Business Statistics 41000

Logistic Regression

Mladen Kolar

Binary response data

Often we want to predict a response variable that is binary: $Y = 0$ or 1.

The goal is generally to predict the probability that $Y = 1$.

You can then do classification based on this estimate.

- Buy or not buy.
- Win or lose.
- Sick or healthy.
- Pay or default.
- Thumbs up or down.

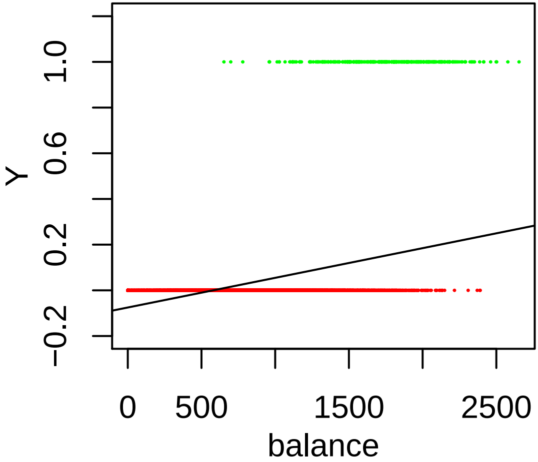
Can we use linear regression for this task?

```r
library(ISLR)
df = Default
df$Y = as.numeric(df$default)-1
head(df)
```

```
##   default student balance income Y
## 1      No      No     730  44362 0
## 2      No     Yes     817  12106 0
## 3      No      No    1074  31767 0
## 4      No      No     529  35704 0
## 5      No      No     786  38463 0
## 6      No     Yes     920   7492 0
```

Can we use linear regression for this task?

```r
par(mar=c(3,3,3,1), mgp=c(2,1,0))
plot(df$balance, df$Y, col=c("red","green")[df$Y+1],
      xlab="balance", ylab="Y", ylim = c(-0.2, 1.2), cex = 0.1)
abline(lm(Y~balance, data=df))
```



Can we use linear regression for this task?

We can, but the fit is very unappealing, as is the interpretation.

```r
summary(lm(Y~balance, data=df))
```

```
##
## Call:
## lm(formula = Y ~ balance, data = df)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -0.2353 -0.0694 -0.0263  0.0200  0.9905
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -7.52e-02   3.35e-03   -22.4   <2e-16 ***
## balance      1.30e-04   3.47e-06    37.4   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.168 on 9998 degrees of freedom
## Multiple R-squared:  0.123,  Adjusted R-squared:  0.122
## F-statistic: 1.4e+03 on 1 and 9998 DF,  p-value: <2e-16
```

This is just a bad fit.

Modelling probability $P(Y = 1 \mid X)$

Y is an indicator: $Y = 0$ or 1.

The conditional mean is thus

$$
\begin{aligned}
\mathbb{E}[Y|X] &= p(Y = 1|X) \times 1 + p(Y = 0|X) \times 0 \\
&= p(Y = 1|X).
\end{aligned}
$$

So the mean function is a probability.

We need a model that gives mean/probability values between 0 and 1.

Logistic regression

Model for conditional distribution of Y given $X = x$

$$p_1(x;\beta) = P(Y = 1 \mid x;\beta) = S(\beta_0 + \sum_{j=1}^{p} \beta_j \cdot x_j)$$

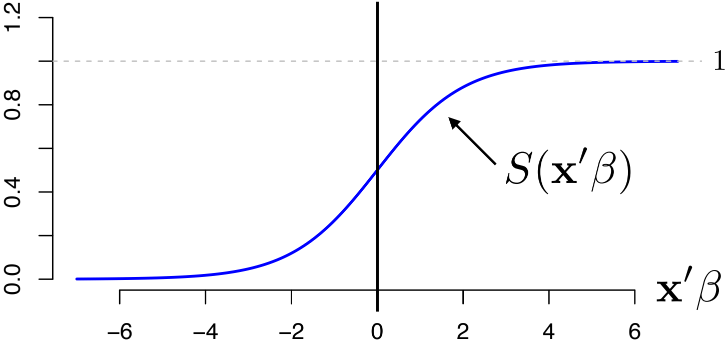
$$p_0(x;\beta) = P(Y = 0 \mid x;\beta) = 1 - p_1(x;\beta)$$

where $S(z) = \dfrac{e^z}{1 + e^z}$ is the logistic sigmoid function.

The log-odds of class 1 is a linear function of X

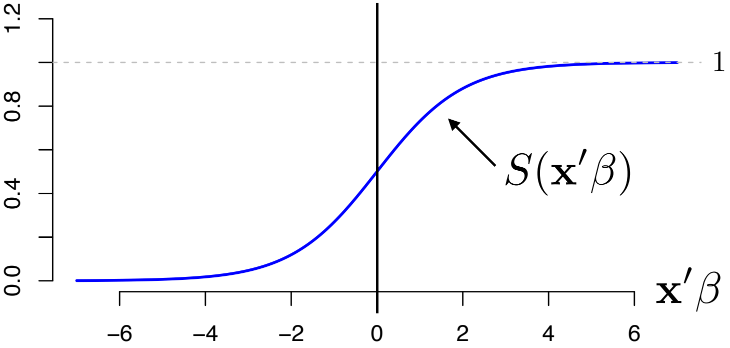
$$\log\left(\frac{p_1(x;\beta)}{p_0(x;\beta)}\right) = \beta_0 + \sum_{j=1}^{p} \beta_j \cdot x_j$$

Representation with logistic regression



A linear function $x'\beta$ can take values from $-\infty$ to ∞.
By passing it through sigmoid, we get values in $[0, 1]$.

These values also sum to 1 to make a probability.



Step 1: Linear combination

$$z = x'\beta$$

Step 2: Nonlinear transformation

$$P(Y = 1 \mid x; \beta) = S(z) = \frac{\exp(z)}{1 + \exp(z)}$$

z	$F(z)$
-3	0.05
-2	0.12
-1	0.27
0	0.5
1	0.73
2	0.88
3	0.95

Logistic regression

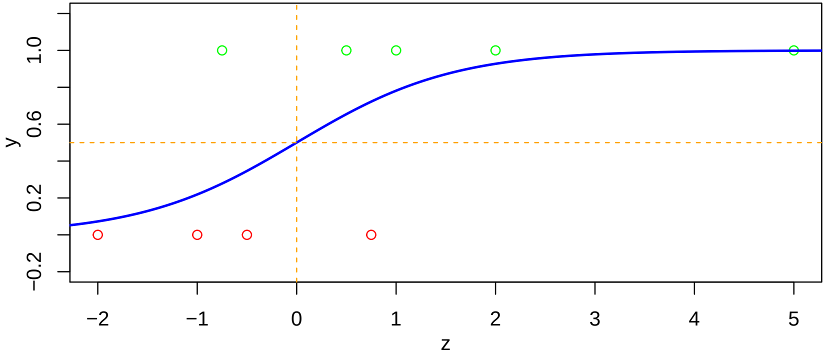
Our model for $Y \mid X$ is the following

$$p(Y=1|X_1 \dots X_d) = S\left(\boldsymbol{X}'\beta\right) = \frac{\exp[\beta_0 + \beta_1 X_1 \dots + \beta_d X_d]}{1 + \exp[\beta_0 + \beta_1 X_1 \dots + \beta_d X_d]}.$$

These models are easy to fit in `R`:

```
glm(Y ~ X1 + X2, family=binomial)
```

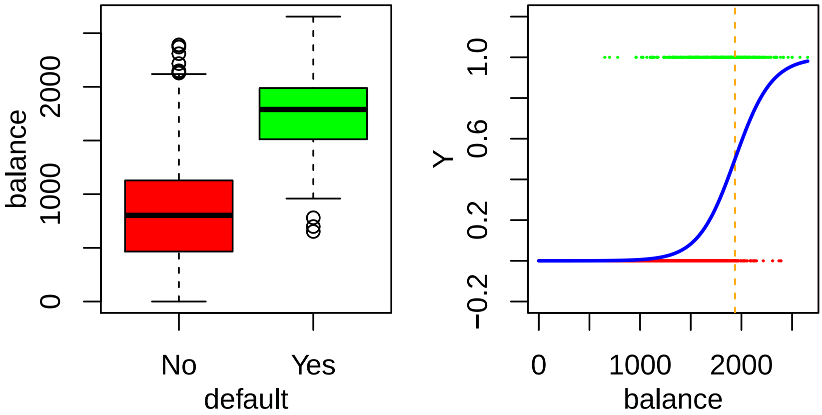
- "g" is for generalized; `binomial` indicates $Y = 0$ or 1.
- Otherwise, `glm` uses the same syntax as `lm`.



Suppose that $\hat{Y}_i = 1$ if $P(Y = 1 \mid z_i) > 0.5$ and $\hat{Y}_i = 0$ otherwise.

When do we assign a new observation to a positive class?
For what values of z do we have $\hat{Y} = 1$?

Example: Predicting default



```r
fit = glm(default~balance, df, family="binomial")
summary(fit)
```

```
## Call:
## glm(formula = default ~ balance, family = "binomial", data = df)
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -10.65133    0.36116   -29.5   <2e-16 ***
## balance       0.00550    0.00022    24.9   <2e-16 ***
```

$$\hat{p}_1(X) = \hat{p}(\text{default} \mid X) = \frac{\exp(-10.65 + X \cdot 0.0055)}{1 + \exp(-10.65 + X \cdot 0.0055)}$$

New predictions

The `predict` function works as before, but add `type = "response"` to get $\hat{p} = \exp[\boldsymbol{x}'b]/(1 + \exp[\boldsymbol{x}'b])$
(otherwise it just returns the linear function $\boldsymbol{x}'b$).

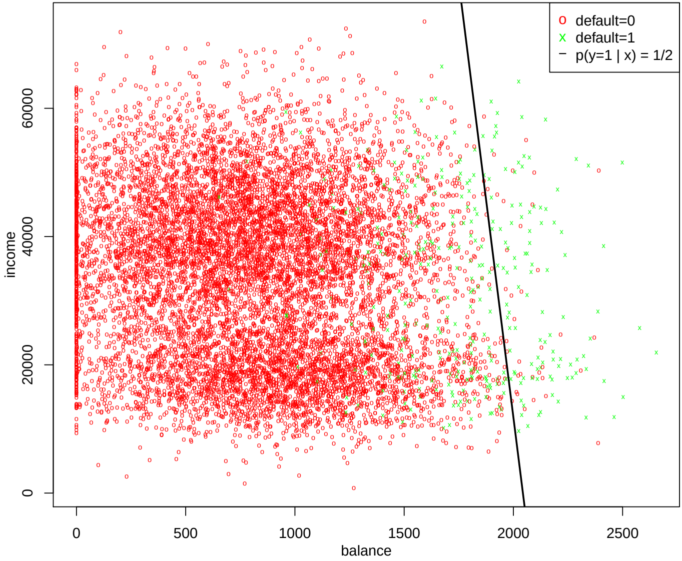
```
xf = data.frame(balance=c(1500, 2000, 2500))
phat = predict(fit, newdata=xf, type="response")
cbind(xf, phat)
```

```
##   balance   phat
## 1    1500 0.0829
## 2    2000 0.5858
## 3    2500 0.9567
```

More features

```r
summary(glm(default~balance+income, data=df, family="binomial"))
```

```
## Call:
## glm(formula = default ~ balance + income, family = "binomial",
##     data = df)
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -1.15e+01   4.35e-01  -26.54   <2e-16 ***
## balance      5.65e-03   2.27e-04   24.84   <2e-16 ***
## income       2.08e-05   4.99e-06    4.17    3e-05 ***
```



o default=0
x default=1
p(y=1 | x) = 1/2
income
0
20000
40000
60000
balance
0
500
1000
1500
2000
2500

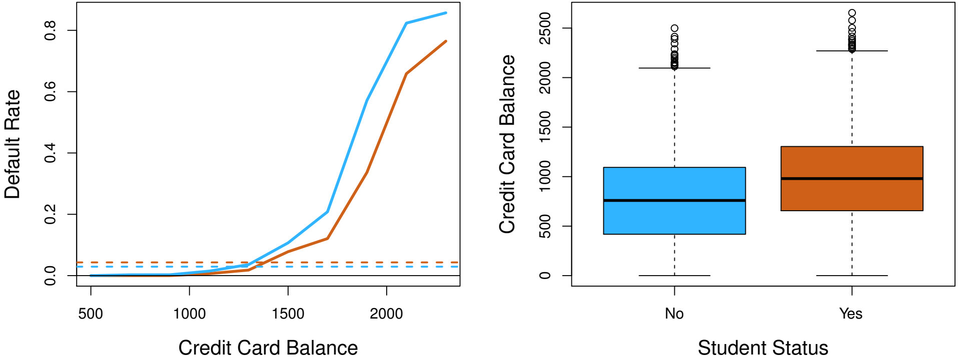
Confounding

```r
summary(glm(default~student, data=df, family="binomial"))
```

```
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  -3.5041     0.0707  -49.55  < 2e-16 ***
## studentYes    0.4049     0.1150    3.52  0.00043 ***
```

```r
summary(glm(default~balance+student, data=df, family="binomial"))
```

```
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -1.07e+01   3.69e-01  -29.12  < 2e-16 ***
## balance      5.74e-03   2.32e-04   24.75  < 2e-16 ***
## studentYes  -7.15e-01   1.48e-01   -4.85  1.3e-06 ***
```


Default Rate
0.0
0.2
0.4
0.6
0.8
500
1000
1500
2000
Credit Card Balance
Credit Card Balance
0
500
1000
1500
2000
2500
No
Yes
Student Status

```r
summary(glm(default~balance+income+student, data=df, family="binomial"))
```

```
## Call:
## glm(formula = default ~ balance + income + student, family = "binomial",
##     data = df)
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -1.09e+01   4.92e-01  -22.08   <2e-16 ***
## balance      5.74e-03   2.32e-04   24.74   <2e-16 ***
## income       3.03e-06   8.20e-06    0.37   0.7115
## studentYes  -6.47e-01   2.36e-01   -2.74   0.0062 **
```