Introduction to Bayesian Econometrics, Second Edition
Answers to Exercises, Chapter 3

1. First, the likelihood function is

$$
\begin{aligned}
f(y_1, \ldots, y_n|\theta_1) &= \theta_1^{y_1}(1-\theta_1)^{1-y_1} \cdots \theta_1^{y_n}(1-\theta_1)^{1-y_n} \\
&= \theta_1^{\sum y_i}(1-\theta_1)^{n-\sum y_i}.
\end{aligned}
$$

Second, since $\theta_1 \sim \text{Beta}(2, 10)$, the prior distribution is

$$
f(\theta_1|2, 10) = \frac{\Gamma(12)}{\Gamma(2)\Gamma(10)}\,\theta_1^{2-1}\,(1-\theta_1)^{10-1}.
$$

Therefore, the posterior distribution is

$$
\pi(\theta_1|y_1, \ldots, y_n) \propto \theta_1^{\sum y_i + 1}\,(1-\theta_1)^{n-\sum y_i + 9}.
$$

For $n = 100$ and $\sum y_i = 19$,

$$
\begin{aligned}
\pi(\theta_1|y_1, \ldots, y_n) &\propto \theta_1^{19+1}\,(1-\theta_1)^{100-19+9} \\
&= \theta_1^{21-1}(1-\theta_1)^{91-1} \\
&\propto \text{Beta}(21, 91).
\end{aligned}
$$

For $n = 1000$ and $\sum y_i = 190$,

$$\begin{aligned} \pi(\theta_1|y_1,\ldots,y_n) &\propto \theta_1^{190+1}\,(1-\theta_1)^{1000-190+9} \\ &= \theta_1^{192-1}\,(1-\theta_1)^{820-1} \\ &\propto \text{Beta}(192, 820). \end{aligned}$$

As the sample size increases, the precision increases. That is, the variance of the posterior distribution becomes smaller. (See figure 1.)

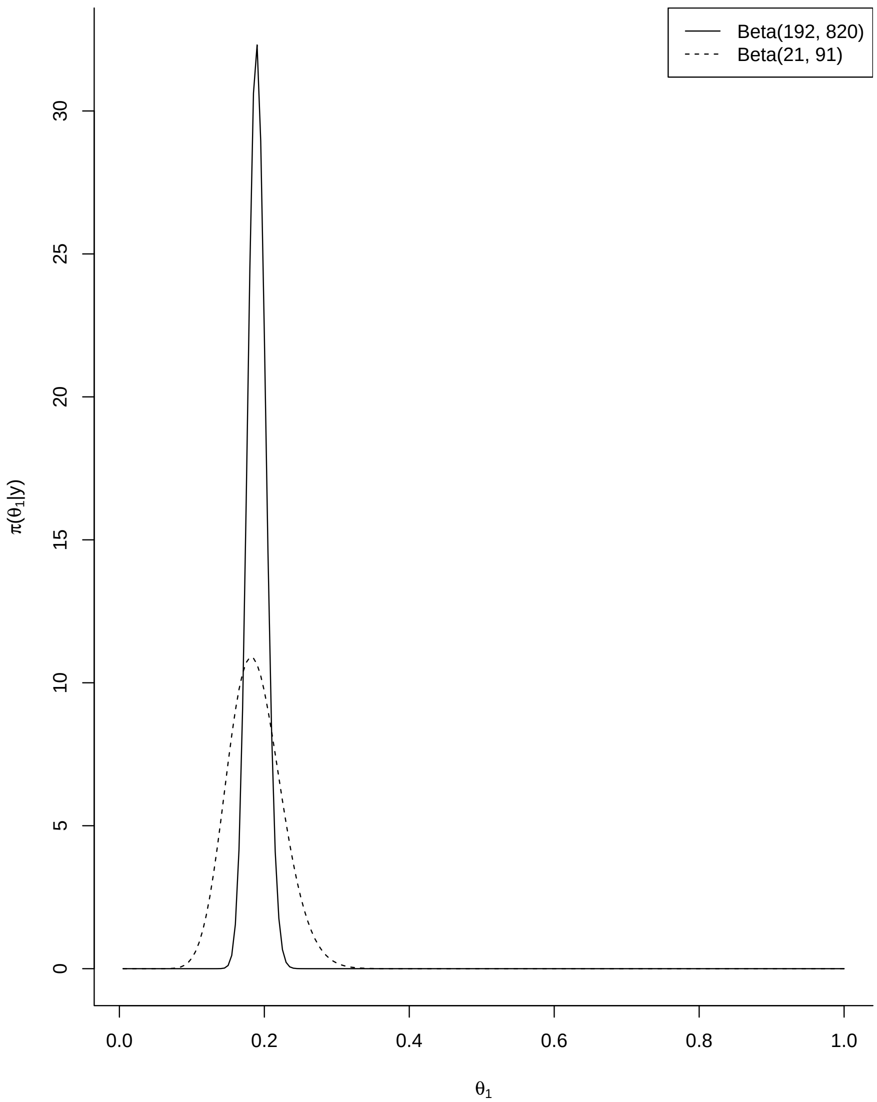

Figure 1: Two beta distributions

2.

$$
\begin{aligned}
p(y_{n+1}|\theta) &= \frac{e^{-\theta}\theta^{y_{n+1}}}{y_{n+1}!}, \\
\pi(\theta|y_1,\ldots,y_n) &= \frac{(\beta+n)^{\alpha+\sum y_i}}{\Gamma(\alpha+\sum y_i)}\,\theta^{\alpha+\sum y_i-1}\,e^{-(\beta+n)\theta}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
p(y_{n+1}|y_1,\ldots,y_n) &= \int p(y_{n+1}|\theta)\,\pi(\theta|y)\,d\theta \\
&= \frac{1}{y_{n+1}!}\,\frac{(\beta+n)^{\alpha+\sum y_i}}{\Gamma(\alpha+\sum y_i)}\int \theta^{\alpha+\sum y_i+y_{n+1}-1}\,e^{-(\beta+n+1)\theta}\,d\theta \\
&= \frac{(\beta+n)^{\alpha+\sum y_i}}{y_{n+1}!\,\Gamma(\alpha+\sum y_i)}\,\frac{\Gamma(\alpha+\sum y_i+y_{n+1})}{(\beta+n+1)^{\alpha+\sum y_i+y_{n+1}}} \\
&= \frac{\Gamma(\alpha+\sum y_i+y_{n+1})}{y_{n+1}!\,\Gamma(\alpha+\sum y_i)}p^{\alpha+\sum y_i}(1-p)^{y_{n+1}},
\end{aligned}
$$

where $(\beta+n)/(\beta+n+1)=p$. This is a $\mathrm{Beta}(\alpha+\sum y_i+1, y_{n+1}+1)$ distribution.

3. Let $y=(y_1,\ldots,y_n)$.

$$
\begin{aligned}
f(y_{n+1}|\mu,1) &= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(y_{n+1}-\mu)^2}{2}\right], \\
\pi(\mu|y) &\propto \exp\left[-\frac{\sum(y_i-\mu)^2}{2}\right]\,\exp\left[-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right].
\end{aligned}
$$

Then, $\mu|y\sim\mathcal{N}(\mu_1,\sigma_1^2)$, where $\mu_1=(n\bar{y}+\sigma_0^{-2}\mu_0)/(n+\sigma_0^{-2})$ and $\sigma_1^2=$

$1/(n+\sigma_0^{-2})$. Therefore,

$$
\begin{aligned}
f(y_{n+1}|y) &= \int f(y_{n+1}|\mu)\,\pi(\mu|y)\,d\mu \\
&\propto \int \exp\left[-\frac{1}{2}(y_{n+1}-\mu)^2\right]\,\exp\left[-\frac{1}{2\sigma_1^2}(\mu-\mu_1)^2\right]\,d\mu.
\end{aligned}
$$

We then complete the square in μ to integrate it out. Consider the term in the exponential multiplied by $-1/2$:

$$
(y_{n+1}-\mu)^2+\sigma_1^{-2}(\mu-\mu_1)^2.
$$

We rewrite this as

$$
(1+\sigma_1^{-2})\left(\mu-\frac{y_{n+1}+\sigma_1^{-2}\mu_1}{1+\sigma_1^{-2}}\right)^2+\sigma_1^{-2}\mu_1^2+y_{n+1}^2-\frac{(y_{n+1}+\sigma_1^{-2}\mu_1)^2}{1+\sigma_1^{-2}}.
$$

Upon integrating out μ, y_{n+1} disappears from the first term because it appears only in the mean. Accordingly, y_{n+1} appears only in the last two terms, so we complete the square in y_{n+1} to conclude that

$$
y_{n+1}|y \sim N\big(\mu_1,\sigma_1^2+1\big).
$$

Intuitively, the result makes sense. It tells us that y_{n+1} is drawn from a Gaussian distribution with mean μ_1, which is our best estimate of the mean based on the data and prior information, and variance equal to the variance of the mean (σ_1^2) plus the variance of an individual observation, which is 1 in this problem.

4. Let $Y_1 = (y_{11}, \ldots, y_{1s_1})$ and $Y_2 = (y_{21}, \ldots, y_{2s_2})$.

Show that $f(\theta|Y_1, Y_2) \propto f(Y_2|Y_1, \theta)\pi(\theta|Y_1)$.

First,

$$\begin{aligned}
\pi(\theta|Y_1, Y_2) &= \frac{e^{-\theta}\theta^{y_{11}}}{y_{11}!} \cdots \frac{e^{-\theta}\theta^{y_{1s_1}}}{y_{1s_1}!} \, \frac{e^{-\theta}\theta^{y_{21}}}{y_{21}!} \cdots \frac{e^{-\theta}\theta^{y_{2s_2}}}{y_{2s_2}!} \\
&= \frac{e^{-\theta s_1}\theta^{\prod_1^{s_1} y_{1i}}}{\prod_1^{s_1} y_{1i}!} \, \frac{e^{-\theta s_2}\theta^{\sum_1^{s_2} y_{2i}}}{\prod_1^{s_2} y_{2i}!}.
\end{aligned}$$

Second,

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \, \theta^{\alpha-1} \, e^{-\beta\theta}.$$

Therefore,

$$\begin{aligned}
\pi(\theta|Y_1, Y_2) &\propto \theta^{\alpha+\sum_1^{s_1} y_{1i}+\sum_1^{s_2} y_{2i}-1} \, e^{-\theta(\beta+s_1+s_2)} \\
&= (\theta^{\sum_1^{s_2} y_{2i}} \, e^{-\theta s_2})(\theta^{\alpha+\sum_1^{s_1} y_{1i}-1} \, e^{-\theta(\beta+s_1)}) \\
&= f(Y_2|\theta) \, \pi(\theta|Y_1) \\
&= f(Y_2|Y_1, \theta) \, \pi(\theta|Y_1) \text{ (due to independence).}
\end{aligned}$$

5.

$$\begin{aligned}
E[L(\hat{\theta}, \theta)] &= \int |\hat{\theta} - \theta| \, \pi(\theta|y) \, d\theta \\
&= \int_{-\infty}^{\hat{\theta}} (\hat{\theta} - \theta) \, \pi(\theta|y) \, d\theta + \int_{\hat{\theta}}^{\infty} (\theta - \hat{\theta}) \, \pi(\theta|y) \, d\theta.
\end{aligned}$$

$$\begin{aligned}\frac{\partial E[L(\hat{\theta},\theta)]}{\partial\hat{\theta}} &= \int_{-\infty}^{\hat{\theta}} \pi(\theta|y)\,d\theta - \int_{\hat{\theta}}^{\infty} \pi(\theta|y)\,d\theta = 0 \\ &\Rightarrow \int_{-\infty}^{\hat{\theta}} \pi(\theta|y)\,d\theta = \int_{\hat{\theta}}^{\infty} \pi(\theta|y)\,d\theta.\end{aligned}$$

Therefore, $\hat{\theta}$ is the median.

6.

$$\begin{aligned}E[L(\hat{\theta},\theta)] &= \int 1(|\hat{\theta}-\theta| > b)\,\pi(\theta|y)\,d\theta \\ &= \int_{-\infty}^{\hat{\theta}-b} \pi(\theta|y)d\theta + \int_{\hat{\theta}+b}^{\infty} \pi(\theta|y)\,d\theta.\end{aligned}$$

$$\begin{aligned}\frac{\partial E[L(\hat{\theta},\theta)]}{\partial\hat{\theta}} &= \pi(\hat{\theta}-b|y) - \pi(\hat{\theta}+b|y) = 0 \\ &\Rightarrow \pi(\hat{\theta}-b|y) = \pi(\hat{\theta}+b|y).\end{aligned}$$

Thus, with the assumptions that $\pi(\theta)$ is unimodal and that the density is strictly increasing below and decreasing above the mode, $\hat{\theta}$ approaches the mode as b goes to zero.

7. Since $\pi(\theta_1,\theta_2) = \pi(\theta_1)\pi(\theta_2) = 1$,

$$\begin{aligned}m_2(y) &= \int\int \theta_1^{m\bar{y}_1}(1-\theta_1)^{m-m\bar{y}_1}\,\theta_2^{m\bar{y}_2}\,(1-\theta_2)^{m-m\bar{y}_2}\,d\theta_1\,d\theta_2 \\ &= \int \theta_1^{m\bar{y}_1}\,(1-\theta_1)^{m-m\bar{y}_1}\,d\theta_1 \int \theta_2^{m\bar{y}_2}\,(1-\theta_2)^{m-m\bar{y}_2}\,d\theta_2 \\ &= \frac{\Gamma(m\bar{y}_1+1)\,\Gamma(m-m\bar{y}_1+1)}{\Gamma(m+2)}\,\frac{\Gamma(m\bar{y}_2+1)\Gamma(m-m\bar{y}_2+1)}{\Gamma(m+2)}.\end{aligned}$$

8. Let y_i be the number of errors in page i, y_{1i} be the number of errors made by Sam in page i of the first m pages, and y_{2i} be the number of errors made by Levi in page i of the last m pages. Also, let $n = 2m$. Then,

$$\sum_{1}^{m} y_{1i} = e_1,$$
$$\sum_{1}^{m} y_{2i} = e_2.$$

Note that $\pi(\theta) = G(1,1) = \frac{1}{\Gamma(1)}\theta^{1-1}e^{-\theta} = e^{-\theta}$.

Then,

$$\begin{aligned} m_1(y) &= \int \frac{e^{-\theta}\,\theta^{y_1}}{y_1!} \dots \frac{e^{-\theta}\,\theta^{y_n}}{y_n!}\, e^{-\theta}\, d\theta \\ &= \frac{1}{\prod y_i!} \int e^{-\theta(n+1)}\,\theta^{\sum_1^n y_i}\, d\theta \end{aligned}$$

Note that $\pi(\theta_1, \theta_2) = \pi(\theta_1)\pi(\theta_2) = e^{-\theta_1 - \theta_2}$.

Then,

$$\begin{aligned} m_2(y) &= \int \frac{e^{-\theta_1}\,\theta_1^{y_{11}}}{y_{11}!} \dots \frac{e^{-\theta_1}\,\theta_1^{y_{1m}}}{y_{1m}!}\,\frac{e^{-\theta_2}\,\theta_2^{y_{21}}}{y_{21}!} \dots \frac{e^{-\theta_2}\,\theta_2^{y_{2m}}}{y_{2m}!}\, e^{-\theta_1-\theta_2}\, d\theta_1\, d\theta_2 \\ &= \frac{1}{\prod y_i!} \int e^{-\theta_1(m+1)}\,\theta_1^{\sum_1^m y_{1i}}\, d\theta_1 \int e^{-\theta_2(m+1)}\,\theta_2^{\sum_1^m y_{2i}}\, d\theta_2 \end{aligned}$$

After substituting $n = 2m$, $\sum_1^m y_{1i} = e_1$, $\sum_1^m y_{2i} = e_2$, and $\sum_1^m y_i = e_1 + e_2$, we have

$$\begin{aligned}\text{Bayes factor} &= \frac{\int e^{-\theta(n+1)}\,\theta^{e_1+e_2}\,d\theta}{\int e^{-\theta_1(m+1)}\,\theta_1^{e_1}\,d\theta_1 \int e^{-\theta_2(m+1)}\,\theta_2^{e_2}\,d\theta_2} \\ &= \frac{\Gamma(e_1+e_2+1)/(n+1)^{e_1+e_2+1}}{[\Gamma(e_1+1)/(m+1)^{e_1+1}]\,[\Gamma(e_2+1)/(m+1)^{e_2+1}]}.\end{aligned}$$

Using a gamma function in a math or stat software (for example, “gamma” in R), we can compute a Bayes factor for each combination of $e_1, e_2,$ and m. (See `AnswerEx3_8.r`.)

Sam	Levi	$\log_{10}$(Bayes factor)	
Error Proportion	Error Proportion	m=100	m=200
0.9	0.1	-0.283	-1.717
0.8	0.2	0.370	-0.310
0.7	0.3	0.796	0.593
0.6	0.4	1.039	1.105
0.5	0.5	1.118	1.271

Notice that the Bayes factor in favor of M_1, in which $\theta_1 = \theta_2$, increases as the observed error proportions become more equal.

9. Given the priors and the likelihood function, we have

$$f(p, N) \propto p^{\alpha+1}\,(1-p)^{\beta+1}\,\frac{1}{N^2}\,\frac{N!}{(N-n)!}\,p^n\,(1-p)^{N-n}.$$

Collect the terms in p and $1-p$, integrate out p as a beta function, and simplify to verify the result in the text. The program `AnswerEx3_9.r` plots $p(N)$. You may substitute other values for n, α, and β. Note that the normalizing constant is unknown. To get a useful figure, you will need to choose N large enough that $p(N)$ is close to zero (you can examine PN1 on line 9 for values of N close to upLimN) and then normalize by sumPN, as is done in lines 10 and 11.

10. You can obtain the product of binomials by writing

$$P(n_1, n_2, m_2) = P(n_1)\, P(m_2|n_1)\, P(n_2|n_1, m_2).$$

You should then verify that

$$\begin{aligned} f(N,p) \propto \frac{1}{N^2}\, p^{\alpha-1}\,(1-p)^{\beta-1} \binom{N}{n_1} p^{n_1}\,(1-p)^{N-n_1}\, p^{m_2}\,(1-p)^{n_1-m_2} \\ \times \binom{N-n_1}{n_2-m_2} p^{n_2-m_2}\,(1-p)^{N-n_1-n_2+m_2}. \end{aligned}$$

Integrate out p as a beta function and simplify to obtain the result in the text. `AnswerEx3_10.r` creates the plot. The posterior is normalized as discussed in the previous answer.