Hwk 2: Bayesian Data Analysis,

Due on March 20th

The time to failure of a light bulb in years, x_i, follows an $Exponential(\lambda)$ model, with a probability density function equal to:

$$p(x_i|\lambda) = \lambda e^{-\lambda x_i}, \tag{1}$$

for $x_i \geq 0$, and hence with $E[x_i|\lambda] = 1/\lambda$, and $Var[x_i|\lambda] = 1/\lambda^2$. When one independently tests n light bulbs, the pdf of the i.i.d. sample will be:

$$p(x|\lambda) = \lambda^n e^{-\lambda \sum_{i=1}^n x_i}. \tag{2}$$

Such an experiment with $n = 10$ was carried out, and the sample of times to failure data obtained were 2.05, 0.37, 0.47, 7.18, 0.36, 1.24, 3.95, 0.49, 1.33 and 0.90 years. The sum of these observations is equal to 18.34.

To save experimenting time, one could start n independent tests at once but stop them as soon as one observed the first light bulb failure, which in our example was at $y = 0.36$ years. In that case, the pdf of this smallest observation is:

$$p(y|\lambda) = n\lambda e^{-\lambda n y}, \quad \text{with} \quad y \geq 0. \tag{3}$$

On the other hand, sometimes out of a sample of n independent times to failure, one only knows the largest waiting time in the sample, which in our example was $z = 7.18$. In that case, the pdf of that largest observation is equal to:

$$p(z|\lambda) = n\lambda(1 - e^{-\lambda z})^{n-1} e^{-\lambda z}, \quad \text{with} \quad z \geq 0. \tag{4}$$

Experts agree on choosing as a prior distribution for λ a $Gamma(a = 5, b = 5)$ distribution, with pdf:

$$\pi(\lambda) = \frac{b^a \lambda^{(a-1)} e^{-b\lambda}}{\Gamma(a)}, \quad \text{with} \quad \lambda \geq 0, \tag{5}$$

and hence with $E[\lambda] = a/b$ and $Var[\lambda] = a/b^2$.

1. What is the statistical model and the Bayesian model when one observes a whole sample of n independent times to failure?

   What is the statistical model and the Bayesian model when only the smallest time to failure is observed?

   What is the statistical model and the Bayesian model when only the largest time to failure is observed?

2. What is the likelihood function when one observes the whole sample of $n = 10$ times to failure, with $\sum_{i=1}^{10} x_i = 18.34$?

   What is the likelihood function when only the smallest time to failure, $y = 0.36$, is observed?

   What is the likelihood function when only the largest time to failure, $z = 7.18$, is observed?

3. What is the posterior distribution for λ when one observes the whole sample of $n = 10$ independent times to failure, with $\sum_{i=1}^{10} x_i = 18.34$?

   What is the posterior distribution for λ when only the smallest time to failure, $y = 0.36$, is observed?

   What is the posterior distribution for λ when only the largest time to failure, $z = 7.18$, is observed?

4. What are the prior predictive and the posterior predictive distributions, when one observes the whole sample of $n = 10$ times to failure, with $\sum_{i=1}^{10} x_i = 18.34$?

   What are the prior predictive and the posterior predictive distributions, when only the smallest time to failure, $y = 0.36$, is observed?

   What are the prior predictive and the posterior predictive distributions, when only the largest time to failure, $z = 7.18$, is observed?

5. How would you compute a point estimate for λ based on the whole sample of $n = 10$ times to failure, with $\sum_{i=1}^{10} x_i = 18.34$?

6. How would you compute a 90% posterior credibility for λ, based on the whole sample of $n = 10$ times to failure, with $\sum_{i=1}^{10} x_i = 18.34$?

7. How would you choose between $H_1 : \lambda < 1.$, $H_2 : 1. \leq \lambda < 2.$ and $H_3 : 2 \leq \lambda$, based on the whole sample of $n = 10$ times to failure, with $\sum_{i=1}^{10} x_i = 18.34$?

   (How would you answer this question if you were not willing to be Bayesian?)

8. How would you compute a point estimate and a 90% posterior credibility for the expected time to failure of such a light bulb, $\theta = 1/\lambda$?

   (How would you compute a 90% confidence interval for $\theta = 1/\lambda$?)

9. How would you compute the probability that the smallest time to failure of a future sample of $n = 10$ times to failure was smaller than 0.20 years, based on the information in the whole sample of $n = 10$ times to failure, with $\sum_{i=1}^{10} x_i = 18.34$? Can this question be answered if you do not adopt the Bayesian paradigm?

10. (Extra) Which one of the three posterior distributions available, (the one based on the whole sample, the one based on the smallest time to failure, and the one based on the largest time to failure), do you anticipate would be a better estimate for λ? In which sense do you expect that posterior distribution to be better than the other two?