Introduction to Bayesian Econometrics, Second Edition
Answers to Exercises, Chapter 4

1. We need to show that

$$y'y+\beta_0'B_0^{-1}\beta_0-\bar{\beta}'B_1^{-1}\bar{\beta} = (y-X\hat{\beta})'(y-X\hat{\beta})+(\beta_0-\hat{\beta})'[(X'X)^{-1}+B_0]^{-1}(\beta_0-\hat{\beta}).$$

We do this by confirming that the expressions between $\hat{\beta}'$ and $\hat{\beta}$, β_0' and β_0, and $\hat{\beta}'$ and β_0 are the same on both sides of the equation. Start with $\hat{\beta}'$ and $\hat{\beta}$. Since $(y - X\hat{\beta})'(y - X\hat{\beta}) = y'y - \hat{\beta}'X'X\hat{\beta}$, we have on the r.h.s.

$$-\hat{\beta}'\big[X'X - [(X'X)^{-1} + B_0]^{-1}\big]\hat{\beta}.$$

On the l.h.s., note that

$$-\bar{\beta}'B_1^{-1}\bar{\beta} = -[B_1(X'X\hat{\beta} + B_0^{-1}\beta_0)]'B_1^{-1}[B_1(X'X\hat{\beta} + B_0^{-1}\beta_0)],$$

so that we have for the quadratic term in $\hat{\beta}$,

$$-\hat{\beta}'[X'XB_1X'X]\hat{\beta}.$$

We then have

$$\begin{aligned}
X'XB_1X'X &= X'X[(X'X) + B_0^{-1}]^{-1}X'X \\
&= X'X\big[(X'X)^{-1} - (X'X)^{-1}[(X'X)^{-1} + B_0]^{-1}(X'X)^{-1}\big]X'X \\
&= X'X - [(X'X)^{-1} + B_0]^{-1},
\end{aligned}$$

where we have used A.17 with $A = X'X$, $B = I$, and $C = B_0^{-1}$.

For the β_0', β_0 term, the r.h.s. is

$$[B_0 + (X'X)^{-1}]^{-1},$$

and the l.h.s. is

$$B_0^{-1} - B_0^{-1}[X'X + B_0^{-1}]^{-1}B_0^{-1},$$

which can be shown to be equal to the r.h.s. by using (A.17) with $A = B_0$, $B = I$, and $C = (X'X)^{-1}$.

Finally, for the $\hat{\beta}', \beta_0$ term, the r.h.s. is $[B_0 + (X'X)^{-1}]^{-1}$ and the l.h.s. is

$$X'X[B_0^{-1} + X'X]^{-1}B_0^{-1} = [B_0(B_0^{-1} + X'X)(X'X)^{-1}]^{-1},$$

which is equal to the r.h.s.

2. From 3.3, we know that $\mu|y \sim \mathcal{N}(\mu_1, \sigma_1^2)$, where $\mu_1 = (n\bar{y} + \sigma_0^{-2}\mu_0)/(n + \sigma_0^{-2})$ and $\sigma_1^2 = 1/(n + \sigma_0^{-2})$.

   Therefore, $\lim_{\sigma_0^2 \to \infty} \mu_1 = \bar{y}$ and $\lim_{\sigma_0^2 \to \infty} \sigma_1^2 = 1/n$. Thus, the posterior distribution is $N(\bar{y}, 1/n)$. This result might be compared to the frequentist treatment of the problem, where the sample mean is the point estimate.

3. Since $\bar{\beta} = (X'X + B_0^{-1})^{-1}(X'y + B_0^{-1}\beta_0)$, $\lim_{\mathrm{diag}(B_0) \to \infty} \bar{\beta} = (X'X)^{-1}X'y = \hat{\beta}$.

Also, note that

$$
\begin{aligned}
\bar{\beta} &= (X'X + B_0^{-1})^{-1}(X'y + B_0^{-1}\beta_0) && (1)\\
&= (X'X + B_0^{-1})^{-1}(X'X)\hat{\beta} + (X'X + B_0^{-1})B_0^{-1}\beta_0. && (2)
\end{aligned}
$$

Thus, the weigh on the prior goes to zero as the variance goes to infinity. In the limiting case,

$$
\begin{aligned}
\lim_{\text{diag}(B_0)\to\infty} \bar{\beta} &= (X'X)^{-1}X'y = \hat{\beta} && (3)\\
\lim_{\text{diag}(B_0)\to\infty} B_1 &= (X'X)^{-1} && (4)\\
\lim_{\text{diag}(B_0)\to\infty} \alpha_1 &= \alpha_0 + n && (5)\\
\lim_{\text{diag}(B_0)\to\infty} \delta_1 &= \delta_0 + y'y - \hat{\beta}'X'X\hat{\beta} && (6)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\pi(\beta|y) &= t_k\left(\alpha_1, \hat{\beta}, \frac{\delta_1}{\alpha_1}(X'X)^{-1}\right) && (7)\\
\pi(\sigma^2|y) &= IG\left(\frac{\alpha_1}{2}, \frac{\delta_1}{2}\right) && (8)
\end{aligned}
$$

4.

5. First,

$$\int \frac{1}{(2\pi\sigma^2)^{K/2}} \exp\left[-\frac{1}{2\sigma^2}(\beta-\bar{\beta})' B_1^{-1}(\beta-\bar{\beta})\right] d\beta$$
$$= |B_1|^{1/2} \int \frac{1}{(2\pi\sigma^2)^{K/2}|B_1|^{1/2}} \exp\left[-\frac{1}{2\sigma^2}(\beta-\bar{\beta})' B_1^{-1}(\beta-\bar{\beta})\right] d\beta$$
$$= |B_1|^{1/2}.$$

Second,

$$\int \left(\frac{1}{\sigma^2}\right)^{\alpha_1/2+1} \exp\left[-\frac{\delta_1}{2\sigma^2}\right] d\sigma^2$$
$$= \frac{\Gamma(\alpha_1/2)}{(\delta_1/2)^{\alpha_1/2}} \int \left(\frac{1}{\sigma^2}\right)^{\alpha_1/2+1} \frac{(\delta_1/2)^{\alpha_1/2}}{\Gamma(\alpha_1/2)} \exp\left[-\frac{\delta_1}{2\sigma^2}\right] d\sigma^2$$
$$= \frac{\Gamma(\alpha_1/2)}{(\delta_1/2)^{\alpha_1/2}}.$$

Therefore,

$$\begin{aligned}\int\int f(y|\beta,\sigma^2)\,\pi(\beta,\sigma^2) d\beta\, d\sigma^2 &= \left(\frac{1}{2\pi}\right)^{n/2} \frac{(\delta_0/2)^{\alpha_0/2}\,|B_1|^{1/2}}{\Gamma(\alpha_0/2)\,|B_0|^{1/2}} \frac{\Gamma(\alpha_1/2)}{(\delta_1/2)^{\alpha_1/2}} \\ &= \left(\frac{1}{\pi}\right)^{n/2} \frac{|B_1|^{1/2}\,\Gamma(\alpha_1/2)\delta_0^{\alpha_0/2}}{|B_0|^{1/2}\,\Gamma(\alpha_0/2)\delta_1^{\alpha_1/2}}.\end{aligned}$$

6. Let $y_1, \ldots, y_n$ be i.i.d. variables. Then, $f(y_1, \ldots, y_n) = f(y_1)\cdots f(y_n)$, which is invariant to permutations in the indices $1, 2, \ldots, n$.

7.

$$
\begin{aligned}
P(R_1, B_2, B_3) &= \frac{R}{R+B} \cdot \frac{B}{R+B-1} \cdot \frac{B-1}{R+B-2}, \\
P(B_1, R_2, B_3) &= \frac{B}{R+B} \cdot \frac{R}{R+B-1} \cdot \frac{B-1}{R+B-2}, \\
P(B_1, B_2, R_3) &= \frac{B}{R+B} \cdot \frac{B-1}{R+B-1} \cdot \frac{R}{R+B-2}.
\end{aligned}
$$

Therefore, $P(R_1, B_2, B_3) = P(B_1, R_2, B_3) = P(B_1, B_2, R_3)$. But

$$
\begin{aligned}
P(R_1) &= \frac{R}{R+B}, \\
P(R_1|B_2) &= \frac{P(R_1, B_2)}{P(B_2)} = \frac{R}{R+B-1} \neq P(R_1).
\end{aligned}
$$