Introduction to Bayesian Econometrics, Second Edition
Answers to Exercises, Chapter 6

1. (a)

i	is accessible to
1	states 1, 2, 3, 4, 5, 6
2	states 1, 2, 3, 4, 5, 6
3	states 1, 2, 3, 4, 5, 6
4	states 4, 5, 6
5	states 5, 6
6	states 5, 6

(b)

$1 \leftrightarrow 1; 1 \leftrightarrow 2; 1 \leftrightarrow 3$

$2 \leftrightarrow 1; 2 \leftrightarrow 2; 2 \leftrightarrow 3$

$3 \leftrightarrow 1; 3 \leftrightarrow 2; 3 \leftrightarrow 3$

$4 \leftrightarrow 4$

$5 \leftrightarrow 5; 5 \leftrightarrow 6$

$6 \leftrightarrow 5; 6 \leftrightarrow 6$

(c) There are three equivalence classes: $\{1, 2, 3\}, \{4\}, \{5, 6\}$.

(d) Not irreducible, because there is more than one equivalence class.

(e)

$$P^{100} = \begin{pmatrix} 2.353064e-17 & 2.284298e-17 & 3.326313e-17 & 1.480373e-17 & 0.4320988 & 0.5679012 \\ 1.247368e-17 & 1.210915e-17 & 1.763291e-17 & 7.847510e-18 & 0.6358025 & 0.3641975 \\ 8.566118e-18 & 8.315784e-18 & 1.210915e-17 & 5.389165e-18 & 0.3641975 & 0.6358025 \\ 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 4.909093e-91 & 0.1111111 & 0.8888889 \\ 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 1.0000000 & 0.0000000 \\ 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 0.0000000 & 1.0000000 \end{pmatrix}$$

$$P^{101} = \begin{pmatrix} 1.615933e-17 & 1.568709e-17 & 2.284298e-17 & 1.016625e-17 & 0.5679012 & 0.4320988 \\ 8.566118e-18 & 8.315784e-18 & 1.210915e-17 & 5.389165e-18 & 0.3641975 & 0.6358025 \\ 5.882660e-18 & 5.710746e-18 & 8.315784e-18 & 3.700932e-18 & 0.6358025 & 0.3641975 \\ 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 6.136367e-92 & 0.8888889 & 0.1111111 \\ 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 0.0000000 & 1.0000000 \\ 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 0.000000e+00 & 1.0000000 & 0.0000000 \end{pmatrix}$$

We already know that the transition matrix does not converge to an invariant distribution. These matrices show the transition probabilities, respectively, after 100 and 101, steps. You can see that they do not appear to be converging to a matrix with identical rows. Note how the last two rows interchange.

2. (a) Since

$$(\pi_1, \pi_2) \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} = (\pi_1, \pi_2)$$

and $\pi_1 + \pi_2 = 1$, we find $\pi_1 = \beta/(\alpha+\beta)$ and $\pi_2 = \alpha/(\alpha+\beta)$.

The process is aperiodic because each state can be reached with positive probability from any state. And the process is irreducible because each state communicates with the other state Accordingly, it has only one equivalence class, and

$$\lim P^n = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}.$$

(b)

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This chain is aperiodic but not irreducible because each state does not communicate with the other state. There is no invariant distribution. Also,

$\lim P^n = P$.

(c) The process

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

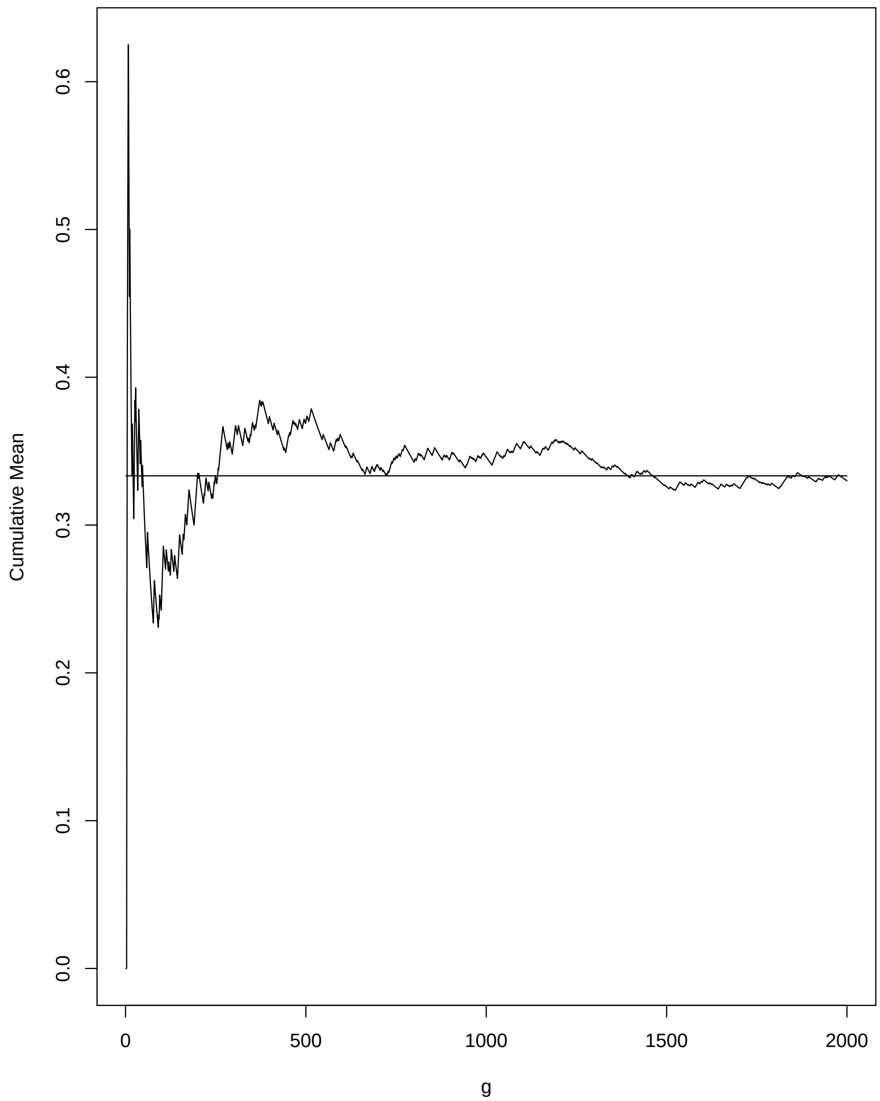
is irreducible and periodic with period 2. The invariant distribution is $\pi = (1/2, 1/2)$. $\lim P^n$ does not exist, because powers of P alternate between P and the identity matrix.

3. (a)

$$P^2 = \begin{pmatrix} 0.5938 & 0.4062 \\ 0.2031 & 0.7969 \end{pmatrix}.$$

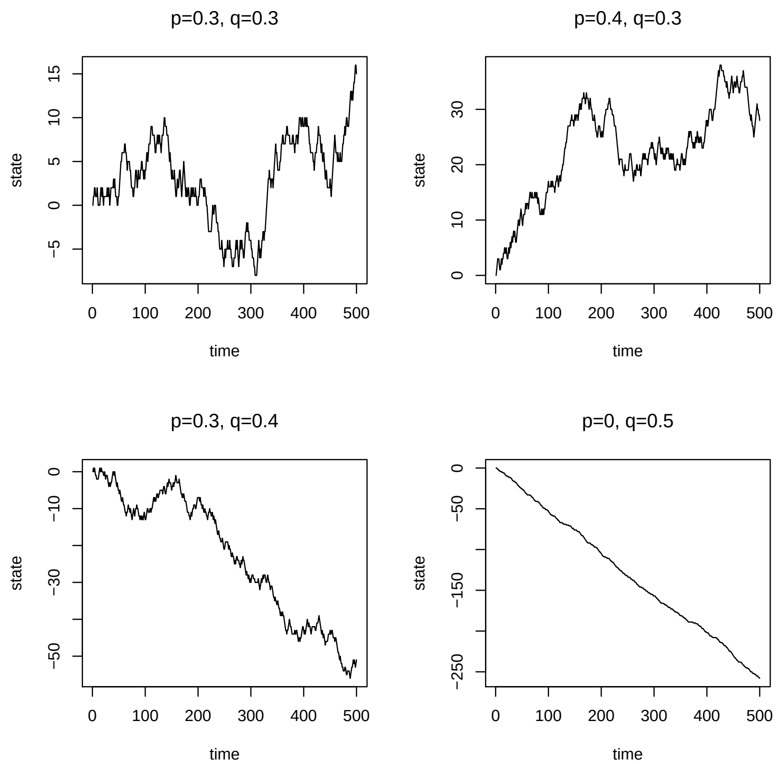
For example, the probability of being in State 1 in two steps given that the current state is State 1 is 0.5938.

(b) R code for the simulation may be found in `AnswerEx6_3b.r`. As you can see from the plot below, as the number of iteration increases, the fraction of time the process is in state 1 converges to 1/3, the true π_1.



Cumulative Mean
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0
500
1000
1500
2000
g

4. R code for a simulation is in `AnswerEx6_4.r`. You should find results like those in the following graphs.



5. (a) By definition, the characteristic vector and the characteristic root satisfy $v_i'P = \lambda_i v_i'$. Thus,

$$\begin{pmatrix} v_1' \\ v_2' \\ \vdots \\ v_S' \end{pmatrix} P = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_S \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \\ \vdots \\ v_S' \end{pmatrix}.$$

In matrix notation, $VP = \Lambda V$.

(b) From (a), $P = V^{-1}\Lambda V$. Thus,

$$\begin{aligned} P^t &= (V^{-1}\Lambda V)^t \\ &= (V^{-1}\Lambda V)(V^{-1}\Lambda V)\cdots(V^{-1}\Lambda V) \\ &= V^{-1}\Lambda^t V. \end{aligned}$$

Since $\Lambda^t = \text{diag}(\lambda_1^t, \lambda_2^t, \cdots, \lambda_S^t)$, we have $p_j^{(t)'} = \sum_k v^{jk}\lambda_k^t v_k'$.

(c) Since we know that P has a unique invariant distribution, we know that every row of P^t, including $p_j^{(t)'}$, converges to π'. Note for later that V can be multiplied by an arbitrary constant, say a, because if $VP = \Lambda V$, then $(aV)P = \Lambda(aV)$. From above, $p_j^{(t)'} = \sum_k v^{jk}\lambda_k^t v_k'$. We renumber the c.r.s to make $\lambda_1 = 1$ (and, therefore, $v_1' = \pi'$) and rewrite the expression to obtain

$$\begin{aligned} p_j^{(t)'} &= v^{j1}\pi' + \sum_{k=2} v^{jk}\lambda_k^t v_k' \\ &= \pi' + \sum_{k=2} v^{jk}\lambda_k^t v_k', \end{aligned}$$

where we have used our arbitrary constant to set $v^{j1} = 1$. Since $\lim_t p_j^{(t)'} = \pi'$, the term in the sum must converge to 0, which implies $|\lambda_k| < 1$, $k = 2, \ldots, S$. Finally, the speed of convergence depends on the largest in absolute value of the λ_k, $k = 2, \ldots, S$. This is the second largest absolute value because $\lambda_1 = 1$ is the largest.

(d) The characteristic roots may be found from the characteristic equation

$$\begin{aligned} \begin{vmatrix} 1-\alpha-\lambda & \alpha \\ \beta & 1-\beta-\lambda \end{vmatrix} &= (1-\alpha-\lambda)(1-\beta-\lambda) - \alpha\beta \\ &= \lambda^2 - (2-\alpha-\beta)\lambda + 1 - \alpha - \beta. \end{aligned}$$

You can now verify that $1-\alpha-\beta$ is the second root by using the quadratic rule or synthetic division, or by verifying that the characteristic equation can be factored as $(\lambda-1)[\lambda-(1-\alpha-\beta)]$. To maximize the speed of convergence, we wish to minimize its absolute value, which may be accomplished by setting $\alpha+\beta=1$, so that the second root is 0. Of course, when this condition is satisfied,

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ 1-\alpha & \alpha \end{pmatrix},$$

so that P is already equal to the invariant distribution.

6. (a) $h_0 = P(\text{hit } 0|W=0) = 1$ because wealth is already zero.

$$\begin{aligned} h_i &= P(W=i+1)P(\text{hit } 0|W=i+1) + P(W=i-1)\,P(\text{hit } 0|W=i-1) \\ &= ph_{i+1} + qh_{i-1}. \end{aligned}$$

(b) From (a), we have

$$ph_{i+1} - h_i + qh_{i-1} = 0.$$

This difference equation has the characteristic equation $p\lambda^2 - \lambda + q = 0$, with roots $\lambda_1 = 1$ and $\lambda_2 = q/p$.

Since $p \neq q$, $h_i = A\lambda_1^i + B\lambda_2^i = A + B(q/p)^i$. For later, note that $h_0 = A + B = 1$.

(c) The probability h_i is bounded by $0 \leq h_i \leq 1$, and, if $q/p > 1$, h_i will be outside those limits for large enough i unless $B = 0$. With $i = 0$, $h_0 = A = 1$.

(d) From (b), we know that $p\lambda^2 - \lambda + q = 0$ has the roots $\lambda_1 = 1, \lambda_2 = q/p$. Now with $p = q = 1/2$, $\lambda_1 = \lambda_2 = 1$. Verify by substituting into $h_i =$

$ph_{i+1} + qh_{i-1}$ that the solution in this case is $h_i = A + Bi$.

From $h_0 = 1$ and $h_i = A + Bi$, we know that $A = 1$. Since $A + B = 1$, we have $B = 0$. Thus, $h_i = 1$.

7. No, they do not. The greatest common divisor of the integers in the set $A = \{n \geq 1 : p_{ii}^{(n)} > 0\}$ is 1. Thus, it is aperiodic.

8. We need to show that $\pi(y) = \int \pi(x)\, p(x, y)\, dx$. Then,

$$\begin{aligned}
\int \pi(x)\, p(x, y)\, dx &= \int_0^y 2(1-x)\,(e^{1-x} - e^{y-x})\, dx + \int_y^1 2(1-x)\, e^{1-x}\, dx \\
&= -2y(1 - e^{1-y}) + 2(1 - ye^{1-y}) \\
&= 2(1-y) \\
&= \pi(y).
\end{aligned}$$