix or a data frame and then aggregate the results into a vector.

```
# Because I can't take the mean of the last names column,
# remove the name column
scores <- grades[,-1]
scores
```

```
##   Exam1 Exam2 Final
## 1    93    98    96
## 2    89    70    85
## 3    80    82    81
## 4    70    85    92
```

```
# Summarize each column by calculating the mean.
apply( scores,      # what object do I want to apply the function to
       MARGIN=2,    # rows = 1, columns = 2, (same order as [rows, cols])
       FUN=mean     # what function do we want to apply
     )
```

```
## Exam1 Exam2 Final
## 83.00 83.75 88.50
```

To apply a function to the rows, we just change which margin we want. We might want to calculate the average exam score for person.

```
apply( scores,      # what object do I want to apply the function to
       MARGIN=1,    # rows = 1, columns = 2, (same order as [rows, cols])
       FUN=mean     # what function do we want to apply
     )
```

```
## [1] 95.66667 81.33333 81.00000 82.33333
```

This is useful, but it would be more useful to concatenate this as a new column in my grades data frame.

```
average <- apply(
  scores,      # what object do I want to apply the function to
  MARGIN=1,    # rows = 1, columns = 2, (same order as [rows, cols])
  FUN=mean     # what function do we want to apply
)
grades <- cbind( grades, average ) # squish together
grades
```



```
##   l.name Exam1 Exam2 Final  average
## 1   Cox     93    98   96 95.66667
## 2 Dorian    89    70   85 81.33333
## 3 Kelso     80    82   81 81.00000
## 4  Turk     70    85   92 82.33333
```

There are several variants of the `apply()` function, and the variant I use most often is the function `sapply()`, which will apply a function to each element of a list or vector and returns a corresponding list or vector of results.

1.2 Package dplyr

```
library(dplyr)  # load the dplyr package!
```

Many of the tools to manipulate data frames in R were written without a consistent syntax and are difficult use together. To remedy this, Hadley Wickham (the writer of `ggplot2`) introduced a package called `plyr` which was quite useful. As with many projects, his first version was good but not great and he introduced an improved version that works exclusively with `data.frames` called `dplyr` which we will investigate. The package `dplyr` strives to provide a convenient and consistent set of functions to handle the most common data frame manipulations and a mechanism for chaining these operations together to perform complex tasks.

The Dr Wickham has put together a very nice introduction to the package that explains in more detail how the various pieces work and I encourage you to read it at some point. [<http://cran.rstudio.com/web/packages/dplyr/vignettes/introduction.html>].

One of the aspects about the `data.frame` object is that R does some simplification for you, but it does not do it in a consistent manner. Somewhat obnoxiously character strings are always converted to factors and subsetting might return a `data.frame` or a `vector` or a `scalar`. This is fine at the command line, but can be problematic when programming. Furthermore, many operations are pretty slow using `data.frame`. To get around this, Dr Wickham introduced a modified version of the `data.frame` called a `tibble`. A `tibble` is a `data.frame` but with a few extra bits. For now we can ignore the differences.

The pipe command `%>%` allows for very readable code. The idea is that the `%>%` operator works by translating the command `a %>% f(b)` to the expression `f(a,b)`. This operator works on any function and was introduced in the `magrittr` package. The beauty of this comes when you have a suite of functions that takes input arguments of the same type as their output.

For example if we wanted to start with `x`, and first apply function `f()`, then `g()`, and then `h()`, the usual R command would be `h(g(f(x)))` which is hard to read because you have to start reading at the innermost set of parentheses. Using the pipe command `%>%`, this sequence of operations becomes `x %>% f() %>% g() %>% h()`.

Dr Wickham gave the following example of readability:

```
bopping(
  scooping_up(
    hopping_through(foo_foo),
    field_mice),
  head)
```

is more readably written:

```
foo_foo %>%
  hopping_through(forest) %>%
  scooping_up( field_mice) %>%
  bopping( head )
```

In `dplyr`, all the functions below take a data set as its first argument and outputs an appropriately modified data set. This will allow me to chain together commands in a readable fashion.

1.2.1 Verbs

The foundational operations to perform on a data set are:

- **Subsetting** - Returns a with only particular columns or rows
 - **select** - Selecting a subset of columns by name or column number.
 - **filter** - Selecting a subset of rows from a data frame based on logical expressions.
 - **slice** - Selecting a subset of rows by row number.
- **arrange** - Re-ordering the rows of a data frame.
- **mutate** - Add a new column that is some function of other columns.
- **summarise** - calculate some summary statistic of a column of data. This collapses a set of rows into a single row.

Each of these operations is a function in the package `dplyr`. These functions all have a similar calling syntax, the first argument is a data set, subsequent arguments describe what to do with the input data frame and you can refer to the columns without using the `df$column` notation. All of these functions will return a data set.

1.2.1.1 Subsetting with `select`, `filter`, and `slice`

These function allows you select certain columns and rows of a data frame.

1.2.1.1.1 `select()`

Often you only want to work with a small number of columns of a data frame. It is relatively easy to do this using the standard `[,col.name]` notation, but is often pretty tedious.

```
# recall what the grades are
grades
```

```
##   l.name Exam1 Exam2 Final  average
## 1   Cox    93    98    96 95.66667
## 2 Dorian   89    70    85 81.33333
## 3  Kelso   80    82    81 81.00000
## 4   Turk   70    85    92 82.33333
```

I could select the columns Exam columns by hand, or by using an extension of the `:` operator

```
grades %>% select( Exam1, Exam2 )    # Exam1 and Exam2
```

```
##   Exam1 Exam2
## 1    93    98
## 2    89    70
## 3    80    82
## 4    70    85
```

```
grades %>% select( Exam1:Final )    # Columns Exam1 through Final
```

```
##      Exam1 Exam2 Final
## 1      93     98    96
## 2      89     70    85
## 3      80     82    81
## 4      70     85    92
```

```
grades %>% select( -Exam1 )      # Negative indexing by name works
```

```
##      l.name Exam2 Final  average
## 1      Cox     98     96 95.66667
## 2 Dorian     70     85 81.33333
## 3  Kelso     82     81 81.00000
## 4   Turk     85     92 82.33333
```

```
grades %>% select( 1:2 )      # Can select column by column position
```

```
##      l.name Exam1
## 1      Cox     93
## 2 Dorian     89
## 3  Kelso     80
## 4   Turk     70
```

The `select()` command has a few other tricks. There are functional calls that describe the columns you wish to select that take advantage of pattern matching. I generally can get by with `starts_with()`, `ends_with()`, and `contains()`, but there is a final operator `matches()` that takes a regular expression.

```
grades %>% select( starts_with('Exam') )  # Exam1 and Exam2
```

```
##      Exam1 Exam2
## 1      93     98
## 2      89     70
## 3      80     82
## 4      70     85
```

1.2.1.1.2 filter()

It is common to want to select particular rows where we have some logical expression to pick the rows.

```
# select students with Final grades greater than 90
grades %>% filter(Final > 90)
```

```
##      l.name Exam1 Exam2 Final  average
## 1      Cox     93     98     96 95.66667
## 2   Turk     70     85     92 82.33333
```

You can have multiple logical expressions to select rows and they will be logically combined so that only rows that satisfy all of the conditions are selected. The logicals are joined together using `&` (and) operator or the `|` (or) operator and you may explicitly use other logicals. For example a factor column type might be used to select rows where type is either one or two via the following: `type==1 | type==2`.

```
# select students with Final grades above 90 and
# average score also above 90
grades %>% filter(Final > 90, average > 90)
```

```
##      l.name Exam1 Exam2 Final  average
## 1      Cox     93     98     96 95.66667
```

```
# we could also use an "and" condition
grades %>% filter(Final > 90 & average > 90)
```

```
##   l.name Exam1 Exam2 Final  average
## 1    Cox    93    98    96 95.66667
```

1.2.1.1.3 slice()

When you want to filter rows based on row number, this is called slicing.

```
# grab the first 2 rows
grades %>% slice(1:2)
```

```
## # A tibble: 2 x 5
##   l.name Exam1 Exam2 Final  average
##   <fctr> <dbl> <dbl> <dbl>    <dbl>
## 1    Cox    93    98    96 95.66667
## 2 Dorian    89    70    85 81.33333
```

1.2.1.2 arrange()

We often need to re-order the rows of a data frame. For example, we might wish to take our grade book and sort the rows by the average score, or perhaps alphabetically. The `arrange()` function does exactly that. The first argument is the data frame to re-order, and the subsequent arguments are the columns to sort on. The order of the sorting column determines the precedent... the first sorting column is first used and the second sorting column is only used to break ties.

```
grades %>% arrange(l.name)
```

```
##   l.name Exam1 Exam2 Final  average
## 1    Cox    93    98    96 95.66667
## 2 Dorian    89    70    85 81.33333
## 3 Kelso    80    82    81 81.00000
## 4   Turk    70    85    92 82.33333
```

The default sorting is in ascending order, so to sort the grades with the highest scoring person in the first row, we must tell `arrange` to do it in descending order using `desc(column.name)`.

```
grades %>% arrange(desc(Final))
```

```
##   l.name Exam1 Exam2 Final  average
## 1    Cox    93    98    96 95.66667
## 2   Turk    70    85    92 82.33333
## 3 Dorian    89    70    85 81.33333
## 4 Kelso    80    82    81 81.00000
```

In a more complicated example, consider the following data and we want to order it first by Treatment Level and secondarily by the y-value. I want the Treatment level in the default ascending order (Low, Medium, High), but the y variable in descending order.

```
# make some data
dd <- data.frame(
  Trt = factor(c("High", "Med", "High", "Low"),
    levels = c("Low", "Med", "High")),
  y = c(8, 3, 9, 9),
  z = c(1, 1, 1, 2))
dd
```

```
##   Trt y z
## 1 High 8 1
```

```
## 2 Med 3 1
## 3 High 9 1
## 4 Low 9 2

# arrange the rows first by treatment, and then by y (y in descending order)
dd %>% arrange(Trt, desc(y))

##   Trt y z
## 1 Low 9 2
## 2 Med 3 1
## 3 High 9 1
## 4 High 8 1
```

1.2.1.3 mutate()

I often need to create a new column that is some function of the old columns. This was often cumbersome. Consider code to calculate the average grade in my grade book example.

```
grades$average <- (grades$Exam1 + grades$Exam2 + grades$Final) / 3
```

Instead, we could use the `mutate()` function and avoid all the `grades$` nonsense.

```
grades %>% mutate( average = (Exam1 + Exam2 + Final)/3 )
```

```
##   l.name Exam1 Exam2 Final average
## 1 Cox      93    98    96 95.66667
## 2 Dorian   89    70    85 81.33333
## 3 Kelso    80    82    81 81.00000
## 4 Turk     70    85    92 82.33333
```

You can do multiple calculations within the same `mutate()` command, and you can even refer to columns that were created in the same `mutate()` command.

```
grades %>% mutate(
  average = (Exam1 + Exam2 + Final)/3,
  grade = cut(average, c(0, 60, 70, 80, 90, 100), # cut takes numeric variable
              c( 'F','D','C','B','A')) ) # and makes a factor
```

```
##   l.name Exam1 Exam2 Final average grade
## 1 Cox      93    98    96 95.66667    A
## 2 Dorian   89    70    85 81.33333    B
## 3 Kelso    80    82    81 81.00000    B
## 4 Turk     70    85    92 82.33333    B
```

1.2.1.4 summarise()

By itself, this function is quite boring, but will become useful later on. Its purpose is to calculate summary statistics using any or all of the data columns. Notice that we get to chose the name of the new column. The way to think about this is that we are collapsing information stored in multiple rows into a single row of values.

```
# calculate the mean of exam 1
grades %>% summarise( mean.E1=mean(Exam1))
```

```
##   mean.E1
## 1      83
```

We could calculate multiple summary statistics if we like.

```
# calculate the mean and standard deviation
grades %>% summarise( mean.E1=mean(Exam1), stddev.E1=sd(Exam2) )

##    mean.E1 stddev.E1
## 1      83      11.5
```

If we want to apply the same statistic to each column, we use the `summarise_each()` command. We have to be a little careful here because the function you use has to work on every column (that isn't part of the grouping structure (see `group_by()`)).

```
# calculate the mean and stddev of each column - Cannot do this to Names!
grades %>%
  select( Exam1:Final ) %>%
  summarise_each( funs(mean, sd) )

## `summarise_each()` is deprecated.
## Use `summarise_all()`, `summarise_at()` or `summarise_if()` instead.
## To map `funs` over all variables, use `summarise_all()`

##    Exam1_mean Exam2_mean Final_mean Exam1_sd Exam2_sd Final_sd
## 1      83      83.75      88.5 10.23067      11.5 6.757712
```

1.2.1.5 Miscellaneous functions

There are some more function that are useful but aren't as commonly used. For sampling the functions `sample_n()` and `sample_frac()` will take a subsample of either `n` rows or of a fraction of the data set. The function `n()` returns the number of rows in the data set. Finally `rename()` will rename a selected column.

1.2.2 Split, apply, combine

Aside from unifying the syntax behind the common operations, the major strength of the `dplyr` package is the ability to split a data frame into a bunch of sub-dataframes, apply a sequence of one or more of the operations we just described, and then combine results back together. We'll consider data from an experiment from spinning wool into yarn. This experiment considered two different types of wool (A or B) and three different levels of tension on the thread. The response variable is the number of breaks in the resulting yarn. For each of the 6 `wool:tension` combinations, there are 9 replicated observations per `wool:tension` level.

```
data(warppbreaks)
str(warppbreaks)

## 'data.frame':    54 obs. of  3 variables:
## $ breaks : num  26 30 54 25 70 52 51 26 67 18 ...
## $ wool : Factor w/ 2 levels "A","B": 1 1 1 1 1 1 1 1 1 1 ...
## $ tension: Factor w/ 3 levels "L","M","H": 1 1 1 1 1 1 1 1 1 2 ...
```

The first we must do is to create a data frame with additional information about how to break the data into sub-dataframes. In this case, I want to break the data up into the 6 wool-by-tension combinations. Initially we will just figure out how many rows are in each wool-by-tension combination.

```
# group_by: what variable(s) shall we group one
# n() is a function that returns how many rows are in the
# currently selected sub-dataframe
warppbreaks %>%
  group_by( wool, tension ) %>%      # grouping
  summarise(n = n() )               # how many in each group
```

```
## # A tibble: 6 x 3
## # Groups:   wool [?]
##   wool tension    n
##   <fctr> <fctr> <int>
## 1     A      L     9
## 2     A      M     9
## 3     A      H     9
## 4     B      L     9
## 5     B      M     9
## 6     B      H     9
```

The `group_by` function takes a data.frame and returns the same data.frame, but with some extra information so that any subsequent function acts on each unique combination defined in the `group_by`. If you wish to remove this behavior, use `group_by()` to reset the grouping to have no grouping variable.

Using the same `summarise` function, we could calculate the group mean and standard deviation for each wool-by-tension group.

```
warpbreaks %>%
  group_by(wool, tension) %>%
  summarise( n           = n(),           # I added some formatting to show the
             mean.breaks = mean(breaks), # reader I am calculating several
             sd.breaks   = sd(breaks))    # statistics.
```

```
## # A tibble: 6 x 5
## # Groups:   wool [?]
##   wool tension    n mean.breaks sd.breaks
##   <fctr> <fctr> <int>      <dbl>    <dbl>
## 1     A      L     9    44.55556  18.097729
## 2     A      M     9    24.00000   8.660254
## 3     A      H     9    24.55556  10.272671
## 4     B      L     9    28.22222   9.858724
## 5     B      M     9    28.77778   9.431036
## 6     B      H     9    18.77778   4.893306
```

If instead of summarizing each split, we might want to just do some calculation and the output should have the same number of rows as the input data frame. In this case I'll tell `dplyr` that we are mutating the data frame instead of summarizing it. For example, suppose that I want to calculate the residual value

$$e_{ijk} = y_{ijk} - \bar{y}_{ij}.$$

where \bar{y}_{ij} is the mean of each `wool:tension` combination.

```
warpbreaks %>%
  group_by(wool, tension) %>%           # group by wool:tension
  mutate(resid = breaks - mean(breaks)) %>% # mean(breaks) of the group!
  head( )                               # show the first couple of rows
```

```
## # A tibble: 6 x 4
## # Groups:   wool, tension [1]
##   breaks wool tension  resid
##   <dbl> <fctr> <fctr>    <dbl>
## 1    26     A      L -18.555556
## 2    30     A      L -14.555556
## 3    54     A      L  9.444444
## 4    25     A      L -19.555556
## 5    70     A      L 25.444444
## 6    52     A      L  7.444444
```

1.2.3 Chaining commands together

In the previous examples we have used the `%>%` operator to make the code more readable but to really appreciate this, we should examine the alternative.

Suppose we have the results of a small 5K race. The data given to us is in the order that the runners signed up but we want to calculate the results for each gender, calculate the placings, and then sort the data frame by gender and then place. We can think of this process as having three steps:

1. Splitting
2. Ranking
3. Re-arranging.

```
# input the initial data
race.results <- data.frame(
  name=c('Bob', 'Jeff', 'Rachel', 'Bonnie', 'Derek', 'April', 'Elise', 'David'),
  time=c(21.23, 19.51, 19.82, 23.45, 20.23, 24.22, 28.83, 15.73),
  gender=c('M', 'M', 'F', 'F', 'M', 'F', 'F', 'M'))
```

We could run all the commands together using the following code:

```
arrange(
  mutate(
    group_by(
      race.results,      # using race.results
      gender),           # group by gender
    place = rank( time ), # mutate to calculate the place column
    gender, place)       # arrange the result by gender and place
```

```
## # A tibble: 8 x 4
## # Groups:   gender [2]
##   name  time gender place
##   <fctr> <dbl> <fctr> <dbl>
## 1 Rachel 19.82      F      1
## 2 Bonnie 23.45      F      2
## 3 April  24.22      F      3
## 4 Elise  28.83      F      4
## 5 David  15.73      M      1
## 6 Jeff   19.51      M      2
## 7 Derek  20.23      M      3
## 8 Bob    21.23      M      4
```

This is very difficult to read because you have to read the code *from the inside out*.

Another (and slightly more readable) way to complete our task is to save each intermediate step of our process and then use that in the next step:

```
temp.df0 <- race.results %>% group_by( gender )
temp.df1 <- temp.df0 %>% mutate( place = rank(time) )
temp.df2 <- temp.df1 %>% arrange( gender, place )
```

It would be nice if I didn't have to save all these intermediate results because keeping track of `temp1` and `temp2` gets pretty annoying if I keep changing the order of how things are calculated or add/subtract steps. This is exactly what `%>%` does for me.

```
race.results %>%
  group_by( gender ) %>%
  mutate( place = rank(time) ) %>%
  arrange( gender, place )
```



```
## # A tibble: 8 x 4
## # Groups:   gender [2]
##   name    time gender place
##   <fctr> <dbl> <fctr> <dbl>
## 1 Rachel 19.82     F      1
## 2 Bonnie 23.45     F      2
## 3 April  24.22     F      3
## 4 Elise  28.83     F      4
## 5 David  15.73     M      1
## 6 Jeff   19.51     M      2
## 7 Derek  20.23     M      3
## 8 Bob    21.23     M      4
```

1.3 Reshaping data

```
library(tidyr) # for the gather/spread commands
library(dplyr) # for the join stuff
```

Most of the time, our data is in the form of a data frame and we are interested in exploring the relationships. However most procedures in R expect the data to show up in a ‘long’ format where each row is an observation and each column is a covariate. In practice, the data is often not stored like that and the data comes to us with repeated observations included on a single row. This is often done as a memory saving technique or because there is some structure in the data that makes the ‘wide’ format attractive. As a result, we need a way to convert data from ‘wide’ to ‘long’ and vice-versa.

1.3.1 tidyr

There is a common issue with obtaining data with many columns that you wish were organized as rows. For example, I might have data in a grade book that has several homework scores and I’d like to produce a nice graph that has assignment number on the x-axis and score on the y-axis. Unfortunately this is incredibly hard to do when the data is arranged in the following way:

```
grade.book <- rbind(
  data.frame(name='Alison', HW.1=8, HW.2=5, HW.3=8),
  data.frame(name='Brandon', HW.1=5, HW.2=3, HW.3=6),
  data.frame(name='Charles', HW.1=9, HW.2=7, HW.3=9))
grade.book
```

```
##   name HW.1 HW.2 HW.3
## 1 Alison   8   5   8
## 2 Brandon   5   3   6
## 3 Charles   9   7   9
```

What we want to do is turn this data frame from a *wide* data frame into a *long* data frame. In MS Excel this is called pivoting. Essentially I’d like to create a data frame with three columns: **name**, **assignment**, and **score**. That is to say that each homework datum really has three pieces of information: who it came from, which homework it was, and what the score was. It doesn’t conceptually matter if I store it as 3 columns or 3 rows so long as there is a way to identify how a student scored on a particular homework. So we want to reshape the HW1 to HW3 columns into two columns (assignment and score).

This package was built by the sample people that created dplyr and ggplot2 and there is a nice introduction at: [<http://blog.rstudio.org/2014/07/22/introducing-tidyr/>]

1.3.1.1 Verbs

As with the dplyr package, there are two main verbs to remember:

1. **gather** - Gather multiple columns that are related into two columns that contain the original column name and the value. For example for columns HW1, HW2, HW3 we would gather them into two column HomeworkNumber and Score. In this case, we refer to HomeworkNumber as the key column and Score as the value column. So for any key:value pair you know everything you need.
2. **spread** - This is the opposite of gather. This takes a key column (or columns) and a results column and forms a new column for each level of the key column(s).

```
# first we gather the score columns into columns we'll name Assesment and Score
tidy.scores <- grade.book %>%
  gather( key=Assesment, # What should I call the key column
          value=Score,   # What should I call the values column
          HW.1:HW.3      # which columns to apply this to
        )
tidy.scores
```

```
##      name Assesment Score
## 1  Alison      HW.1      8
## 2 Brandon      HW.1      5
## 3 Charles      HW.1      9
## 4  Alison      HW.2      5
## 5 Brandon      HW.2      3
## 6 Charles      HW.2      7
## 7  Alison      HW.3      8
## 8 Brandon      HW.3      6
## 9 Charles      HW.3      9
```

To spread the key:value pairs out into a matrix, we use the **spread()** command.

```
# Turn the Assessment/Score pair of columns into one column per factor level of Assessment
tidy.scores %>% spread( key=Assesment, value=Score )
```

```
##      name HW.1 HW.2 HW.3
## 1  Alison      8      5      8
## 2 Brandon      5      3      6
## 3 Charles      9      7      9
```

One way to keep straight which is the key column is that the key is the category, while **value** is the numerical value or response.

1.4 Storing Data in Multiple Tables

In many datasets it is common to store data across multiple tables, usually with the goal of minimizing memory used as well as providing minimal duplication of information so any change that must be made is only made in a single place.

To see the rational why we might do this, consider building a data set of blood donations by a variety of donors across several years. For each blood donation, we will perform some assay and measure certain qualities about the blood and the patients health at the donation.

```
## Donor Hemoglobin Systolic Diastolic
## 1 Derek      17.4      121      80
## 2 Jeff       16.9      145     101
```

But now we have to ask, what happens when we have a donor that has given blood multiple times? In this case we should just have multiple rows per person along with a date column to uniquely identify a particular donation.

donations

```
## Donor      Date Hemoglobin Systolic Diastolic
## 1 Derek 2017-04-14      17.4      120      79
## 2 Derek 2017-06-20      16.5      121      80
## 3 Jeff 2017-08-14      16.9      145      101
```

I would like to include additional information about the donor where that information doesn't change overtime. For example we might want to have information about the donor's birthdate, sex, blood type. However, I don't want that information in *every single donation line*. Otherwise if I mistype a birthday and have to correct it, I would have to correct it *everywhere*. For information about the donor, should live in a **donors** table, while information about a particular donation should live in the **donations** table.

Furthermore, there are many Jeffs and Dereks in the world and to maintain a unique identifier (without using Social Security numbers) I will just create a **Donor_ID** code that will uniquely identify a person. Similarly I will create a **Donation_ID** that will uniquely identify a donation.

donors

```
## Donor_ID F_Name L_Name B_Type      Birth      Street      City State
## 1 Donor_1 Derek   Lee      0+ 1976-09-17 7392 Willard Flagstaff AZ
## 2 Donor_2 Jeff    Smith    A 1974-06-23 873 Vine   Bozeman  MT
```

donations

```
## Donation_ID Donor_ID      Date Hemoglobin Systolic Diastolic
## 1 Donation_1 Donor_1 2017-04-14      17.4      120      79
## 2 Donation_2 Donor_1 2017-06-20      16.5      121      80
## 3 Donation_3 Donor_2 2017-08-14      16.9      145      101
```

If we have a new donor walk in and give blood, then we'll have to create a new entry in the **donors** table as well as a new entry in the **donations** table. If an experienced donor gives again, we just have to create a new entry in the **donations** table.

donors

```
## Donor_ID F_Name L_Name B_Type      Birth      Street      City State
## 1 Donor_1 Derek   Lee      0+ 1976-09-17 7392 Willard Flagstaff AZ
## 2 Donor_2 Jeff    Smith    A 1974-06-23 873 Vine   Bozeman  MT
## 3 Donor_3 Aubrey   Lee      0+ 1980-12-15 7392 Willard Flagstaff AZ
```

donations

```
## Donation_ID Donor_ID      Date Hemoglobin Systolic Diastolic
## 1 Donation_1 Donor_1 2017-04-14      17.4      120      79
## 2 Donation_2 Donor_1 2017-06-20      16.5      121      80
## 3 Donation_3 Donor_2 2017-08-14      16.9      145      101
## 4 Donation_4 Donor_1 2017-08-26      17.6      120      79
## 5 Donation_5 Donor_4 2017-08-26      16.1      137      90
```

This data storage set-up might be flexible enough for us. However what happens if somebody moves? If we don't want to keep the historical information, then we could just change the person's **Street_Address**, **City**, and **State** values. If we do want to keep that, then we could create **donor_addresses** table that contains a **Start_Date** and **End_Date** that denotes the period of time that the address was valid.

donor_addresses

```
## Donor_ID      Street      City State Start_Date End_Date
```

```
## 1 Donor_1 346 Treeline Pullman WA 2015-01-26 2016-06-27
## 2 Donor_1 645 Main Flagstsf AZ 2016-06-28 2017-07-02
## 3 Donor_1 7392 Willard Flagstaff AZ 2017-07-03 <NA>
## 4 Donor_2 873 Vine Bozeman MT 2015-03-17 <NA>
## 5 Donor_3 7392 Willard Flagstaff AZ 2017-06-01 <NA>
```

Given this data structure, we can now easily create new donations as well as store donor information. In the event that we need to change something about a donor, there is only *one* place to make that change.

However, having data spread across multiple tables is challenging because I often want that information squished back together. For example, the blood donations services might want to find all ‘O’ or ‘O+’ donors in Flagstaff and their current mailing address and send them some notification about blood supplies being low. So we need someway to join the `donors` and `donor_addresses` tables together in a sensible manner.

1.4.1 Table Joins

Often we need to squish together two data frames but they do not have the same number of rows. Consider the case where we have a data frame of observations of fish and a separate data frame that contains information about lake (perhaps surface area, max depth, pH, etc). I want to store them as two separate tables so that when I have to record a lake level observation, I only input it *one* place. This decreases the chance that I make a copy/paste error.

To illustrate the different types of table joins, we’ll consider two different tables.

```
# tibbles are just data.frames that print a bit nicer and don't automaticall
# convert character columns into factors. They behave a bit more consistently
# in a wide variety of situations compared to data.frames.
```

```
Fish.Data <- tibble(
  Lake_ID = c('A','A','B','B','C','C'),
  Fish.Weight=rnorm(6, mean=260, sd=25) ) # make up some data
Lake.Data <- tibble(
  Lake_ID = c('B','C','D'),
  Lake_Name = c('Lake Elaine', 'Mormon Lake', 'Lake Mary'),
  pH=c(6.5, 6.3, 6.1),
  area = c(40, 210, 240),
  avg_depth = c(8, 10, 38))
```

Fish.Data

```
## # A tibble: 6 x 2
##   Lake_ID Fish.Weight
##   <chr>      <dbl>
## 1      A    238.8015
## 2      A    261.9207
## 3      B    316.0419
## 4      B    267.4112
## 5      C    248.0787
## 6      C    247.5060
```

Lake.Data

```
## # A tibble: 3 x 5
##   Lake_ID Lake_Name    pH  area avg_depth
##   <chr>      <chr> <dbl> <dbl>    <dbl>
## 1      B Lake Elaine  6.5   40      8
## 2      C Mormon Lake  6.3  210     10
## 3      D Lake Mary   6.1  240     38
```

Notice that each of these tables has a column labeled `Lake_ID`. When we join these two tables, the row that describes lake A should be duplicated for each row in the `Fish.Data` that corresponds with fish caught from lake A.

```
full_join(Fish.Data, Lake.Data)
```

```
## Joining, by = "Lake_ID"
```

```
## # A tibble: 7 x 6
```

	Lake_ID	Fish.Weight	Lake_Name	pH	area	avg_depth
	<chr>	<dbl>	<chr>	<dbl>	<dbl>	<dbl>
## 1	A	238.8015	<NA>	NA	NA	NA
## 2	A	261.9207	<NA>	NA	NA	NA
## 3	B	316.0419	Lake Elaine	6.5	40	8
## 4	B	267.4112	Lake Elaine	6.5	40	8
## 5	C	248.0787	Mormon Lake	6.3	210	10
## 6	C	247.5060	Mormon Lake	6.3	210	10
## 7	D	NA	Lake Mary	6.1	240	38

Notice that because we didn't have any fish caught in lake D and we don't have any Lake information about lake A, when we join these two tables, we end up introducing missing observations into the resulting data frame.

The other types of joins govern the behavior of these missing data.

`left_join(A, B)` For each row in A, match with a row in B, but don't create any more rows than what was already in A.

`inner_join(A,B)` Only match row values where both data frames have a value.

```
left_join(Fish.Data, Lake.Data)
```

```
## Joining, by = "Lake_ID"
```

```
## # A tibble: 6 x 6
```

	Lake_ID	Fish.Weight	Lake_Name	pH	area	avg_depth
	<chr>	<dbl>	<chr>	<dbl>	<dbl>	<dbl>
## 1	A	238.8015	<NA>	NA	NA	NA
## 2	A	261.9207	<NA>	NA	NA	NA
## 3	B	316.0419	Lake Elaine	6.5	40	8
## 4	B	267.4112	Lake Elaine	6.5	40	8
## 5	C	248.0787	Mormon Lake	6.3	210	10
## 6	C	247.5060	Mormon Lake	6.3	210	10

```
inner_join(Fish.Data, Lake.Data)
```

```
## Joining, by = "Lake_ID"
```

```
## # A tibble: 4 x 6
```

	Lake_ID	Fish.Weight	Lake_Name	pH	area	avg_depth
	<chr>	<dbl>	<chr>	<dbl>	<dbl>	<dbl>
## 1	B	316.0419	Lake Elaine	6.5	40	8
## 2	B	267.4112	Lake Elaine	6.5	40	8
## 3	C	248.0787	Mormon Lake	6.3	210	10
## 4	C	247.5060	Mormon Lake	6.3	210	10

The above examples assumed that the column used to join the two tables was named the same in both tables. This is good practice to try to do, but sometimes you have to work with data where that isn't the case. In that situation you can use the `by=c("ColName.A"="ColName.B")` syntax where `ColName.A` represents the name of the column in the first data frame and `ColName.B` is the equivalent column in the second data frame.

1.5 Exercises

1. The dataset `ChickWeight` tracks the weights of 48 baby chickens (chicks) feed four different diets.

- a. Load the dataset using

```
data(ChickWeight)
```

- b. Look at the help files for the description of the columns.
- c) Remove all the observations except for the weights on day 10 and day 20.
- d) Calculate the mean and standard deviation for each diet group on days 10 and 20.

2. The OpenIntro textbook on statistics includes a data set on body dimensions.

- a) Load the file using

```
Body <- read.csv('http://www.openintro.org/stat/data/bdims.csv')
```

- b) The column `sex` is coded as a 1 if the individual is male and 0 if female. This is a non-intuitive labeling system. Create a new column `sex.MF` that uses labels Male and Female.
- c) The columns `wgt` and `hgt` measure weight and height in kilograms and centimeters (respectively). Use these to calculate the Body Mass Index (BMI) for each individual where

$$BMI = \frac{Weight(kg)}{[Height(m)]^2}$$

- d) Double check that your calculated BMI column is correct by examining the summary statistics of the column. BMI values should be between 18 to 40 or so. Did you make an error in your calculation?
- e) The function `cut` takes a vector of continuous numerical data and creates a factor based on your give cut-points.

```
# Define a continuous vector to convert to a factor
x <- 1:10

# divide range of x into three groups of equal length
cut(x, breaks=3)

# divide x into four groups, where I specify all 5 break points
cut(x, breaks = c(0, 2.5, 5.0, 7.5, 10))
# (0,2.5] (2.5,5] means 2.5 is included in first group
# right=FALSE changes this to make 2.5 included in the second

# divide x into 3 groups, but give them a nicer
# set of group names
cut(x, breaks=3, labels=c('Low','Medium','High'))
```

Create a new column of in the data frame that divides the age into decades (10-19, 20-29, 30-39, etc). Notice the oldest person in the study is 67.

```
Body <- Body %>%
  mutate( Age.Grp = cut(age,
                        breaks=c(10,20,30,40,50,60,70),
                        right=FALSE))
```

- f) Find the average BMI for each Sex and Age group.

3. Suppose we are given information about the maximum daily temperature from a weather station in Flagstaff, AZ. The file is available at the GitHub site that this book is hosted on.

```
FlagTemp <- read.csv(
  'https://github.com/dereksonderegger/570L/raw/master/data-raw/FlagMaxTemp.csv',
  header=TRUE, sep=',')
```

This file is in a wide format, where each row represents a month and the columns X1, X2, ..., X31 represent the day of the month the observation was made.

- Convert data set to the long format where the data has only four columns: **Year**, **Month**, **Day**, **Tmax**.
 - Calculate the average monthly maximum temperature for each Month in the dataset (So there will be 365 mean maximum temperatures). *You'll probably have some issues taking the mean because there are a number of values that are missing and by default R refuses to take means and sums when there is missing data. The argument `na.rm=TRUE` to `mean()` allows you to force R to remove the missing observations before calculating the mean.*
 - Convert the average month maximums back to a wide data format where each line represents a year and there are 12 columns of temperature data (one for each month) along with a column for the year. *There will be a couple of months that still have missing data because the weather station was out of commission for those months and there was NO data for the entire month.*
4. A common task is to take a set of data that has multiple categorical variables and create a table of the number of cases for each combination. An introductory statistics textbook contains a dataset summarizing student surveys from several sections of an intro class. The two variables of interest for us are **Gender** and **Year** which are the students gender and year in college.

- Download the dataset and correctly order the **Year** variable using the following:

```
Survey <- read.csv('http://www.lock5stat.com/datasets/StudentSurvey.csv', na.strings=c('', ' '))
mutate(Year = factor(Year, levels=c('FirstYear', 'Sophomore', 'Junior', 'Senior')))
```

- Using some combination of **dplyr** functions, produce a data set with eight rows that contains the number of responses for each gender:year combination. *Notice there are two females that neglected to give their Year and you should remove them first. The function `is.na(Year)` will return logical values indicating if the Year value was missing and you can flip those values using the negation operator `!`. So you might consider using `!is.na(Year)` as the argument to a `filter()` command. Alternatively you sort on Year and remove the first two rows using `slice(-2:-1)`. Next you'll want to summarize each Year/Gender group using the `n()` function which gives the number of rows in a data set.*
- Using **tidyr** commands, produce a table of the number of responses in the following form:

Gender	First Year	Sophomore	Junior	Senior
Female				
Male				

5. The package **nycflights** contains information about all the flights that arrived in or left from New York City in 2013. This package contains five data tables, but there are three data tables we will work with. The data table **flights** gives information about a particular flight, **airports** gives information about a particular airport, and **airlines** gives information about each airline. Create a table of all the flights on February 14th by Virgin America that has columns for the carrier, destination, departure time, and flight duration. Join this table with the airports information for the destination. Notice that because the column for the destination airport code doesn't match up between **flights** and **airports**, you'll have to use the `by=c("TableA.Col"="TableB.Col")` argument where you insert the correct

names for `TableA.Col` and `TableB.Col`.

Chapter 2

Markov Chain Monte Carlo

```
library(ggplot2)
library(dplyr)
library(devtools)
install_github('dereksonderegger/STA578') # Some routines I created for this
library(STA578)
```

Modern statistical methods often rely on being able to produce random numbers from an arbitrary distribution. In this chapter we will explore how this is done.

2.1 Generating $U \sim \text{Uniform}(0, 1)$

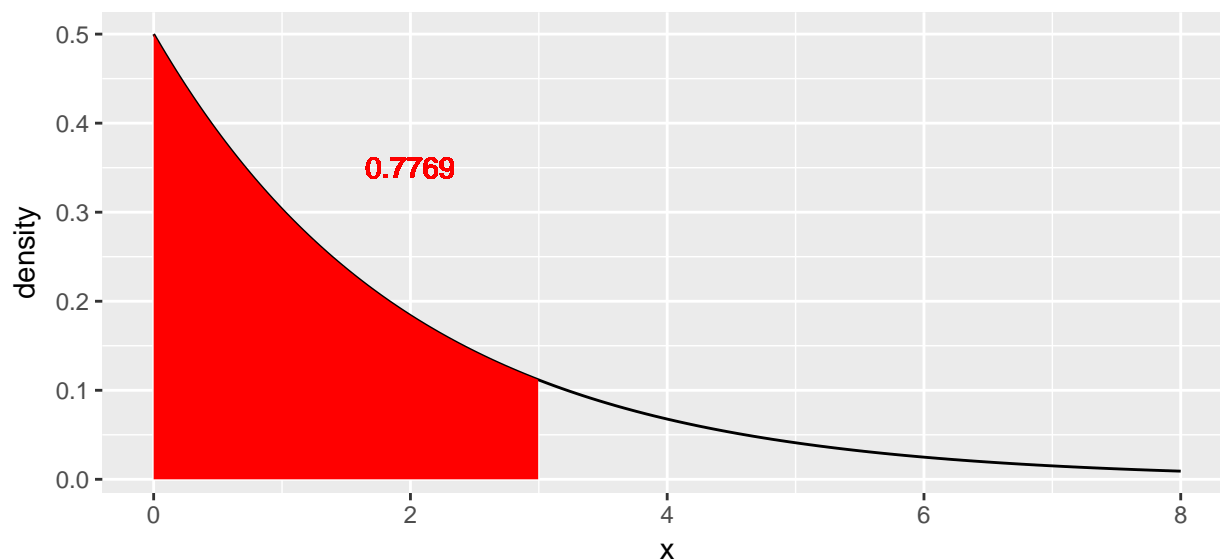
It is extremely difficult to generate actual random numbers but it is possible to generate pseudo-random numbers. That is, we'll generate a sequence of digits, say 1,000,000,000 long such that any sub-sequence looks like we are just drawing digits [0-9] randomly. Then given this sequence, we chose a starting point (perhaps something based off the clock time of the computer we are using). From the starting point, we generate $U \sim \text{Uniform}(0, 1)$ numbers by using just reading off successive digits.

In practice there are many details of the above algorithm that are quite tricky, but we will ignore those issues and assume we have some method for producing random samples from $\text{Uniform}(0, 1)$ distribution.

2.2 Inverse CDF Method

Suppose that we wish to generate a random sample from a given distribution, say, $X \sim \text{Exp}(\lambda = 1/2)$. This distribution is pretty well understood and it is easy to calculate various probabilities. The density function is $f(x) = \lambda \cdot e^{-x\lambda}$ and we can easily calculate probabilities such as

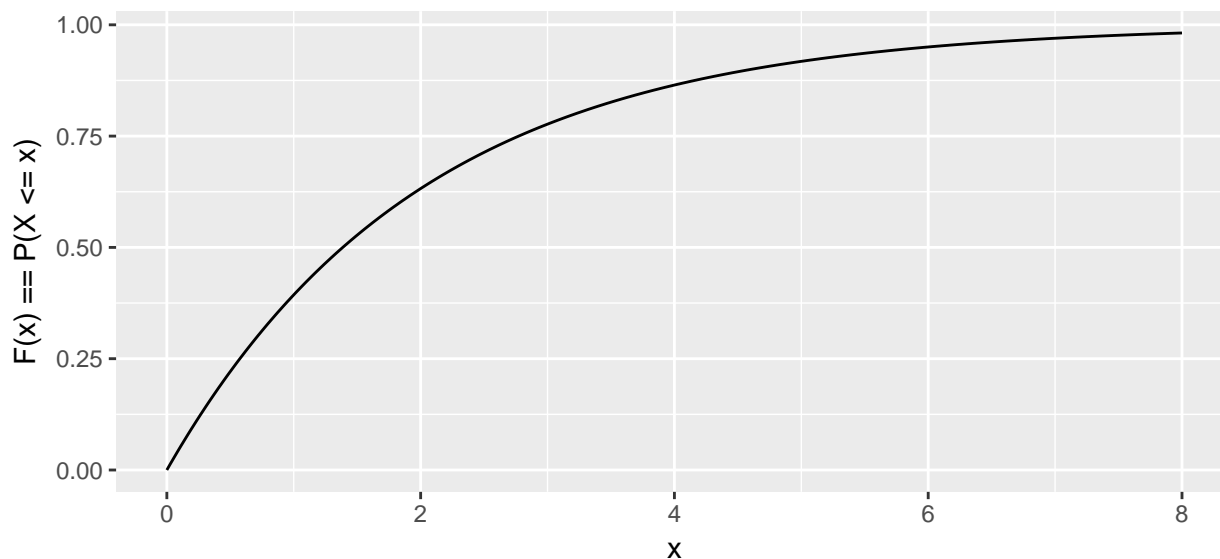
$$\begin{aligned} P(X \leq 3) &= \int_0^3 \lambda e^{-x\lambda} dx \\ &= e^{-0\lambda} - e^{-3\lambda} \\ &= 1 - e^{-3\lambda} \\ &= 0.7769 \end{aligned}$$



Given this result, it is possible to figure out $P(X \leq x)$ for any value of x . (Here the capital X represents the random variable and the lower case x represents a particular value that this variable can take on.) Now thinking of these probabilities as a function, we define the cumulative distribution function (CDF) as

$$F(x) = P(X \leq x) = 1 - e^{-x\lambda}$$

If we make a graph of this function we have



Given this CDF, we if we can generate a $U \sim \text{Uniform}(0,1)$, we can just use the CDF function in reverse (i.e the inverse CDF) and transform the U to be an $X \sim \text{Exp}(\lambda)$ random variable. In R, most of the common distributions have a function that calculates the inverse CDF, in the exponential distribution it is `qexp(x, rate)` and for the normal it would be `qnorm()`, etc.

```
U <- runif(10000, min=0, max=1) # 10,000 Uniform(0,1) values
X <- qexp(U, rate=1/2)
par(mfrow=c(1,2))
hist(U)
hist(X)
```

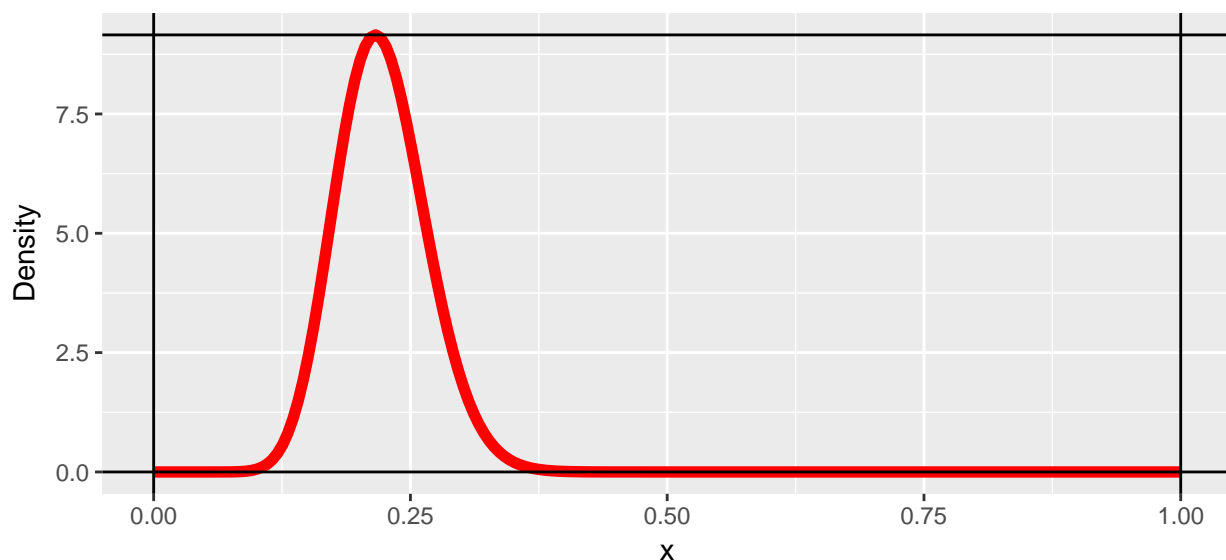


This is the type of trick that Statistics students might learn in a probability course, but this is hardly interesting from a computationally intensive perspective, so if you didn't follow the calculus, don't fret.

2.3 Accept/Reject Algorithm

We now consider a case where we don't know the CDF (or it is really hard to work with). Let the random variable X which can take on values from $0 \leq X \leq 1$ and has probability density function $f(x)$. Furthermore, suppose that we know what the maximum value of the density function is, which we'll denote $M = \max f(x)$.

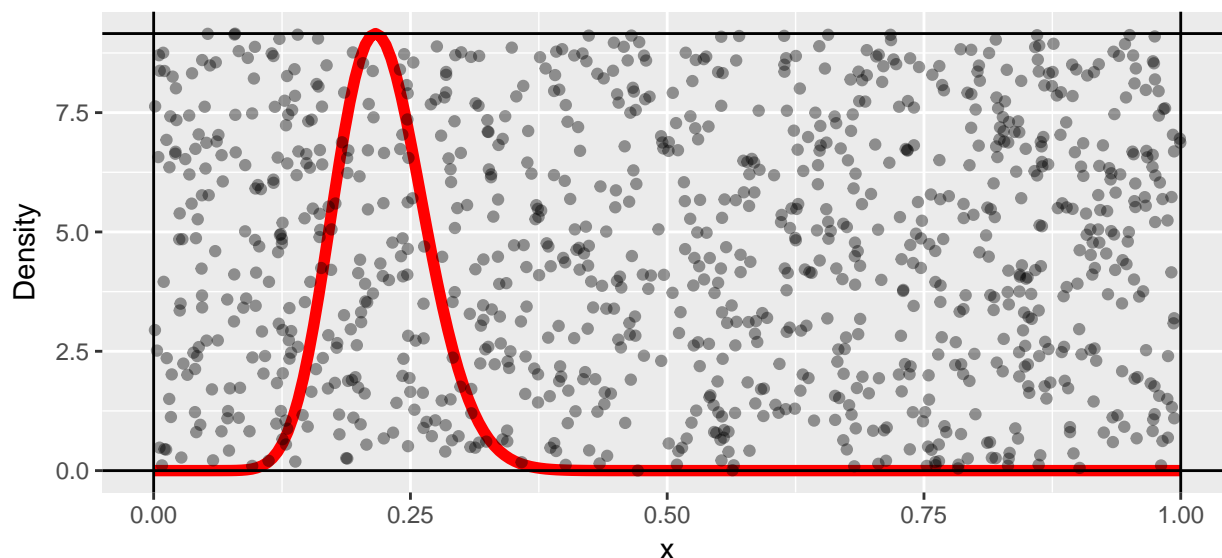
```
x <- seq(0,1, length=200)
y <- dbeta(x, 20, 70)
M <- max(y)
data.line <- data.frame(x=x, y=y)
p <- ggplot(data.line, aes(x=x, y=y)) + geom_line(color='red', size=2) +
  labs(y='Density') +
  geom_vline(xintercept=c(0,1)) +
  geom_hline(yintercept=c(0, max(y)))
p
```



It is trivial to generate points that are uniformly distributed in the rectangle by randomly selecting points

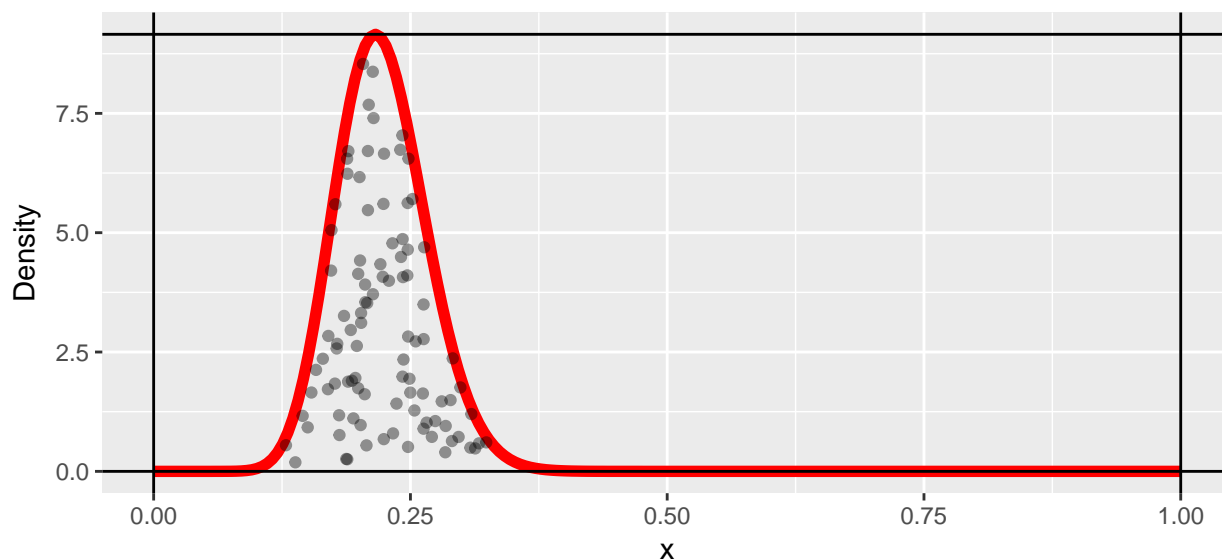
(x_i, y_i) by letting x_i be randomly sampled from a $Uniform(0,1)$ distribution and y_i be sampled from a $Uniform(0,M)$ distribution. Below we sample a thousand points.

```
N <- 1000
x <- runif(N, 0,1)
y <- runif(N, 0, M)
proposed <- data.frame(x=x, y=y)
p + geom_point(data=proposed, alpha=.4)
```



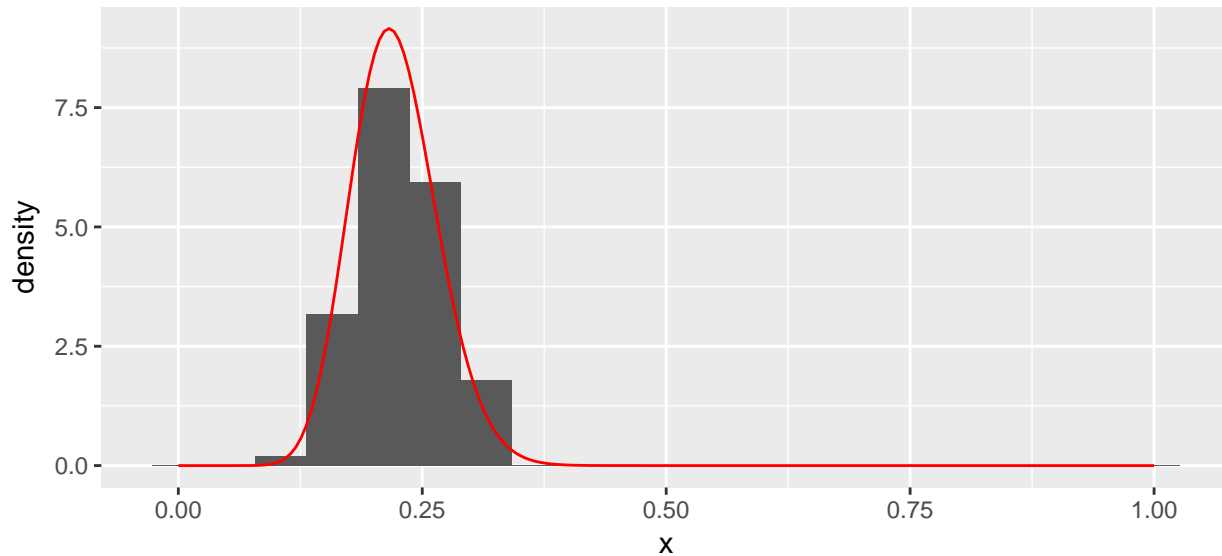
Since we want to select a random sample from the curved distribution (and not uniformly from the box), I will reject pairs (x_i, y_i) if $y_i \geq f(x_i)$. This leaves us with the following points.

```
accepted <- proposed %>%
  filter( y <= dbeta(x, 20,70) )
p + geom_point(data=accepted, alpha=.4)
```



We can now regard those x_i values as a random sample from the distribution $f(x)$ and the histogram of those values is a good approximation to the distribution $f(x)$.

```
ggplot(data=accepted, aes(x=x)) +
  geom_histogram(aes(y=..density..), bins=20) +
  geom_line(data=data.line, aes(x=x,y=y), color='red')
```



Clearly this is a fairly inefficient way to generate points from a distribution, but it contains a key concept

$$x_i \text{ is accepted if } u_i < \frac{f(x_i)}{M}$$

where $y_i = u_i \cdot M$ is the height of the point and u_i is a random observation from the $Uniform(0,1)$ distribution.

Pros/Cons

- Pro: This technique is very simple to program.
- Cons: Need to know the maximum height of the distribution.
- Cons: This can be a very inefficient algorithm as most of the proposed values are thrown away.

2.4 MCMC algorithm

Suppose that we have one observation, x_1 , from the distribution $f(x)$ and we wish to create a whole sequence of observations x_2, \dots, x_n that are also from that distribution. The following algorithm will generate x_{i+1} given the value of x_i .

1. Generate a proposed x_{i+1}^* from a distribution that is symmetric about x_i . For example $x_{i+1}^* \sim N(x_i, \sigma = 1)$.
2. Generate a random variable u_{i+1} from a $Uniform(0,1)$ distribution.
3. If

$$u_{i+1} \leq \frac{f(x_{i+1}^*)}{f(x_i)}$$

then we accept x_{i+1}^* and define $x_{i+1} = x_{i+1}^*$, otherwise reject x_{i+1}^* and define $x_{i+1} = x_i$

The idea of this algorithm is that if we propose an x_{i+1}^* value that has a higher density than x_i , we should always accept that value as the next observation in our chain. If we propose a value that has lower density, then we should *sometimes* accept it, and the correct probability of accepting it is the ratio of the probability

density function. If we propose a value that has a much lower density, then we should rarely accept it, but if we propose a value that has only a slightly lower density, then we should usually accept it.

There is theory that shows that a large sample drawn as described will have the desired proportions that match the distribution of interest. However, the question of “how large is large enough” is a very difficult question.

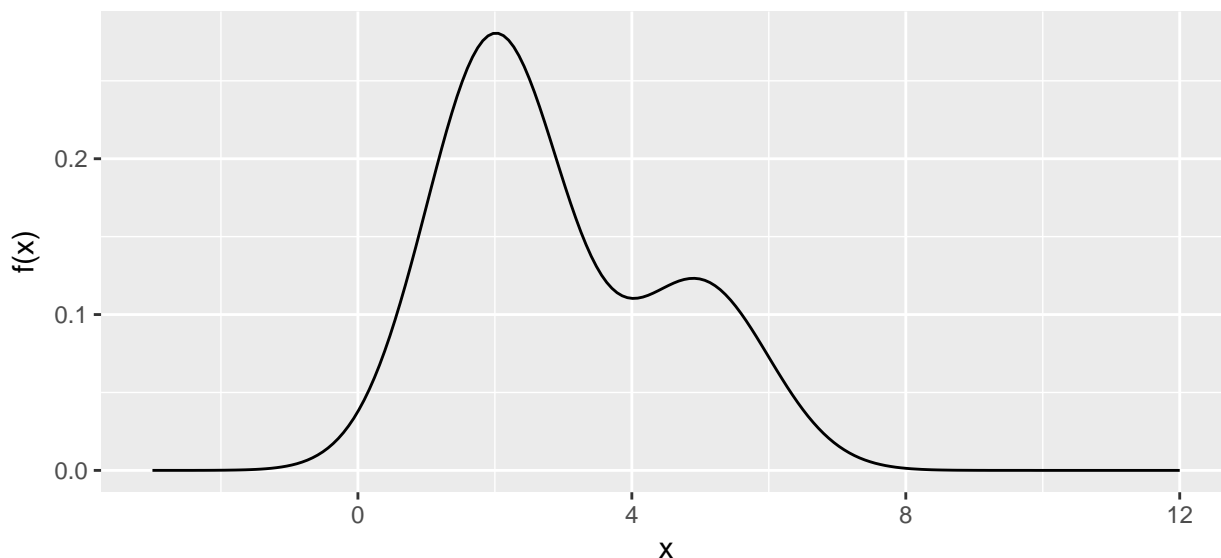
Actually we only need to know the distribution up to a scaling constant. Because we’ll look at the ratio $f(x^*)/f(x)$ any constants that don’t depend on x will cancel. For Bayesians, this is a crucial point.

2.4.1 Mixture of normals

We consider the problem of generating a random sample from a mixture of normal distributions. We first define our distribution $f(x)$

```
df <- function(x){
  return(.7*dnorm(x, mean=2, sd=1) + .3*dnorm(x, mean=5, sd=1))
}
```

```
x <- seq(-3,12, length=200)
density.data <- data.frame(x=x, y=df(x))
density <- ggplot( density.data, aes(x=x, y=y)) +
  geom_line() +
  labs(y='f(x)', x='x')
density
```



It is worth noting at this point that the variance of this distribution is $\sigma = 2.89$.

```
mu <- integrate( function(x){ x * df(x) }, lower=-Inf, upper=Inf)
mu <- mu$value
sigma2 <- integrate( function(x){ (x-mu)^2 * df(x) }, lower=-Inf, upper=Inf)
c( mu=mu, sigma2=sigma2$value )
```

```
##      mu sigma2
##    2.90    2.89
```

Next we define our proposal distribution, which will be $Uniform(x_i - 2, x_i + 2)$.

```
rproposal <- function(x.i){
  out <- x.i + runif(1, -2, 2) # x.i + 1 observation from Uniform(-2, 2)
  return(out)
}
```

Starting from $x_1 = 3$, we will randomly propose a new value.

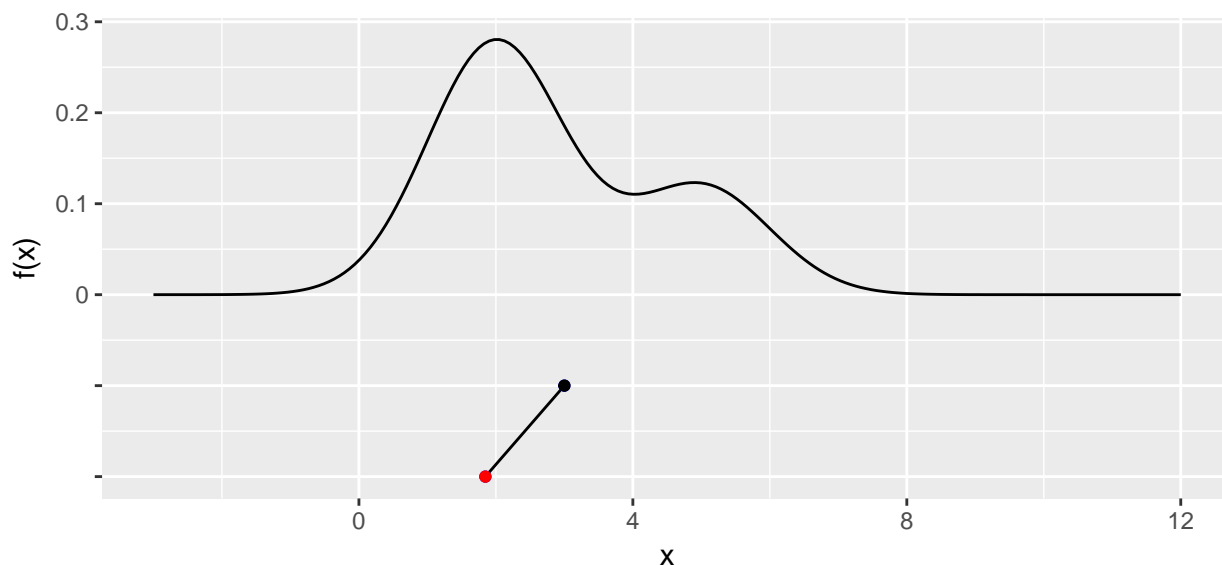
```
x <- 3;           # starting value for the chain
x.star <- 3       # initialize the vector of proposal values
x.star[2] <- rproposal( x[1] )
x.star[2]
```

```
## [1] 1.846296
```

```
if( df(x.star[2]) / df(x[1]) > runif(1) ){
  x[2] <- x.star[2]
}else{
  x[2] <- x[1]
}
```

We proposed a value of $x = 1.846$ and because $f(1.846) \geq f(3)$ then this proposed value must be accepted (because the ratio $f(x_2^*)/f(x_1) \geq 1$ and is therefore greater than $u_2 \in [0, 1]$). We plot the chain below the density plot to help visualize the process.

```
STA578::plot_chain(density, x, x.star)
```



Next we will repeat the process for ten iterations. If a proposed point is rejected, it will be shown in blue and not connected to the chain. The connected line will be called a traceplot and is usually seen rotated counter-clockwise by 90 degrees.

```
for( i in 2:10 ){
  x.star[i+1] <- rproposal( x[i] )
  if( df(x.star[i+1]) / df(x[i]) > runif(1) ){
    x[i+1] <- x.star[i+1]
  }else{
    x[i+1] <- x[i]
  }
}
```

```
STA578::plot_chain(density, x, x.star)
```



We see that the third step in the chain, we proposed a value near $x = 1$ and it was rejected. Similarly steps 5 and 9 were rejected because $f(x_{i+1}^*)$ of the proposed value was very small compared to $f(x_i)$ and thus the proposed value was unlikely to be accepted.

At this point, we should make one final adjustment to the algorithm. In general, it is not necessary for the proposal distribution to be symmetric about x_i . Let $g(x_{i+1}^*|x_i)$ be the density function of the proposal distribution, then the appropriate accept/reject criteria is modified to be

- If

$$u_{i+1} \leq \frac{f(x_{i+1}^*)}{f(x_i)} \frac{g(x_i|x_{i+1}^*)}{g(x_{i+1}^*|x_i)}$$

then we accept x_{i+1}^* and define $x_{i+1} = x_{i+1}^*$, otherwise reject x_{i+1}^* and define $x_{i+1} = x_i$

Notice that if $g(\cdot)$ is symmetric about x_i then $g(x_i|x_{i+1}^*) = g(x_{i+1}^*|x_i)$ and the second fraction is simply 1.

It will be convenient to hide the looping part of the MCMC, so we'll create a simple function to handle the details. We'll also make a program to make a traceplot of the chain.

```
?MCMC
```

```
# You can look at the source code of many functions by just typing the name.  
# This trick unfortunately doesn't work with functions that are just wrappers  
# to C or C++ programs or are hidden due to Object Oriented inheritance.
```

```
MCMC
```

```
## function (df, start, rprop, dprop = NULL, N = 1000)  
## {  
##   if (is.null(dprop)) {  
##     dprop <- function(to, from) {  
##       1  
##     }  
##   }  
##   chain <- rep(NA, N)  
##   chain[1] <- start  
##   for (i in 1:(N - 1)) {  
##     x.star <- rprop(chain[i])  
##     r1 <- df(x.star)/df(chain[i])
```



```
##      r2 <- dprop(chain[i], x.star)/dprop(x.star, chain[i])
##      if (r1 * r2 > runif(1)) {
##          chain[i + 1] <- x.star
##      }
##      else {
##          chain[i + 1] <- chain[i]
##      }
##  }
##  return(chain)
## }
## <environment: namespace:STA578>
```

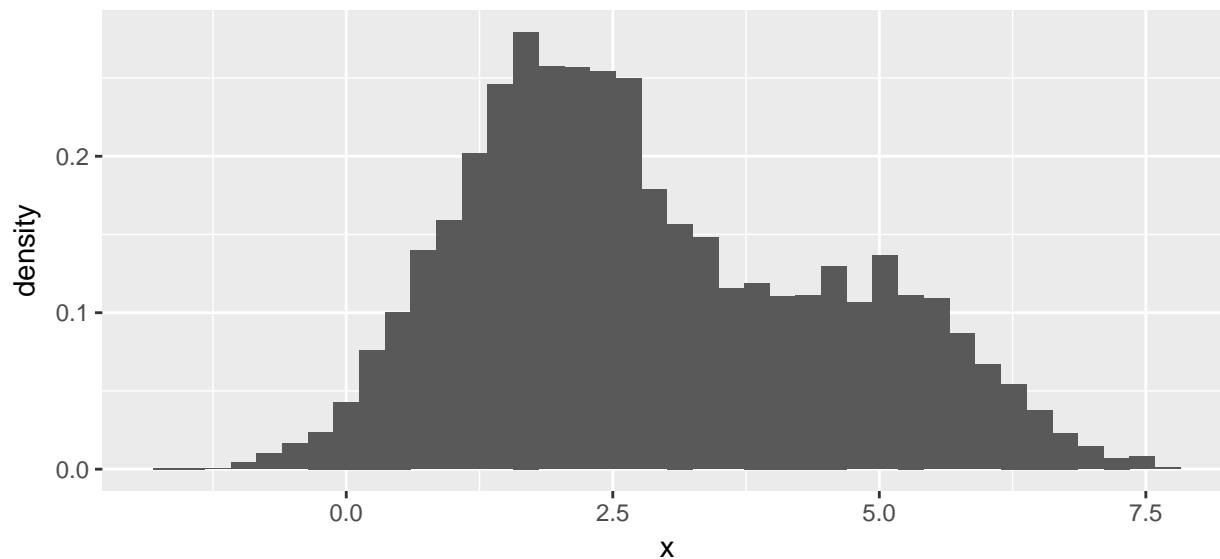
We should let this process run for a long time, and we will just look the traceplot.

```
chain <- STA578::MCMC(df, 2, rproposal, N=1000) # do this new!
STA578::trace_plot(chain)
```



This seems like it is working as the chain has occasional excursions out to the tails of the distribution, but is mostly concentrated in the peaks of the distribution. We'll run it longer and then examine a histogram of the resulting chain.

```
chain2 <- MCMC(df, chain[1000], rproposal, N=10000)
long.chain <- c(chain, chain2)
ggplot(data.frame(x=long.chain), aes(x=x, y=..density..)) +
  geom_histogram(bins=40)
```



All in all this looks quite good.

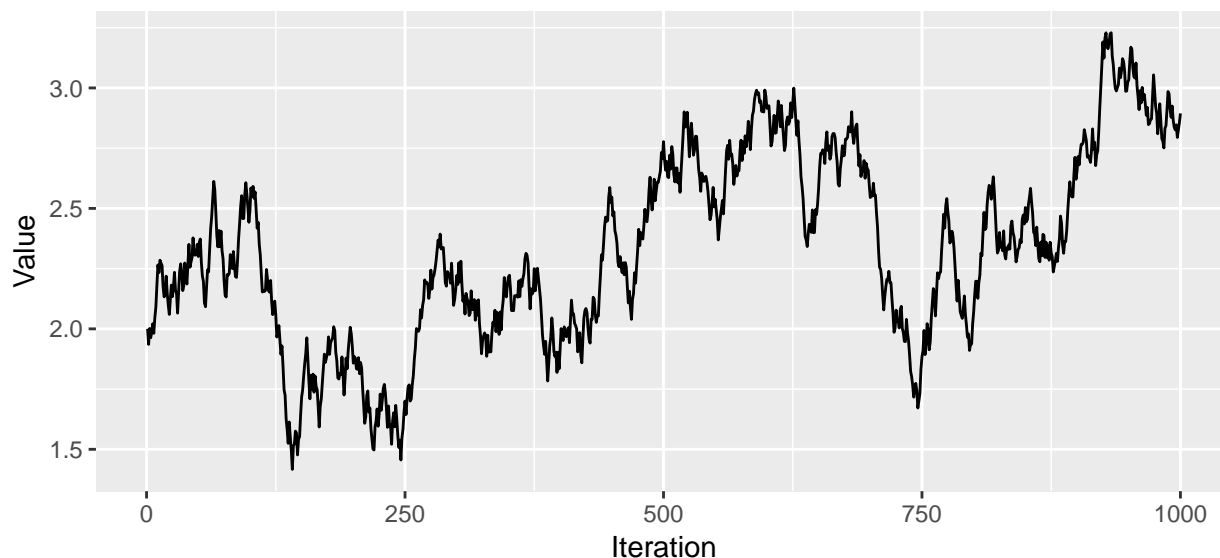
2.4.2 Common problems

Proposal Distribution

The proposal distribution plays a critical role in how well the MCMC simulation explores the distribution of interest. If the variance of the proposal distribution is too small, then the number of steps to required to move a significant distance across the distribution becomes large. If the variance is too large, then a large percentage of the proposed values will be far from the center of the distribution and will be rejected.

First, we create a proposal distribution with a small variance and examine its behavior.

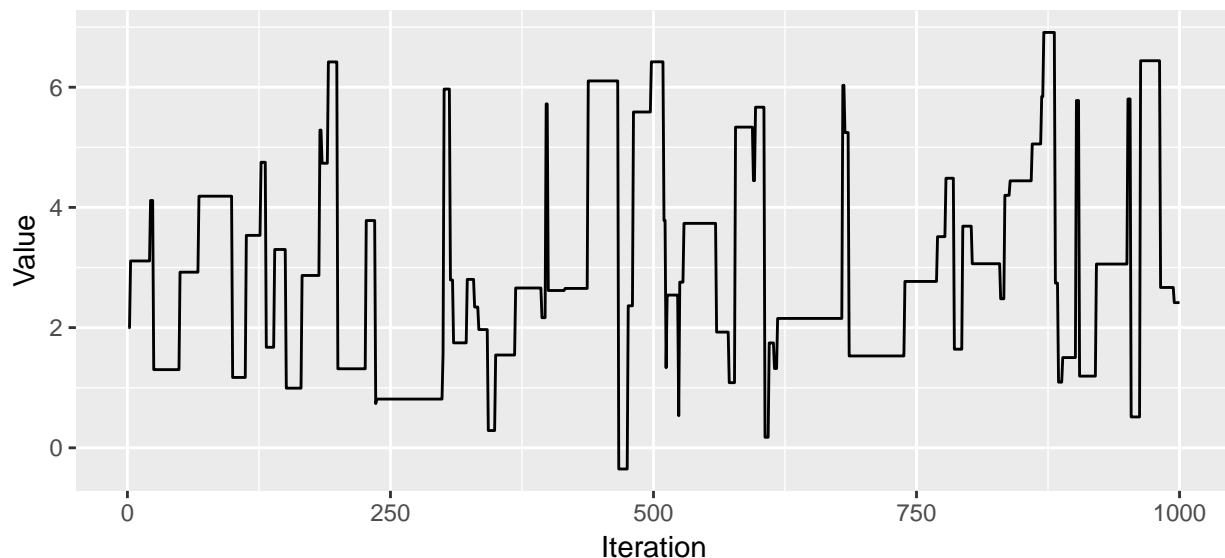
```
p.small <- function(x){
  return( x + runif(1, -.1, +.1) )
}
chain <- MCMC(df, 2, p.small, N=1000)
trace_plot(chain)
```



In these 1000 steps, the chain has not gone smaller than 1 or larger than 3.5 and hasn't explored the second "hump" of the distribution yet. If the valley between the two humps was deeper, it is possible that the chain would never manage to cross the valley and find the second hump.

The effect of too large of variance is also troublesome.

```
p.large <- function(x){
  return( x + runif(1, -30, +30) )
}
chain <- MCMC(df, 2, p.large, N=1000)
trace_plot(chain)
```



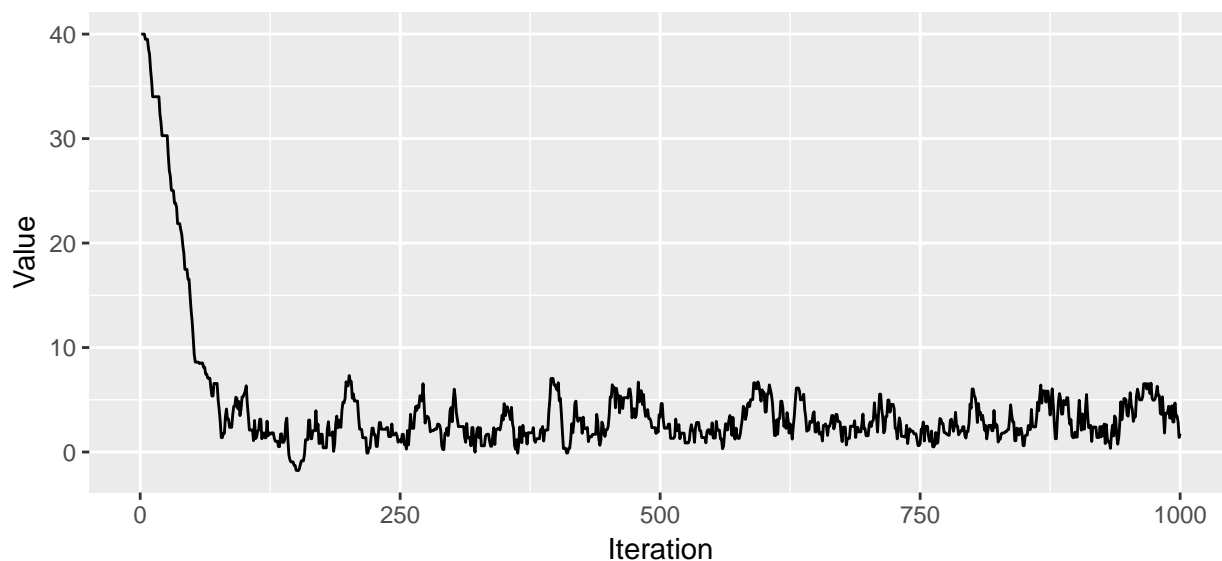
This chain is not good because it stays in place for too long before finding another suitable value. It will take a very long time to suitably explore the distribution.

Given these two examples, it is clear that the choice of the variance parameter is a critical choice. A commonly used rule of thumb that balances these two issues is that the variance parameter should be chosen such that approximately 40 – 60% of the proposed values are accepted. To go about finding that variance parameter, we will have a initialization period where we tweak the variance parameter until we are satisfied with the acceptance rate.

Burn-in

Another problem with the MCMC algorithm is that it requires that we already have a sample drawn from the distribution in question. This is clearly not possible, so what we do is use a random start and hope that the chain eventually finds the center of the distribution.

```
chain <- MCMC(df, 40, rproposal, N=1000) # start at x = 40
trace_plot(chain)
```



Given this example, it seems reasonable to start the chain and then disregard the initial “burn-in” period. However, how can we tell the difference between a proposal distribution with too small of variance, and a starting value that is far from the center of the distribution? One solution is to create multiple chains with different starting points. When all the chains become well mixed and indistinguishable, we’ll use the remaining observations in the chains as the sample.

```
chain1 <- MCMC(df, -30, rproposal, N=1000)
chain2 <- MCMC(df, 0, rproposal, N=1000)
chain3 <- MCMC(df, 30, rproposal, N=1000)
chains <- cbind(chain1, chain2, chain3)
trace_plot(chains)
```



2.4.3 Assessing Chain Convergence

Perhaps the most convenient way to assess convergence is just looking at the traceplots, but it is also good to derive some quantitative measurements of convergence. One idea from ANOVA is that the variance of the distribution we are sampling from, σ^2 , could be estimated using the within chain variability or it could be estimated using the variance between the chains.

Let m be the number of chains and n be the number of samples per chain. We define W be the average of the within chain sample variances. Formally we let s_i^2 be the sample variance of the i^{th} chain and

$$W = \frac{1}{m} \sum_{i=1}^m s_i^2$$

We can interpret W as the amount of wiggle within each chain.

Next we recognize that under perfect sampling and large sample sizes, the chains should be right on top of each other, and the mean of each chain should be very very close to the mean of all the other chains. Mathematically, we'd say the mean of each chain has a distribution $\bar{x}_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ where μ is the mean of the distribution we are approximating. Doing some probability, this implies that $n\bar{x}_i \sim N(n\mu, \sigma^2)$. Next, using the m chain means, we could estimate σ^2 by looking at the variances of the chain means! Let $\bar{x}_{..}$ be the mean of all observation and $\bar{x}_{i.}$ be the mean of the i^{th} chain. Let

$$B = \frac{n}{m-1} \sum_{i=1}^m (\bar{x}_{i.} - \bar{x}_{..})^2$$

Consider chains after we have removed the initial burn-in period where the chains obviously don't overlap. Notice that both W and B are estimating σ^2 , but if the MCMC has not converged (i.e. the space explored by each chains doesn't perfectly overlap the space explored by the other chains), then the difference in means will be big and as a result, B will be much larger than σ^2 . Likewise, W will be much smaller than σ^2 because the chains won't have fully explored the distribution. We'll put these two estimators together using a weighted average and define.

$$\hat{\sigma}^2 = \frac{n-1}{n} W + \frac{1}{n} B$$

Notice that if the chains have not yet converged, then $\hat{\sigma}^2$ should overestimate σ^2 because the between chain variance is too large. By itself W will underestimate σ^2 because the chains haven't fully explored the distribution. This suggests

$$\hat{R} = \sqrt{\frac{\hat{\sigma}^2}{W}}$$

is a useful ratio which decreases to 1 as $n \rightarrow \infty$. This ratio (Gelman and Rubin, 1992) can be interpreted as the potential reduction in the estimate of $\hat{\sigma}^2$ if we continued sampling.

A closely related is the number of effective samples from the posterior distribution

$$n_{eff} = mn \frac{\hat{\sigma}^2}{B}$$

To demonstrate these, we'll calculate \hat{R} for the above chains

```
# remove the first 200 iterations
trimmed.chains <- chains[201:1000, ]      # Both do the
#trimmed.chains <- chains[-(1:200), ]      # same thing.

# covert matrix to data.frame with 2 columns - Chain, Value
X <- as.data.frame(trimmed.chains)
colnames(X) <- paste('Chain.', 1:ncol(X), sep='')
data <- tidyr::gather(X, key='Chain') %>% # A function from the tidyr package
  mutate(Chain = factor(Chain))
str(data)
```

```
## 'data.frame':   2400 obs. of  2 variables:
## $ Chain: Factor w/ 3 levels "Chain.1","Chain.2",...: 1 1 1 1 1 1 1 1 1 1 ...
## $ value: num  3.45 2.77 1.98 1.22 1.72 ...
```

```
# Calculate a few easy things.
```

```
n <- nrow(X)
m <- ncol(X)
overall.mean <- mean( data$value )
```

```
# Use the package dplyr to summarise the data frame into W,B
```

```
temp <- data %>% group_by(Chain) %>%
  summarise( s2_i = var(value),
             xbar_i = mean(value)) %>%
  summarise( W = mean(s2_i),
             B = n/(m-1) * sum( (xbar_i - overall.mean)^2 ) )
```

```
W <- temp$W
```

```
B <- temp$B
```

```
sigma2.hat <- (n-1)/n * W + (1/n)*B
```

```
R <- sqrt(sigma2.hat/W)
```

```
n.eff <- m*n * sigma2.hat / B
```

```
cbind( W, B, sigma2.hat, R, n.eff ) # print with nice labeling on top!
```

```
##           W           B sigma2.hat           R       n.eff
## [1,] 2.635861 37.45643   2.679386 1.008223 171.6802
```

```
# Or just use the Gelman() function in the STA578 package
```

```
STA578::Gelman(trimmed.chains)
```

```
## # A tibble: 1 x 6
```

```
##   Parameter           W           B sigma2.hat       R.hat       n.eff
##   <fctr>       <dbl>       <dbl>       <dbl>       <dbl>       <dbl>
## 1      x1 2.635861 37.45643   2.679386 1.008223 171.6802
```

A useful rule-of-thumb for assessing convergence is if the \hat{R} value(s) are less than 1.05, then we are close enough. However, this does not imply that we have enough independent draws from the sample to trust our results. In the above example, even though we have $3 \cdot 800 = 2400$ observations, because of the Markov property of our chain, we actually only have about 171 independent draws from the distribution. This suggests that it takes around 14 MCMC steps before we get independence between two observations.

To further explore this idea, let's look at what happens when the chains don't mix as well.

```
chains <- mMCMC(df, list(3,4,5,2),
               rprop=function(x){ return(x+runif(1,-.3,.3)) },
               N=1000, num.chains=4)
trace_plot(chains)
```



```
Gelman(chains)
```

```
## # A tibble: 1 x 6
##   Parameter      W      B sigma2.hat  R.hat  n.eff
##   <fctr>    <dbl> <dbl>    <dbl>  <dbl>  <dbl>
## 1      X1 1.295232 466.1368  1.760074 1.165713 15.10349
```

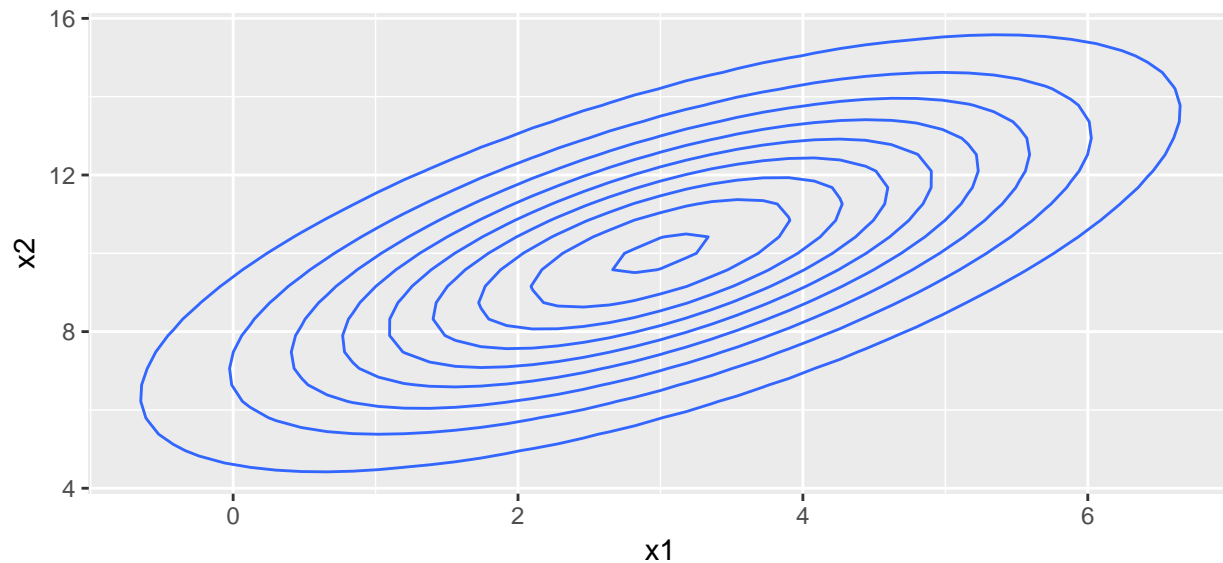
Notice that I didn't have to trim the burn-in part of the chain because I started the chains in the distribution, but because the step-size is too small, the means of the chains are very different from each other and therefore the *between* estimate of the target distribution variance is huge compared to the *within* estimates.

2.5 Multi-variate MCMC

We now consider the case of doing MCMC in multiple dimensions. This is where MCMC is vastly superior to other methods of sampling, however we still have some substantial issues to address. In principle, instead of our proposed value x_{i+1}^* being a slightly perturbed version of x_i in 1-D, we could perturb it in multiple dimensions.

Suppose we wish to sample from the multivariate normal distribution pictured below:

```
dtarget <- function(x){
  dmvmnorm(x, mean=c(3,10), sigma=matrix(c(3,3,3,7), nrow=2))
}
x1 <- seq(-6,12,length=101)
x2 <- seq(-11,31, length=101)
contour.data <- expand.grid(x1=x1, x2=x2)
contour.data$Z <- apply(contour.data, MARGIN=1, dtarget)
target.map <- ggplot(contour.data, aes(x=x1, y=x2, z=Z)) +
  stat_contour()
target.map
```



We'll now consider a proposal distribution that is nice and simple, a multivariate normal with no correlation. We start the chain at (0,0) and it takes awhile to find the distribution.

```
rprop <- function(x){
  rmvnorm(1, mean=x, sigma=diag(c(1,1)))
}
start <- c(x1=0, x2=0)
chain <- mMCMC(df=dtarget, start, rprop, N=20, num.chains=1)
target.map +
  geom_path(data=data.frame(chain[[1]]), aes(z=0), color='red', alpha=.6) +
  geom_point(data=data.frame(chain[[1]]), aes(z=0), color='red', alpha=.6, size=2)
```



```
chain <- mMCMC(df=dtarget, start, rprop, N=1000, num.chains=1)
target.map +
  geom_path(data=data.frame(chain[[1]]), aes(z=0), color='red', alpha=.4) +
  geom_point(data=data.frame(chain[[1]]), aes(z=0), color='red', alpha=.2, size=2)
```




```
# four chains, all starting at different places
chains <- mMCMC(df=dtarget,
  start = list( c(10,10), c(-10,-10), c(-10,10), c(10,-10) ),
  rprop, N=1000, num.chains=4)
trace_plot(chains)
```



```
Gelman(window(chains, 200, 1000)) # Diagnostics after removing first 200
```

```
## # A tibble: 2 x 6
##   Parameter      W      B sigma2.hat  R.hat  n.eff
##   <fctr>    <dbl>    <dbl>    <dbl>  <dbl>
## 1      X1 3.260507 23.71835  3.286048 1.003909 443.8966
## 2      X2 6.440546 21.65755  6.459544 1.001474 955.6196
```

That worked quite nicely, but everything works nicely in the normal case. Lets try a distribution that is a bit uglier, which I'll call the *banana* distribution.

```
dtarget <- function(x){
  B <- .05
  exp( -x[1]^2 / 200 - (1/2)*(x[2]+B*x[1]^2 -100*B)^2 )
}
```

```

}
x1 <- seq(-20,20, length=201)
x2 <- seq(-15,10, length=201)
contour.data <- expand.grid(x1=x1, x2=x2)
contour.data$Z <- apply(contour.data, MARGIN=1, dtarget)

target.map <- ggplot(contour.data, aes(x=x1, y=x2)) +
  stat_contour(aes(z=Z))
target.map

```



```

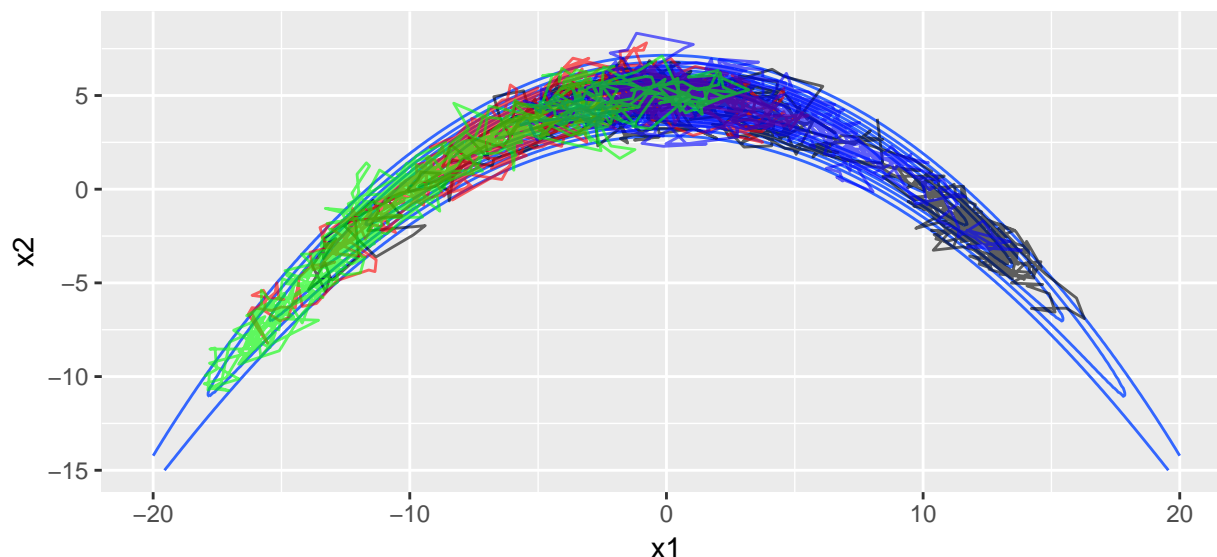
# four chains, all starting in the peak of the distribution
chains <- mcmc(df=dtarget,
  start = c(0,5),
  rprop, N=1000, num.chains=4)
trace_plot(chains)

```



The convergence doesn't look great, and when we overlay the chain paths on the distribution, we see why the convergence isn't good.

```
target.map +
  geom_path(data=data.frame(chains[[1]]), aes(x=X1, y=X2, z=0),
            color='black', alpha=.6) +
  geom_path(data=data.frame(chains[[2]]), aes(x=X1, y=X2, z=0),
            color='red', alpha=.6) +
  geom_path(data=data.frame(chains[[3]]), aes(x=X1, y=X2, z=0),
            color='blue', alpha=.6) +
  geom_path(data=data.frame(chains[[4]]), aes(x=X1, y=X2, z=0),
            color='green', alpha=.6)
```



The chains are having a very hard time moving into the ends of the banana. Unfortunately there is little we can do about this other than run our chains for a very long time. Anything you do to optimize the proposal distribution at the one end of the banana will fail at the other end.

2.6 Hamiltonian MCMC

One of the biggest developments in Bayesian computing over the last decade has been the introduction of Hamiltonian Monte Carlo (usually denoted as HMC). This method works when the distribution we wish to explore is continuous and we (or an algorithm can) evaluate the gradient of the distribution.

Most of the information here comes from Radford Neal's book chapter [<http://arxiv.org/pdf/1206.1901.pdf> || MCMC using Hamiltonian Dynamics]. We will use his notation and the information in this section is mostly his work, though I've tried not to use his exact verbiage.

Hamiltonian dynamics can be thought of as describing the motion of a hockey puck sliding across a frictionless (possibly non-flat) surface. In this system there are two things to know: the position of the puck (which we'll denote as \mathbf{q}) and its momentum which is the puck's mass*velocity and which we'll denote as \mathbf{p} . (I assume there is some horrible physics convention that we are following for this notation). Given this, we can look at the total energy of the puck as it's potential energy $U(\mathbf{q})$ which is its height at the current position \mathbf{q} and it's kinetic energy $K(\mathbf{p})$. The Hamiltonian part of this is to say that

$$H(\mathbf{q}, \mathbf{p}) = U(\mathbf{q}) + K(\mathbf{p})$$

remains constant as the puck moves around the surface.

- As the puck slides on a level surface, the potential energy remains the same and so does the kinetic and the puck just slides with constant velocity.
- As the puck slides up an incline, the potential energy increases and the kinetic energy decreases (i.e. the puck slows down).
- As the puck slides down an incline, the potential energy decreases and the kinetic energy increases (i.e. the puck speeds up).

In our case, the position \mathbf{q} will correspond to the variables of interest and the momentum \mathbf{p} will be auxiliary variables that we add to the problem. Somewhat paradoxically, by making the problem harder (i.e. more variables) we actually make the problem much more tractable. In order to move the puck around the space and stay near the peak of the distribution, we'll define

$$U(\mathbf{q}) = -\log f(\mathbf{q})$$

so that the edges of the distribution have high potential energy (that is $U(\mathbf{q})$ is a bathtub shaped surface and the puck is making paths around the tub, but never exiting the bathtub). Likewise we'll define the kinetic energy as some standard form involving the velocity and mass.

$$K(\mathbf{p}) = \mathbf{p}^T \mathbf{p} / 2m$$

where m is the mass of the puck.

Fortunately Hamiltonian dynamics are well understood and we can approximately solve the system of equations (letting p_i and q_i be the i th component of the \mathbf{p} and \mathbf{q} vectors)

$$\begin{aligned} \frac{dq_i}{dt} &= m^{-1} p_i \\ \frac{dp_i}{dt} &= -\frac{\partial U}{\partial q_i} \end{aligned}$$

which can be easily discretized to estimate $p_i(t + \epsilon)$ and $q_i(t + \epsilon)$ from a given momentum and location $p_i(t)$ and $q_i(t)$. The most accurate numerical discretization is called "leapfrog" where we take a half-step in p and then a full step in q and a second half step in p .

$$\begin{aligned} p_i\left(t + \frac{\epsilon}{2}\right) &= p_i(t) + \left(\frac{\epsilon}{2}\right) \frac{\partial}{\partial q_i} U(q(t)) \\ q_i(t + \epsilon) &= q_i(t) + \frac{\epsilon}{m} p_i\left(t + \frac{\epsilon}{2}\right) \\ p_i(t + \epsilon) &= p_i\left(t + \frac{\epsilon}{2}\right) + \left(\frac{\epsilon}{2}\right) \frac{\partial}{\partial q_i} U(q(t + \epsilon)) \end{aligned}$$

Our plan for MCMC is given a current chain location \mathbf{q} , we will generate a random momentum vector \mathbf{p} and then let the Hamiltonian dynamics run for a little while and then use the resulting location as the proposal \mathbf{q}^* . Before we do the MCMC, let's first investigate how well the Hamiltonian dynamics work.

To see how well this works, we'll consider our multivariate normal example where \mathbf{x}_1 and \mathbf{x}_2 are correlated. Given a starting point, we'll generate a random starting point and random momentum and see what happens.

```
dtarget <- function(x, log=FALSE){
  dmvnorm(x, mean=c(3,10), sigma=matrix(c(3,3,3,7), nrow=2), log)
}
x1 <- seq(-6,12,length=101)
x2 <- seq(-11,31, length=101)
contour.data <- expand.grid(x1=x1, x2=x2)
contour.data$Z <- apply(contour.data, MARGIN=1, dtarget)
target.map <- ggplot(contour.data, aes(x=x1, y=x2)) +
  stat_contour(aes(z=Z))
```

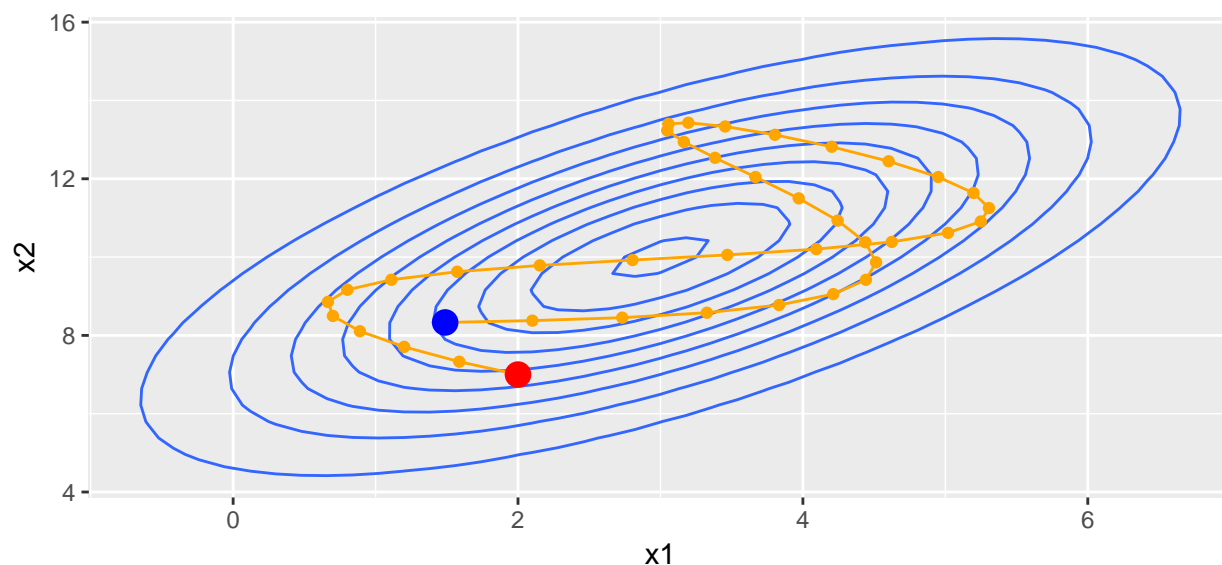
```
# set.seed() starts R's uniform random number generator at a set location so that
# the rest of this example doesn't change every time I recreate these notes.
```

```
set.seed(8675309)
U <- function(x){
  return( -(dtarget(x, log=TRUE)))
}
steps <- HMC.one.step(U, current_q=c(2,7), Eps=.4, L=40, m=1 )
```

```
str(steps)
```

```
## List of 3
## $ q      : num [1:40, 1:2] 2 1.588 1.201 0.89 0.701 ...
## $ p      : num [1:40, 1:2] -0.9966 -0.9676 -0.777 -0.4741 -0.0879 ...
## $ value: num [1, 1:2] 1.49 8.33
```

```
plot_HMC_proposal_path(target.map, steps)
```



Lets look at a few more of these:

```
set.seed(86709) # so it is reproducible
steps1 <- HMC.one.step(U, current_q=c(3,7), Eps=.4, L=40, m=1 )
steps2 <- HMC.one.step(U, current_q=c(5,7), Eps=.4, L=40, m=1 )
multiplot( plot_HMC_proposal_path(target.map, steps1),
            plot_HMC_proposal_path(target.map, steps2), ncol=2 )
```

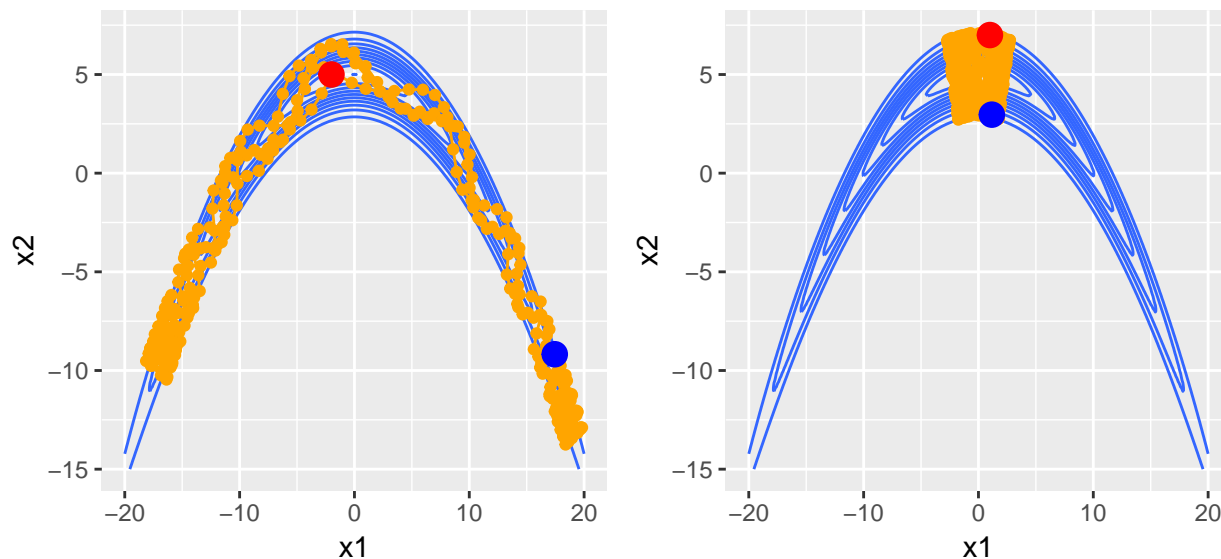


Now let's look at some proposed circuits in the banana distribution.

```
dtarget <- function(x, log=FALSE){
  B <- .05
  power <- -x[1]^2 / 200 - (1/2)*(x[2]+B*x[1]^2 -100*B)^2
  if( log ){
    return( power )
  }else{
    exp( power )
  }
}
U <- function(x){
  return( -(dtarget(x, log=TRUE)))
}
x1 <- seq(-20,20, length=201)
x2 <- seq(-15,10, length=201)
contour.data <- expand.grid(x1=x1, x2=x2)
contour.data$Z <- apply(contour.data, MARGIN=1, dtarget)

target.map <- ggplot(contour.data, aes(x=x1, y=x2)) +
  stat_contour(aes(z=Z))

set.seed(427) # so it is reproducible
steps1 <- HMC.one.step(U, current_q=c(-2,5), Eps=.5, L=300, m=1)
steps2 <- HMC.one.step(U, current_q=c(1,7), Eps=.5, L=300, m=1)
multiplot( plot_HMC_proposal_path(target.map, steps1),
            plot_HMC_proposal_path(target.map, steps2), ncol=2)
```



So now our proposal mechanism has three parameters to tune the MCMC with - the step size ϵ , the number of leapfrog steps to take L , and the mass of the particle m .

If we are able to exactly follow the Hamiltonian path, then

$$H(\mathbf{q}, \mathbf{p}) = H(\mathbf{q}^*, \mathbf{p}^*)$$

and we should always accept the proposed value. However in practice there are small changes due to numerical issues and we'll add an accept/reject criteria. In the following rule, notice the order of subtraction is opposite of what we normally would do because $U()$ is $-\log f()$.

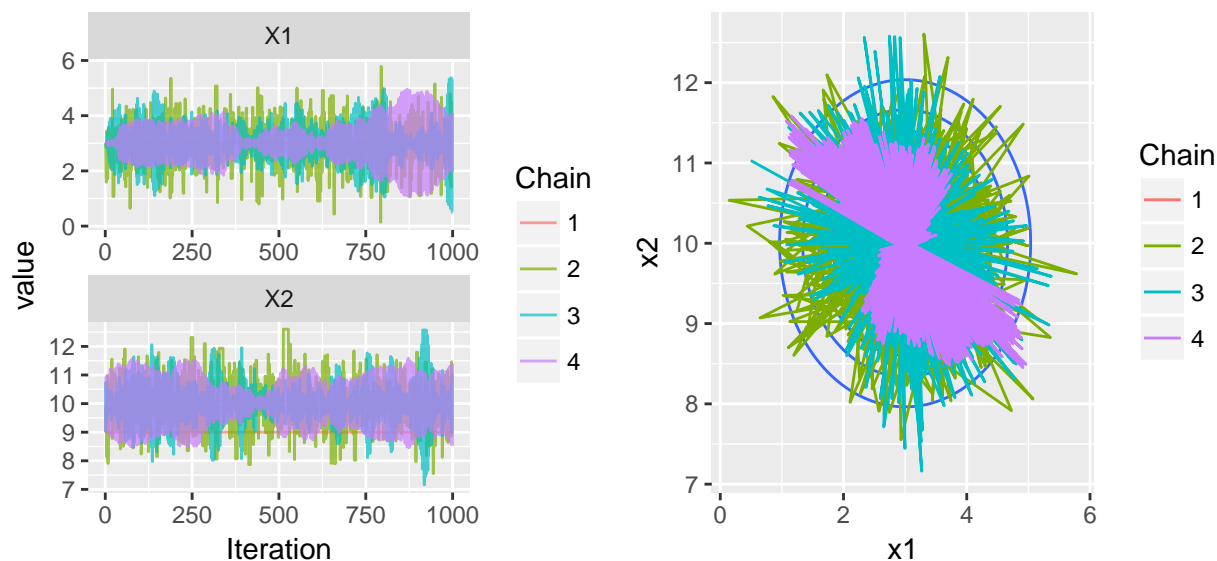
$$\text{accept if } \text{uniform}(0,1) < \exp(H(\mathbf{q}, \mathbf{p}) - H(\mathbf{q}^*, \mathbf{p}^*))$$

We will now investigate how sensitive this sampler is to its tuning parameters. We'll first investigate the simple uncorrelated multivariate Normal.

```
dtarget <- function(x){
  dmvnorm(x, mean=c(3,10), sigma=matrix(c(1,0,0,1), nrow=2))
}
x1 <- seq(0,6,length=101)
x2 <- seq(7,13, length=101)
contour.data <- expand.grid(x1=x1, x2=x2)
contour.data$Z <- apply(contour.data, MARGIN=1, dtarget)
target.map <- ggplot(contour.data, aes(x=x1, y=x2)) +
  stat_contour(aes(z=Z))
```

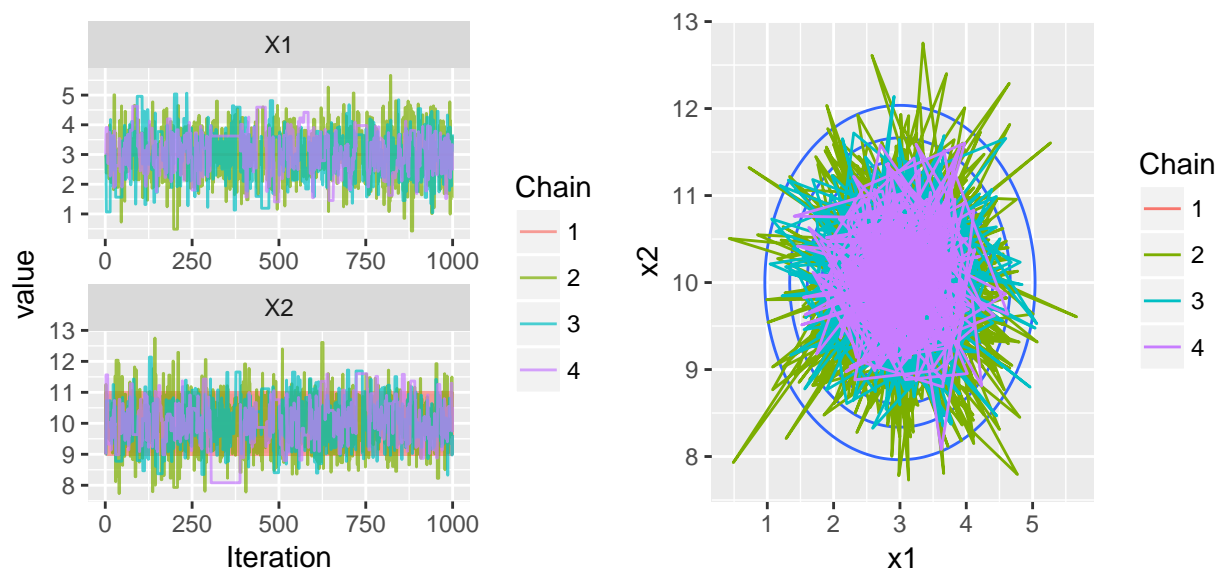
Our first run has chosen the number of leapfrog steps in an unfortunate fashion and our proposed value is often nearly a rotation of 180° from the current value.

```
# Poor behavior of the chain
set.seed(712)
chains <- HMC(dtarget, start=c(3,9), Eps=1, L=7, N=1000, num.chains=4)
multiplot( trace_plot(chains), plot_2D_chains(target.map, chains), ncol=2 )
```



Making an adjustment to the number of leapfrog steps easily fixes this problem, but we have another issue in this next example. Now the within chain variances aren't the same, chain is stuck within a much smaller radius than the other chains.

```
# Poor behavior variances
set.seed(7345)
chains <- HMC(dtargget, start=c(3,9), Eps=1, L=4, N=1000, num.chains=4)
multiplot( trace_plot(chains), plot_2D_chains(target.map, chains), ncol=2 )
```



To try to fix this, lets mess with the mass parameter, step size, and number of leapfrog steps...

```
set.seed(214)
chains <- HMC(dtargget, start=c(3,9), Eps=.5, L=4, m=.5, N=1000, num.chains=4)
multiplot( trace_plot(chains), plot_2D_chains(target.map, chains), ncol=2 )
```




Next we'll go to the banana distribution

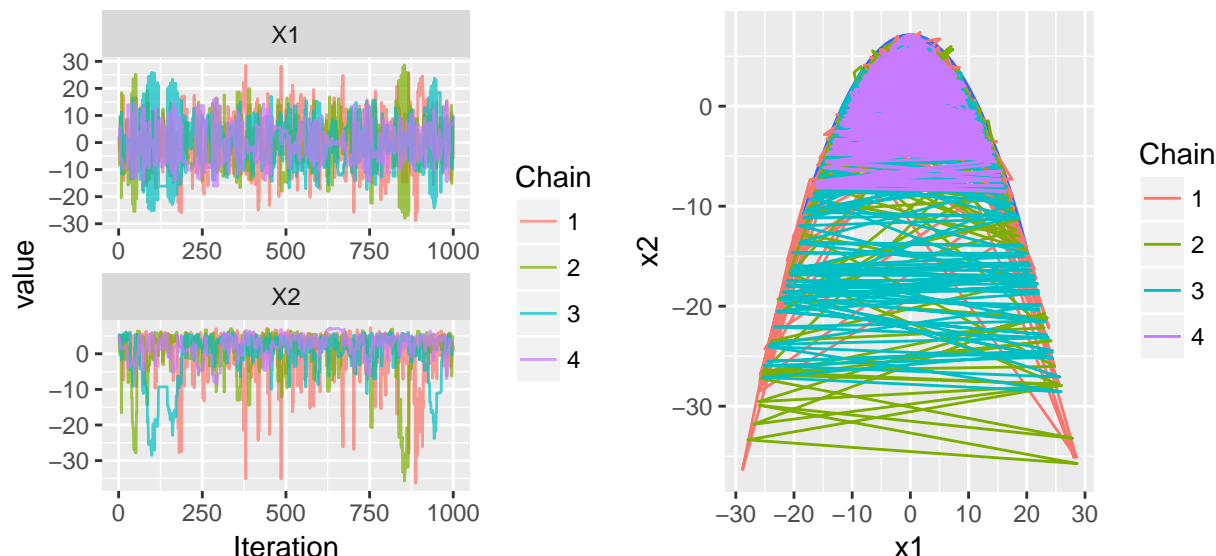
```
dtarget <- function(x){
  B <- .05
  exp( -x[1]^2 / 200 - (1/2)*(x[2]+B*x[1]^2 -100*B)^2 )
}
x1 <- seq(-20,20, length=201)
x2 <- seq(-15,10, length=201)
contour.data <- expand.grid(x1=x1, x2=x2)
contour.data$Z <- apply(contour.data, MARGIN=1, dtarget)
target.map <- ggplot(contour.data, aes(x=x1, y=x2)) +
  stat_contour(aes(z=Z))

set.seed(21434)
chains <- HMC(dtarget, start=c(2,5), Eps=.5, L=10, m=1)
multiplot( trace_plot(chains), plot_2D_chains(target.map, chains), ncol=2 )
```



That was my first pass with hardly thinking about the tuning parameters. Lets increase the number of leapfrog steps and decrease the mass.

```
set.seed(780)
chains <- HMC(dtargget, start=c(2,5), Eps=.5, L=200, m=.5)
multiplot( trace_plot(chains), plot_2D_chains(target.map, chains), ncol=2 )
```



This is exactly what we want to see. The chains are jumping between the tails and show very little correlation between steps.

These results show that the tuning of the MCMC parameters is very important, but with properly selected values, the MCMC sampler can quite effectively explore distributions. While we haven't considered more interesting examples, the HMC method is quite effective in high dimensional spaces (100s of dimensions) with complicated geometries.

2.7 Exercises

1. The $Beta(\alpha, \beta)$ distribution is a distribution that lives on the interval $[0, 1]$ (for more info, check out the wikipedia page. Write a function called `my.rbeta(n,a,b)` to generate a random sample of size n from the $Beta(a,b)$ distribution using the accept-reject algorithm. Your function should mimic the usual `rbeta()` function and return a vector of size n . Generate a random sample of size 1000 from the $Beta(3,2)$ distribution. Graph the histogram of the sample with the theoretical $Beta(3,2)$ density superimposed.
 - You may use the function `dbeta()` for the density function.
 - You may generate the maximum height of the distribution by evaluating the density function along a grid of points in $[0, 1]$, then choosing the maximum height and multiplying by something slightly larger than 1 (say 1.10) to get a safe value for M .
 - Because you don't know how many points will be accepted, you might need employ some sort of looping or recursion to make sure you generate n observations. If you accidentally generate more than n data points, you should trim out the excess observations. For a brief introduction to decisions statements and loops in R, see Chapter 11 in A Sufficient Introduction to R.

```
# Some hints about the graphing
library(ggplot2) # my favorite graphing package

# make a graph of the density
x <- seq(0, 1, length=1001)
density.data <- data.frame( x = x, y=dbeta(x, 3,2) )
```

```
density.plot <- ggplot(density.data, aes(x=x, y=y)) +
  geom_line()
density.plot

# Draw some samples from your function
my.samples <- my.rbeta(1000, 3,2)

# Add a histogram of my.samples to the density.plot
# (this is one of the most obnoxiously hard things to do in ggplot2)
density.plot + geom_histogram(data=data.frame(x=my.samples),
                              aes(x=x, y=..density..),
                              alpha=.5, color='salmon')
```

2. The Rayleigh distribution has density function

$$f(x) = \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} \quad x \geq 0, \sigma > 0.$$

It is defined for positive values of x and has one parameter (a scale parameter) which we denote as σ . We will consider the problem of using MCMC to draw from this distribution.

- a) Create a function `drayleigh(x, scale)` that evaluates $f(x)$ using $\sigma = \text{scale}$. Make sure that if you submit a negative value for x , the function returns 0. Likewise, make sure that your function can accept vectors of x values, as in the following example of what you should get.

```
> drayleigh( x=c(-1, 2, 5), scale=3 )
[1] 0.0000000 0.1779416 0.1385290
```

Hint: to handle negative values, you can just round the negative x values values up to zero. The functions `pmax()` or `ifelse()` could be very helpful here.

- b) Plot the Rayleigh density function for $\sigma = \{2, 5, 10\}$. *Hint: if you don't want to use a `facet_grid` in `ggplot2` you could just make three separate plots and squish them onto the same figure using the `multiplot()` function available in the `STA578` package. Check out the help file to see how to use it and some examples.*
- c) You will use the `mMCMC()` function that is available in the `STA578` library to pull samples from the Rayleigh distribution. Consider the case where `scale=2` and we will use a proposal distribution that is `Uniform(-0.1, +0.1)` about the current chain value. Run four chains for 1000 steps each. Assess convergence and comment on your results. Start each chain at $x = 1$.
- d) Repeat part (c) but now consider a proposal distribution that is `Uniform(-1, 1)` about the current chain value. Assess convergence and comment on your results.
- e) Repeat part (c) but now consider a proposal distribution that is `Uniform(-10, 10)` about the current chain value. Assess convergence and comment on your results.
- f) Repeat part (c) but now consider a proposal distribution that is `Uniform(-50, 50)` about the current chain value. Assess convergence and comment on your results.
- g) Repeat steps (c)-(f) but now the target distribution to sample from is the Rayleigh distribution with `scale=15`.
- h) Comment on the need to tune the width of the proposal distribution relative to the distribution of interest.

Chapter 3

Overview of Statistical Learning

Chapter 2 of *Introduction to Statistical Learning* is very conceptual and there isn't much code to mess with, at least until the end. At the end of the chapter, they introduce the machine learning algorithm K-nearest Neighbors but don't mention anything about how to do an analysis with it.

```
library(ggplot2)
library(dplyr)
library(caret) # for the train() function
```

There are many packages in R that implement a wide variety of machine learning techniques, and each package works in slightly different ways based on the peculiarities of the individual package authors. Unfortunately makes working with these algorithms a pain because we have to learn how each package does things.

The R package `caret` attempts to resolve this by providing a uniform interface to the most widely used machine learning packages. Whenever possible we will use the `caret` interface, but there will be times where we have to resort to figuring out the peculiarities of specific packages.

If you are interested in more information about how the `caret` package works, and what methods are available, I've found the vignette and manual very helpful. If video tutorials are your thing, there is a DataCamp course and the introductory chapter is free.

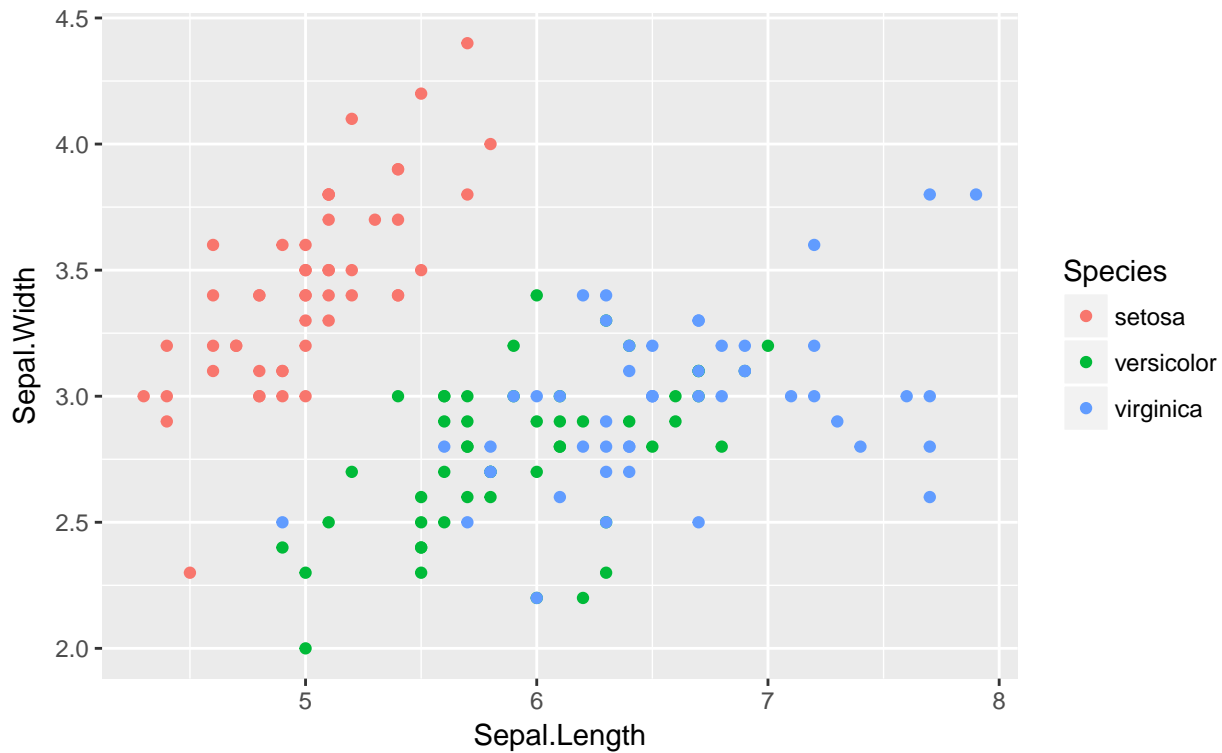
3.1 K-Nearest Neighbors

K-nearest neighbors (KNN) is a technique in machine learning that is extremely simple to implement and extremely easy to understand. We will use this as a prototypical machine learning technique for supervised learning. The idea is that if we wish to predict a response, \hat{y}_0 given some covariate(s) x_0 , then we look at the k nearest data points to x_0 and take the average of those nearest y values (or in the classification setting, the group with highest representation within the k nearest neighbors).

3.1.1 KNN for Classification

Suppose we consider the `iris` dataset and we wish to provide a classifier for iris species based on the sepal length and sepal width.

```
ggplot(iris, aes(x=Sepal.Length, y=Sepal.Width, color=Species)) +
  geom_point()
```



Clearly anything in the upper left should be Setosa, anything in the lower left should be Versicolor, and anything on the right half should be Virginica.

So lets consider KNN.

```
model <- train(Species ~ Sepal.Length + Sepal.Width, data=iris,
               method='knn', tuneGrid=data.frame(k=3))

# expand.grid() makes a data.frame() with all possible combinations
Pred.grid <- expand.grid( Sepal.Length = seq(4, 8, length=101),
                          Sepal.Width  = seq(2, 4.5, length=101) )

Pred.grid$yhat <- predict(model, newdata=Pred.grid)
```

Now that we have the predicted classifications for the grid, lets graph it.

```
ggplot(Pred.grid, aes(x=Sepal.Length, y=Sepal.Width)) +
  geom_tile(aes(fill=yhat), alpha=.4) +
  geom_point(data=iris, aes(color=Species, shape=Species)) +
  scale_color_manual(values=c('dark red', 'dark blue', 'forestgreen')) +
  scale_fill_manual(values=c('pink', 'light blue', 'cornsilk')) +
  ggtitle('KNN: k=3')
```



Now we want to see what happens as we vary k .

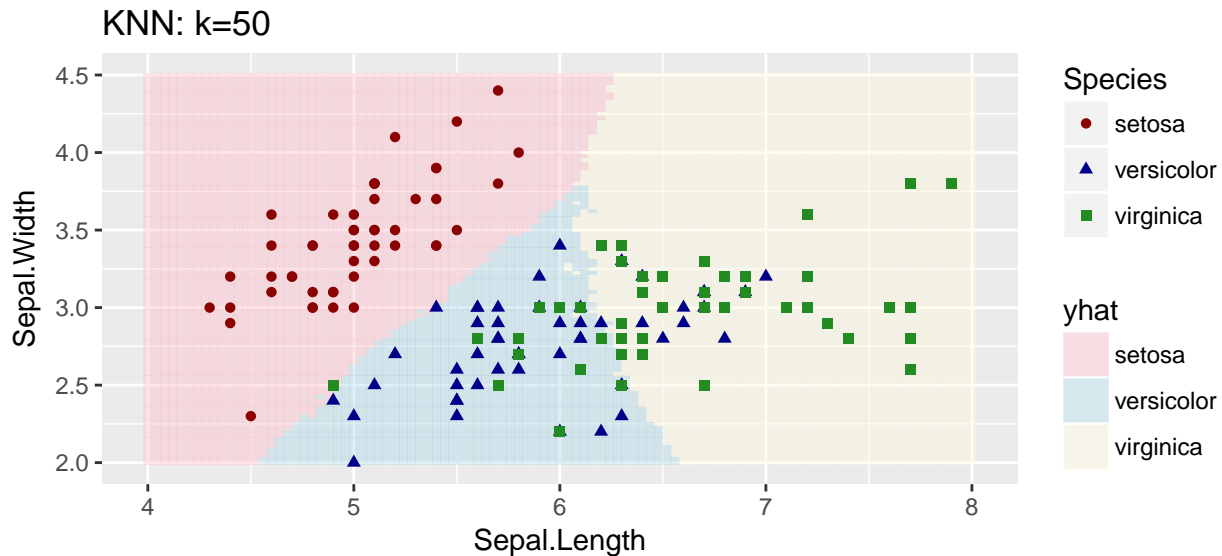
```
Current.K <- 10
model <- train(Species ~ Sepal.Length + Sepal.Width, data=iris,
               method='knn', tuneGrid=data.frame(k=Current.K))
Pred.grid$yhat <- predict(model, Pred.grid)

ggplot(Pred.grid, aes(x=Sepal.Length, y=Sepal.Width)) +
  geom_tile(aes(fill=yhat), alpha=.4) +
  geom_point(data=iris, aes(color=Species, shape=Species)) +
  scale_color_manual(values=c('dark red', 'dark blue', 'forestgreen')) +
  scale_fill_manual(values=c('pink', 'light blue', 'cornsilk')) +
  ggtitle(paste('KNN: k=', Current.K, sep=''))
```



```
Current.K <- 50
model <- train(Species ~ Sepal.Length + Sepal.Width, data=iris,
               method='knn', tuneGrid=data.frame(k=Current.K))
Pred.grid$yhat <- predict(model, Pred.grid)
```

```
ggplot(Pred.grid, aes(x=Sepal.Length, y=Sepal.Width)) +
  geom_tile(aes(fill=yhat), alpha=.4) +
  geom_point(data=iris, aes(color=Species, shape=Species)) +
  scale_color_manual(values=c('dark red', 'dark blue', 'forestgreen')) +
  scale_fill_manual(values=c('pink', 'light blue', 'cornsilk')) +
  ggtitle(paste('KNN: k=', Current.K, sep=''))
```



3.1.2 KNN for Regression

```
set.seed(8675309)
n <- 50
data <- data.frame( x=seq(.5,3, length=n) ) %>%
  mutate( Ey = 2 - 4*x + 3*x^2 - .55*x^3 + cos(x),
           y = Ey + rnorm(n, sd=.1))
ggplot(data, aes(x=x)) +
  geom_line(aes(y=Ey)) +
  geom_point(aes(y=y)) +
  labs(main='foo', y='y', title='Made up data')
```


Made up data



```
# values of x where we wish to make a prediction
Pred.grid <- data.frame(x=seq(0.5, 3, length=1001))
```

```
Current.K <- 3
model <- train(y~x, data=data, method='knn', tuneGrid=data.frame(k=Current.K))
Pred.grid$yhat <- predict(model, Pred.grid)
```

```
ggplot(data, aes(x=x)) +
  geom_line(aes(y=Ey)) +
  geom_point(aes(y=y)) +
  geom_line(data=Pred.grid, aes(y=yhat), color='red') +
  labs(main='foo', y='y', title=paste('KNN: k=', Current.K, sep=''))
```

KNN: k=3



```
Current.K <- 10
model <- train(y~x, data=data, method='knn', tuneGrid=data.frame(k=Current.K))
Pred.grid$yhat <- predict(model, Pred.grid)
```

```
ggplot(data, aes(x=x)) +
  geom_line(aes(y=Ey)) +
```

```
geom_point(aes(y=y)) +
geom_line( data=Pred.grid, aes(y=yhat), color='red') +
labs(main='foo', y='y', title=paste('KNN: k=', Current.K, sep=''))
```



```
Current.K <- 50
model <- train(y~x, data=data, method='knn', tuneGrid=data.frame(k=Current.K))
Pred.grid$yhat <- predict(model, Pred.grid)

ggplot(data, aes(x=x)) +
  geom_line(aes(y=Ey)) +
  geom_point(aes(y=y)) +
  geom_line( data=Pred.grid, aes(y=yhat), color='red') +
  labs(main='foo', y='y', title=paste('KNN: k=', Current.K, sep=''))
```



3.2 Splitting into a test and training sets

We continue to consider the Regression problem but we now consider splitting the data into Training and Test sets and comparing the Root Mean Squared Error (RMSE) of Test sets for various values of K .

```
Train <- data %>% sample_frac(.5)
Test  <- setdiff(data, Train)

K <- 5
model <- train(y ~ x, data=Train, method='knn', tuneGrid=data.frame(k=K) )
Test$yhat <- predict(model, Test)
Test %>% mutate(resid = y-yhat) %>%
  summarise( Test.RMSE = sqrt(mean( resid^2 )), K = K)

##      Test.RMSE K
## 1 0.09594326 5
```

Now that we see how to calculate the Test Mean Squared Error, lets look at a range of values for K .

```
Results <- NULL
for(K in 1:25){
  model <- train(y ~ x, data=Train, method='knn', tuneGrid=data.frame(k=K) )
  Test$yhat <- predict(model, Test)

  Temp <- Test %>% mutate(resid = y-yhat) %>%
    summarise( Test.RMSE = sqrt(mean( resid^2 )), K = K)
  Results <- rbind(Results, Temp)
}
```

```
## Warning in nominalTrainWorkflow(x = x, y = y, wts = weights, info =
## trainInfo, : There were missing values in resampled performance measures.
```

```
ggplot(Results, aes(x=K, y=Test.RMSE)) +
  geom_point() +
  scale_x_reverse() # so model complexity increases along x-axis.
```



3.3 Exercises

1. ISL 2.1: For each part below, indicate whether we would generally expect the performance of a flexible statistical learning method to be better or worse than an inflexible method. Justify your answer.
 - a) The sample size n is extremely large, and the number of predictors p is small.
 - b) The number of predictors p is extremely large, and the number of observations n is small.
 - c) Relationship between the predictors and response is highly non-linear.
 - d) The variance of the error terms, i.e. $\sigma^2 = \text{Var}(\epsilon)$, is extremely high.
2. ISL 2.2: Explain whether each scenario is a classification or regression problem, and indicate whether we are most interested in inference or prediction. Finally, provide n and p .
 - a) We collect a set of data on the top 500 firms in the US. For each firm we record profit, number of employees, industry and the CEO salary. We are interested in understanding which factors affect CEO salary.
 - b) We are considering launching a new product and wish to know whether it will be a **success** or **failure**. We collect data on 20 similar products that were previously launched. For each product we have record whether it was a success or failure, price charged for the product, marketing budget, competition price, and ten other variables.
 - c) We are interested in predicting the % change in the US dollar in relation to the weekly changes in the world stock markets. Hence we collect weekly data for all of 2012. For each week we record the % change in the dollar, the % change in the US market, the % change in the British market, and the % change in the German market.
3. ISL 2.3: We now revisit the bias-variance decomposition.
 - a) Provide a sketch of typical (squared) bias, variance, training error, test error, and Bayes (or irreducible) error curves, on a single plot, as we go from less flexible statistical learning methods towards more flexible approaches. The x-axis should represent the amount of flexibility in the method, and the y-axis should represent the values for each curve. There should be five curves. Make sure to label each one.
 - b) Explain why each of the five curves has the shape displayed in part (a).

Chapter 4

Classification with LDA, QDA, and KNN

```
library(caret)
library(MASS)      # lda function used by caret
library(dplyr)     # data frame manipulations
library(ggplot2)   # plotting
library(STA578)    # for my multiplot() function
library(pROC)      # All the ROC stuff
```

In this chapter we look at several forms of classification. In particular we will focus on the classical techniques of logistic regression, Linear Discriminant Analysis and Quadratic Discriminant Analysis.

4.1 Logistic Regression

We won't cover the theory of logistic regression here, but you can find it elsewhere.

The dataset `faraway::wbca` comes from a study of breast cancer in Wisconsin. There are 681 cases of potentially cancerous tumors of which 238 are actually malignant (ie cancerous). Determining whether a tumor is really malignant is traditionally determined by an invasive surgical procedure. The purpose of this study was to determine whether a new procedure called 'fine needle aspiration', which draws only a small sample of tissue, could be effective in determining tumor status.

```
data('wbca', package='faraway')

# clean up the data
wbca <- wbca %>%
  mutate(Class = ifelse(Class==0, 'malignant', 'benign')) %>%
  dplyr::select(Class, BNucl, UShap, USize)

# Fit the model
# Malignant is considered a success
model <- glm( I(Class=='malignant') ~ ., data=wbca, family='binomial' )

# Get the response values
# type='response' gives phat values which live in [0,1]
# type='link' gives the Xbeta values whice live in (-infinity, infinity)
```

```

wbca <- wbca %>%
  mutate(phat = predict(model, type='response'),
         yhat = ifelse(phat > .5, 'malignant', 'benign'))

# Calculate the confusion matrix
table( Truth=wbca$Class, Predicted=wbca$yhat )

```

```

##           Predicted
## Truth      benign malignant
##  benign      432         11
##  malignant   15         223

```

As usual we can calculate the summary tables...

```

summary(model)

##
## Call:
## glm(formula = I(Class == "malignant") ~ ., family = "binomial",
##      data = wbca)
##
## Deviance Residuals:
##      Min       1Q   Median       3Q      Max
## -3.8890  -0.1409  -0.1409   0.0287   2.2284
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -6.35433    0.54076 -11.751  < 2e-16 ***
## BNucl       0.55297    0.08041   6.877 6.13e-12 ***
## UShap       0.62583    0.17506   3.575 0.000350 ***
## USize       0.56793    0.15910   3.570 0.000358 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##      Null deviance: 881.39  on 680  degrees of freedom
## Residual deviance: 148.43  on 677  degrees of freedom
## AIC: 156.43
##
## Number of Fisher Scoring iterations: 7

```

From this table, we see that for a breast tumor, the larger values of BNucl, UShap, and USize imply a greater probability of it being malignant. So for a tumor with

```

##  BNucl UShap USize
## 1     2     1     2

```

We would calculate

$$\begin{aligned}
 X\hat{\beta} &= \hat{\beta}_0 + 2 \cdot \hat{\beta}_1 + 1 \cdot \hat{\beta}_2 + 2 \cdot \hat{\beta}_3 \\
 &= -6.35 + 2 * (0.553) + 1 * (0.626) + 2 * (0.568) \\
 &= -3.482
 \end{aligned}$$

and therefore

$$\hat{p} = \frac{1}{1 + e^{-X\hat{\beta}}} = \frac{1}{1 + e^{-3.482}} = 0.0297$$

```
newdata = data.frame( BNucl=2, UShap=1, USize=2 )
predict(model, newdata=newdata)
```

```
##          1
## -3.486719
```

```
predict(model, newdata=newdata, type='response')
```

```
##          1
## 0.0296925
```

So for a tumor with these covariates, we would classify it as most likely to be benign.

4.2 ROC Curves

In the medical scenario where we have to decide if a tumor is malignant or benign, we shouldn't treat the misclassification errors as being the same. If we incorrectly identify a tumor as malignant when it is not, that will cause a patient to undergo a somewhat invasive surgery to remove the tumor. However if we incorrectly identify a tumor as being benign, then the cancerous tumor will likely grow and eventually kill the patient. While the first error is regrettable, the second is far worse.

Given that reasoning, perhaps we shouldn't use the rule: If $\hat{p} \geq 0.5$ classify as malignant. Instead perhaps we should use $\hat{p} \geq 0.3$ or $\hat{p} \geq 0.05$.

Whatever decision rule we make, we should consider how many of each type of error we make. Consider the following confusion matrix:

	Predict Positive	Predict Negative	Total
Positive	True Pos (TP)	False Neg (FN)	P
Negative	False Pos (FP)	True Neg (TN)	N
Total	P^*	N^*	

where P is the number of positive cases, N is the number of negative cases, P^* is the number of observations predicted to be positive, and N^* is the number predicted to be negative.

Quantity	Formula	Synonyms
False Positive Rate	FP/N	Type I Error; 1-Specificity
True Positive Rate	TP/P	Power; Sensitivity; Recall
Pos. Pred. Value	TP/P^*	Precision

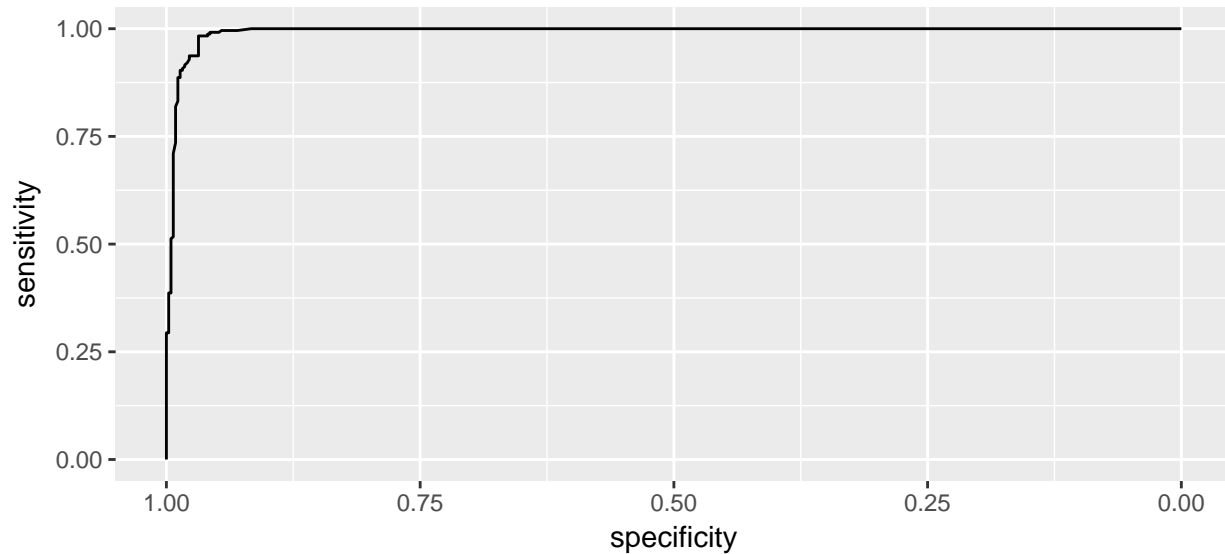
We can think of the True Positive rate as the probability that a Positive case will be correctly classified as a positive. Similarly a False Positive Rate is the probability that a Negative case will be incorrectly classified as a positive.

I wish to examine the relationship between the False Positive Rate and the True Positive Rate for any decision rule. So what we could do is select a sequence of decision rules and for each calculate the (FPR, TPR) pair, and then make a plot where we play connect the dots with the (FPR, TPR) pairs.

Of course we don't want to have to do this by hand, so we'll use the package `pROC` to do it for us.

```
# Calculate the ROC information
myROC <- roc( wbcA$Class, wbcA$phat )
```

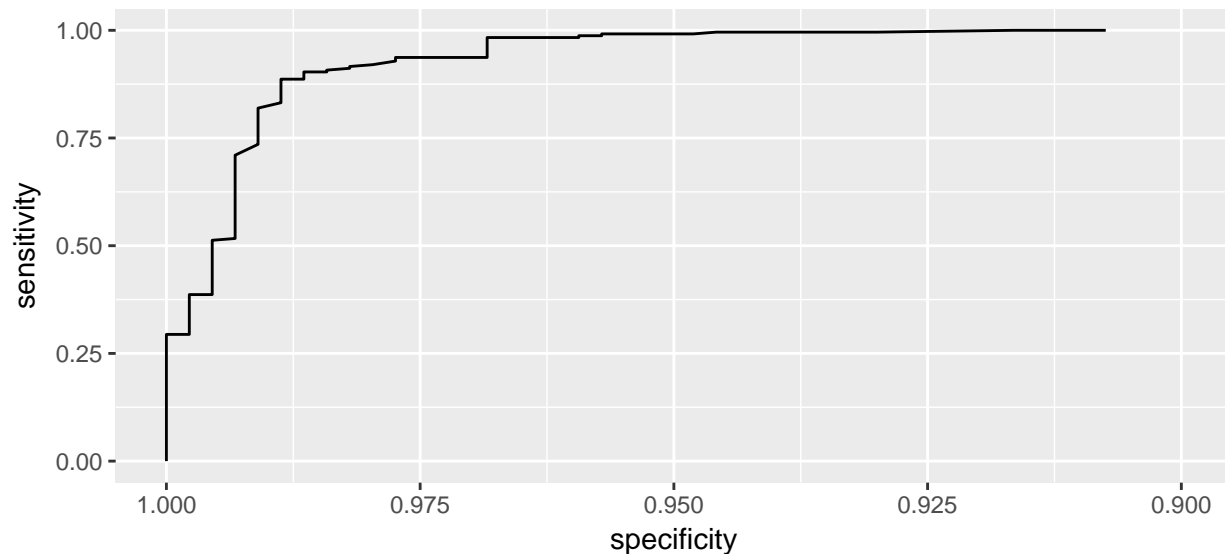
```
# make a nice plot using ggplot2
ggroc( myROC )
```



This looks pretty good and in an ideal classifier that makes prefect predictions, this would be a perfect right angle at the bend.

Lets zoom in a little on the high specificity values (i.e. low false positive rates)

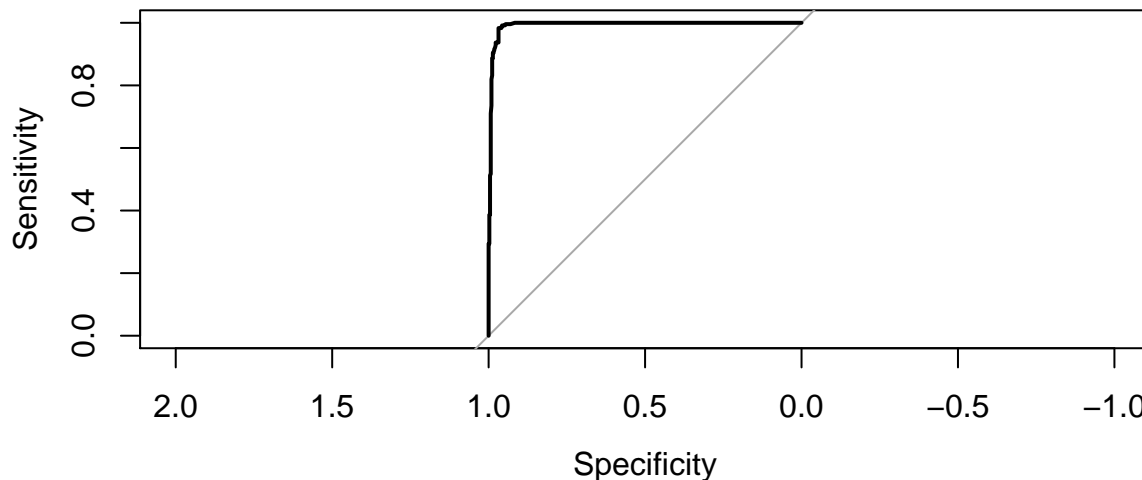
```
ggroc( myROC ) + xlim(1, .9)
```



We see that if we want to correctly identify about 99% of malignant tumors, we will have a false positive rate of about $1 - 0.95 = 0.05$. So about 5% of benign tumors would be incorrectly classified as malignant.

It is a little challenging to read the graph to see what the Sensitivity is for a particular value of Specificity. The in addition to displaying the ROC curve, the function `plot.roc()` also returns a list of the information it used to create the graph.

```
foo <- plot.roc( myROC )
```

```
index <- which.min( abs( foo$sensitivities - 0.99 ) )
foo$specificities[index]
```

```
## [1] 0.9480813
```

While you can obtain the estimated Specificity using this trick, the authors of `pROC` don't want you to do this because they would prefer you to also estimate the confidence intervals for the quantity using bootstrapping.

```
ci(myROC, of='sp', sensitivities=.99)
```

```
## 95% CI (2000 stratified bootstrap replicates):
```

```
##      se sp.low sp.median sp.high
## 0.99 0.9222    0.9571 0.9797
```

```
ci(myROC, of='se', specificities=.975)
```

```
## 95% CI (2000 stratified bootstrap replicates):
```

```
##      sp se.low se.median se.high
## 0.975 0.8824    0.937 0.9958
```

One measure of how far we are from the perfect predictor is the area under the curve. The perfect model would have an area under the curve of 1. For this model the area under the curve is:

```
auc(myROC)
```

```
## Area under the curve: 0.9929
```

```
ci(myROC, of='auc')
```

```
## 95% CI: 0.9878-0.9981 (DeLong)
```

```
ci(myROC, of='auc', method='bootstrap')
```

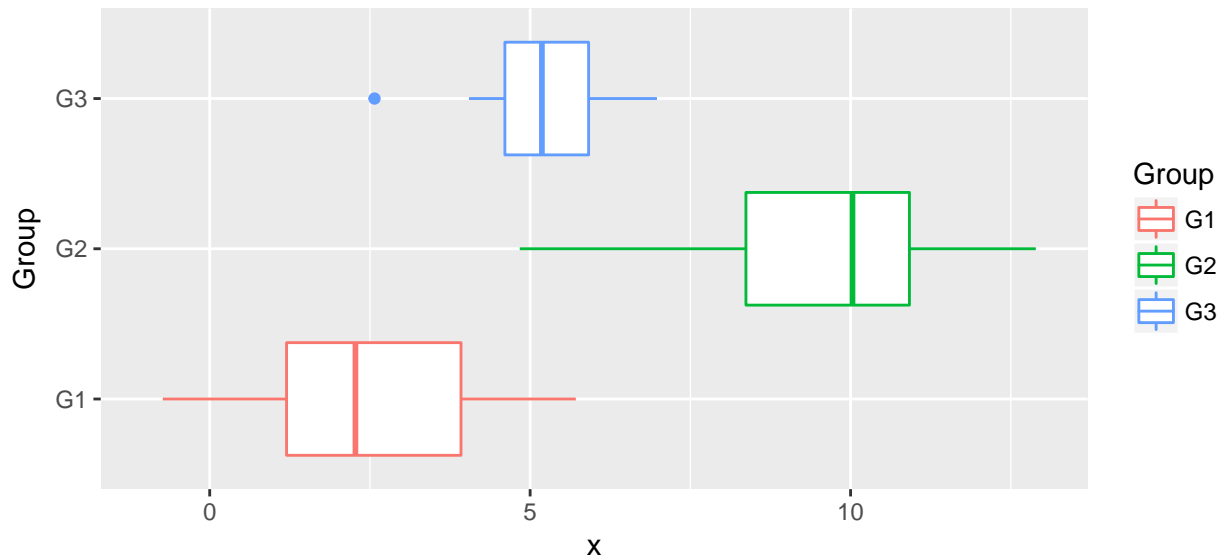
```
## 95% CI: 0.9875-0.9975 (2000 stratified bootstrap replicates)
```

which seems pretty good and Area Under the Curve (AUC) is often used as a way of comparing the quality of binary classifiers.

4.3 Linear Discriminant Analysis

Discriminant analysis classification techniques assume that we have a categorical (possibly unordered) response y and a continuous covariate predictor x . In many ways we can consider this the inverse problem of

a 1-way anova.



Because we are doing ANOVA in reverse, we assume the same relationship between the groups and the continuous covariate, namely that

$$x_{ij} \stackrel{iid}{\sim} N(\mu_i, \sigma^2)$$

So to fit normal distributions, we just need to estimate values for the mean of each group $\hat{\mu}_i$ and something for the overall variance, $\hat{\sigma}^2$.

We will use the individual sample means for $\hat{\mu}_i$ and the pooled variance estimator for $\hat{\sigma}^2$. Let k be the number of groups, n_i be the number of samples in group i , and $n = n_1 + \dots + n_k$ be the total number of observations.

$$\hat{\mu}_i = \bar{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}$$

$$\begin{aligned} \hat{\sigma}^2 &= \frac{1}{n-k} \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \hat{\mu}_i)^2 \\ &= \frac{1}{n-k} \sum_{i=1}^k (n_i - 1) s_i^2 \end{aligned}$$

```
str(data)
```

```
## 'data.frame':  90 obs. of  2 variables:
## $ x      : num  4.41 8.27 6 3.94 10.77 ...
## $ Group: Factor w/ 3 levels "G1","G2","G3": 1 2 3 1 2 3 1 2 3 1 ...
```

```
n <- nrow(data)
```

```
k <- length( levels( data$Group ))
```

```
# calculate group level statistics and pooled variance
```

```
params <- data %>%
```

```
  group_by(Group) %>%
```

```
  summarise( xbar = mean(x),
```

```
             s2_i = var(x),
```

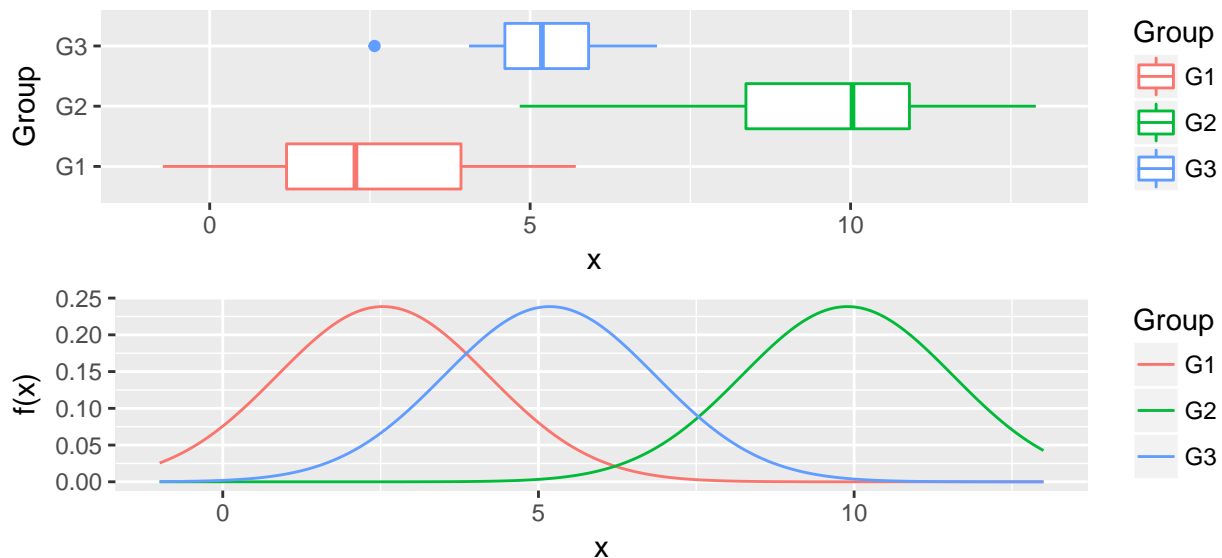
```

      n_i = n() ) %>%
mutate( s2 = sum( (n_i -1)*s2_i ) / (n-k) )

# Calculate the k curves
newdata <- expand.grid( x=seq(-1,13, by=0.01), Group=paste('G', 1:3, sep='') ) %>%
  left_join(params, by='Group') %>%
  mutate( f = dnorm(x, mean=xbar, sd=sqrt(s2) ) )
P2 <- ggplot(newdata, aes(x=x, y=f, color=Group)) +
  geom_line() + xlim(-1, 13) +
  labs( y='f(x)' )

multiplot(P1, P2, ncol=1)

```



Now for any value along the x-axis, say x^* , we predict y^* as whichever group with the largest $\hat{f}_i(x^*)$.

4.4 Quadratic Discriminant Analysis

The only difference between linear discriminant analysis and quadratic, is that we now allow for the groups to have different variance terms.

$$x_{ij} \stackrel{iid}{\sim} N(\mu_i, \sigma_i^2)$$

```

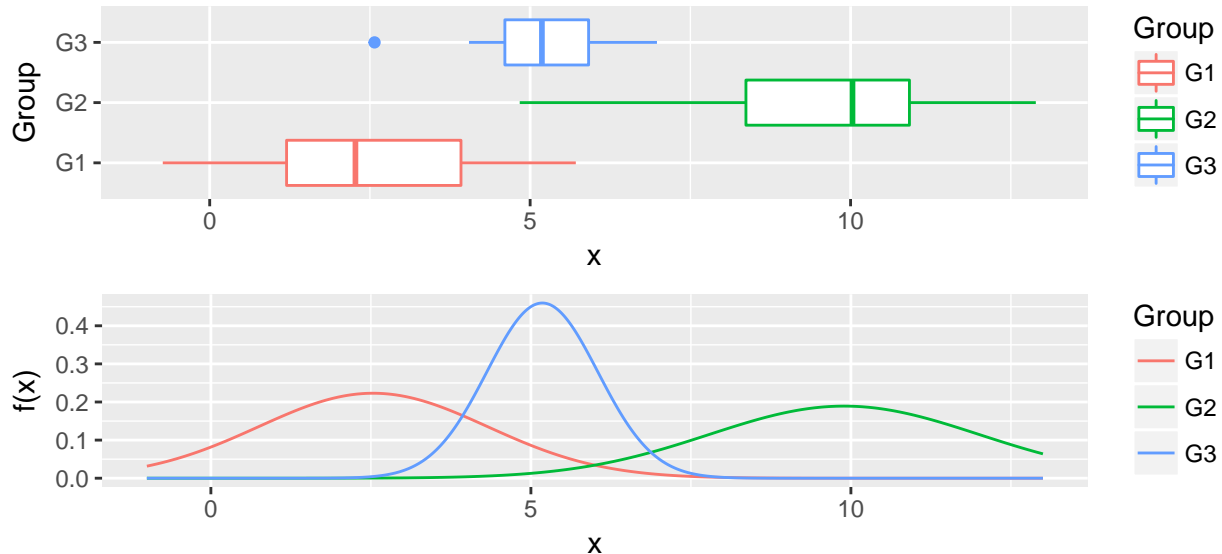
n <- nrow(data)
k <- length( levels( data$Group ) )

# calculate group level statistics and pooled variance
params <- data %>%
  group_by(Group) %>%
  summarise( xbar = mean(x),
             s2_i = var(x),
             n_i = n() )

# Calculate the k curves
newdata <- expand.grid( x=seq(-1,13, by=0.01), Group=paste('G', 1:3, sep='') ) %>%
  left_join(params, by='Group') %>%
  mutate( f = dnorm(x, mean=xbar, sd=sqrt(s2_i) ) )

```

```
P2 <- ggplot(newdata, aes(x=x, y=f, color=Group)) +
  geom_line() + xlim(-1, 13) +
  labs( y='f(x)' )
multiplot(P1, P2, ncol=1)
```



The decision to classify a point x^*, y^* as in a particular group is often too stark of a decision, and we would like to assess the certainty of the assignment. We define the *posterior probability* of the point as coming from group i as

$$P(y^* = i | x^*) = \frac{f_i(x^*)}{\sum_j f_j(x^*)}$$

That is, add up the heights of all the curves at x^* and then use the percent contribution of each to the total as the probability.

4.5 Examples

4.5.1 Iris Data

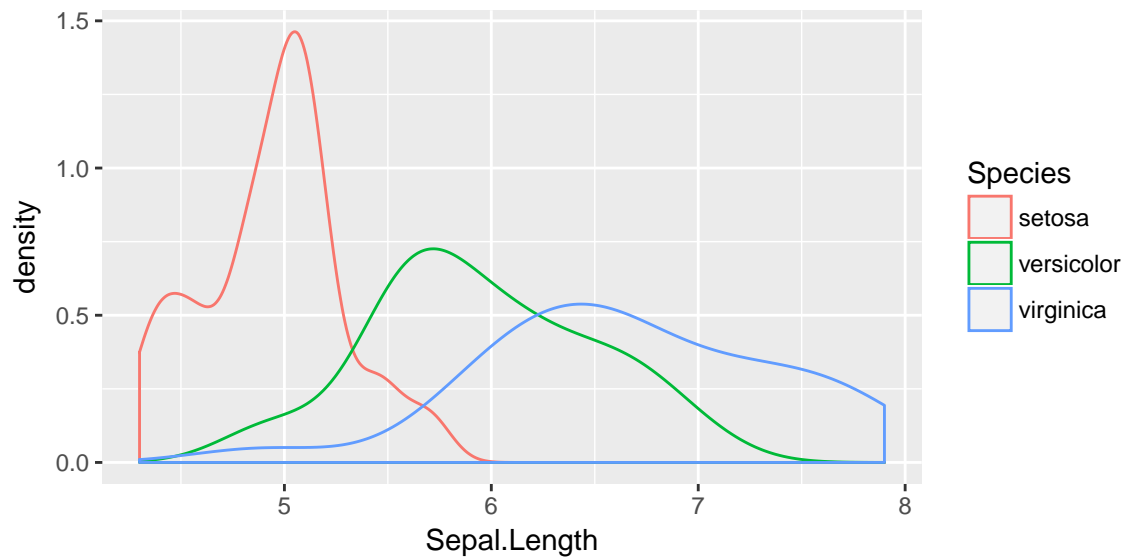
Recall the iris data set is 150 observations that measure leaf and sepal characteristics for three different species of iris. We'll use the leaf characteristics to try to produce a classification rule.

We will first split this into a training data set and a test data set.

```
# set.seed( 8675309 ) # so that the same training set is chosen every time...
iris$obs_ID <- 1:150 # there are a couple of identical rows, we don't like that

# random select 1/2 from each species to be the training set
train <- iris %>% group_by(Species) %>% sample_n( 25 )
test  <- setdiff(iris, train)
```

```
ggplot(train, aes(x=Sepal.Length, color=Species)) +
  geom_density()
```



While these certainly aren't normal and it isn't clear that the equal variance amongst groups is accurate, there is nothing that prevents us from assuming so and just doing LDA.

```
train %>% group_by(Species) %>%
  summarize(xbar = mean(Sepal.Length),
            sd   = sd(Sepal.Length))
```

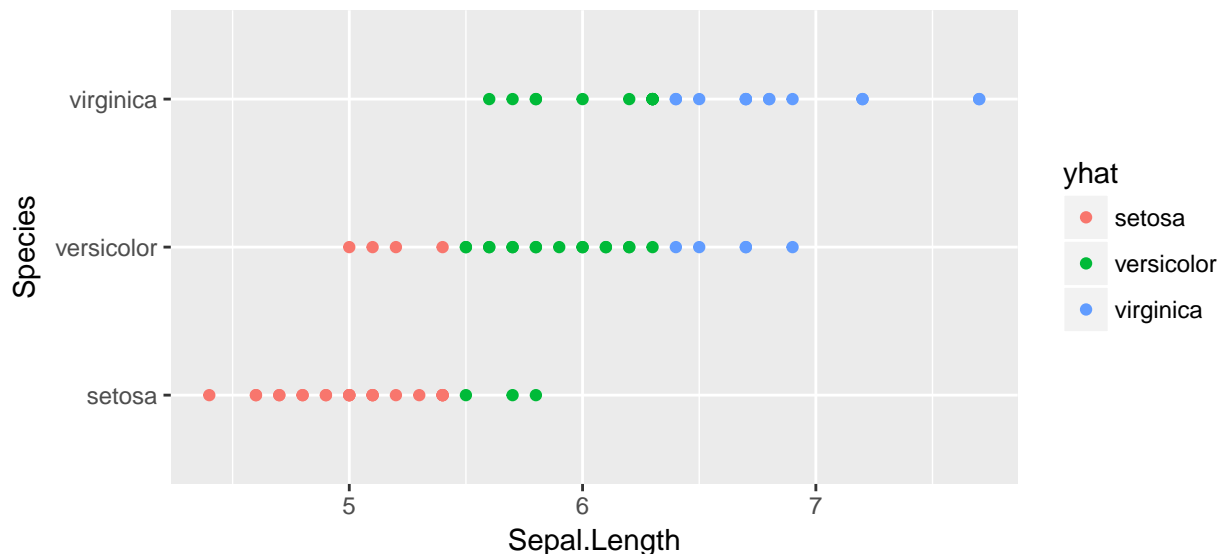
```
## # A tibble: 3 x 3
##   Species xbar      sd
##   <fctr> <dbl>   <dbl>
## 1  setosa 4.940 0.3452053
## 2 versicolor 5.956 0.5370289
## 3  virginica 6.672 0.7103520
```

We'll predict setosa if $x < (4.988 + 5.812)/2 = 5.4$. We'll predict versicolor if $5.4 < x < (5.812 + 6.412)/2 = 6.1$. We'll predict virginica if $6.1 < x$.

```
# have R do this
model <- train(Species ~ Sepal.Length, method='lda', data=train)
test$yhat <- predict(model, newdata=test)
table( Truth=test$Species, Prediction=test$yhat )
```

```
##           Prediction
## Truth      setosa versicolor virginica
## setosa         22          3          0
## versicolor      4         16          5
## virginica       0         12         13

ggplot(test, aes(x=Sepal.Length, y=Species, color=yhat) ) + geom_point()
```



This is a little hard to read, but of the 25 observations that we know are setosa, 5 of them have been misclassified as versicolor. Likewise, of the 25 observations that are virginica, 22 are correctly identified as virginica while 3 are misclassified as versicolor. So out of 75 test cases, we correctly classified 57, and incorrectly classified 18.

```
# Calculate the Misclassification rate
18/75
```

```
## [1] 0.24
```

```
test %>%
  summarise( misclassification_rate = mean( Species != yhat ) )
```

```
##   misclassification_rate
## 1                      0.32
```

Next, we will relax the assumption that the distributions have equal variance.

```
# have R do this
model <- train(Species ~ Sepal.Length, method='qda', data=train)
test$yhat <- predict(model, newdata=test)
table( Truth=test$Species, Prediction=test$yhat )
```

```
##           Prediction
## Truth      setosa versicolor virginica
## setosa       22         3         0
## versicolor   4        17         4
## virginica    0        14        11
```

```
test %>%
  summarise( misclassification_rate = mean( Species != yhat ) )
```

```
##   misclassification_rate
## 1                      0.333333
```

This has improved our accuracy as 4 setosa observations that were previously misclassified are now correctly classified.

```
# What about KNN?
model <- train(Species ~ Sepal.Length, method='knn', tuneGrid=data.frame(k=3), data=train)
test$yhat <- predict(model, newdata=test)
```

```
table( Truth=test$Species, Prediction=test$yhat )

##           Prediction
## Truth      setosa versicolor virginica
##  setosa         18          7          0
##  versicolor      3         15          7
##  virginica       0         11         14

test %>%
  summarise( misclassification_rate = mean( Species != yhat ) )

##  misclassification_rate
## 1                0.3733333
```

4.5.2 Detecting Blood Doping

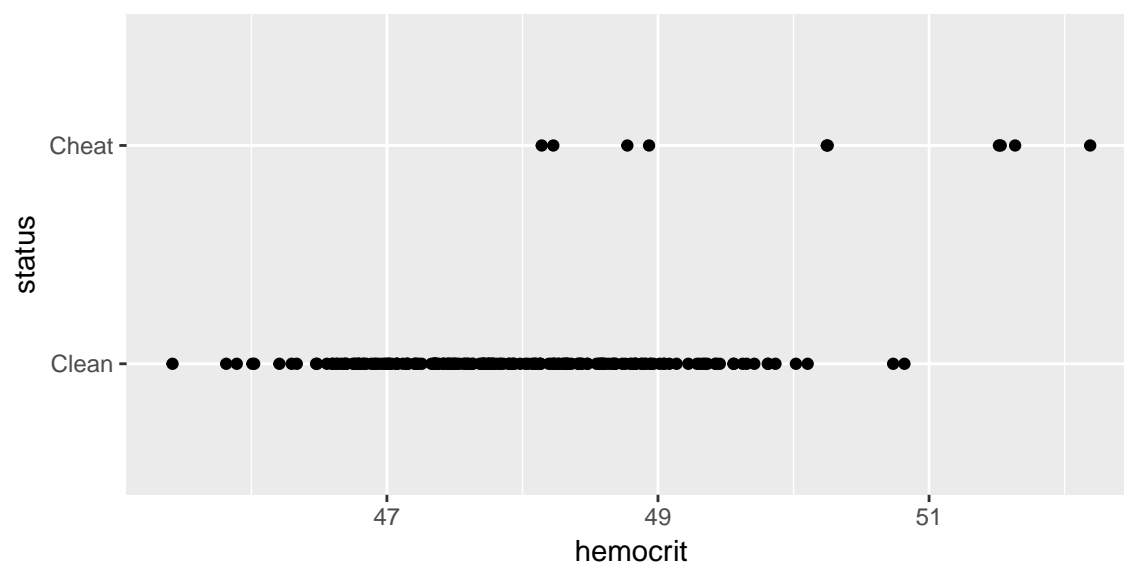
```
library(devtools)
install_github('dereksonderegger/dsData')
```

```
## Skipping install of 'dsData' from a github remote, the SHA1 (208be54a) has not changed since last install.
## Use `force = TRUE` to force installation
```

```
library(dsData)
```

We now consider a case where the number of observations is not the same between groups. Here we consider the case where we are interested in using hemocrit levels to detect if a cyclist is cheating.

```
data('Hemocrit', package='dsData')
ggplot(Hemocrit, aes(x=hemocrit, y=status)) +
  geom_point()
```



What if I just naively assume that all professional cyclists are clean? How accurate is this prediction scheme?

```
mean( Hemocrit$status == 'Clean' )
```

```
## [1] 0.95
```

In this case, I am pretty accurate because we correctly classify 95% of the cases! Clearly we should be more intelligent. Lets use the LDA to fit a model that uses hemocrit.

```
model <- train( status ~ hemocrit, method='lda', data=Hemocrit)
Hemocrit$yhat <- predict(model)
table( Truth=Hemocrit$status, Predicted=Hemocrit$yhat)
```

```
##           Predicted
## Truth   Clean Cheat
##   Clean   188    2
##   Cheat    6    4
```

So this method basically looks to see if the hemocrit level is greater than

```
Hemocrit %>% group_by(status) %>%
  summarise(xbar=mean(hemocrit))
```

```
## # A tibble: 2 x 2
##   status   xbar
##   <fctr>   <dbl>
## 1   Clean 47.91675
## 2   Cheat 50.14432
```

```
(47.917 + 50.144)/2
```

```
## [1] 49.0305
```

and calls them a cheater. Can we choose something a bit more clever? The `predict` function has an option where it returns the class probabilities. By default, it chooses the category with the highest probability (for two classes than means whichever is greater than 0.50). We can create a different rule that labels somebody a cheater only if the posterior probability is greater than 0.8 or whatever.

```
pred <- predict(model, type='prob')
Hemocrit <- Hemocrit %>%
  mutate( phat = pred$Cheat,
           yhat = ifelse(phat <= .8, 'Clean', 'Cheat' ))
table( Truth=Hemocrit$status, Predicted=Hemocrit$yhat)
```

```
##           Predicted
## Truth   Cheat Clean
##   Clean    0   190
##   Cheat    4    6
```

```
mean( Hemocrit$status == Hemocrit$yhat )
```

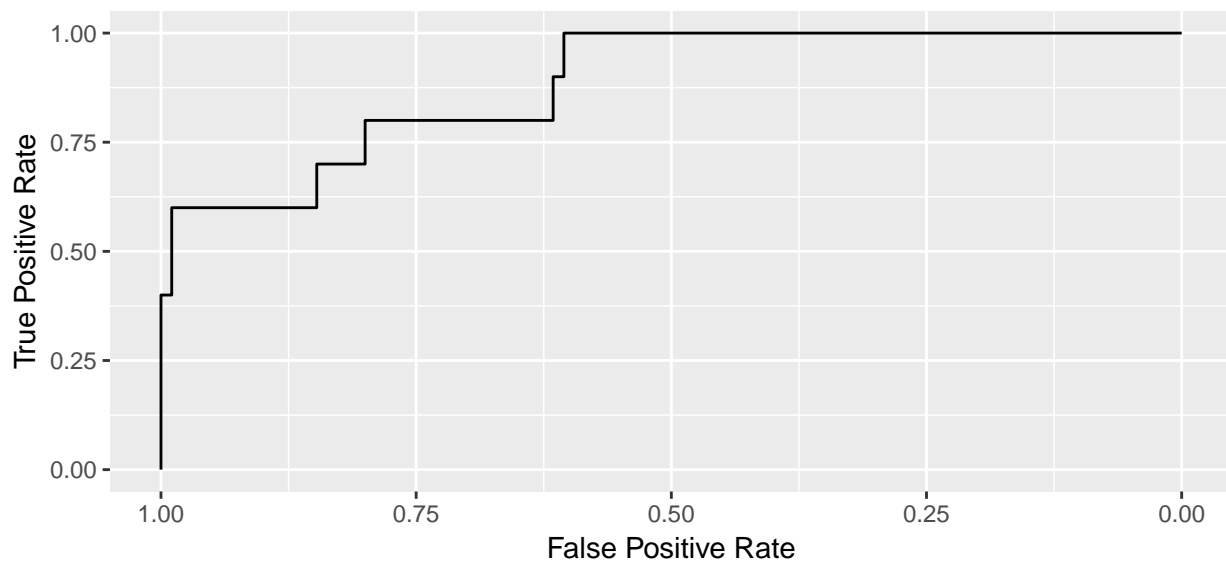
```
## [1] 0.97
```

```
ggplot(Hemocrit, aes(x=hemocrit, y=status, color=yhat)) +
  geom_point()
```




Great, now we have no false-positives, but a number of folks are getting away with cheating. But what if we back that up, how many false positives do we get... What we want is a graph that compares my false-positive numbers to the true-positives.

```
P3 <- pROC::roc(Hemocrit$status, Hemocrit$phat)
pROC::ggroc(P3) +
  labs(x='False Positive Rate', y='True Positive Rate')
```



4.6 Exercises

1. ISL Chapter 4, problem 4: When the number of features p is large, there tends to be a deterioration in the performance of KNN and other local approaches that perform prediction using only observations that are near the test observation for which a prediction must be made. This phenomenon is known as *the curse of dimensionality*, and it ties into the fact that non-parametric approaches often perform poorly when p is large. We will now investigate this curse.
 - a) Suppose that we have a set of observations, each with measurements on $p = 1$ feature, X . We assume that X is uniformly (evenly) distributed on $[0, 1]$. Associated with each observation

is a response value. Suppose that we wish to predict a test observation's response using only observations that are within 10% of the range of X closest to that test observation. For instance, in order to predict the response for a test observation with $X = 0.6$, we will use observations in the range $[0.55, 0.65]$. On average, what fraction of the available observations will we use to make the prediction?

- b) Now suppose that we have a set of observations, each with measurements on $p = 2$ features, X_1 and X_2 . We assume that (X_1, X_2) are uniformly distributed on $[0, 1] \times [0, 1]$. We wish to predict a test observation's response using only observations that are within 10% of the range of X_1 and within 10% of the range of X_2 closest to that test observation. For instance, in order to predict the response for a test observation with $X_1 = 0.6$ and $X_2 = 0.35$, we will use observations in the range $[0.55, 0.65]$ for X_1 and in the range $[0.3, 0.4]$ for X_2 . On average, what fraction of the available observations will we use to make the prediction?
 - c) Now suppose that we have a set of observations on $p = 100$ features. Again the observations are uniformly distributed on each feature, and again each feature ranges in value from 0 to 1. We wish to predict a test observation's response using observations within the 10% of each feature's range that is closest to that test observation. What fraction of the available observations will we use to make the prediction?
 - d) Using your answers to parts (a)–(c), argue that a drawback of KNN when p is large is that there are very few training observations “near” any given test observation. Even for 3 dimensions, the scheme proposed for KNN can only use approximately 1/1000th of the data, and adding more dimensions just makes this worse. In very high dimensions, the idea of “near” doesn't really work because in order to be “near” two points must be near in all dimensions and if they are different in any one dimension, then they get classified as being far apart, even if they are identical in the other 99 dimensions.
 - e) Now suppose that we wish to make a prediction for a test observation by creating a p -dimensional hypercube centered around the test observation that contains, on average, 10% of the training observations. For $p = 1, 2$, and 100, what is the length of each side of the hypercube? Comment on your answer. *Hint: Use the equation from part (c).*
2. ISL Chapter 4, problem 5: We now examine the differences between LDA and QDA.
- a) If the Bayes decision boundary is linear, do we expect LDA or QDA to perform better on the training set? On the test set?
 - b) If the Bayes decision boundary is non-linear, do we expect LDA or QDA to perform better on the training set? On the test set?
 - c) In general, as the sample size n increases, do we expect the test prediction accuracy of QDA relative to LDA to improve, decline, or be unchanged? Why?
 - d) True or False: Even if the Bayes decision boundary for a given problem is linear, we will probably achieve a superior test error rate using QDA rather than LDA because QDA is flexible enough to model a linear decision boundary. Justify your answer.
3. ISL Chapter 4, problem 6: Suppose we collect data for a group of students in a statistics class with variables X_1 = hours studied, X_2 = undergrad GPA, and Y = receive an A. We fit a logistic regression and produce estimated coefficient, $\hat{\beta}_0 = -6$, $\hat{\beta}_1 = 0.05$, $\hat{\beta}_2 = 1$.
- a) Estimate the probability that a student who studies for 40 hours and has an undergrad GPA of 3.5 gets an A in the class.
 - b) How many hours would the student in part (a) need to study to have a 50% chance of getting an A in the class?
4. ISL Chapter 4, problem 8: Suppose that we take a data set, divide it into equally-sized training and test sets, and then try out two different classification procedures. First we use logistic regression and get an error rate of 20% on the training data and 30% on the test data. Next we use 1-nearest neighbors (i.e. $K = 1$) and get an average error rate (averaged over both test and training data sets) of 18%.

Based on these results, which method should we prefer to use for classification of new observations? Why?

5. ISL Chapter 4, problem 11: In this problem, you will develop a model to predict whether a given car gets high or low gas mileage based on the Auto data set.
 - a) Create a binary variable, `mpg01`, that contains a 1 if `mpg` contains a value above its median, and a 0 if `mpg` contains a value below its median. You can compute the median using the `median()` function. Note you may find it helpful to use the `data.frame()` function to create a single data set containing both `mpg01` and the other Auto variables.
 - b) Explore the data graphically in order to investigate the association between `mpg01` and the other features. Which of the other features seem most likely to be useful in predicting `mpg01`? Scatterplots and boxplots may be useful tools to answer this question. Describe your findings.
 - c) Split the data into equally sized training and test sets.
 - d) Perform LDA on the training data in order to predict `mpg01` using the variables that seemed most associated with `mpg01` in (b). What is the test error of the model obtained?
 - e) Perform QDA on the training data in order to predict `mpg01` using the variables that seemed most associated with `mpg01` in (b). What is the test error of the model obtained?
 - f) Perform logistic regression on the training data in order to predict `mpg01` using the variables that seemed most associated with `mpg01` in (b). What is the test error of the model obtained?
 - g) Perform KNN on the training data, with several values of `K`, in order to predict `mpg01`. Use only the variables that seemed most associated with `mpg01` in (b). What test errors do you obtain? Which value of `K` seems to perform the best on this data set?
6. ISL Chapter 4, problem 13: Using the Boston data set, fit classification models in order to predict whether a given suburb has a crime rate above or below the median. Explore logistic regression, LDA, and KNN models using various subsets of the predictors. Describe your findings.

Bibliography