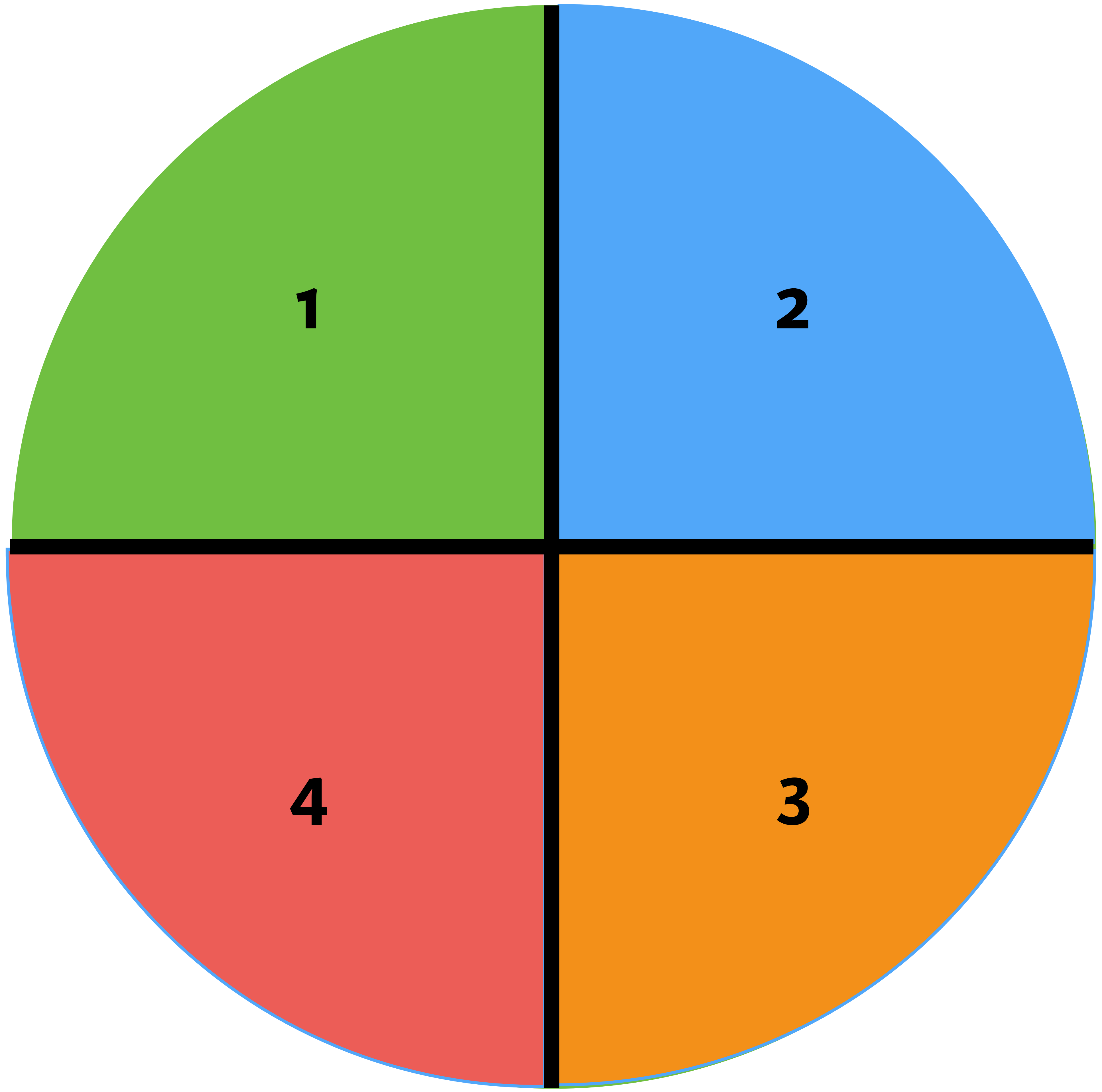
Discrete probability distributions

Course overview

- Two schools of thought: frequentist and Bayesian
- Introduction to the Bayesian way of thinking
- Subjective probability:
  - Probability describes beliefs about unknown quantities
  - Done through probability distributions

A spinner

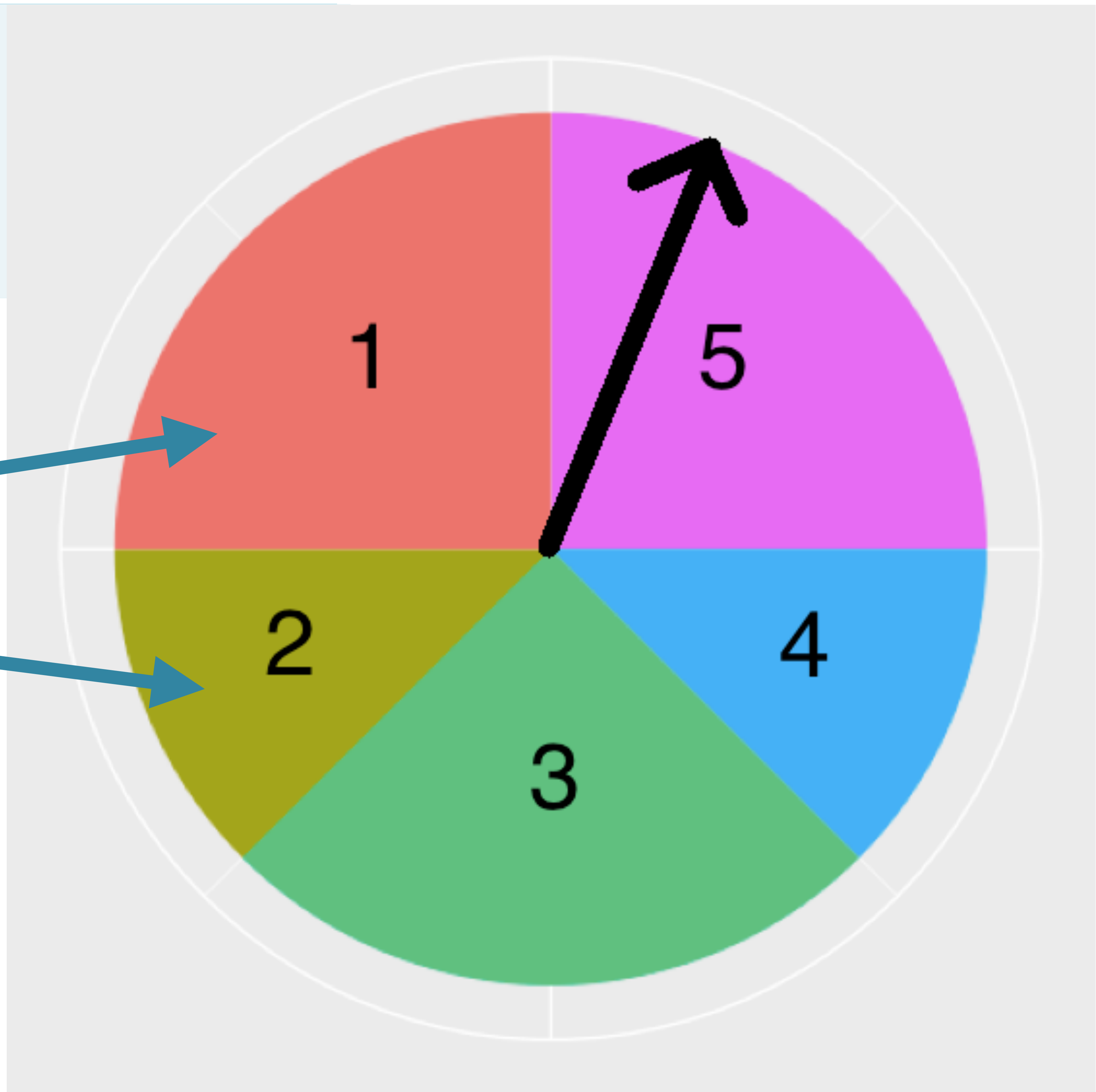


TeachBayes package

- Package (available on CRAN) for helping teach Bayesian thinking
- Special functions for:
  - Bayes' rule (spinners)
  - Learning about a proportion and a mean
  - Comparing two proportions

Construct my spinner

```
> library(TeachBayes)
> areas <- c(2, 1, 2, 1, 2)
> spinner_plot(areas)
```



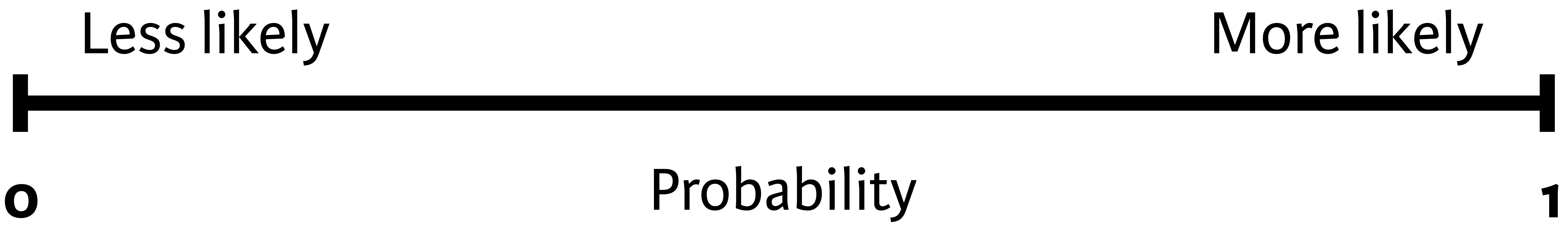
For example, region 1 is twice the size of region 2

Construct probability distribution

```
> (df <- data.frame(Region = 1:5, areas,
                    Probability = areas / sum(areas)))

  Region areas Probability
1      1     2       0.250
2      2     1       0.125
3      3     2       0.250
4      4     1       0.125
5      5     2       0.250
```

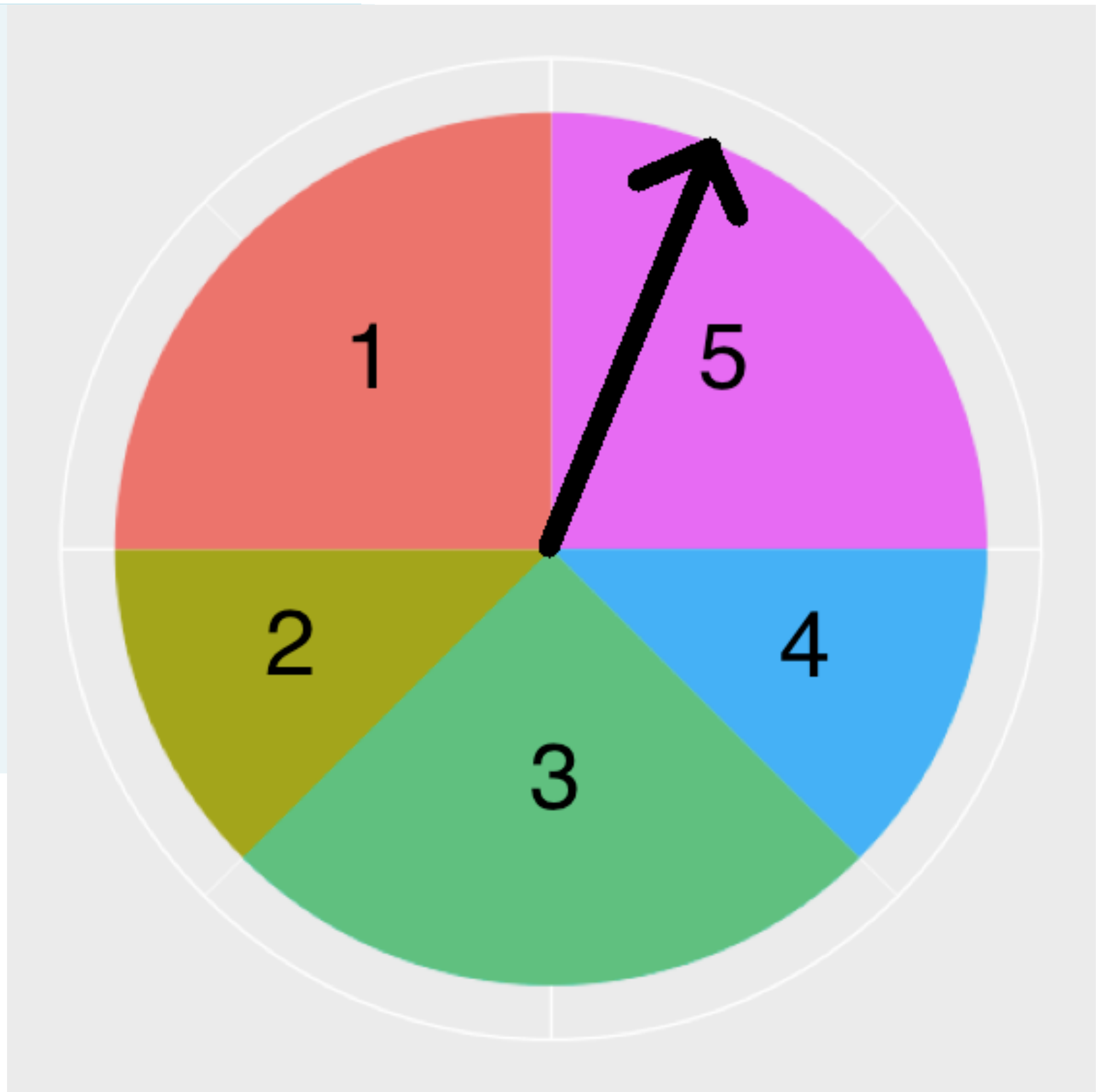
Numerical scale for probabilities



My spinner

```
> df
  Region areas Probability
1      1     2       0.250
2      2     1       0.125
3      3     2       0.250
4      4     1       0.125
5      5     2       0.250

> sum(df$Probability)
[1] 1
```



Probabilities of compound events

- Identify outcomes of interest
- Sum the probabilities of individual outcomes

Probabilities of compound events

```
> library(dplyr)
> filter(df, Region %in% c(1, 3, 5))
  Region areas Probability
1      1     2        0.25
2      3     2        0.25
3      5     2        0.25

> filter(df, Region > 3)
  Region areas Probability
1      4     1       0.125
2      5     2       0.250
```

$$Prob(Odd) = 0.25 + 0.25 + 0.25 = 0.75$$

$$Prob(larger\ than\ 3) = 0.125 + 0.25 = 0.375$$

Find probabilities by simulation

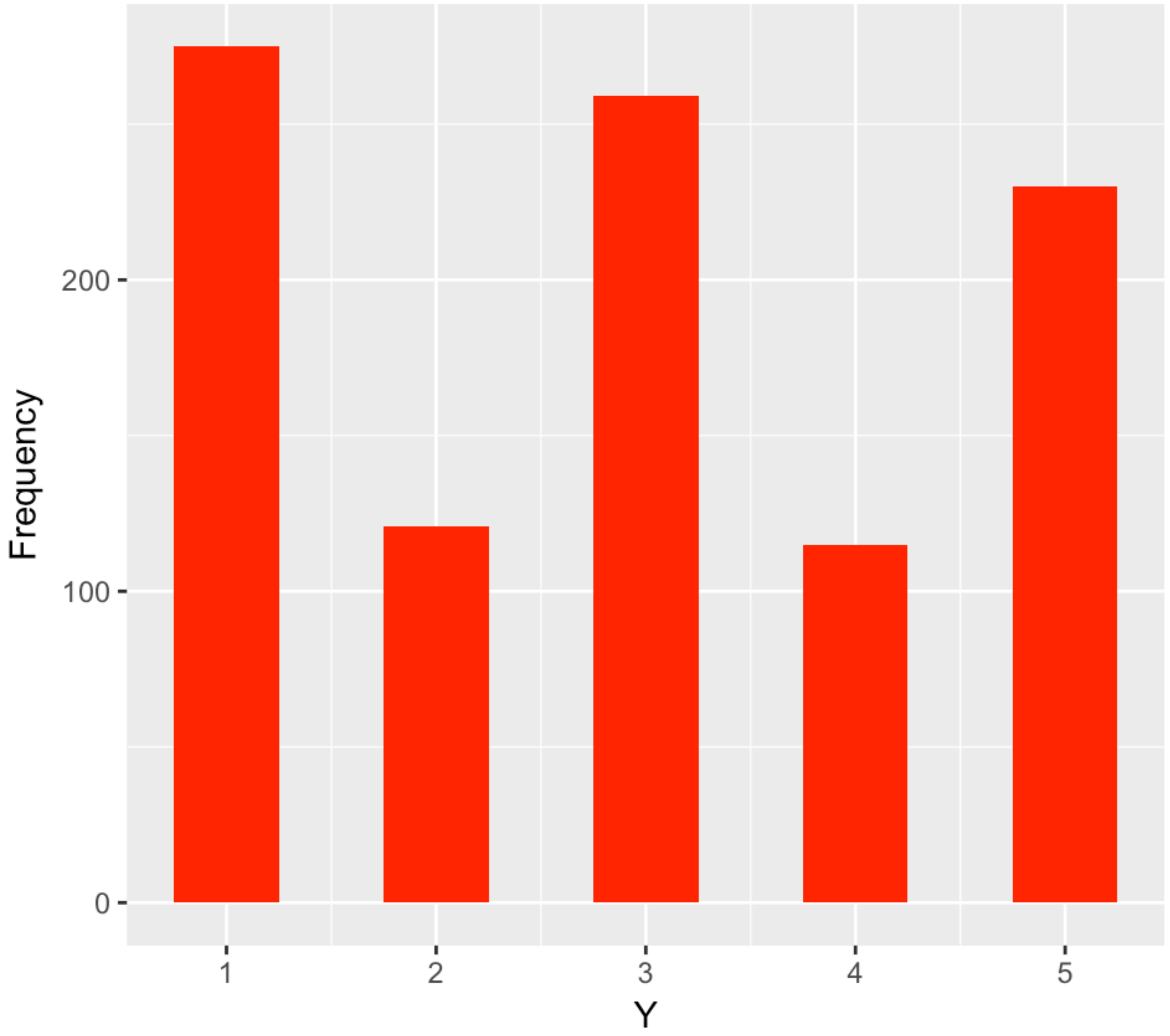
- Spin spinner many times
- Summarize simulated outcomes

Plotting simulation data

```
> library(TeachBayes)
> (ten_spins <- spinner_data(areas, 10))
[1] 4 3 2 3 1 3 3 2 1 1

> many_spins <- spinner_data(areas, 1000)

> bar_plot(many_spins)
```



Simulation data displayed as a table

```
> (S <- summarize(group_by(data.frame(Region = many_spins), Region),
                  N = n()))
# A tibble: 5 × 2
  Region     N
   <int> <int>
1      1   262
2      2   115
3      3   258
4      4   122
5      5   243
```

Probability of Region = 1

```
> (Freq_1 <- sum(S$N[S$Region == 1]))
[1] 255

> (Prob_1 <- Freq_1 / 1000)
[1] 0.255
```

An approximation to the actual probability of 0.25

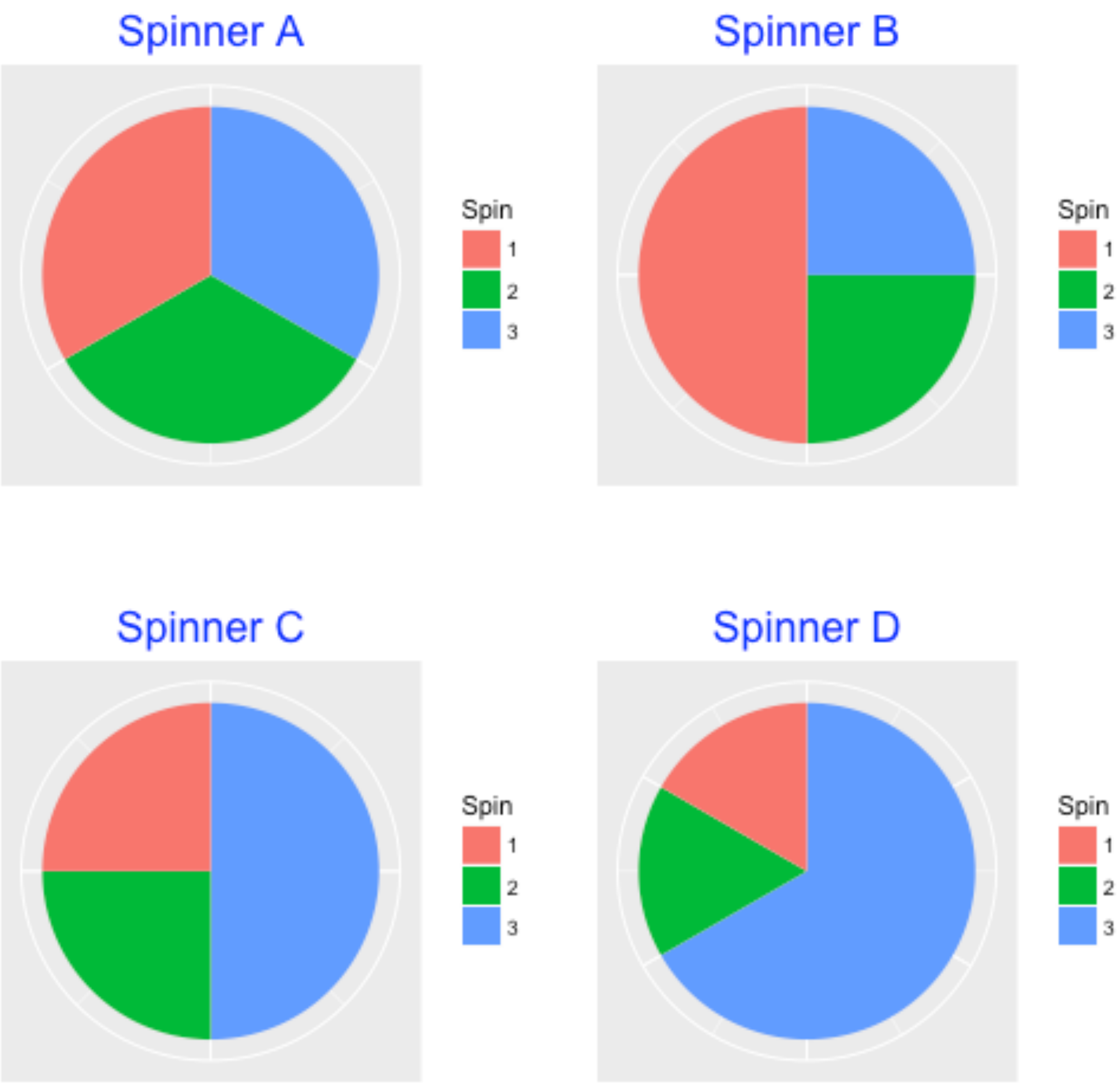
Let's practice!

Bayes' rule

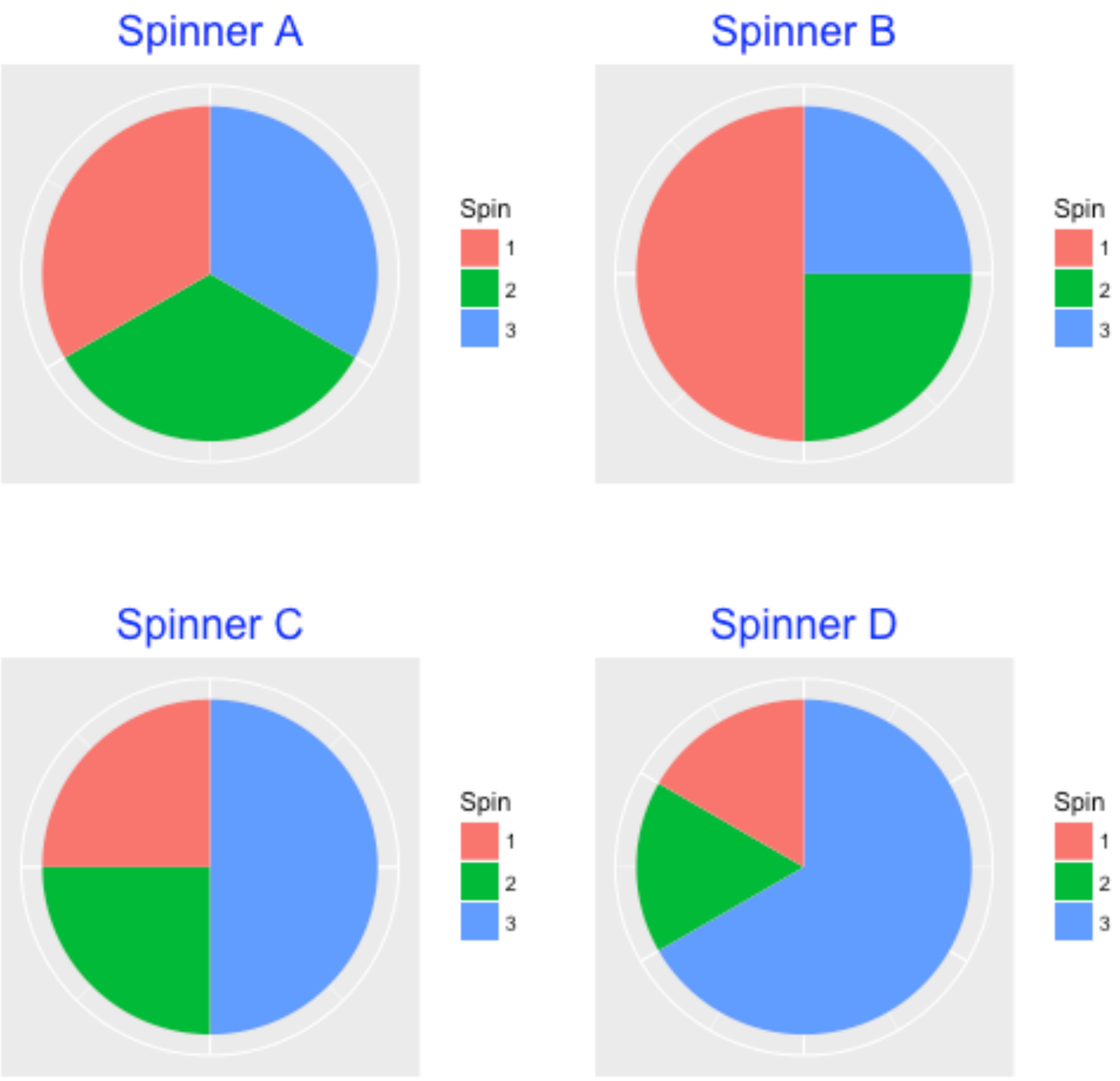
Thomas Bayes

- Presbyterian minister (1702 - 1761)
- Mathematician in his spare time
- *Essay Towards Solving a Problem in the Doctrine of Chances* (1763)

Define four spinners



Choose one spinner from box



Choose one spinner from box

Which spinner is she holding?

Outline of Bayesian approach

- Identify possible **models** and construct **prior probabilities** which reflect your knowledge about the models
- Collect **data** — think of **likelihoods**, the chance of getting this data for each model
- Use **Bayes' rule** to find **posterior probabilities**, updated knowledge about models

Identity of the spinner is a model

```
> (bayes_df <- data.frame(Model = paste("Spinner",
                                        c("A", "B", "C", "D"))))
      Model
1 Spinner A
2 Spinner B
3 Spinner C
4 Spinner D
```

The prior

- Don't know what spinner she's holding
- Assign probabilities that reflect belief about likelihoods of her holding each of these spinners (i.e. assign priors)

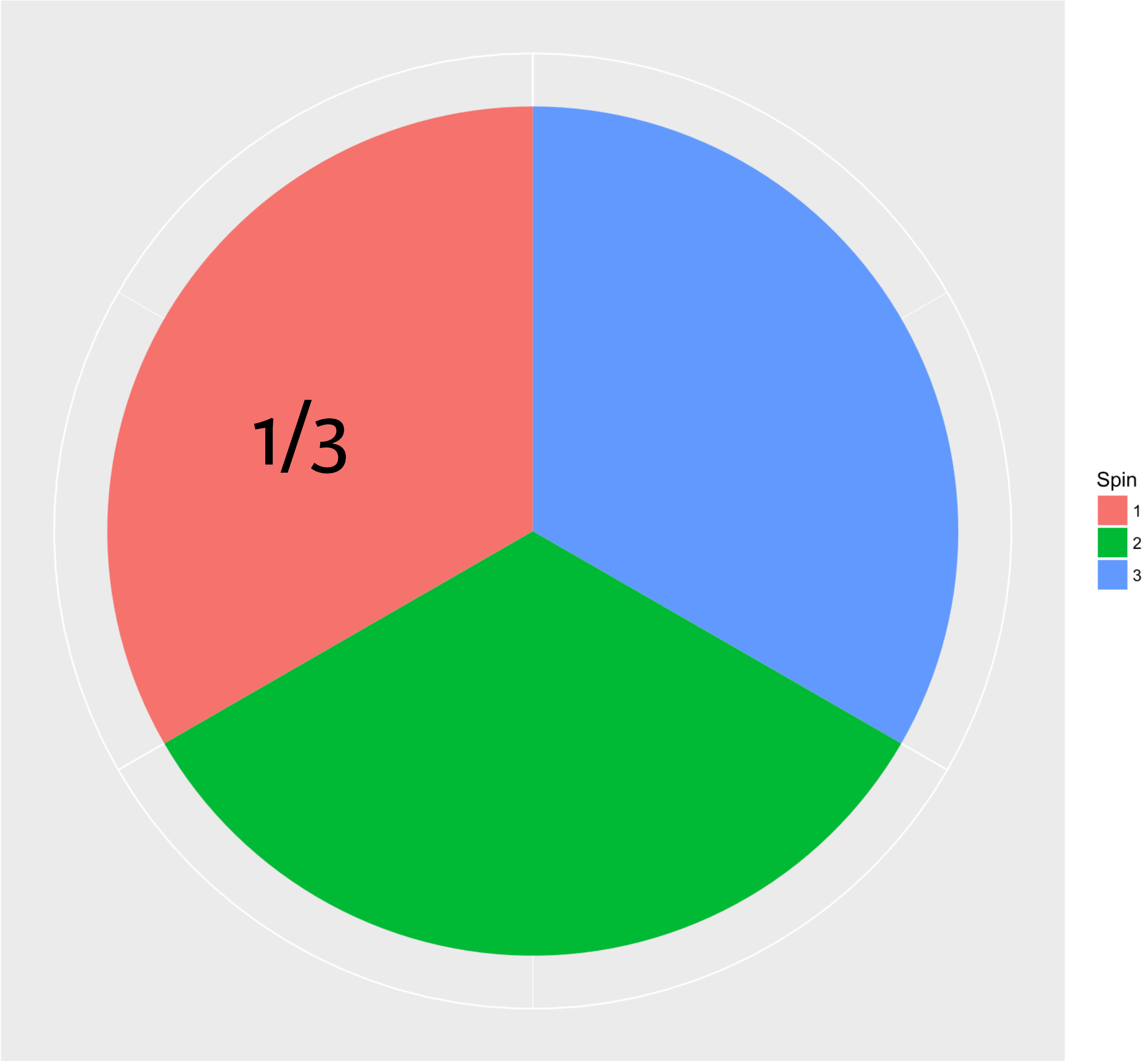
```
> bayes_df$Prior <- rep(0.25, 4)
> bayes_df
      Model Prior
1 Spinner A  0.25
2 Spinner B  0.25
3 Spinner C  0.25
4 Spinner D  0.25
```

This is an example of a uniform prior since prior probabilities are spread out uniformly

Finding the likelihood of observing red

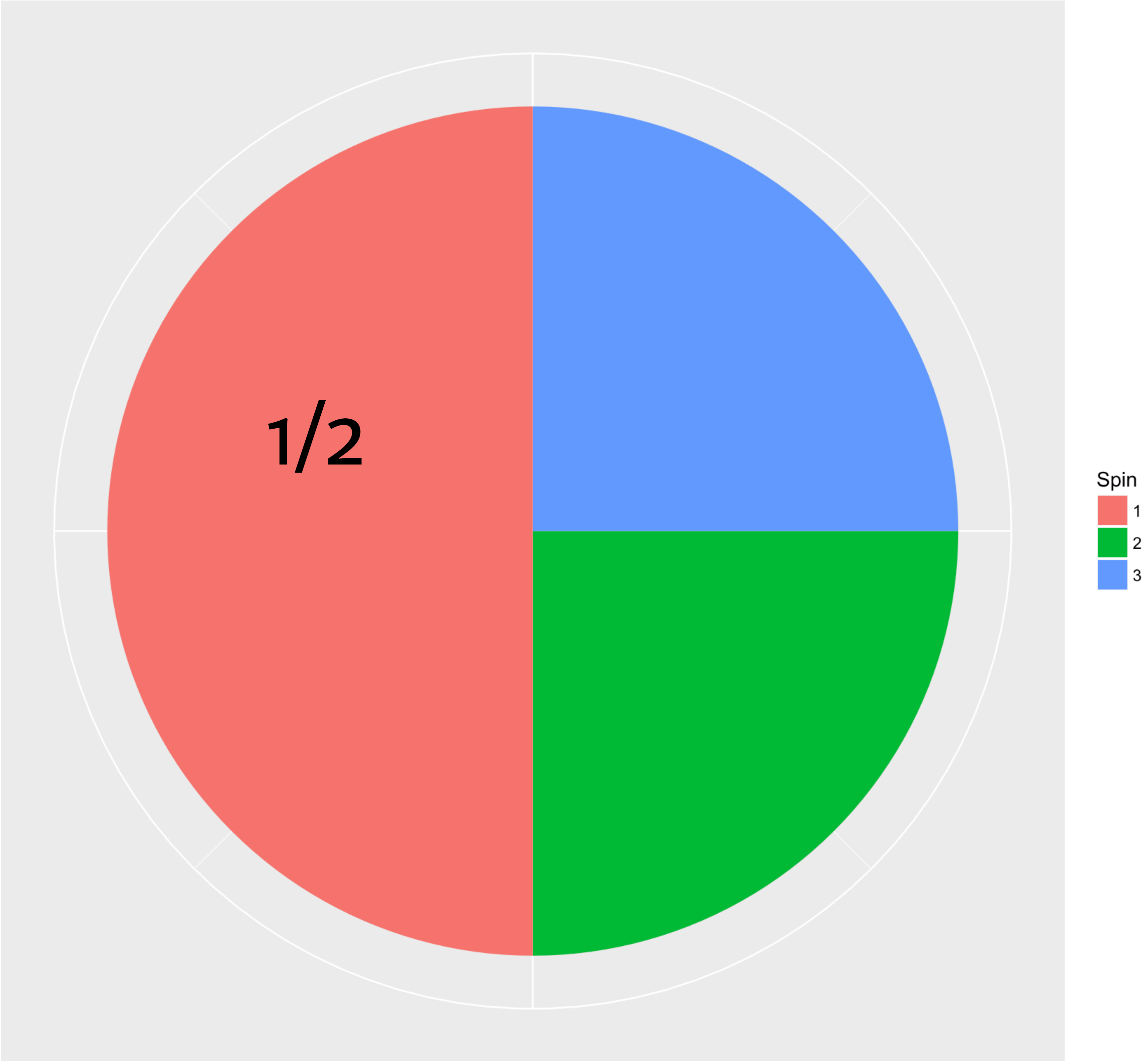
- She spins her spinner once
- Lands on RED
- For each spinner, what is the probability of observing RED?

Likelihood of Spinner A?



Spinner A

Likelihood of Spinner B?



Spinner B

Add likelihoods to table

```
> bayes_df$Likelihood <- round(c(1/3, 1/2, 1/4, 1/6), 2)
> bayes_df
Model        Prior  Likelihood
1 Spinner A  0.25        0.33
2 Spinner B  0.25        0.50
3 Spinner C  0.25        0.25
4 Spinner D  0.25        0.17
```

Likelihoods for Spinner C and Spinner D

Bayes' rule

- Posterior probability is proportional to
  Prior Probability x Likelihood
- "Turn the Bayesian Crank" means to compute posterior probabilities using Bayes' rule

Add products and posterior to table

```
> library(TeachBayes)
> bayesian_crank(bayes_df)
      Model Prior Likelihood Product Posterior
1 Spinner A  0.25       0.33  0.0825     0.264
2 Spinner B  0.25       0.50  0.1250     0.400
3 Spinner C  0.25       0.25  0.0625     0.200
4 Spinner D  0.25       0.17  0.0425     0.136
```

bayes_df contains the prior and likelihood probabilities

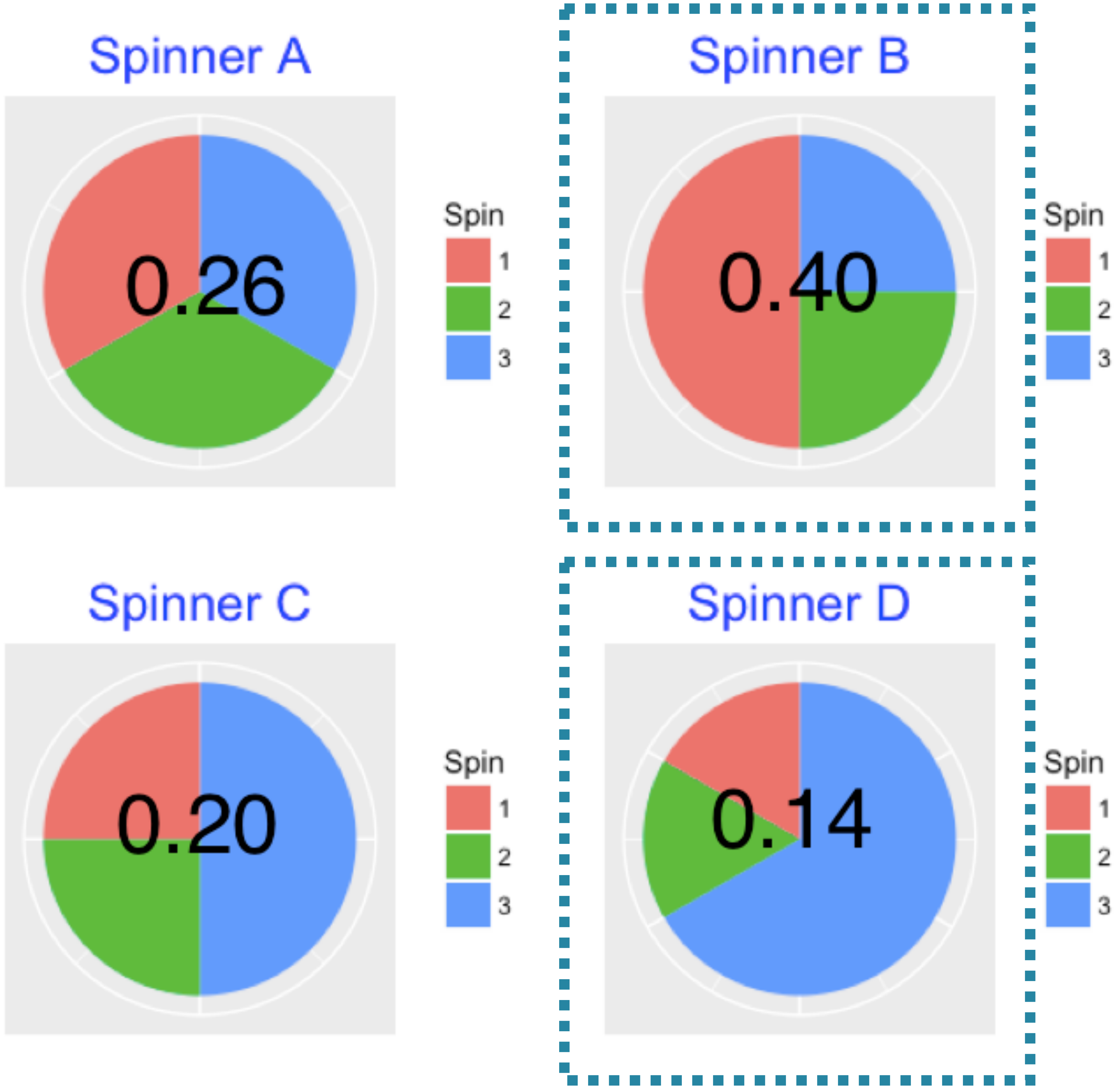
Prior x Likelihood = Product

Product / sum(Product) = Posterior

Review: the prior

Review: the posterior

Interpreting the posterior probabilities



We can learn more about the identity of spinners by simulating more spins

Let's practice!

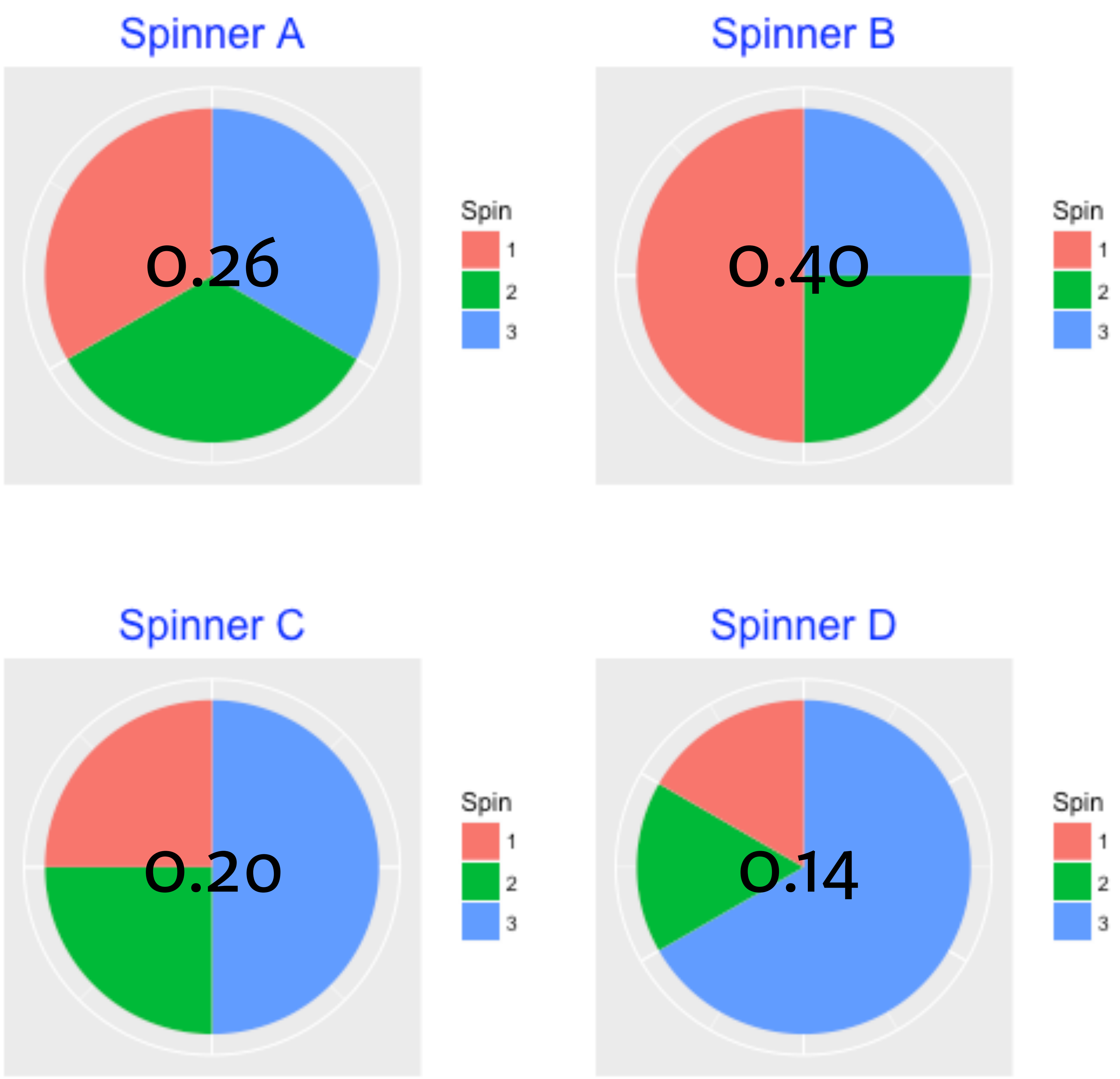
Sequential Bayes and looking ahead

Four spinners example

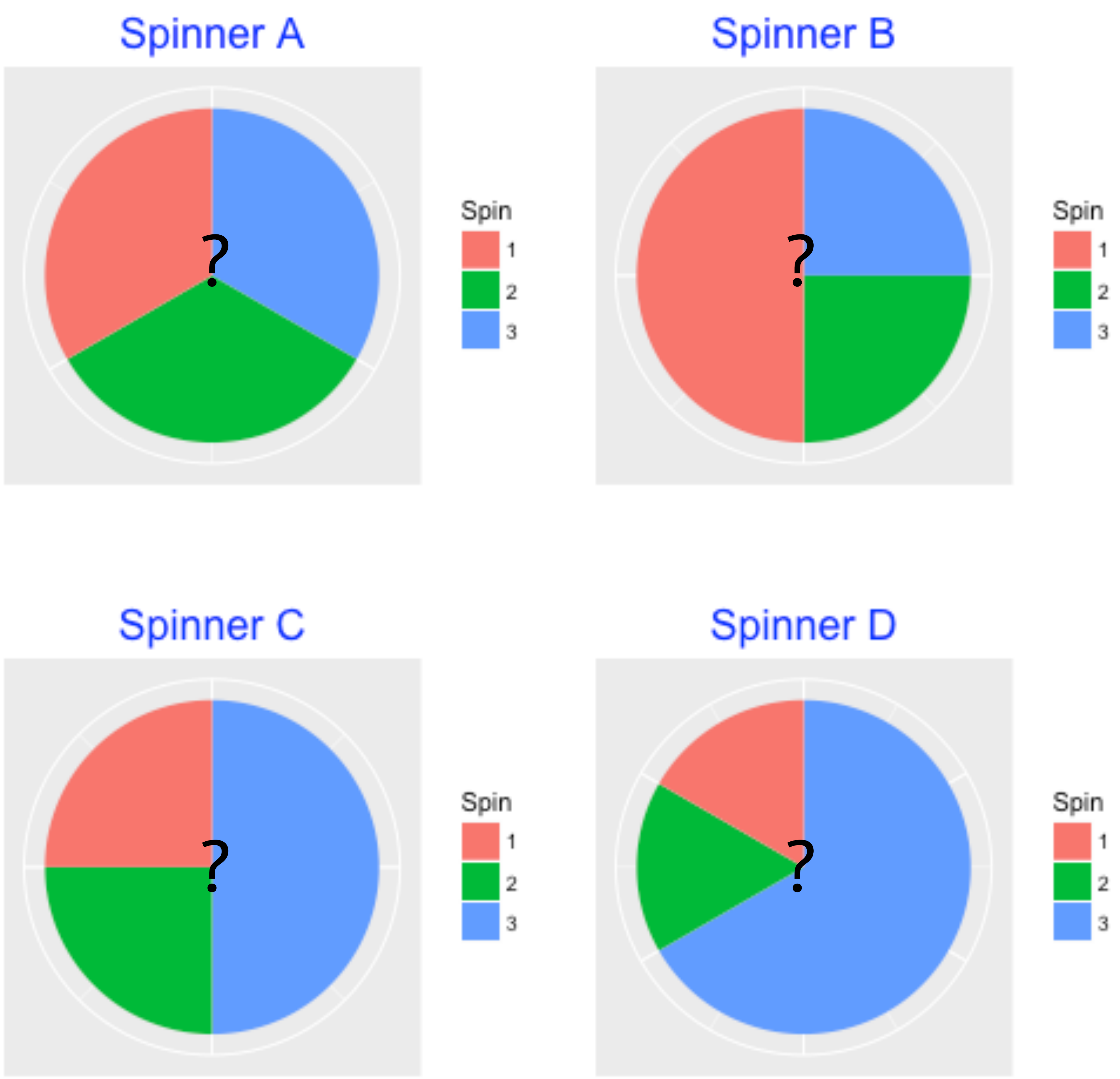
Spun spinner once

Observed RED

The posterior



Suppose you observe a second spin



Spin spinner again

Observe **BLUE**

Prior?

Likelihood?

Prior?

- Prior represents our current beliefs about the spinners
- After first spin, the posterior becomes my new prior: (.264, .400, .200, .136)

```
> # Old posterior becomes new prior
> bayes_df
      Model Prior
1 Spinner A 0.264
2 Spinner B 0.400
3 Spinner C 0.200
4 Spinner D 0.136
```

Likelihood?

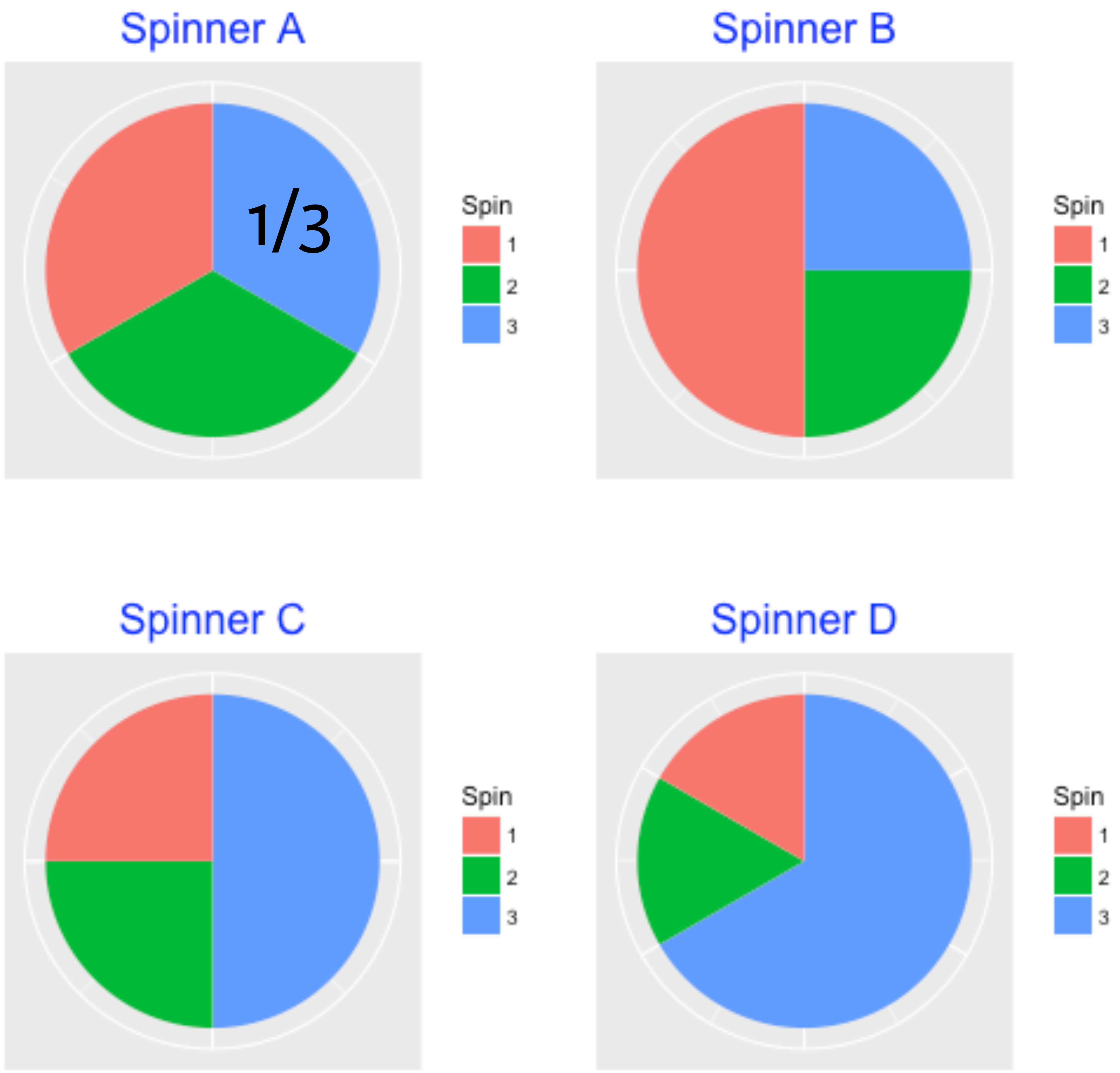
For each spinner, find chance of **BLUE**

```
> bayes_df$Likelihood <- c(1/3, 1/4, 1/2, 2/3)
> bayes_df
      Model Prior Likelihood
1 Spinner A 0.264  0.3333333
2 Spinner B 0.400  0.2500000
3 Spinner C 0.200  0.5000000
4 Spinner D 0.136  0.6666667
```

Likelihoods of all four spinners

Likelihood?

For each spinner, find chance of **BLUE**



Update probabilities

After observing two spins (RED, BLUE), Spinners B and C are each slightly more likely than Spinners A and D

```
> library(TeachBayes)
> bayesian_crank(bayes_df)
      Model Prior Likelihood    Product Posterior
1 Spinner A 0.264  0.3333333 0.08800000 0.2323944
2 Spinner B 0.400  0.2500000 0.10000000 0.2640845
3 Spinner C 0.200  0.5000000 0.10000000 0.2640845
4 Spinner D 0.136  0.6666667 0.09066667 0.2394366
```

Looking ahead

- Bayesian methods for one proportion and one normal mean inference
- Introduce continuous priors
  - Input prior information?
  - Obtain posterior?
  - Compare with frequentist inferences?
- Use simulation to summarize posterior distributions
- Simulation for prediction and Bayesian regression models

Let's practice!