Introduction to Bayesian Econometrics, 2nd Edition
Answers to Exercises, Chapter 9

1. First, evaluate $\Phi_m(\tau_m)$. From the third line of the definition of $\Phi_m()$, we obtain

$$\begin{aligned}\Phi_m(\tau_m) &= (2/h_{m+1}^3)(\tau_m - \tau_{m+1})^2(\tau_m - \tau_m + 0.5h_{m+1})\\ &= (2/h_{m+1})^3(-h_{m+1})^2(0.5h_{m+1})\\ &= 1.\end{aligned}$$

From the third line of $\Psi_m()$, we obtain

$$\begin{aligned}\Psi_m(\tau_m) &= (1/h_{m+1})^2(\tau_m - \tau_{m+1})^2(\tau_m - \tau_m)\\ &= 0.\end{aligned}$$

Accordingly,

$$\begin{aligned}f(\tau_m) &= \sum_m (\Phi_m(\tau_m)b_m + \Psi_m(\tau_m)s_m)\\ &= b_m.\end{aligned}$$

For the derivative, start with

$$\Phi_m'(x) = (2/h_{m+1}^3)\big[(x - \tau_{m+1})^2 + 2(x - \tau_m + 0.5h_{m+1})(x - \tau_{m+1})\big].$$

Then, with $x = \tau_m$, we have

$$\Phi_m'(\tau_m) = (2/h_{m+1}^3)\big[(-h_{m+1})^2 + 2(0.5h_{m+1})(-h_{m+1})\big] = 0,$$

and

$$\Psi'_m(x) = (1/h^2_{m+1})\big[(x-\tau_{m+1})^2 + 2(x-\tau_m)(x-\tau_{m+1})\big],$$
$$\Psi'_m(\tau_m) = (1/h^2_{m+1})\big[(-h_{m+1})^2\big] = 1.$$

It follows that $f'(\tau_m) = s_m$.

To show that the derivatives are computed with the same coefficients as the functions, start from

$$f(x_i) = \sum_{m=1}^{M} \big(\Phi_m(x_i)b_m + \Psi_m(x_i)s_m\big),$$

where the parameters are the b_m and s_m, it follows that

$$f'(x_i) = \sum_{m=1}^{M} \big(\Phi'_m(x_i)b_m + \Psi'_m(x_i)s_m\big),$$

which depends on the same b_m and s_m.

2. Let $z_i = cx_i + d$. Then show that the knot points (τ_m) for x go into $c\tau_m + d$ for the z_i and the h_m for x go into ch_m for the z. We can now verify that $\Phi_m(x_i) = \Phi_m(z_i)$ and that $\Psi_m(x_i) = \Psi_m(z_i)$. As an example, take the second line of $\Phi_m(x_i)$:

$$\begin{aligned}
\Phi_m(z_i) &= -\frac{2}{(ch_m)^3}[z_i - (c\tau_{m-1}+d)]^2[z_i - (c\tau_m + d) - 0.5ch_m] \\
&= -\frac{2}{(ch_m)^3}[cx_i + d - (c\tau_{m-1}+d)]^2[cx_i + d - (c\tau_m + d) - 0.5ch_m] \\
&= -\frac{2}{(ch_m)^3}\big[[c(x_i - \tau_{m-1})]^2 c[x_i - \tau_m - 0.5h_m]\big] \\
&= \Phi_m(x_i).
\end{aligned}$$

3. From the likelihood function and the prior for β, which are the only expressions

in which β appears, we have

$$\pi(\beta|y, \sigma^2, \sigma_e^2, \sigma_d^2) \propto \exp\left[-\frac{1}{2\sigma^2}(y - W\beta)'(y - W\beta)\right] \exp\left[-\frac{1}{2}\,\beta' B_0^{-1}\beta\right].$$

The standard completing the square computation will verify the draw in step 2 of algorithm 9.1.

For σ^2, we have

$$\pi(\sigma^2|y, \beta) \propto \left(\frac{1}{\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}(y - W\beta)'(y - W\beta)\right] \left(\frac{1}{2\sigma^2}\right)^{\alpha_0/2+1} \exp\left[-\frac{\delta_0}{2\sigma^2}\right],$$

which justifies step 3.

For step 4, we first isolate the terms involving σ_{ek}^2. These are the prior and the conditional prior $\pi(\beta_k|\sigma_{ek}^2, \sigma_{dk}^2)$, so that

$$\pi(\sigma_{ek}^2|\beta_k) \propto \left(\frac{1}{\sigma_{ek}^2}\right) \exp\left[-\frac{1}{2}\,\beta_k' \Delta_k' T_k^{-1} \Delta_k \beta_k\right] \left(\frac{1}{\sigma_{ek}^2}\right)^{\alpha_{ek0}/2+1} \exp\left[-\frac{\delta_{ek0}}{2\sigma_{ek}^2}\right].$$

Since σ_{ek}^2 appears only in the 1,1 and $M_k - 1, M_k - 1$ positions of T_k, we can rewrite the previous expression as

$$\pi(\sigma_{ek}^2|\beta_k) \propto \exp\left[-\frac{1}{2\sigma_{ek}^2}\beta_k' \Delta_k' D_{0k} \Delta_k \beta_k\right] \left(\frac{1}{\sigma_{ek}^2}\right)^{(\alpha_{ek0}+2)/2+1} \exp\left[-\frac{\delta_{ek0}}{2\sigma_{ek}^2}\right],$$

where D_{0k} is defined in the text.

The sampling for σ_{dk}^2 is analogous to that of σ_{ek}^2.

4. Following the discussion in the text, we have that

$$\begin{aligned}
\pi(p_i|y_i) &\propto f(y_i|p_i)\,\pi(p_i) \\
&\propto p_i^{y_i}(1 - p_i)^{n-y_i}\, p_i^{a_1-1}(1 - p_i)^{b_1-1} \\
&= p_i^{a_1+y_i-1}\,(1 - p_i)^{b_1+n-y_i}.
\end{aligned}$$

This expression is recognized as a $\mathrm{Beta}(a_1 + y_i, b_1 + n - y_i)$ distribution, and the result in the text follows.

5. The full conditional distribution in this case is

$$\begin{aligned}\pi(\mu_i|y_i) &\propto f(y_i|\mu_i)\,\pi(\mu_i)\\ &\propto \exp\left[-\frac{1}{2}(y_i-\mu_i)^2\right]\,\exp\left[-\frac{1}{2\sigma_0^2}(\mu_i-\mu_0)^2\right].\end{aligned}$$

Standard computations show that $\mu_i|y_i$ has a normal distribution with mean

$$E(\mu_i|y_i) = \left(\frac{\sigma_0^2}{1+\sigma_0^2}\right)\left(y_i + \frac{\mu_0}{\sigma_0^2}\right)$$

and variance

$$\mathrm{Var}(\mu_i|y_i) = \frac{\sigma_0^2}{1+\sigma_0^2}.$$

Accordingly, the normalizing constant is

$$K_i = \sqrt{2\pi}\left(\frac{\sigma_0^2}{1+\sigma_0^2}\right)^{1/2}.$$

We therefore have that μ_i is drawn from

$$N\left(\left[\frac{\sigma_0^2}{1+\sigma_0^2}\right]\left[y_i+\frac{\mu_0}{\sigma_0^2}\right], \frac{\sigma_0^2}{1+\sigma_0^2}\right)$$

with probability

$$\frac{\alpha\sqrt{2\pi}\left(\frac{\sigma_0^2}{1+\sigma_0^2}\right)^{1/2}}{\alpha\sqrt{2\pi}\left(\frac{\sigma_0^2}{1+\sigma_0^2}\right)^{1/2} + \sum_{k\neq i} n_k N(y_i|\mu_k)},$$

and from μ_k with probability

$$\frac{N(y_i|\mu_k, 1)n_k}{\alpha\sqrt{2\pi}\left(\frac{\sigma_0^2}{1+\sigma_0^2}\right)^{1/2} + \sum_{k\neq i} n_k N(y_i|\mu_k)},$$

where n_k is the number of times μ_k is repeated in μ_{-i}.

6. The algorithm is similar to that for binary data with an inverse t link function discussed in section 8.2.2 and in exercise 8.8. The sampling for y^* and β are exactly the same as that specified in the exercise. The only difference is the sampling for $\lambda = (\lambda_1, \ldots, \lambda_n)$, which is now drawn from a DP(αG_0) rather than from a gamma distribution. To determine the sampling for λ, we first need the normalizing constant in $\pi(\lambda_i|y_i^*, \beta)$. By the usual computations, we have

$$\begin{aligned}\pi(\lambda_i|y_i^*, \beta) &\propto \lambda_i^{1/2} \exp\left[-\frac{\lambda_i}{2}(y_i^* - x_i'\beta)^2\right] \lambda_i^{\nu/2-1} \exp\left[-\frac{\nu\,\lambda_i}{2}\right] \\ &= \lambda_i^{(\nu+1)/2-1} \exp\left[-\frac{\lambda_i\left[\nu + (y_i^* - x_i'\beta)^2\right]}{2}\right].\end{aligned}$$

We recognize this expression as the kernel of a gamma distribution,

$$\mathrm{Ga}\left(\frac{\nu+1}{2}, \frac{\nu + (y_i^* - x_i'\beta)^2}{2}\right),$$

with normalizing constant

$$K_i = \Gamma\left(\frac{\nu+1}{2}, \frac{\nu + (y_i^* - x_i'\beta)^2}{2}\right)$$

and $A_i = \alpha K_i$. This explains the sampling for λ_i that appears in the text.