

# Bias variance decomposition

ETC3250

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Let  $y = f(x) + \varepsilon$  where  $\varepsilon$  is iid noise with zero mean and variance  $\sigma^2$ .

We estimate  $f$  using  $\hat{f}$ . Then the expected MSE for a new  $y$  at  $x_0$  will be equal to

$$E[(y - \hat{f}(x_0))^2] = [\text{Bias}(\hat{f}(x_0))]^2 + \text{Var}(\hat{f}(x_0)) + \sigma^2$$

where

$$\begin{aligned} \text{Bias}(\hat{f}(x_0)) &= E[\hat{f}(x_0)] - f(x_0) \\ \text{and} \quad \text{Var}(\hat{f}(x_0)) &= E\left[\left(\hat{f}(x_0) - E[\hat{f}(x_0)]\right)^2\right]. \end{aligned}$$

## Proof

We will abbreviate  $f = f(x_0)$  and  $\hat{f} = \hat{f}(x_0)$ .

Since  $f$  is deterministic,  $E[f] = f$  and  $\text{Var}[f] = 0$ .

$$\begin{aligned} E[(y - \hat{f})^2] &= E[(y - f + f - \hat{f})^2] \\ &= E[(y - f)^2 + (f - \hat{f})^2 + 2(y - f)(f - \hat{f})] \\ &= \sigma^2 + E[(\hat{f} - f)^2] + 2E[(y - f)(f - \hat{f})] \end{aligned}$$

Now

$$\begin{aligned} E[(\hat{f} - f)^2] &= E[(\hat{f} - E[\hat{f}] + E[\hat{f}] - f)^2] \\ &= E[(\hat{f} - E[\hat{f}])^2] + (E[\hat{f}] - f)^2 + 2E[(\hat{f} - E[\hat{f}])(E[\hat{f}] - f)] \\ &= \text{Var}[\hat{f}] + \text{Bias}^2[\hat{f}] + 2E[(\hat{f} - E[\hat{f}])(E[\hat{f}] - f)] \end{aligned}$$

Both cross-product terms are equal to zero as can be shown by expansion:

$$\begin{aligned} E[(y - f)(f - \hat{f})] &= E[yf - f^2 - y\hat{f} + f\hat{f}] \\ &= f^2 - f^2 - E[y\hat{f}] + fE[\hat{f}] \\ &= -E[(f + \varepsilon)\hat{f}] + fE[\hat{f}] \\ &= -E[f\hat{f}] - E[\varepsilon\hat{f}] + fE[\hat{f}] \\ &= 0 \end{aligned}$$

$$\begin{aligned} E[(\hat{f} - E[\hat{f}])(E[\hat{f}] - f)] &= E\left[\hat{f}E[\hat{f}] - E[\hat{f}]E[\hat{f}] - \hat{f}f + E[\hat{f}]f\right] \\ &= E[\hat{f}]E[\hat{f}] - E[\hat{f}]E[\hat{f}] - E[\hat{f}]f + E[\hat{f}]f \\ &= 0 \end{aligned}$$