Business Statistics Midterm Exam

Spring 2018: BUS41000

This is a closed-book, closed-notes exam. You may use any calculator.

Please answer all problems in the space provided on the exam.

Read each question carefully and clearly present your answers.

Honor Code Pledge: "I pledge my honor that I have not violated the University Honor Code during this examination."

Sign: _____

Name: _____

Useful formulas

- $E(aX + bY) = aE(X) + bE(Y)$
- $Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2ab \times Cov(X, Y)$
- The standard error of $\bar{X}$ is defined as $s_{\bar{X}} = \sqrt{\frac{s_X^2}{n}}$, where s_X^2 denotes the sample variance of X.
- The standard error for the difference in the averages between groups a and b is defined as:

$$s_{(\bar{X}_a - \bar{X}_b)} = \sqrt{\frac{s_a^2}{n_a} + \frac{s_b^2}{n_b}}$$

  where s_a^2 denotes the sample variance of group a and n_a the number of observations in group a.
- The standard error for a proportion is defined by: $s_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
- The standard error for difference in proportion is defined by:

$$s_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

  where $\hat{p}_1$ and $\hat{p}_2$ denote two independent proportions, and n_1 and n_2 are the number of trials.
- Bayes formula:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

  where A, B are two events.
- For $Z \sim N(0, 1)$, $P(-1 \leq Z \leq 1) = 68\%, P(-2 \leq Z \leq 2) = 95\%, P(-3 \leq Z \leq 3) = 99\%$.
- Similarly, $X \sim N(\mu, \sigma^2)$, $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 95\%$.
- Standardization to standard normal: assume $X \sim N(\mu, \sigma^2)$, $Z \sim N(0, 1)$, then

$$P(a \leq X \leq b) = P(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}).$$

Problem 1: To build or not to build?

You live in a house that is somewhat prone to mud slides. In the coming rainy season there is a 3% chance of a mud slide occurring and you estimate that a mud slide would do $400,000 in damages.

Before the rainy season you are considering building a retaining wall that would potentially stop the damages from a mud slide. The wall costs $10,000 to build and, if the slide occurs, the wall will hold with 80% probability.

1. If you build the wall, what is the probability that you will lose $400,000 in damages this coming raining season? [10 points]

If build the wall,

$$P(damage) = P(damage \mid mudslide) \times P(mudslide) = 0.2 \times 0.03 = 0.006$$

2. Considering the potential damages on average (in expectation), should you build the wall? (Justify your answer) [5 points]

If build the wall, expected cost

$$E[cost] = 10,000 + 400,000 \times 0.06 = 12,400$$

If not build the wall, expected cost

$$E[cost] = 400,000 \times 0.03 = 12,000$$

So in terms of expectation, do not build the wall

3. Now considering instead the potential damages of only one rainy season, should you build the wall? (Justify your answer) [5 points] If build the wall,

$$cost = \begin{cases} 10,000 & \text{with probability } 0.994 \\ 12,400 & \text{with probability } 0.006 \end{cases}$$

If do not build the wall

$$cost = \begin{cases} 0 & \text{with probability } 0.97 \\ 400,400 & \text{with probability } 0.03 \end{cases}$$

In terms of probability, do not build the wall.

Problem 2: Who's to blame?

In manufacturing a new generation of iPhone, Apple buys a particular kind of microchip from 3 suppliers: 30% from Qualcomm, 50% from Intel, and 20% from Samsung.

Apple has extensive histories on the reliability of the chips and knows that 3% of the chips from Qualcomm are defective, 4% of the Intel are defective and 5% from Samsung are defective.

1. What is the probability that Apple choosed the chip from Intel and that the chip is defective? [5 points]

$$P(defectiveandIntel) = P(defective \mid Intel)P(Intel) = 0.04 \times 0.5 = 0.02$$

2. Write out the joint probability table of two random varibles: Brand of the chip (Brand = {Qualcomm, Intel, Samsung}), and whether the chip is defective (Defect = {Yes, No}). [5 points]

Table	Qualcomm	Intel	Samsung
Defect	$0.03 \times 0.3 = 0.009$	$0.04 \times 0.5 = 0.02$	$0.05 \times 0.2 = 0.01$
Not defect	$0.97 \times 0.3 = 0.291$	$0.96 \times 0.5 = 0.48$	$0.95 \times 0.2 = 0.19$

3. In testing a newly assembled iPhone, Apple found the microchip to be defective. Which provider is most likely to blame? (Justify your answer) [10 points]

$$P(Qualcomm \mid defective) = \frac{0.03 \times 0.3}{0.03 \times 0.3 + 0.04 \times 0.5 + 0.05 \times 0.2} = \frac{0.009}{0.039}$$

$$P(Qualcomm \mid defective) = \frac{0.04 \times 0.5}{0.03 \times 0.3 + 0.04 \times 0.5 + 0.05 \times 0.2} = \frac{0.02}{0.039}$$

$$P(Qualcomm \mid defective) = \frac{0.05 \times 0.2}{0.03 \times 0.3 + 0.04 \times 0.5 + 0.05 \times 0.2} = \frac{0.01}{0.039}$$

Intel has highest probability given the chip is defective. It's most likely to blame.

Problem 3: Breaking bad...

Two chemists working for a chicken fast food company, have been producing a very popular sauce. Let's call then Jesse and Mr. White. Gus, their boss, is tired of Mr. White's negative attitude and is thinking about "firing" him and keeping only Jesse on payroll. The problem, however, is that Mr. White seems to produce a higher quality sauce whenever he is in charge of production if compared to Jesse. Before making a final decision, Gus collected some data measuring the quality of different batches of sauce produced by Mr. White and Jesse. The results, measured on a quality scale, are listed below:

Table	average	std. deviation	sample size
Mr. White	97	1	7
Jess	94	3	10

1. Based in this data, can we tell for sure which one is the better chemist? [15 points]

Method 1

$$97 - 94 \pm 2\sqrt{\frac{1^2}{7} + \frac{3^2}{10}} = 3 \pm 2 \times 1.0212 = [0.96, 5.04]$$

At 95% confidence level, Mr. White is better.

Method 2

$$97 \pm 2\sqrt{\frac{1^2}{7}} = [96.24.97.75]$$

$$94 \pm 2\sqrt{\frac{3^2}{10}} = [92.10, 95.89]$$

At 95% confidence level, the two confidence intervals do not overlap, so we conclude that Mr. White is better.

2. Gus wants to keep the mean quality score for the sauce above 90. In this case, can he get rid of Mr. White, i.e., is Jesse good enough to run the sauce production? [15 points]

$$94 \pm 2\sqrt{\frac{3^2}{10}} = [92.10, 95, 89] > 90$$

Since the 95% confidence interval is higher than 90, he can.

Problem 4

I am trying to build a portfolio composed of SP500 and Bonds. Assume $SP500 \sim N(11, 19^2)$ and $Bonds \sim N(4, 6^2)$.

1. Consider the 50-50 split between SP500 and Bonds, assume the standard deviation of this 50-50 portfolio is
$$sd(0.5SP500 + 0.5Bonds) = 9.71.$$
Can you figure out the covariance between SP500 and Bonds, as well as the correlation? [10 points]

$$Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2 \times a \times b \times Cov(X, Y)$$

So
$$9.71^2 = 0.5^2 \times 19^2 + 0.5^2 \times 6^2 + 2 \times 0.5 \times 0.5 \times Cov(X, Y)$$

Solve the equation above, we have $Cov(X, Y) = -9.9318$

$$Corr(X, Y) = \frac{Cov(X, Y)}{SD(X) \times SD(Y)} = \frac{-9.9318}{19 \times 6} = -0.0871$$

2. Using the covariance you calculated in sub-problem 1, can you show me why
$$sd(0.8SP500 + 0.2Bonds) = 15.14.$$
In addition, which portfolio is better: a 80-20 split between SP500 and Bonds, or 50-50 split? Justify your criteria for a comparison. [10 points]

The variance of 80/20 portfolio is

$$0.8^2 \times 19^2 + 0.2^2 \times 6^2 + 2 \times 0.8 \times 0.2 \times (-9.9318) = 229.3 \approx 15.14^2$$

Sharp ratio of 50/50 portfolio is $\frac{7.5}{9.71} = 0.7724$ and 80/20 portfolio is $\frac{9.6}{15.14} = 0.6341$. 50/50 portfolio has larger Sharp ratio so we prefer it.

3. Which one of the two portolios considered in the above sub-problem 2 has a larger probability of delivering a positive return? [5 points]

$$P(\text{return of 50/50 portfolio} > 0) = P\left(Z > \frac{0 - 7.5}{9.71}\right) = P(Z > -0.77)$$

$$P(\text{return of 80/20 portfolio} > 0) = P\left(Z > \frac{0 - 9.6}{15.14}\right) = P(Z > -0.63)$$

where Z is standard normal distribution. Since $-0.77 < -0.63$, we have $P(Z > -0.77) > P(Z > -0.63)$. The 50/50 portfolio has higher probability of positive return.

Problem 5

Assume the model: $Y = 5 - 2X + \epsilon$, $\epsilon \sim N(0, 3^2)$.

1. Give a 95% prediction interval for Y given $X = 2$. [5 points]

$$5 - 2 \times 2 \pm 2 \times 3 = [-5, 7]$$

2. What is the mean of the distribution for Y when $X = 3$? How about the variance? [5 points]

Mean: $5 - 2 \times 3 = 01$, variance 9

3. What is the probability $P(Y > 7|X = 0.5)$, given that $X = 0.5$? [5 points]

$$P(Y > 7) = P(5 - 2 \times 0.5 + \epsilon > 7) = P(\epsilon > 3) = P\left(Z > \frac{3-0}{3}\right) = P(Z > 1) = 0.16$$

4. What is the probability of $P(Y > 10|X = 0.5)$ and $P(7 < Y < 10|X = 0.5)$, given that $X = 0.5$? [5 points]

$$P(Y > 10 \mid X = 0.5) = P(5 - 2 \times 0.5 + \epsilon > 10) = P(\epsilon > 6) = P\left(Z > \frac{6-0}{3}\right) = P(Z > 2) = 0.025$$

$$P(7 < Y < 10) = P(7 < 5 - 2 \times 0.5 + \epsilon < 10) = P(3 < \epsilon < 6) = P(1 < Z < 2) = 0.16 - 0.025 = 0.135$$

Problem 6:

The following table summarizes the annual returns on the SP500 from 1900 until the end of 2015, in total 116 years (in percentage terms):

116 years of SP500	
Sample average	7.2
Sample std. deviation	13.0

1. Based on these results, what is the probability of the SP500 returning less than 20% next year? [15 points]

$$P(X < 0.2) = P\left(Z < \frac{0.2 - 0.072}{0.13}\right) = P(Z < 0.98) \approx 0.84$$

2. Using a 99% confidence interval, to test the hypothesis that the expected return (true mean) of the SP500 is equal to 4% a year. In addition, what is the statistical meaning of a 99% confidence interval? [15 points]

The standard deviation is $\sqrt{13^2/116} = 1.21$, confidence interval is $7.2 \pm 3 \times 1.21 = [3.58, 10.82]$. We see that 4% is inside the interval so cannot reject the null hypothesis.

Statistical meaning of 99% confidence interval: If we simulate data and calculate confidence interval many times, on average, 99% of them can cover the true mean. Note that the true mean is fixed and never change, but the confidence interval moves around in different simulations. About 99% confidence intervals can cover the trun mean.

Common incorrect interprtations. 1. With probability 99%, the true mean lies in the confidence interval. 2. We are 99% confident that the true mean is in the confidence interval.

Problem 7: Google

Google is testing a new algorithm on personalized recommendation based on state-of-the-art deep learning research. Here is the data from the experiment: for each experiment, there are only two outcomes, success or failure.

Algorithm	current	new
success	1843	1920
failure	657	580

Is the new algorithm better? Justify your answer using either hypothesis testing, or confidence interval. [20 points]

Success rate of current method

$$\frac{1843}{1843+657} = 0.7372$$

Success rate of new method

$$\frac{1920}{1920+580} = 0.768$$

Difference of success rate

$$0.768 - 0.7372 = 0.0308$$

Standard deviation of the difference of success rates

$$s_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} = \sqrt{\frac{0.7372(1-0.7372)}{2500} + \frac{0.768(1-0.768)}{2500}} = 0.0122$$

Confidence interval. The confidence interval of difference at 95% level

$$0.0308 \pm 2 \times 0.0122 = [0.0064, 0.0552]$$

The confidence interval is postive and doesn't contain 0. So we conclude that the new approach is significantly better at 95% level.

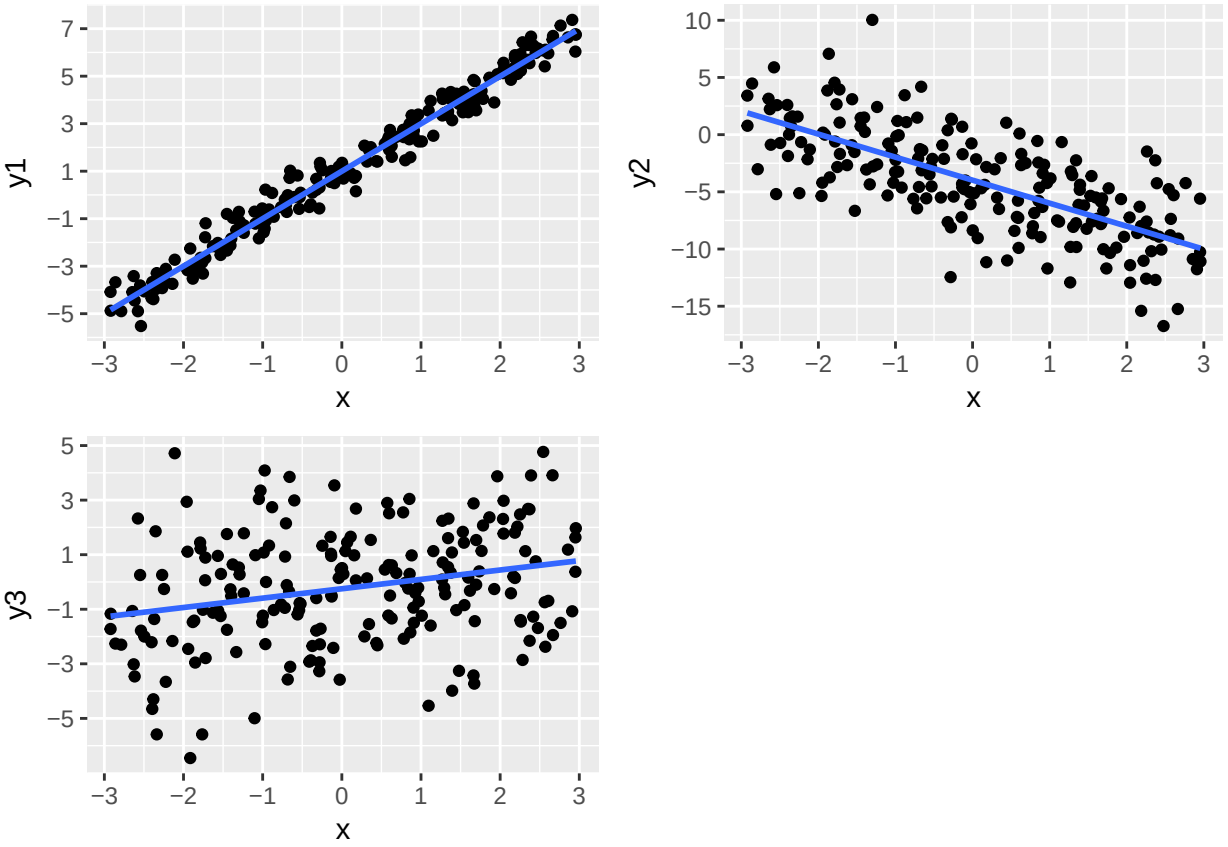
Hypothesis testing. Null hypothesis is $\hat{p}_{new} - \hat{p}_{current} = 0$.

$$\frac{0.0308}{0.0122} = 2.52 > 2$$

reject the null hypothesis at 95% level.

Note that because of rounding errors, some of you might get non significant result. I give full credict if your equations are correct.

Problem 8:



In the above scatterplots, three different variables $Y1, Y2, Y3$ are regressed onto the same X (in all three scatterplot we have the exact same n = 200 values for X). The line is the least square regression line. Carefully examine the plots and answer the questions below:

1. Which of the following is the least square estimates of the slope (b_1) and intercept (b_0) for the regression of $Y3$ on X? [5 points]

- (a) $b_1 = 0.34, b_0 = -0.24$ correct
- (b) $b_1 = 0.78, b_0 = 0.02$
- (c) $b_1 = 2.53, b_0 = -0.05$

2. Which of the following is the least square estimates of the slope (b_1) and residual standard error (s), for regression $Y2$ on X? [5 points]

- (a) $b_1 = -0.9, s = 6.3$
- (b) $b_1 = -2.0, s = 3.2$ correct
- (c) $b_1 = -4.3, s = 3.1$

3. Which of the following is the R^2 and residual standard error (s), for regression $Y1$ on X? [5 points]

- (a) $R^2 = 0.98, s = 0.5$ correct
- (b) $R^2 = 0.63, s = 0.97$
- (c) $R^2 = 0.88, s = 0.1$

4. What is the correlation between $Y2$ and X? [5 points]

- (a) -0.71 correct
- (b) -0.97
- (c) -0.26

5. Using all the information provided so far, give a rough approximation for the 99% prediction interval for $Y1$ given $X = 0$. [5 points]

$$Y1 \mid X = 0 \sim N(1, 0.5)$$

The confidence interval is

$$[1 - 3 \times 0.5, 1 + 3 \times 0.5] = [-0.5, 2.5]$$

6. What is the residual standard error s for $Y3$?

- (a) 2.05 correct
- (b) 0.49
- (c) 3.20

In addition, give an approximation for $P(Y3 > 4|X = 3)$. [5 points]

$$Y3 \mid X = 3 \sim N(1, 2^2)$$

$$P(Y3 > 4) = P\left(\frac{Y3 - 1}{2} > \frac{4 - 1}{2}\right) = P(Z > 1.5)$$

Problem 9: Olympics medal and GDP

Using data from Beijing 2008 and London 2012 I run a regression trying to understand the impact of **GDP** (gross domestic product measured in billions of US$) on the **total number of medals** won by a country in the summer Olympics. The results are

Regression Statistics	
Multiple R	0.816
R Square	0.666
Adjusted R Square	0.664
Standard Error	11.034
Observations	170.000

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1.000	40849.807	40849.807	335.525	0.000
Residual	168.000	20453.816	121.749		
Total	169.000	61303.624			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	5.16916	0.90899		0.000		
GDP	0.00836	0.00046		0.000		

1. What is the percentage of variation of total medals explained by **GDP**? [4 points]

R squared is the percentage of variation of total medals explained by GDP, it's 0.666 from the table.

2. From the results, what is your prediction (best guess) for the total number of medals for the U.S. in the Rio 2016 Olympics, given that the U.S. current GDP is of 18.5 trillion of dollars? [4 points]

$$5.16916 + 0.00836 \times 18500 = 159.829 \approx 160$$

The following table shows the total medal count for a few countries in Rio 2016 Olympics along with their current GDP:

Country	Total Medals	GDP (in US$ billions)
US	121	18,500
China	70	11,300
Brazil	19	1,600
UK	67	2,800

Using the results from the regression presented, answer the following questions:

3. Conditional on their GDP, which of these countries performance in the Rio 2016 is not surprising? Why? (Hint: use prediction interval) [4 points]

Confidence intervals at 95% confidence level

$US : 159.829 \pm 2 \times 11.034 China : 99.639 \pm 2 \times 11.034 Brazil : 18.549 \pm 2 \times 11.034 UK : 28.577 \pm 2 \times 11.034$

19 lies in the confidence interval of Brazil, so it is not surprising.

4. Conditional on their GDP, which of these countries looks like a clear overachiever? [4 points]

UK has 67 medals, which is above it's 95% confidence interval.

5. Based on the predictions from this model, who did better, China or the U.S.? (Hint: use how many standard deviation away the performance is from predicted value) [4 points]

US has 121 medals, which is 38 below its predicted mean and China has 70 medals, 29 below its predicted mean. So China does better.

Problem 10: Making the B-17 flying fortress stronger!

During a period in World War II, the U.S. Army Air Forces (AAF) would send over 300 B-17 bombers daily to raid factories in Germany. These missions, originating in the U.K., were very dangerous and, in the peak of the campaign, the return probability for a B-17 crew was only 84%.

In trying to reduce the probability of a failed mission, a Navy statistician (Abraham Wald) was put in charge of studying the damage patterns in the B-17's that successfully made back from a mission. His ultimate goal was to decide where to add extra armor in the planes (you couldn't just add heavy armor everywhere, as the planes would be too heavy to fly!). Wald was able to learn that if a plane made back from a mission there was a 67% probability they were shot in the fuselage, 15% in the fuel systems, 10% in the cockpit area and 8% in the engines.

From experiments, Wald was also able to deduce that during combat, a B-17 would be shot in the fuselage with 58% probability, in the fuel systems with 14%, in the cockpit area 14% and engine 14%.

Based on this information what was Wald's recommendation to the AAF, i.e., if they had to choose one area of the plane, where should they add extra armor to the B-17's? (Hint: Wald suggested to improve on the weakest area — the area with the smallest $P(\textit{success return}|\textit{area being shot})$, the probability of success return condition on that area is shot.) [20 points]

$$\begin{aligned} P(\text{success return} \mid \text{area being shot}) &= \frac{P(\text{success return\&area being shot})}{P(\text{area being shot})} \\ &= \frac{P(\text{area being shot} \mid \text{success return})P(\text{success return})}{P(\text{area being shot})} \end{aligned}$$

So

$$\begin{aligned} \text{fuselage}: &\frac{0.67 \times 0.84}{0.58} = 0.97 \\ \text{fuel}: &\frac{0.15 \times 0.84}{0.14} = 0.90 \\ \text{cockpit}: &\frac{0.1 \times 0.84}{0.14} = 0.60 \\ \text{engines}: &\frac{0.08 \times 0.84}{0.14} = 0.48 \end{aligned}$$

So engines are most weak part, needs extra armor.

(Bonus) Problem 11: Envelope game

At the end of BUS41000 class, your Professor decided to reward you for your hard work, but also to test your probability skills. He placed two checks (one check is \$30, the other is \$70) into two envelopes. Note you have no idea about the value of the checks at all.

1. You decided to randomly pick one envelope, how much is your reward, in expectation? [2 points]

$$30 \times 0.5 + 70 \times 0.5 = 50$$

2. Suppose that the rule changed slightly: You are allowed to choose only one envelope, open it, review the check value. And then decide whether to stick with the opened envelope, or to swap to the other envelope. You recalled that Professor said one can use probability to get more money. Suppose you are going to do the following: draw a random number X using R/Excel from a normal distribution $X \sim N(50, 10^2)$, then compare this X with the value of the check you opened, and only to keep the check if and only if its value is larger than X, otherwise swap to the other emvelope. Using this strategy, how much is your reward, in expectation? How much more money you are going to get compared to sub-problem 1? [10 points]

Suppose Y is value of check you first pick

Table	$X > Y$	$X < Y$
$Y = 30$	70, with probability 0.5×0.975	30, with probability 0.5×0.025
$Y = 70$	30, with probability 0.5×0.025	70, with probability 0.5×0.975

So the expectation is

$$70 \times 0.5 \times 0.975 + 30 \times 0.5 \times 0.025 \times 2 = 69$$

and $69 - 50 = 19$

Summary of midterm scores