# Section 5: Dummy Variables and Interactions

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Suggested Reading:
Statistics for Business, Part IV

# Example: Detecting Sex Discrimination

Imagine you are a trial lawyer and you want to file a suit against a company for salary discrimination... you gather the following data...

```
 Gender    Salary
1    Male   32.0
2  Female   39.1
3  Female   33.2
4  Female   30.6
5    Male   29.0
... ... ...
208 Female  30.0
```

# Detecting Sex Discrimination

You want to relate salary($Y$) to gender($X$)... how can we do that?

Gender is an example of a categorical variable. The variable gender separates our data into 2 groups or categories. The question we want to answer is: *"how is your salary related to which group you belong to..."*

Could we think about additional examples of categories potentially associated with salary?

- MBA education vs. not
- legal vs. illegal immigrant
- quarterback vs wide receiver

# Detecting Sex Discrimination

We can use regression to answer these question but we need to recode the categorical variable into a dummy variable

```
 Gender     Salary  Sex
1     Male  32.00   1
2   Female  39.10   0
3   Female  33.20   0
4   Female  30.60   0
5     Male  29.00   1
... ... ...
208  Female  30.00  0
```

Note: In Excel you can create the dummy variable using the formula:

=IF(Gender="Male",1,0)

# Detecting Sex Discrimination

Now you can present the following model in court:

$$Salary_i = \beta_0 + \beta_1 Sex_i + \epsilon_i$$

How do you interpret $\beta_1$?

$$\begin{aligned} E[Salary|Sex = 0] &= \beta_0 \\ E[Salary|Sex = 1] &= \beta_0 + \beta_1 \end{aligned}$$

$\beta_1$ is the male/female difference

# Detecting Sex Discrimination

$$Salary_i = \beta_0 + \beta_1 Sex_i + \epsilon_i$$

| *Regression Statistics* | |
|---|---|
| Multiple R | 0.346541 |
| R Square | 0.120091 |
| Adjusted R Square | 0.115819 |
| Standard Error | 10.58426 |
| Observations | 208 |

ANOVA

| | *df* | *SS* | *MS* | *F* | *Significance F* |
|---|---|---|---|---|---|
| Regression | 1 | 3149.634 | 3149.6 | 28.1151 | 2.93545E-07 |
| Residual | 206 | 23077.47 | 112.03 | | |
| Total | 207 | 26227.11 | | | |

| | *Coefficients* | *tandard Erro* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
|---|---|---|---|---|---|---|
| Intercept | 37.20993 | 0.894533 | 41.597 | 3E-102 | 35.44631451 | 38.9735426 |
| Gender | 8.295513 | 1.564493 | 5.3024 | 2.9E-07 | 5.211041089 | 11.3799841 |

$\hat{\beta}_1 = b_1 = 8.29...$ on average, a male makes approximately \$8,300 more than a female in this firm.

How should the plaintiff's lawyer use the confidence interval in his presentation?

# Detecting Sex Discrimination

How can the defense attorney try to counteract the plaintiff's argument?

Perhaps, the observed difference in salaries is related to other variables in the background and NOT to policy discrimination...

Obviously, there are many other factors which we can legitimately use in determining salaries:

- education
- job productivity
- experience

How can we use regression to incorporate additional information?

# Detecting Sex Discrimination

Let's add a measure of experience...

$$Salary_i = \beta_0 + \beta_1 Sex_i + \beta_2 Exp_i + \epsilon_i$$

What does that mean?

$$\begin{aligned} E[Salary|Sex = 0, Exp] &= \beta_0 + \beta_2 Exp \\ E[Salary|Sex = 1, Exp] &= (\beta_0 + \beta_1) + \beta_2 Exp \end{aligned}$$

# Detecting Sex Discrimination

The data gives us the "year hired" as a measure of experience...

```
    Exp   Gender   Salary  Sex
1        3   Male  32.00   1
2       14 Female  39.10   0
3       12 Female  33.20   0
4        8 Female  30.60   0
5        3   Male  29.00   1
... ... ...
208      33 Female  30.00  0
```

# Detecting Sex Discrimination

$$Salary_i = \beta_0 + \beta_1 Sex_i + \beta_2 Exp + \epsilon_i$$

SUMMARY OUTPUT

| *Regression Statistics* | |
|---|---|
| Multiple R | 0.70068016 |
| R Square | 0.49095268 |
| Adjusted R Sq | 0.48598637 |
| Standard Erro | 8.07007076 |
| Observations | 208 |

ANOVA

| | *df* | *SS* | *MS* | *F* | *Significance F* |
|---|---|---|---|---|---|
| Regression | 2 | 12876.2686 | 6438.13431 | 98.8565267 | 8.7642E-31 |
| Residual | 205 | 13350.8386 | 65.126042 | | |
| Total | 207 | 26227.1072 | | | |

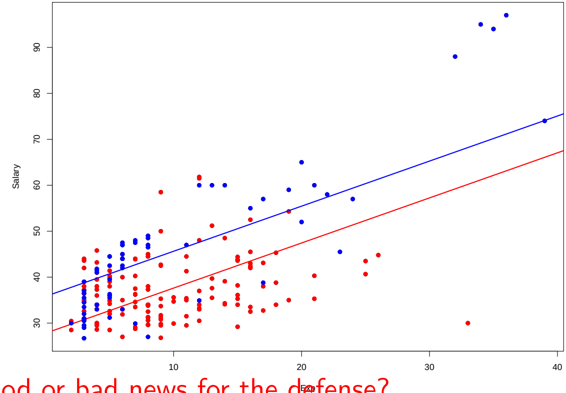
| | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
|---|---|---|---|---|---|---|
| Intercept | 27.8119041 | 1.02789303 | 27.0571969 | 1.3985E-69 | 25.7853066 | 29.8385016 |
| Exp | 0.98115095 | 0.08028453 | 12.2209217 | 3.6995E-26 | 0.82286169 | 1.13944021 |
| Sex | 8.01188578 | 1.19308866 | 6.71524761 | 1.8094E-10 | 5.659588 | 10.3641836 |

$$Salary_i = 27 + 8Sex_i + 0.98Exp_i + \epsilon_i$$

Is this good or bad news for the defense?

# Detecting Sex Discrimination

$$Salary_i = \begin{cases} 27 + 0.98 Exp_i + \epsilon_i & \text{females} \\ 35 + 0.98 Exp_i + \epsilon_i & \text{males} \end{cases}$$




Is this good or bad news for the defense?

# More than Two Categories

We can use dummy variables in situations in which there are more than two categories. Dummy variables are needed for each category except one, designated as the "base" category.

Why? *Remember that the numerical value of each category has no quantitative meaning!*

# Example: House Prices

We want to evaluate the difference in house prices in a couple of different neighborhoods.

```
   Nbhd SqFt  Price
1     2 1.79  114.3
2     2 2.03  114.2
3     2 1.74  114.8
4     2 1.98   94.7
5     2 2.13  119.8
6     1 1.78  114.6
7     3 1.83  151.6
8     3 2.16  150.7
... ... ... ...
```

# Example: House Prices

Let's create the dummy variables $dn1$, $dn2$ and $dn3$...

| | Nbhd | SqFt | Price | dn1 | dn2 | dn3 |
|---|---|---|---|---|---|---|
| 1 | 2 | 1.79 | 114.3 | 0 | 1 | 0 |
| 2 | 2 | 2.03 | 114.2 | 0 | 1 | 0 |
| 3 | 2 | 1.74 | 114.8 | 0 | 1 | 0 |
| 4 | 2 | 1.98 | 94.7 | 0 | 1 | 0 |
| 5 | 2 | 2.13 | 119.8 | 0 | 1 | 0 |
| 6 | 1 | 1.78 | 114.6 | 1 | 0 | 0 |
| 7 | 3 | 1.83 | 151.6 | 0 | 0 | 1 |
| 8 | 3 | 2.16 | 150.7 | 0 | 0 | 1 |
| ... | ... | ... | | | | |

# Example: House Prices

$$Price_i = \beta_0 + \beta_1 dn1_i + \beta_2 dn2_i + \beta_3 Size_i + \epsilon_i$$

$$\begin{aligned}
E[Price|dn1 = 1, Size] &= \beta_0 + \beta_1 + \beta_3 Size \quad \text{(Nbhd 1)} \\
E[Price|dn2 = 1, Size] &= \beta_0 + \beta_2 + \beta_3 Size \quad \text{(Nbhd 2)} \\
E[Price|dn1 = 0, dn2 = 0, Size] &= \beta_0 + \beta_3 Size \qquad\qquad \text{(Nbhd 3)}
\end{aligned}$$

# Example: House Prices

$$Price = \beta_0 + \beta_1 dn1 + \beta_2 dn2 + \beta_3 Size + \epsilon$$

| *Regression Statistics* | |
|---|---|
| Multiple R | 0.828 |
| R Square | 0.685 |
| Adjusted R Square | 0.677 |
| Standard Error | 15.260 |
| Observations | 128 |

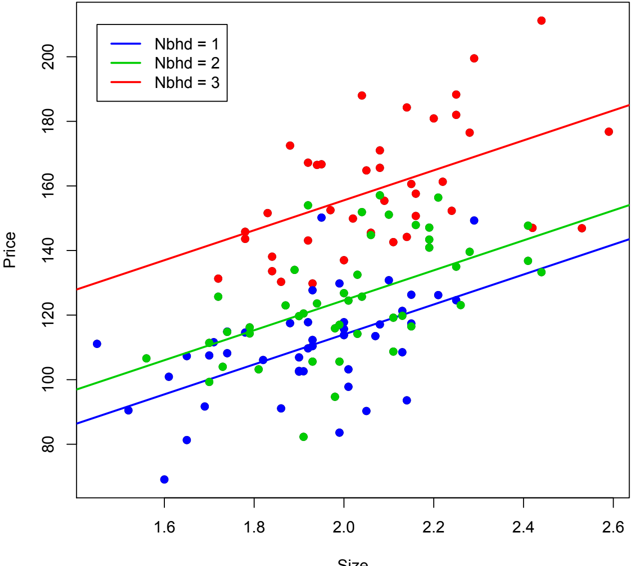
ANOVA

| | *df* | *SS* | *MS* | *F* | *gnificance F* |
|---|---|---|---|---|---|
| Regression | 3 | 62809.1504 | 20936 | 89.9053 | 5.8E-31 |
| Residual | 124 | 28876.0639 | 232.87 | | |
| Total | 127 | 91685.2143 | | | |

| | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
|---|---|---|---|---|---|---|
| Intercept | 62.78 | 14.25 | 4.41 | 0.00 | 34.58 | 90.98 |
| dn1 | -41.54 | 3.53 | -11.75 | 0.00 | -48.53 | -34.54 |
| dn2 | -30.97 | 3.37 | -9.19 | 0.00 | -37.63 | -24.30 |
| size | 46.39 | 6.75 | 6.88 | 0.00 | 33.03 | 59.74 |

$$Price = 62.78 - 41.54 dn1 - 30.97 dn2 + 46.39 Size + \epsilon$$

# Example: House Prices



# Example: House Prices

$$Price = \beta_0 + \beta_1 Size + \epsilon$$

| *Regression Statistics* | |
|---|---|
| Multiple R | 0.553 |
| R Square | 0.306 |
| Adjusted R Square | 0.300 |
| Standard Error | 22.476 |
| Observations | 128 |

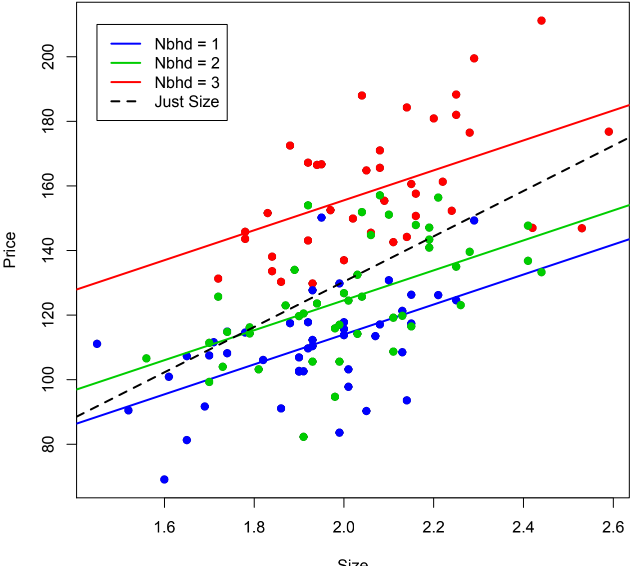
ANOVA

| | *df* | *SS* | *MS* | *F* | *gnificance F* |
|---|---|---|---|---|---|
| Regression | 1 | 28036.4 | 28036.36 | 55.501 | 1E-11 |
| Residual | 126 | 63648.9 | 505.1496 | | |
| Total | 127 | 91685.2 | | | |

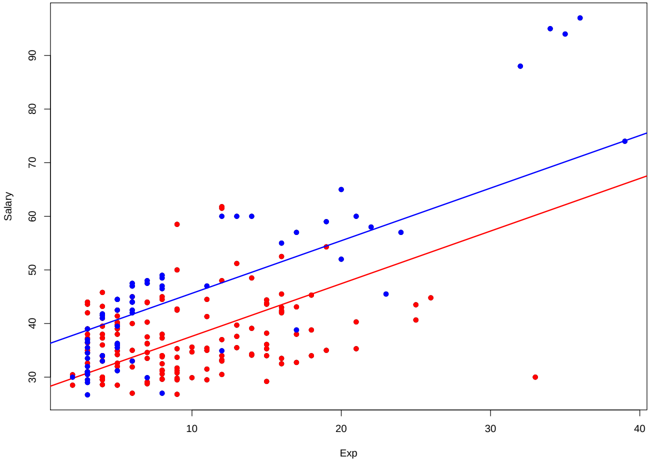
| | *Coefficients* | *andard Err* | *t Stat* | *P-value* | *ower 95%* | *pper 95%* |
|---|---|---|---|---|---|---|
| Intercept | -10.09 | 18.97 | -0.53 | 0.60 | -47.62 | 27.44 |
| size | 70.23 | 9.43 | 7.45 | 0.00 | 51.57 | 88.88 |

$$Price = -10.09 + 70.23 Size + \epsilon$$

# Example: House Prices



# Back to the Sex Discrimination Case




Does it look like the effect of experience on salary is the same for males and females?

# Back to the Sex Discrimination Case

Could we try to expand our analysis by allowing a different slope for each group?

Yes... Consider the following model:

$$Salary_i = \beta_0 + \beta_1 Exp_i + \beta_2 Sex_i + \beta_3 Exp_i \times Sex_i + \epsilon_i$$

For Females:

$$Salary_i = \beta_0 + \beta_1 Exp_i + \epsilon_i$$

For Males:

$$Salary_i = (\beta_0 + \beta_2) + (\beta_1 + \beta_3) Exp_i + \epsilon_i$$

How does the data look like?

| | Exp | Gender | Salary | Sex | Exp*Sex |
|---|---|---|---|---|---|
| 1 | 3 | Male | 32.00 | 1 | 3 |
| 2 | 14 | Female | 39.10 | 0 | 0 |
| 3 | 12 | Female | 33.20 | 0 | 0 |
| 4 | 8 | Female | 30.60 | 0 | 0 |
| 5 | 3 | Male | 29.00 | 1 | 3 |
| ... | ... | ... | | | |
| 208 | 33 | Female | 30.00 | 0 | 0 |

# Sex Discrimination Case

$$Salary = \beta_0 + \beta_1 Sex + \beta_2 Exp + \beta_3 Exp * Sex + \epsilon$$

SUMMARY OUTPUT

| *Regression Statistics* | |
|---|---|
| Multiple R | 0.79913035 |
| R Square | 0.63860932 |
| Adjusted R Sq | 0.63329475 |
| Standard Erro | 6.81629829 |
| Observations | 208 |

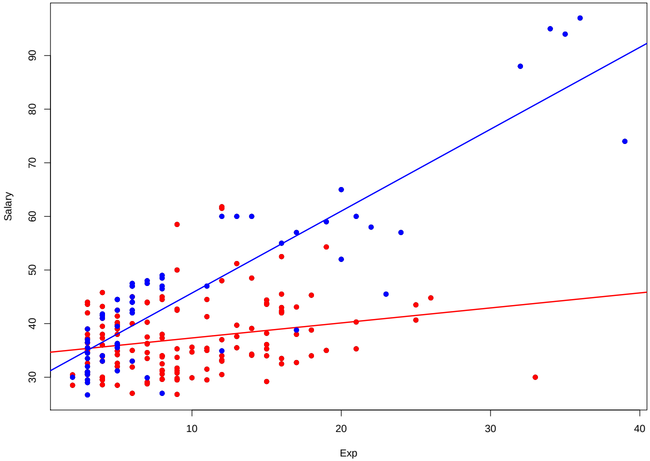
ANOVA

| | *df* | *SS* | *MS* | *F* | *Significance F* |
|---|---|---|---|---|---|
| Regression | 3 | 16748.8751 | 5582.95836 | 120.162018 | 7.5128E-45 |
| Residual | 204 | 9478.23216 | 46.4619224 | | |
| Total | 207 | 26227.1072 | | | |

| | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
|---|---|---|---|---|---|---|
| Intercept | 34.5282796 | 1.13797036 | 30.3419852 | 1.4745E-77 | 32.284588 | 36.7719713 |
| Exp | 0.27996335 | 0.10245572 | 2.73253013 | 0.00683654 | 0.07795541 | 0.48197129 |
| Sex | -4.0982519 | 1.66584202 | -2.4601684 | 0.01471882 | -7.3827274 | -0.8137763 |
| ExpSex | 1.24779837 | 0.1366757 | 9.12962828 | 6.8335E-17 | 0.97832023 | 1.51727651 |

$$Salary = 34 - 4Sex + 0.28Exp + 1.24Exp * Sex + \epsilon$$

# Sex Discrimination Case




Is this good or bad news for the plaintiff?

# Variable Interaction

So, the effect of experience on salary is different for males and females... in general, when the effect of the variable $X_1$ onto $Y$ depends on another variable $X_2$ we say that $X_1$ and $X_2$ interact with each other.

We can extend this notion by the inclusion of multiplicative effects through interaction terms.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 X_2) + \varepsilon$$

$$\frac{\partial E[Y|X_1, X_2]}{\partial X_1} = \beta_1 + \beta_3 X_2$$

We will pick this up in our next section...

# Example: College GPA and Age

Consider the connection between college and MBA grades:

A model to predict MBA GPA from college GPA could be

$$GPA^{MBA} = \beta_0 + \beta_1 GPA^{Bach} + \varepsilon$$

| | Estimate | Std.Error | t value | $Pr(>\|t\|)$ |
|---|---|---|---|---|
| BachGPA | 0.26269 | 0.09244 | 2.842 | 0.00607 ** |

For every 1 point increase in college GPA, your expected MBA GPA increases by about .26 points.

# College GPA and Age

However, this model assumes that the marginal effect of College GPA is the same for any age.

It seems that how you did in college should have less effect on your MBA GPA as you get older (further from college).

We can account for this intuition with an interaction term:

$$GPA^{MBA} = \beta_0 + \beta_1 GPA^{Bach} + \beta_2 (Age \times GPA^{Bach}) + \varepsilon$$

Now, the college effect is $\frac{\partial E[GPA^{MBA} | GPA^{Bach} \ Age]}{\partial GPA^{Bach}} = \beta_1 + \beta_2 Age$.

Depends on Age!

# College GPA and Age

$$GPA^{MBA} = \beta_0 + \beta_1 GPA^{Bach} + \beta_2 (Age \times GPA^{Bach}) + \varepsilon$$

Here, we have the interaction term but do not the main effect of age... what are we assuming?

| | Estimate | Std.Error | t value | $Pr(>\|t\|)$ |
|---|---|---|---|---|
| BachGPA | 0.455750 | 0.103026 | 4.424 | 4.07e-05 *** |
| BachGPA:Age | -0.009377 | 0.002786 | -3.366 | 0.00132 ** |

# College GPA and Age

Without the interaction term

- Marginal effect of College GPA is $b_1 = 0.26$.

With the interaction term:

- Marginal effect is $b_1 + b_2 Age = 0.46 - 0.0094 Age$.

| Age | Marginal Effect |
|---|---|
| 25 | 0.22 |
| 30 | 0.17 |
| 35 | 0.13 |
| 40 | 0.08 |