Week 3: Lag Operators Cointegration (continued), Forecasting

Advanced Econometrics 4EK608

Vysoká škola ekonomická v Praze

Outline

Czech terminology

Operátory zpoždění
Superkonzistence
Grangerův reprezentační teorém
Engle-Grangerova dvoustupňová procedura
Kointegrační vektor

Lag operators

Lag operators:

$$
\begin{aligned}
Lx_t &= x_{t-1} \\
L(Lx_t) = L^2 x_t &= x_{t-2} \\
&\cdots \\
L^p x_t &= x_{t-p}
\end{aligned}
$$

Using lag operators,

$$AR(p) : x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + u_t$$

can be rewritten as:

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) x_t = \alpha + u_t$$

Lag operators

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) x_t = \alpha + u_t \tag{1}$$

Stochastic process (1) will only be stationary if the roots of corresponding equation (2) are all greater than unity in absolute value

$$1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p = 0 \tag{2}$$

Illustration 1 - AR(1) process:

$$x_t = \alpha + \phi x_{t-1} + u_t \tag{3}$$

$$(1 - \phi L) x_t = \alpha + u_t$$

$$1 - \phi L = 0$$

$$L = 1/\phi$$

$$\text{For (3) to be stationary}, |L| > 1 \leftrightarrow -1 < \phi < 1$$

Lag operators

Illustration 2:

$$x_t = 2 + 3.9x_{t-1} + 0.6x_{t-2} - 0.8x_{t-3} + u_t \tag{4}$$

To evaluate stationarity of x_t, we use

$$1 - 3.9L - 0.6L^2 + 0.8L^3 = 0,$$

which can be factorized:

$$(1 - 0.4L)(1 + 0.5L)(1 - 4L) = 0$$

$$\begin{aligned}
1^{st} root :& L = 2.5 \\
2^{nd} root :& L = -2 \\
3^{rd} root :& L = 0.25 \Rightarrow \text{(4) is non-stationary}
\end{aligned}$$

Cointegration between two variables

Superconsistency:[1] $y_t = \beta_0 + \beta_1 x_t + u_t$

1. Provided x_t and y_t are cointegrated, the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ will be consistent.
2. They converge in probability to their true values more quickly in the non-stationary case than in the stationary case. They are therefore very asymptotically efficient.

Consequences:

For simple static regression between two cointegrated variables: $y_t, x_t \sim C(1,1)$, super-consistency applies (with deterministic regressors such as intercept and trend added upon relevance). Dynamic misspecifications do not necessarily have serious consequences. This is a large sample property - in small samples, OLS estimators are biased.
(Specific statistical inference applies to cointegrating vectors.)

[1] Stock 1987

Cointegration between two variables

Granger representation theorem:[2]
If two TS x_t and y_t are cointegrated, the short-term disequilibrium relationship between them can be expressed in the ECM form

$$\Delta y_t = lagged(\Delta y, \Delta x) - \delta u_{t-1} + \varepsilon_t \tag{5}$$

where $u_{t-1} = y_{t-1} - \beta_0 - \beta_1 x_{t-1}$ is the disequilibrium error and δ is a short-run adjustment parameter.
Note: as u is on the scale of y, δ can be interpreted in percentages.
Example: $\delta = 0.8 \rightarrow 80\%$ of the disequilibrium error gets corrected between $t-1$ and t (on average).

Two implications:

1. The general-to-specific approach can focus on ECMs
2. Engle-Granger two-stage procedure

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[2] Engle and Granger (1987)

Cointegration between two variables

Engle-Granger two-stage procedure:
Way of short-cutting the search of an ECM model from general model

1^{st} stage: Estimation of the cointegrating (static) regression and saving residuals

$$\hat{u}_t = y_t - \hat{\beta}_0 - \hat{\beta}_1 x_t$$

2^{nd} stage: Use residuals $\hat{u}_{t-1}$ in (5) instead of u_{t-1} and estimate by OLS

Estimator are consistent and asymptotically efficient, but biased in small samples.

Cointegration among more than two variables

Possibility of more cointegrating vectors
long-run: $r_t = \beta_0 + \beta_1 w_t + \beta_2 x_t + \beta_3 y_t$, all variables are $I(1)$
If this long-run relationship exists, then the disequilibrium error

$$u_t = r_t - \beta_0 - \beta_1 w_t - \beta_2 x_t - \beta_3 y_t \text{ should be } I(0) \tag{6}$$

In the multivariate case, there may be more then one stationary linear combination linking cointegrated variables.

If a linear combination of variables such as (6) is stationary, then the coefficients in this relationship form a cointegrating vector, e.g. $(1, -\beta_1, -\beta_2, -\beta_3)$.

Cointegration implies the existence of at least one cointegrating vector.

Cointegration among more than two variables

Testing and estimation

Testing is analogical to the two-variable case.

If there is just one cointegrating vector then estimation can still proceed by the Engle-Granger two-stage method.

With more then one cointegrating vectors, the preceding method is no more applicable. In such cases, Johansen (1988) suggests a maximum likelihood approach.

Predictions - basics

- LRM and its estimate:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + u$$
$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \cdots + \hat{\beta}_k x_k$$

- Prediction of expected value:

$$\hat{y}_p = E(y|x_1 = c_1, x_2 = c_2, \ldots, x_k = c_k)$$
$$\hat{y}_p = \hat{\beta}_0 + \hat{\beta}_1 c_1 + \hat{\beta}_2 c_2 + \cdots + \hat{\beta}_k c_k$$

- Rough (underestimated) confidence interval for the expected value prediction: (95%): $\hat{y}_p \pm 2 \times s.e.(\hat{y}_p)$. (Rule of thumb)

Predictions - basics

$s.e.(\hat{y}_p)$ can be obtained by reparametrization:

- Reparametrized LRM:

$$y^* = \beta_0^* + \beta_1^*(x_1 - c_1) + \beta_2^*(x_2 - c_2) + \cdots + u$$

- The following holds:

$$\begin{aligned} \hat{y}_p &= \hat{\beta}_0^* \\ s.e.(\hat{y}_p) &= s.e.(\hat{\beta}_0^*), \quad i.e. \\ Var(\hat{y}_p) &= Var(\hat{\beta}_0^*) \end{aligned}$$

Predictions - basics

- Predicted and actual values of y_p:

$$\hat{y}_p = \hat{\beta}_0 + \hat{\beta}_1 c_1 + \hat{\beta}_2 c_2 + \cdots + \hat{\beta}_k c_k$$
$$y_p = \beta_0 + \beta_1 c_1 + \beta_2 c_2 + \cdots + \beta_k c_k + u_p$$

- Prediction error

$$\hat{e}_p = y_p - \hat{y}_p = (\beta_0 + \beta_1 c_1 + \beta_2 c_2 + \cdots + \beta_k c_k) + u_p - \hat{y}_p$$

- Prediction error variance

$$Var(\hat{e}_p) = Var(u_p) + Var(\hat{y}_p)$$

because $Var(\beta_0 + \beta_1 c_1 + \beta_2 c_2 + \cdots + \beta_k c_k) = 0$

Predictions - basics

- If homoskedasticity holds, $\sigma^2 = Var(u_p)$:
  - $Var(\hat{e}_p) = \sigma^2 + Var(\hat{y}_p)$
  - We estimate σ^2 from the original LRM as $(SSR/(n-k-1))$
  - We get $Var(\hat{y}_p)$ from the reparametrized LRM

- Standard prediction error:
  - $s.e.(\hat{e}_p) = \sqrt{Var(\hat{e}_p)}$

- Prediction interval (95%)
  - $\hat{y}_p \pm t_{0.025} \times s.e.(\hat{e}_p)$

Predictions - basics

- Prediction with logarithmic dependent variable

$$\log(y) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + u$$
$$\widehat{\log(y)} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \cdots + \hat{\beta}_k x_k$$

$\hat{y} = e^{\widehat{\log(y)}}$ systematically underestimates $\hat{y}$,

we can use a correction: $\hat{y} = \widehat{\alpha}_0 e^{\widehat{\log(y)}}$

where $\widehat{\alpha}_0 = n^{-1} \sum_{i=1}^{n} \exp(\hat{u}_i)$

is a consistent (not unbiased) estimator of $\exp(u)$.

Predictions - basics (Matrix form)

Prediction based on estimated model:

$$\hat{y}_p = \boldsymbol{x}'_p \hat{\boldsymbol{\beta}}$$

Difference between prediction and actual y_p value:

$$\hat{e}_p = \hat{y}_p - y_p = \boldsymbol{x}'_p \hat{\boldsymbol{\beta}} - \boldsymbol{x}'_p \boldsymbol{\beta} - u_p = \boldsymbol{x}'_p (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) - u_p$$

If $\hat{\boldsymbol{\beta}}$ is unbiased estimator for $\boldsymbol{\beta}$,
$\hat{y}_p$ is an unbiased estimator for y_p value:

$$E(\hat{e}_p) = E(\hat{y}_p - y_p) = \boldsymbol{x}'_p E(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) + E(-u_p) = 0$$

and the variance of $\hat{e}_p$ can be expressed as:

$$E(\hat{e}_p^2) = var(\hat{e}_p) = \boldsymbol{x}'_p var(\hat{\boldsymbol{\beta}}) \boldsymbol{x}_p + var(u_p)$$

Predictions - basics (Matrix form)

Variance of $\hat{e}_p$ (continued):

$$\begin{aligned} var(\hat{e}_p) &= \boldsymbol{x}_p' var(\hat{\boldsymbol{\beta}}) \boldsymbol{x}_p + var(u_p) \\ &= \boldsymbol{x}_p' \left[\sigma^2 \left(\boldsymbol{X}'\boldsymbol{X} \right)^{-1} \right] \boldsymbol{x}_p + var(u_p) \\ &\text{substitute } \sigma^2, var(u_p) \text{ with } \hat{\sigma}^2 \text{ (homoskedasticity)} \\ &= \underbrace{\boldsymbol{x}_p' \left[\hat{\sigma}^2 \left(\boldsymbol{X}'\boldsymbol{X} \right)^{-1} \right] \boldsymbol{x}_p}_{\hat{\sigma}_p^2} + \hat{\sigma}^2 \end{aligned}$$

With growing sample size (asymptotically),
$var(u_p) = \hat{\sigma}_p^2 + \hat{\sigma}^2$ converges to $\hat{\sigma}^2$
$\ldots \; plim\, \hat{\boldsymbol{\beta}} = \boldsymbol{\beta} \quad \Rightarrow \quad plim\, \hat{\sigma}_p^2 = 0$

Predictions - basics (Matrix form)

Variance of $\hat{e}_p$ (continued):

$$var(\hat{e}_p) = \boldsymbol{x}_p' \left[\hat{\sigma}^2 \left(\boldsymbol{X}'\boldsymbol{X} \right)^{-1} \right] \boldsymbol{x}_p + \hat{\sigma}^2$$

after re-arranging, $s.e.(\hat{e}_p)$ may be written as

$$s.e.(\hat{e}_p) = \hat{\sigma} \cdot \sqrt{1 + \boldsymbol{x}_p' \left(\boldsymbol{X}'\boldsymbol{X} \right)^{-1} \boldsymbol{x}_p} \ ,$$

which relates to the individual prediction error.

For mean prediction errors (considering $\hat{\sigma}_p^2$ only):

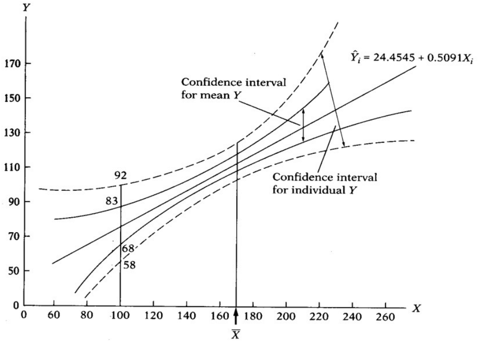
$$s.e.(\widetilde{e}_p) = \hat{\sigma} \cdot \sqrt{\quad \boldsymbol{x}_p' \left(\boldsymbol{X}'\boldsymbol{X} \right)^{-1} \boldsymbol{x}_p} \ .$$

Predictions - basics (Matrix form)

Prediction intervals: individual vs. mean value predictions:

Individual prediction: $y_p \in \hat{y}_p \pm t^*_{\alpha/2} \times s.e.(\hat{e}_p)$

Mean value: $y_p \in \hat{y}_p \pm t^*_{\alpha/2} \times s.e.(\widetilde{e}_p)$



Predictions - basics

- Why is it difficult to predict individual values?
  - they include random errors
  - we work with estimated parameters
  - parameters can change in time
  - (look at the formula for prediction error.)
- Prediction of expected values
  - parameters are estimated
  - parameters can change in time
- Impacts of random errors on predictions of individual values are usually much bigger than the impacts of (variance in) estimated parameters.

Chow tests

For any LRM: $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}$

- Chow tests are used to determine whether the regression function differs for different time periods (respondent groups).
- Time-stability of the estimated coefficients is a necessary condition for forecasting from an estimated model.
- Groups can be formed by different time periods.
- Chow tests can be defined for cross-sectional units as well. (wages for male/female individuals, etc.)

Chow tests

For any LRM: $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}$

- Say, the sample (time series) for a period $t = 1, 2, \ldots, T$ may be conveniently divided into two groups: $T_1 + T_2 = T$.
[consider two periods: fixed vs. floating F/X rates]
[pre-EU accession vs. post-EU accession period]

- Now, the LRM's vectors and matrices may be partitioned as follows:

$$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} \boldsymbol{\beta} + \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}$$

where $\boldsymbol{y}_1' = (y_1, \ldots, y_{T_1})$, $\boldsymbol{y}_2' = (y_{T_1+1}, \ldots, y_T)$, etc.
i.e. $\{\boldsymbol{y}_1, \boldsymbol{X}_1\} \in T_1$, $\{\boldsymbol{y}_2, \boldsymbol{X}_2\} \in T_2$.

Chow tests

For any LRM: $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}$, Chow test can be based on an auxiliary regression (unrestricted model for the F test):

- $$\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} \boldsymbol{\beta} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{X}_2 \end{bmatrix} \boldsymbol{\gamma} + \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}$$

  where $\mathbf{0}$ is a zero-matrix of the same dimensions as $\boldsymbol{X}_1$, i.e. $(T_1 \times k)$.

Also, we can see that:

- $T_1 : \quad \hat{\boldsymbol{y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}}$
- $T_2 : \quad \hat{\boldsymbol{y}} = \boldsymbol{X}(\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{\gamma}})$

Note: Power of the test depends on proper T_1 vs. T_2 cutoff. Chow test may be generalized for 3+ time periods (groups).

Chow tests

For our unrestricted model:

- $\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} \boldsymbol{\beta} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{X}_2 \end{bmatrix} \boldsymbol{\gamma} + \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}$

We can formulate the null of no structural change in model dynamics between the two time periods (groups) as follows:

- $H_0 : \quad \boldsymbol{\gamma} = \boldsymbol{0}$, i.e.: $\gamma_0 = \gamma_1 = \gamma_2 = \cdots = \gamma_k = 0$
- $H_1 : \quad \neg H_0$

This can be tested using an F-test (or its HC version):

- $F = \frac{SSR_r - SSR_{ur}}{SSR_{ur}} \times \frac{n-2k}{k} \underset{H_0}{\sim} F[k, (n-2k)]$

Chow test - Example

A simple Chow test example for CS data:
(to assess whether parameters are equal for M/F students.)

- Original model (Chow test restricted model):
  ...based on the well known Wooldridge dataset.

  $$cumgpa = \beta_0 + \beta_1 sat + \beta_2 hsperc + \beta_3 tothrs + u$$

- Auxiliary model (Chow test unrestricted model):

$$\begin{aligned} cumgpa = \beta_0 \quad & + \gamma_0 female \\ + \beta_1 sat \quad & + \gamma_1 (female \times sat) \\ + \beta_2 hsperc & + \gamma_2 (female \times hsperc) \\ + \beta_3 tothrs & + \gamma_3 (female \times tothrs) + u \end{aligned}$$

Chow test - Example (contd.)

- Null hypothesis $H_0 : \gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = 0$

  If all interactions effects are zero, we have the same regression function for both groups.

- Estimate of the unrestricted model

$$
\begin{aligned}
\widehat{cumgpa} = \underset{(.21)}{1.48} - \underset{(.411)}{.353}\,female + \underset{(.0002)}{.0011}\,sat + \underset{(.00039)}{.0075}\,(female \times sat) \\
- \underset{(.0014)}{.0085}\,hsperc - \underset{(.00316)}{.00055}(female \times hsperc) \\
+ \underset{(.0009)}{.0023}\,tothrs - \underset{(.00163)}{.00012}(female \times tothrs)
\end{aligned}
$$

…t-tests cannot be used to evaluate the joint H_0.

Chow test - Example (contd.)

- F-statistic:

$$F = \frac{(SSR_r - SSR_{ur})/k}{SSR_{ur}/(n-2k)} = \frac{(85.515 - 78.355)/4}{78.355/(366-8)} \approx 8.18$$

...using p-value, we reject the null hypothesis

- **Important:** Chow tests (all types) assume constant error variance across groups.

Chow 1: stability test for TS

Here, the F-statistic for the Chow test is calculated in an alternative way (Chow 1):

- For a suitable (potential) "breakpoint", we divide our sample $\{t = 1, 2, \ldots, T\}$ in two groups:
  "T_1" with $\{t = 1, 2, \ldots, T_1\}$ and
  "T_2" with $\{t = T_1+1,\ T_1+2, \ldots, T\}$
  . . . note that the choice of T_1 is arbitrary
  . . . (breakpoint-searching algorithms can be used)
- Run separate regressions for both T_1, T_2 groups; the SSR_{ur} is given by the sum of the SSRs of the two separately estimated regression models.
  . . . sufficient observations in T_1 and T_2 are required (d.f.)
- Run the original (restricted) regression model on the whole sample T and store SSR_r.

Chow 1: stability test for TS

$$F = \frac{SSR_r - SSR_{ur}}{SSR_{ur}} \cdot \frac{T-2k}{k} \underset{H_0}{\sim} F(\,k\,,\,T-2k\,)$$

where
$SSR_{ur} = SSR_{T_1} + SSR_{T_2}$
$SSR_r \; = SSR_T$
k is the number of parameters (including intercept) in LRM

H_0: stable structure of coefficients - no statistically significant differences between T_1 and T_2.
H_1: $\neg H_0$ (assume structural change in parameters over time)

Note: Chow 1 can be generalized for G time periods ($G-1$ "breakpoints").
... In such case, $SSR_{ur} = \sum_{g=1}^{G} SSR_g$, d.f. $= T - Gk$
... and we assume $T_g > k$ for all time groups.
... (only usable for small G-values, problematic setup of breakpoints)

Chow 2: prediction test for TS

Sometimes, we do not have enough observations to estimate the LRM separately for T_1 and T_2 as in the Chow 1 test.

In such case, we can use Chow 2: test of prediction unsuitability (slightly different F-statistics).

- The whole period is again divided into two subsets: $T = T_1 + T_2$.
- T_1 is the "base" period (sample size)
- T_2 is the number of "additional" observations, it usually corresponds to an ex-post prediction period

Chow 2: prediction test for TS

$$F = \frac{SSR_r - SSR_{ur}}{SSR_{ur}} \cdot \frac{T_1 - k}{T_2} \underset{H_0}{\sim} F(T_2, T_1 - k)$$

where

$SSR_{ur} = SSR_{T_1}$ (from LRM estimated for "base" period)

$SSR_r = SSR_T$ (from LRM estimated for the whole period)

k is the number of parameters (including intercept) in LRM

H_0: additional (T_2) observations come from the same DGP as in T_1.

H_1: $\neg H_0$ (assume significant differences between samples)

...If H_0 is rejected, we would expect large differences

...between predictions and actual observations of y_t.

If enough T_1 and T_2 observations are available, Chow 1 is preferred (compared to Chow 2) as it has more "power".

Forecasting time series

- **One-step-ahead forecast** f_t
  Forecast error $e_{t+1} = y_{t+1} - f_t$
  Information set: I_t
  Loss function: e_{t+1}^2 or $|e_{t+1}|$
  In forecasting, we minimize $E(e_{t+1}^2|I_t) = E[(y_{t+1} - f_t)^2|I_t]$
  Solution: $E(y_{t+1}|I_t)$

- **Multiple-step-ahead forecast** $f_{t,h}$
  Solution: $E(y_{t+h}|I_t)$

Forecasting time series

For some processes, $E(y_{t+1}|I_t)$ is easy to obtain:

1. **Martingale process (MP):**
   If $E(y_{t+1}|y_t, y_{t-1}, \ldots, y_0) = y_t, \forall t \geq 0$ then $\{y_t\}$ is MP
   $f_t = y_t$
   If a process $\{y_t\}$ is a martingale then $\{\Delta y_t\}$ is martingale difference sequence (MDS)

$$E(\Delta y_{t+1}|y_t, y_{t-1}, \ldots, y_0) = 0$$

2. **Process with exponential smoothing:**

$$E(y_{t+1}|I_t) = \alpha y_t + \alpha(1-\alpha)y_{t-1} + \cdots + \alpha(1-\alpha)^t y_0; \;\; 0 < \alpha < 1.$$

   Set $f_0 = y_0$, then for $t \geq 1: \; f_t = \alpha y_t + (1-\alpha)f_{t-1}$

Forecasting time series

3. **Regression models**
   - Static model: $y_t = \beta_0 + \beta_1 x_t + u_t$
     $E(y_{t+1}|I_t) = \beta_0 + \beta_1 x_{t+1} \rightarrow$ Conditional forecasting
     I_t contains $x_{t+1}, y_t, x_t, \ldots, y_1, x_1$
     $E(y_{t+1}|I_t) = \beta_0 + \beta_1 E(x_{t+1}|I_t) \rightarrow$ Unconditional forecasting
     I_t contains $y_t, x_t, \ldots, y_1, x_1$
   - More sense makes: $y_t = \delta_0 + \alpha_1 y_{t-1} + \gamma_1 x_{t-1} + u_t$
     $E(u_t|I_{t-1}) = 0$
     $E(y_{t+1}|I_t) = \delta_0 + \alpha_1 y_{t-1} + \gamma_1 x_{t-1}$
     We can use more lags, drop x or add more variables

Forecasting time series

One-Step-Ahead Forecasting with

$y_t = \delta_0 + \alpha_1 y_{t-1} + \gamma_1 x_{t-1} + u_t$:

point forecast: $\hat{f}_t = \hat{\delta}_0 + \hat{\alpha}_1 y_t + \hat{\gamma}_1 x_t$

forecast error: $\hat{e}_{t+1} = y_{t+1} - \hat{f}_t$

$s.e.$ of forecast: $s.e.(\hat{e}_{t+1}) = \{[s.e.(\hat{f}_t)]^2 + \hat{\sigma}^2\}^{1/2}$

forecast interval: essentially the same as prediction interval

approximate 95% forecast interval is: $\hat{f}_t \pm 1.96 \times s.e.(\hat{e}_{t+1})$

Forecasting time series

Example: File PHILLIPS

Forecasting US unemployment rate

$$\widehat{unem}_t = \underset{(.577)}{1.572} + \underset{(.097)}{.732}\, unem_{t-1}$$

$$n = 48, \overline{R}^2 = .544$$

$$\widehat{unem}_t = \underset{(.490)}{1.304} + \underset{(.084)}{.647}\, unem_{t-1} + \underset{(.041)}{.184}\, inf_{t-1}$$

$$n = 48, \overline{R}^2 = .677$$

Note that these regressions are not meant as causal equations. The hope is that the linear regressions approximate well the conditional expectation.

Forecasting time series

Evaluating forecast quality

- We can measure how forecasted values fit to actual observations (in-sample criteria, e.g. R^2)
- It is better, however, to evaluate the forecasting performance when forecasting out-of-sample values (out-of-sample criteria). For this purpose, use first n observations for estimation, and the remaining m observations to calculate the forecast errors $\hat{e}_{n+h}$
- Forecast evaluation measures:

  Mean Absolute Error $MAE = m^{-1} \sum_{h=1}^{m} |\hat{e}_{n+h}|$,

  Root Mean Squared Error $RMSE = (m^{-1} \sum_{h=1}^{m} \hat{e}^2_{n+h})^{1/2}$

  k-Fold Cross-Validation (kFCV) approach

Forecasting time series

- **Some comments**
  - Multiple-step-ahead forecasts are possible, but necessarily less precise.
  - Forecasts may make use of deterministic trends, but the error made by extrapolating time trends too far into the future may be large.
  - Similarly, seasonal patterns may be incorporated into forecasts.
  - It is possible to calculate confidence intervals for the point multiple-step-ahead forecasts.
  - Forecasting $I(1)$ time series can be based on adding predicted changes (which are $I(0)$) to base levels.
  - Forecast intervals for $I(0)$ series converge to the unconditional variance, whereas for integrated series, they are unbounded.