812 Section # 4

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1 Conditional Independence

Markov Property: The fundamental component of Markov is **memorylessness**. For later, this will pertain to the fact that the conditional probability distribution depends *only* on the current state of the world, rather than on past states of the world.

1.1 Markov Property Example

Last weekend, the Badgers went up against Bowling Green (in football). The final score was 68–17, with the Badgers winning. College (American) football is divided into two 30-minute halfs consisting of 2 15-minute quarters each. At the end of the 1st quarter of that game, Bowling Green had scored 10 points, and the Badgers had scored 14. By halftime, however, the Badgers had scored 41 (total) points, and Bowling Green still only had 10 points. That is, in the 2nd quarter, the Badgers scored 27 points, while Bowling Green scored 0 points. In this case, who wins the game doesn't depend on *both* who had the most points in the 1st quarter *and* who had the most points in the 2nd quarter, etc. The probability that the Badgers were going to win the game given their score in the 4th quarter doesn't depend on their score in the 1st quarter, or 2nd quarter, or 3rd quarter; it depends on their score in the final quarter. That final quarter score will be comprised of points scored throughout the game, but it is "memoryless" because it doesn't matter when those points were scored, now that we know we scored them.

Conditional Independence: Given an event Z, events X and Y are independent. Note that this differs from independence that is not conditional, in which knowing that X occurred does not influence the probability of Y.

Formal definition: Events $A_1, ..., A_k$ are conditionally independent given B if for every subset of $A_{i1}, ..., A_{ij}$ of j of these events, where $j = 1, 2, ..., k$,

$$P\left(A_{i1} \cap ... \cap A_{ij} \mid B\right) = P(A_{i1} \mid B)...P(A_{ij} \mid B)$$

1.2 Conditional Independence Examples

1.2.1 Example 1: Minnesota Winter

Riley and Shawn both live in Minneapolis and work at the University of Minnesota. At the end of every long day, each of them makes the long trek home. There is some probability that each makes it home in time for dinner; the probability for Riley is independent of the probability for Shawn (remember that Minneapolis isn't a small city, and if it makes it easier, assume they're driving in opposite directions away from the University). Call the probability that Riley makes it home in time for dinner at 6 pm p, and the probability that Shawn makes it home in time for dinner at 6 pm q.

Now imagine that, hypothetically (or, last January), a huge snowstorm hits Minneapolis and then the temperature drops to -45°F (with windchill). This occurrence produces traffic standstills in every direction with probability t. Probability t influences p and q, the probabilities that Riley and Shawn each make it home by 6 pm that day, but p and q remain independent of each other, conditional on t.

Why does this represent conditional independence? Because conditional on t, knowing that Riley made it home in time for dinner does not give us any more information about whether Shawn did.

1.2.2 Example 2: Sale, No Sale

Suppose that both Avery and Jayden like to shop. On any given Saturday, the probability that Avery goes shopping is $p = .8$ and the probability that Jayden goes shopping is $q = .6$. Avery and Jayden don't know each other.

On October 11th, 2013, there was some probability $s = .9$ that the store Avery and Jayden often go to will have sales. This may or may not have to do with Columbus Day. We know that the store is not JC Penney's, but other than that, what can we say about the probability that Avery goes shopping given the sale? What about the probability that Jayden goes shopping? Are these probabilities conditionally independent (i.e., once we know about the sale)?

1.2.3 Example 3: Rolling Dice

I have two die that I'm rolling. The numbers that come face up on each die are independent of each other. Let's say I'm playing an rpg with a group of friends. We come upon 2 orcs, and in order to determine whether I can inflict damage on each, I roll 2 d6 (that is, regular dice). Assume that in order to hit each orc I must roll at least a "4." The first die I roll ends up near me, but I have a spastic right wrist so the other one ends up behind the fruit bowl across the table. I can see that the one near me turned up as a "2." Does that give me any information about the one behind the fruit bowl? Are these rolls conditionally independent?

Now let's say that in the course of exploring a set of ruins I come upon a treasure chest, and I roll 2d10 (pentagonal trapazohedrons for the win!) to generate a percentage. This percentage determines what loot I get out of the chest (from a table enumerating 100 possible items). If I roll two 10's, I get 100%, but otherwise the number on one die denote the 10s place and the number on the other denotes the units place of the percentage.

The same thing happens as before, with one die classically hidden just out of sight behind the fruit bowl. I can see the first die, though, and the number face up is a "5." Does this give me information about the one behind the fruit bowl? Does it give me information about the ultimate outcome? Are the die conditionally independent?[1]

2 Random Variables and Distributions

Random Variable: A real valued function for which the domain is a sample space:

$$X : \Omega \to \mathbb{R} \text{ or } \mathbb{R}^n$$

[1]Thanks to Nat for suggesting several additional interesting, albeit unused, examples involving wizards and orcs.

2.1 Random Variable Example

Let's go back to the example from Problem Set 1. Recall that we surveyed 4 households to ask whether their income was greater than or equal to the median household income for the country as a whole. In this case, assume that we've taken multiple samples of 4 households. Their responses are enumerated in Table 1 below.

	H_1	H_2	H_3	H_4
1	Y	Y	Y	Y
2	N	N	N	N
3	Y	Y	N	N
4	Y	N	Y	N
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
16	.	.	.	.

Table 1: Household Income Survey

What is an example of a random variable that pertains to this data? That is, what is a function that maps this sample space into $\mathbb{R}$? Recall that the sample space will have 2^4 elements because there are two options for each of 4 households responding, where Y = above median income and N = equal to or below median income.

Random Sample: Assume a population of size N and a sample of size n. A random sample is one in which there is an equal probability of the $\binom{N}{n}$ samples.

Probability Mass Function: The pmf gives the probability that a random variable from that distribution takes on a given value.

$$P(Y = y) = p(y), \text{ sometimes written as } f(y)^2$$

[2]Note here that capital letters generally denote random variables, while lower case letters generally denote a particular value.

The pmf is used to discuss *discrete* distributions. Foreshadowing: with continuous distributions, the probability that a random variable takes on any given value is 0, so we will have to do something else. Note that all the values of the pmf must sum to 1.

2.2 Bernoulli Example

A Bernoulli distribution is a discrete distribution that can only take on values of $(0, 1)$. Typically we will define a "success" with a Bernoulli as a "1," where p will be the probability of success. That is:

$$P(Y = y) = \begin{cases} 1 - p & \text{if} \quad y = 0 \\ p & \text{if} \quad y = 1 \end{cases}$$

Cumulative Distribution Functions: The cdf gives the probability that the random variable Y will be $\leq y$:

$$P(Y \leq y) = F(y)$$

The cdf is always a nondecreasing function.

2.3 Bernoulli Example, continued

$F(0) = 1 - p$
$F(1) = 1$

Many thanks to Emily Sellars for past years' section materials!