Bayesian Data Analysis
Assignment (I)

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1 Introduction

Consider the problem that we have a six faced dice and that each of the six faces of the given six faced fair die is painted either black or white, but we do not know how many of its six faces are painted black. To learn about the number of black painted faces, and hence about the probability that a black face is uppermost when it comes to rest after being rolled, the die is rolled 20 times and the number of times the uppermost face is black, Y , is observed.

2 Methodology

Since upon rolling, front face of our dice can take only one of the two possible colors i.e. black or white. Let us consider the appearance of black as a success and the appearance of white as a failure. Now, when we perform the act of rolling the dice for a given number of times, say n, we can have a certain number of success, say k, and some certain number of failures which will be equal to $1 - k$. This is similar to the coin toss experiment in which we have only two possibilities i.e. either a success (for e.g. getting a head) or a failure (for e.g. getting a tail). Therefore, it becomes preferable to go ahead with the binomial model.

2.1 *Problem 1: Choosing a statistical model*

In case of our experiment, it is recommended to proceed with the binomial model. Since we are tossing the die 20 times, we have the number of trials =20 and so we can describe our chosen statistical model as-

$$Y \sim binomial(20, \theta), \theta \in [0, 1]$$

where θ represents the probability that a success will occur in a single trial i.e. getting a black faced die.

We can represent the PMF of our statistical model as-

$$p(y|\theta) = \binom{20}{y} \theta^{y}(1-\theta)^{(20-y)} \tag{1}$$

where $y \in \{0, 1, 2, ...20\}$

2.2 *Problem 2: Prior distribution for the probability that the face that is uppermost after rolling that die is black*

Now, based on the information we have so far, we do not not know about the number of painted black faces. So, we assume that the number of painted black faces is equal to λ, where λ can take values between $\{0, 1, 2, 3, 4, 5, 6\}$ and since each of the values are equally likely, we can denote the probability of having a success as follows-

$$\pi(\theta) = \frac{1}{7} \tag{2}$$

where, $\theta \in \Omega = \{0, \frac{1}{6}, \frac{2}{6}, ..., \frac{6}{6}\}$

So, we chose to have our prior distribution as a discrete uniform such that, $\theta \sim Uniform(0, 1)$, where $a = 0$, $b = 1$

2.3 Problem 3

We define our prior predictive distribution as follows-

$$p(\tilde{y}) = \sum_{\Omega} p(\tilde{y}|\theta)\pi(\theta) = \frac{1}{7} \sum_{\Omega} \binom{20}{y} \theta^{y}(1-\theta)^{(20-y)} \tag{3}$$

2.4 Problem 4

The likelihood function as defined previously can be expressed as,

$$p(y|\theta) = \binom{20}{y} \theta^{y}(1-\theta)^{(20-y)} \tag{4}$$

So, in case of $Y = 14$,

$$p(14|\theta) = \binom{20}{14} \theta^{(14)}(1-\theta)^{(6)} \tag{5}$$

and in case of $Y = 2$,

$$p(2|\theta) = \binom{20}{2} \theta^{(2)}(1-\theta)^{(18)} \tag{6}$$

2.5 Problem 5:

We define the posterior distribution as follows-

$$p(\theta|y) = \frac{p(y|\theta).p(\theta)}{p(y)} = \frac{p(y|\theta).p(\theta)}{\sum_{\Omega} p(y|\theta)p(\theta)} \tag{7}$$

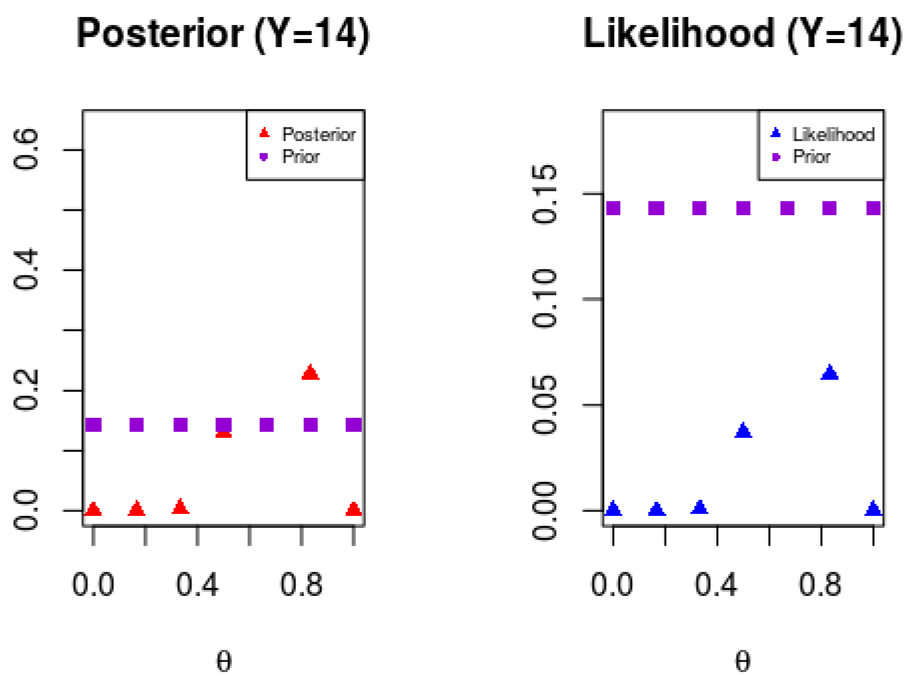
For y=14 and y=2,

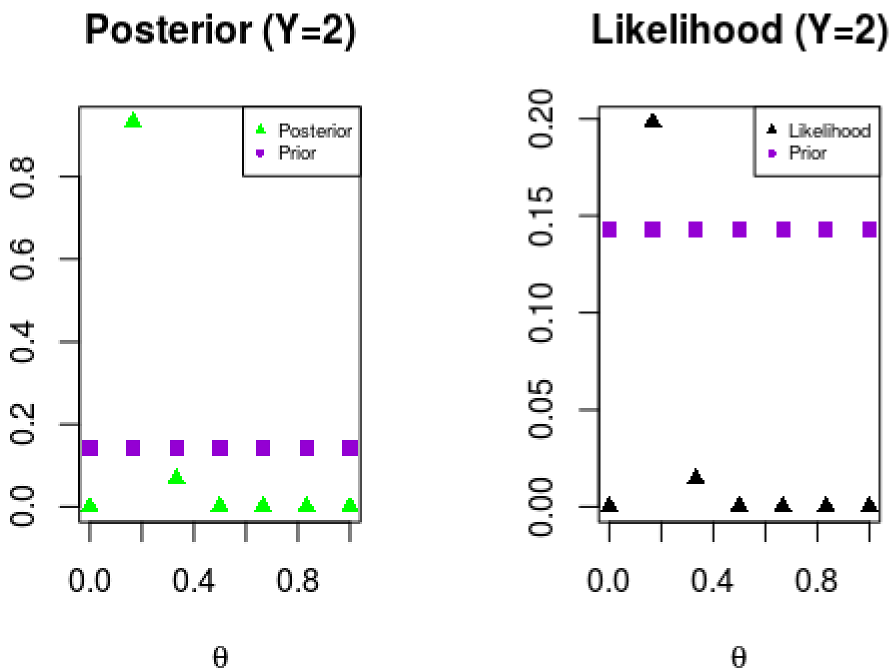
$$p(\theta|14) = \frac{\theta^{(14)}(1-\theta)^{(6)}}{\sum_{\Omega} \theta^{(14)}(1-\theta)^{(6)}} \tag{8}$$

$$p(\theta|2) = \frac{\theta^{(2)}(1-\theta)^{(18)}}{\sum_{\Omega} \theta^{(2)}(1-\theta)^{(18)}} \tag{9}$$

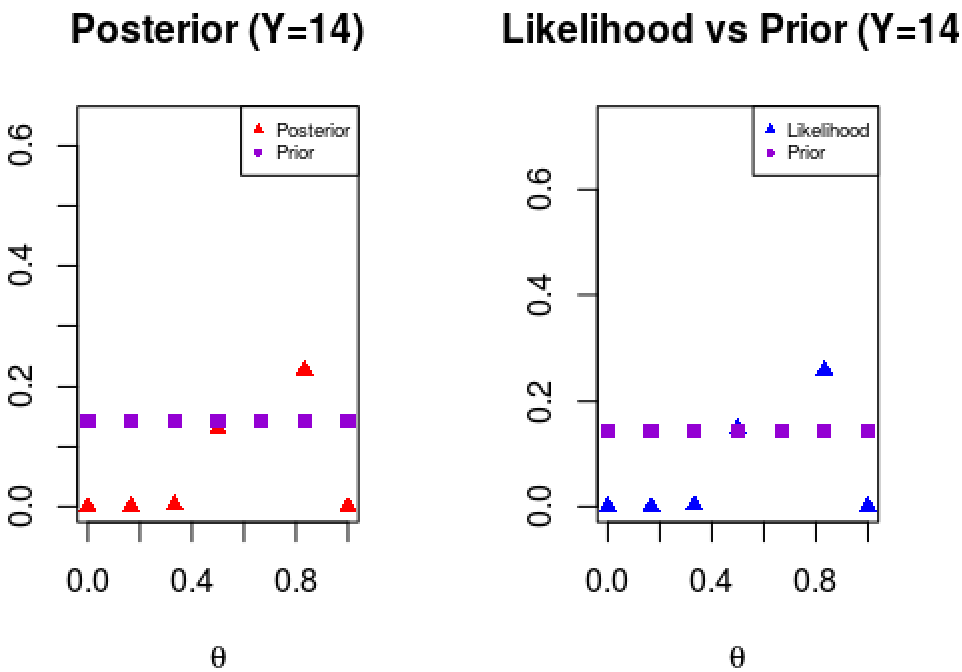
3 Results

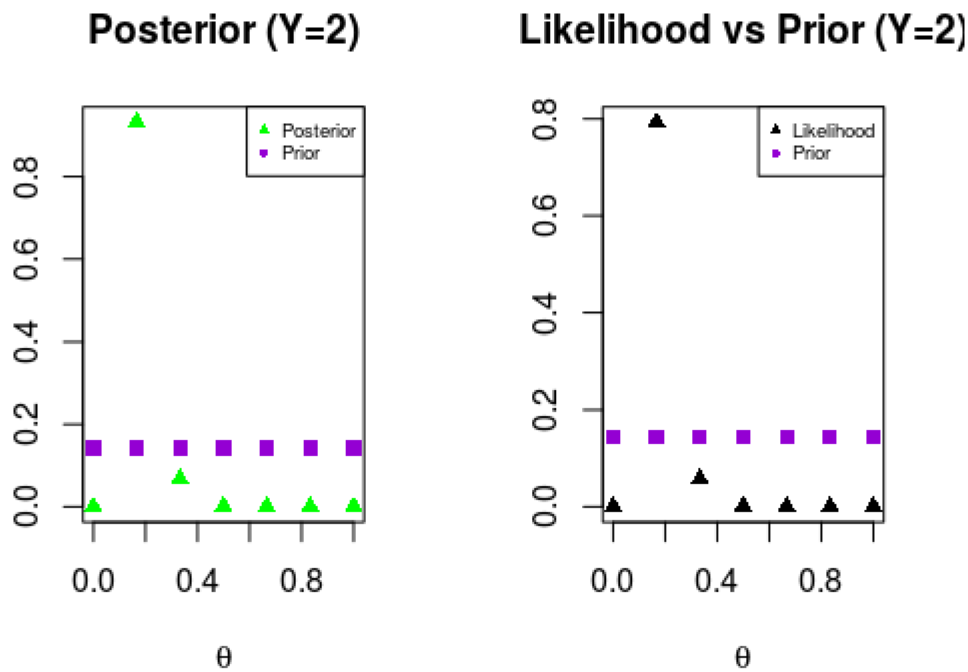
3.1 *Problem 6: Compare your two posterior distributions with the two likelihood functions and with your prior distribution, by plotting all of them*





Posterior (Y=2)
Posterior
Prior
0.0
0.2
0.4
0.6
0.8
0.0
0.4
0.8
θ
Likelihood (Y=2)
Likelihood
Prior
0.00
0.05
0.10
0.15
0.20
0.0
0.4
0.8
θ


Posterior (Y=14)
Posterior
Prior
0.0
0.2
0.4
0.6
0.0
0.4
0.8
θ
Likelihood vs Prior (Y=14
Likelihood
Prior
0.0
0.2
0.4
0.6
0.0
0.4
0.8
θ



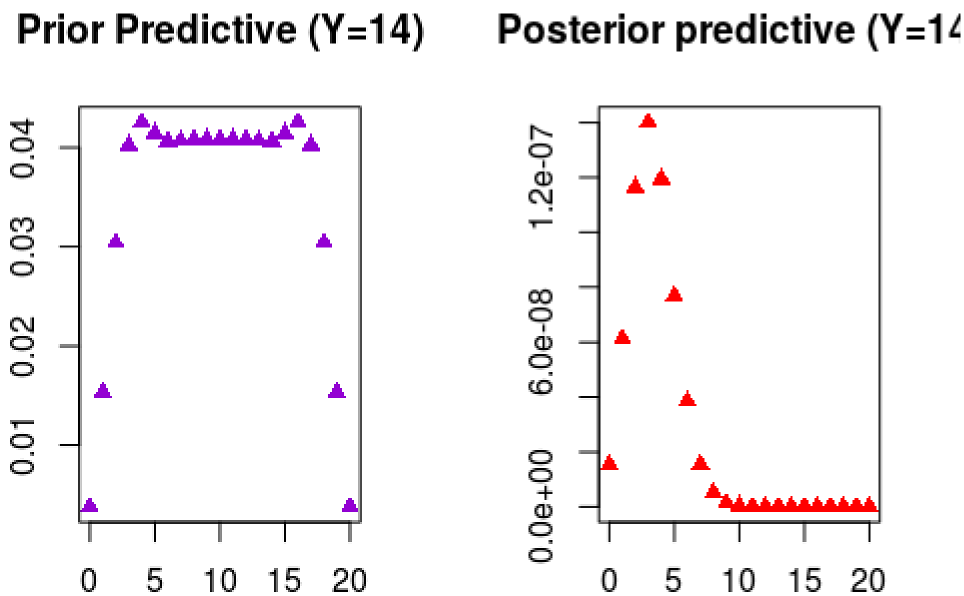
3.2 *Problem 7: Computing the posterior predictive distribution when $Y = 14$ and we intend to roll the die again 20 times, and compare it with your prior predictive distribution by plotting them*

The posterior-predictive distribution of is the set of marginal probabilities obtained when the dependency on θ is integrated/summed out by weighting the probability mass function $f(y|\theta)$ with the posterior distribution of θ given an already obtained set of observations.

$$p(\tilde{y}|y) = \sum_{\theta \in \Omega} p(\tilde{y}|\theta, y)p(\theta|y) = \sum_{\theta \in \Omega} p(\tilde{y}|\theta)p(\theta|y) \tag{10}$$

So, in our particular case,

$$p(\tilde{y} = 14|y) = \sum_{\theta \in \Omega} \binom{20}{\tilde{y}} \theta^{\tilde{y}}(1-\theta)^{(20-\tilde{y})} \frac{\binom{20}{14}\theta^{(14)}(1-\theta)^{(6)}}{\sum_{\Omega}\binom{20}{14}\theta^{(14)}(1-\theta)^{(6)}} \tag{11}$$



3.3 *Problem 8: Now, you are told that in order to decide whether each face was painted black or white, a fair coin was tossed and the face was painted black if it came heads, and it was painted white otherwise. In this case, what would be your prior for the probability that the face that is uppermost after rolling the die is black? With the information given in this last question, is a non-Bayesian entitled to chose a prior and use the corresponding posterior, without the need of "being Bayesian"?*

If a fair coin was used to decide whether a side of the dice is black or not, it would be a probability of 1/2 for a certain side to be black. Letting be the number of black sides, I set the prior as If a non-Bayesian chooses a prior (for example the on suggested) and uses the corresponding posterior, then he's already being a Bayesian. He does not need to do this, as he could for example just assume that takes the expected value (= 3) and make a model from this assumption. However, that would not have been as good as a model, because it does not take in to account that is a parameter that is not fixed.