DECISION THEORY UNDER AMBIGUITY

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Abstract. We review recent advances in the field of decision making under uncertainty or ambiguity. We start with a presentation of the general approach to a decision problem under uncertainty, as well as the 'standard' Bayesian treatment and issues with this treatment. We present more general approaches (Choquet expected utility, maximin expected utility, smooth ambiguity and so forth) that have been developed in the literature under the name of models of ambiguity sensitive preferences. We draw a distinction between fully subjective models and models incorporating explicitly some information. We review definitions and characterizations of ambiguity aversion in these models. We mention the challenges posed by some of the models presented. We end with a review of part of the experimental literature and applications of these models to economic settings.

Keywords: Ambiguity; Ambiguity aversion; Decision; Uncertainty.

1. Introduction

We survey recent advances in decision theory under uncertainty with a focus on so-called models of ambiguity sensitive preferences. This field has seen a surge of research in the past 20 years or so, with both theoretical and experimental advances. This literature still produces hot debates as can be seen in the criticism expressed by Al Najjar and Weinstein (2009) and the replies by important authors in the field, published in *Economics and Philosophy* in 2009[1] or, to a lesser extent, in the recent exchange between Epstein (2009) and Klibanoff *et al*. (2009a).

The non-specialist observer might be overwhelmed by the number of different models aimed at capturing how ambiguity can affect decisions. And even before that, he might be baffled by terminology issues. Indeed, in the literature ambiguity and uncertainty are not always distinguished, nor are they clearly defined. In this survey, we will use both terms equivalently. Uncertainty or ambiguity is then meant to represent 'non-probabilized' uncertainty – situations in which the decision maker is not given a probabilistic information about the external events that might affect the outcome of a decision – as opposed to risk which is 'probabilized' uncertainty. We will thus concentrate on situations in which there is too little information to pin down easily probabilistic beliefs (as opposed to risky situations, in which objects of choice – lotteries – are already formulated in terms of probability distributions).

The aim of this survey is to provide a (more or less) user friendly guide of these models, their properties and how they relate to one another, although we won't for the most part discuss axioms in details. We will also provide a (quick) presentation of some of the major empirical findings in this field. Although our aim is certainly not to be exhaustive, we feel the references gathered here are largely sufficient and recent for the interested reader to pursue his own view of what the field is about.

We feel the time is ripe for more applications of these models. For instance, the Stern review explicitly calls for using models of ambiguity when doing cost-benefit analysis related to climate change. This survey is thus designed to provide a toolkit for economists willing to incorporate ambiguity in their models.

We should mention at the outset that we will not deal with the recent literature on unforeseen contingencies and more generally the issue of subjective state space. We feel that, while the contributions are worthy of interest, this field is still at a too early stage to report its main findings. We will not either review the huge literature in the domain of decision under risk. Nor will we do justice to the literature in psychology that touches upon similar issues.

The paper proceeds as follows. We start with a rather didactic section in which we present the general approach to a decision problem under uncertainty, as well as the 'standard' Bayesian treatment. We end this section with a series of 'problems' with the Bayesian/Savagian approach. We then discuss in Section 3 recent, more general approaches that have been developed in the literature. We draw a distinction between fully subjective models and models incorporating explicitly some information. Next, we move to the tricky issue of defining what is exactly ambiguity and ambiguity aversion (Section 4) and how it is characterized in the various existing models. Even more problematic is the dynamic of these types of models. We mention the problem and some possible ways out in Section 5. Section 6 takes a glimpse at the huge experimental literature and is meant more as a way to enter the literature than a well thought, thorough recension of the various issues raised by and results obtained in this literature. Section 7 gathers a limited number of applications of these models to various fields of economics.

2. Setting the Scene

This survey is concerned with decision under uncertainty, as 'opposed' to risk.[2] Risk in the decision theoretical jargon represents situations in which information is available, in the form of probability distributions. The decision maker thus chooses among *lotteries* that is, probability distributions on some outcome space. The leading model under risk is the expected utility model of von Neumann and Morgenstern (1947). Starting with Allais (1953) experiments, it has been challenged over the last 30 years by various forms of rank dependent utility. Loosely speaking, rank dependent utility models assume that agents do not use probability in a linear manner, as does the expected utility model. These models have been shown to fare better descriptively in experiments.

By contrast, uncertainty represents a situation in which no probabilistic information is available to the decision maker. We discuss in this section the way traditional decision theory has been dealing with such choice situations.

2.1 *Modelling a Decision Problem under Uncertainty*

Formally, defining a choice problem is simple: there is a state space S, whose elements are called states of nature and represent all the possible realizations of uncertainty; there is an outcome space $\mathcal{X}$, whose elements represent the possible results of any conceivable situation; and there is a preference relation $\succsim$ over the mappings from S to $\mathcal{X}$.

2.1.1 *State Space*

Let us examine in more details this concept. The state space is given and is supposed to include all relevant possible futures. This is of course an important limitation of the classical approach. It leaves aside the issue of unforeseen contingencies, that is, events that cannot be described or even thought of at the moment the decision is taken. Modelling unforeseen contingencies has proved to be a challenge to decision theory. Various attempts have been made recently [e.g. Dekel *et al*. (2001), or in a different vein Epstein *et al*. (2007)] following Kreps (1979) lead.

One conceptual (logical) issue when dealing with unforeseen contingencies is the following: how can one take into account, when making a decision today, that things can happen that the decision maker cannot even conceive of?

Although of interest especially when studying emerging risks, we will not deal with this issue here[3] and, for the rest of this survey, will assume that the state space S covers all possible sources of uncertainty. An element s of S is a state of nature, sometimes also called state of the world. Sets of states of nature, $E \subset S$ are called events. Denote $\mathcal{A}$ a σ-algebra of events.

2.1.2 *Outcome Space*

The outcome space $\mathcal{X}$ can be rather abstract. It captures all possible aspects of a decision that affect the decision maker well being. For most of the interpretation, we can take $\mathcal{X}$ to be simply the set of real numbers (money for instance). However, it can also include state of health or any relevant aspect of the problem considered.

It will also be convenient to sometimes assume that $\mathcal{X}$ is a set of lotteries. Thus, the result from an act could for instance be, 'if state s realizes, get a lottery that yields some amount x with probability p and some amount y with probability $1 - p$'. When the outcome space has this convex structure, things can be technically simpler. This approach has been advocated by Anscombe and Aumann (1963).[4]

2.1.3 *Preferences*

In all the material we will survey, a decision maker is assumed to have preferences over 'acts'. An act is a mapping from S to $\mathcal{X}$. It associates to each state of nature a possible consequence. Thus, the decision maker compares acts that are rather extensive descriptions of what would occur if the decision was taken. Call D the set of all possible (conceivable) acts. $f(s)$ is the outcome of act f in state s.

The preference relation is denoted $\succsim$ and $f \succsim g$ means that the decision maker prefers act f to act g. Sticking to the (challenged) behaviourist approach so central in economics under the label of 'revealed preferences', this relation is unknown to an observer, who can only observe choices made by the decision maker. Thus the interpretation should be reversed: when I, as an outside observer, see that a decision maker chooses f over g, then I can infer that he prefers f over g.

All preferences we consider in the rest of the survey are assumed to be complete (the decision maker is always able to rank acts – we will only allude to what can be done if one drops this assumption), reflexive and transitive (if the decision maker prefers f over g and g over h, then he also prefers f over h).

Example 1. We illustrate the concepts introduced in this section using a toy insurance model that will be used throughout the paper. This example is not supposed to be realistic but is the simplest one can think of to illustrate the abstract analysis.

Assume that an individual with initial wealth w is facing a risk of loss d. We assume that this risk is the only relevant source of uncertainty for the problem studied. The state space is then $S = \{L, \overline{L}\}$ with

Table 1. An Illustration of the Sure Thing Principle.

	E		E^c	
	1	2	3	4
f	1	5	−4	3
g	−3	8	−4	3
f'	1	5	5	6
g'	−3	8	5	6

$L = \{loss\}$ and $\overline{L} = \{no\ loss\}$. If we take the outcome to be money, an act specifies what amount of money the decision maker has in each state of nature. An act is then simply a couple (a, b) specifying the amount of money a if loss occurs and b when there is no loss.

One possible decision for the individual would be not to buy any insurance. This can be represented by the act $f = (w - d, w)$. Another decision would be to buy full coverage at a premium π, yielding $g = (w - \pi, w - \pi)$. A third possible decision would be to buy partial coverage at a premium π', yielding $h = (w - d + I - \pi', w - \pi')$ where I is the indemnity paid in case of damage.

The decision maker has well-defined preferences over these three acts and will choose his most preferred option.

2.2 Savage

Savage (1954)'s book is still considered today to be one of the major achievements in decision theory. With a scarcity of input, he delivers the classic subjective expected utility representation of preferences. Savage thus ties together the idea of subjective probability advocated by Ramsey and de Finetti with the idea of expected utility derived (with given probabilities) by von Neumann and Morgenstern.

Since it is the starting point of all the models we will review next, it seems indispensable to spend some time presenting it. The general idea of the line of research we survey in this paper is to impose reasonable restrictions, 'axioms', on preferences and derive from them some particular representations that will hopefully be more tractable than mere preferences to investigate various choice problems, e.g. in the insurance domain.

We describe Savage's core axioms. The most important one is often referred to as the 'sure thing principle'.

Axiom 1. *Let f, g, f', g' be decisions and E an event. Assume*

$$f(s) = f'(s)\ g(s) = g(s')\ s \in E$$

and

$$f(s) = g(s)\ f'(s) = g(s')\ s \notin E$$

then, $f \succeq g \Leftrightarrow f' \succeq g'$.

This seems quite complicated. It is not. This principle is a separability principle that has practical meaning: it says that when comparing two decisions, it is not necessary to consider states of nature in which these decisions yield the same outcome. This is illustrated in Table 1. In this table, there are four states (1–4) and two events, $E = \{1, 2\}$, $E^c = \{3, 4\}$. The sure thing principle says that if for whatever reason the decision maker prefers decision f to decision g, then he *has to* prefer f' over g'.

The logic behind this is the fact that, in event E^c, f and g yield the same outcomes. They thus differ only in case event E occurs (that is, either state 1 or state 2 occurs).

Now look at f' and g'. They too differ only in case event E occurs. And if event E occurs, then f is identical to f' and g is identical to g'. The axiom concludes that the ranking of f' and g' should be the same as that of f and g.

Note that this abstract principle has practical value since it allows to trim down the set of states to 'study' to discard the set of states in which the decisions all have the same consequences. We will see in the next section that this principle prevents to model complementarity across states that could be based for instance on lack of information.

Another core axiom in Savage's construction is the idea that the likelihood ranking of events does not depend on the consequences. More concretely, if you prefer receiving 100 euros if event E occurs and nothing if it does not occur to receiving 100 euros if event F occurs and nothing if it does not occur, then you should also prefer receiving, for example, a car if event E occurs and nothing if if it does not occur, to receiving a car if event F occurs and nothing if not.

Savage's theorem can be expressed as follows. Assume $\succsim$ satisfies a number of axioms (including the sure thing principle) then, there exists a probability measure μ on S and a function $u : \mathcal{X} \to \mathbb{R}$ such that for any decision f and g

$$f \succsim g \Leftrightarrow \int_S u(f(s)) \, d\mu(s) \geq \int_S u(g(s)) \, d\mu(s)$$

In this formula, μ is interpreted as a subjective probability distribution over the state space. It represents the decision maker's beliefs. u is a utility function over outcome, and represents the decision maker's taste. Furthermore, μ is unique and u is uniquely defined up to a positive affine transformation.

Note that this formula is, formally, nothing but an expected utility and thus resembles von Neumann and Morgenstern expected utility under risk. This means in particular that, when the outcome is simply money, the concavity of u is a measure of the decision maker's risk aversion (for given beliefs). This also means more generally, that decision under uncertainty can in some sense be reduced to decision under risk, with one important caveat: beliefs are here a purely subjective construct.

Example 2. In our insurance example, a decision maker abiding by Savage's axioms would choose among the three decisions mentioned according to the subjective expected utility criterion. He would have a probability distribution on the state space, i.e. $(p, 1 - p)$ where p is the probability of loss (state L) and a utility function defined on money and would compute $pu(w - d) + (1 - p)u(w)$, $pu(w - \pi) + (1 - p)u(w - \pi)$, $pu(w - d + I - \pi') + (1 - p)u(w - \pi')$ to assess whether he chooses f, g or h.

Note that Savage's construction delivers two a priori unrelated things. The first is that the decision maker has probabilistic beliefs and the second is that he uses them in a linear manner (by taking expectations). Machina and Schmeidler (1992) characterizes decision makers that satisfy only the first property, i.e. decision makers that have probabilistic beliefs but do not necessarily use expected utility as a decision criterion. These decision makers are probabilistically sophisticated, without being expected utility maximizers.

2.3 *Issues with Savage*

Although Savage's beautiful construction is widely adopted in economics, it is not without raising a number of issues. We will mention two of them here.

2.3.1 *Where Do Beliefs Come From?*

The issue of subjective belief formation is not new and has been extensively discussed. See for instance Gilboa *et al*. (2008b) for a presentation of some issues (and potential solutions) that are most relevant to economics.[5]

Savage's result simply tells us that the behaviour of the decision maker can be represented as if he maximized a subjective expected utility. The only restrictions it places on beliefs is that they be represented by a probability distribution. So it allows strange or unreasonable beliefs, as long as they are probabilistic. One could argue that such strange beliefs would converge to 'true probabilities' upon learning and Bayesian updating. This however cannot be a full answer for first, learning can be taking quite a long time and second, Bayes' law is silent when conditioning on zero probability event.

More generally, one would probably gain a lot of insights if one could provide a more precise account of the way individuals form their beliefs, when for instance, frequentist evidence is not available but some relevant information is indeed available.

For emerging risks, on which data are scarce, the issue might be daunting. Think of events related to climate change. Or risk associated with new technologies such as nano-technologies. At one extreme, if no information about the possibility of occurrence of some new risk is available, a decision maker still 'needs to' come up with a probability distribution μ. His behaviour would then be indistinguishable from his behaviour in face of a well documented risk with the same distribution μ. This somehow does not feel right. Alternative decision models discussed below are meant to provide a (non-probabilistic) way out of this conundrum.

2.3.2 *Ellsberg Experiment*

Ellsberg (1961) proposed thoughts experiments that illustrate the fact that the sure thing principle is not, in the end, such an intuitive principle. His experiments go not only against subjective expected utilities but more generally against probabilistic sophistication. In the experiments he proposes, it is simply not possible to assume that subjects act based on a probability distribution, no matter what the associated decision criterion is.

In one experiment (the so-called two-urn experiments) a subject is presented with two urns containing 100 balls each. In urn I, he can check that there are 50 white balls and 50 black balls. He, on the other hand, is not informed of the composition of urn II. Placed in a choice situation, it seems plausible that a subject would strictly prefer receiving the prize upon drawing a white ball from urn I than receiving the same prize upon drawing a white ball from urn II, and similarly if one replace white by black. This behaviour which seems reasonable, is not compatible with the idea that the subject has probabilistic beliefs on the composition of urn II. Indeed, the choice pattern observed creates a difficulty to any decision criterion based on probability. No matter what the decision criterion used, if it is probabilistically sophisticated and satisfies simple dominance principle, the choice of betting on white in urn I rather than in urn II reveals that the subject thinks there are less than 50 white balls in urn II. The choice of betting on black in urn I reveals that the subject thinks there are less than 50 black balls in urn II. These two statements create a contradiction.[6]

In another experiment, the subject is facing an urn with 90 balls in it. He is told (and can check) that there are 30 red balls. The remaining 60 are blue or yellow, in unknown proportion. The subject is then asked to choose between f and g described in Table 2, and then between h and i. For instance, f is a bet yielding 100 euros if the ball drawn is red and nothing otherwise.

A modal choice observed in this type of experiment is that subjects prefer f over g and g' over f'. This is a direct violation of the sure thing principle since f' and g' are obtained from f and g by changing their common outcome of 0 in case a yellow ball is drawn, to a common outcome of 100.

Table 2. Ellsberg Three Colour Urn.

| | 30 | 60 | |
	Red	Blue	Yellow
f	100	0	0
g	0	100	0
f′	100	0	100
g′	0	100	100

Again, a subject making these choices cannot have probabilistic beliefs. This is easily seen under subjective utility, but is also true more generally. Imagine that he had a distribution (p_R, p_B, p_Y) on the state space (the colour of the ball drawn). Then, normalizing utility so that $u(100) = 1$ and $u(0) = 0$, direct computation shows that, under subjective expected utility, $f \succ g$ implies $p_R > p_B$, while $g′ \succ f′$ implies $p_R + p_Y < p_B + p_Y$, a contradiction.

Thus, Ellsberg experiments show that there exist (simple) situations where lack of information about relevant aspects of the problem might lead the decision maker to behave in a way incompatible with probabilistic sophistication and thus, a fortiori, subjective expected utility. The separability principle entailed by the sure thing principle is here violated because of some complementarity among events. In Ellsberg three colour example, the event 'Blue' and the event 'Yellow' are complement because some information is given on their union, that cannot be further decomposed.

In this section, we have provided a fairly general setting in which one can think of decision under uncertainty and then studied one possible (and indeed predominantly assumed in economics) way of behaving when facing uncertainty. We also presented issues raised by this type of behaviour.

3. Non-Bayesian Decision Criteria

Spurred by cognitive unease due to the similar treatment of situations with no available information and situations in which information is precise, as well as experimental results of the type reported in Ellsberg (1961), a number of models generalizing Savage expected utility have been proposed over the last twenty years.

3.1 *Wald Maxmin and Arrow and Hurwicz α Maxmin*

An ancestor to this literature is the Wald maxmin criterion, generalized by Arrow and Hurwicz (1972). This criterion is intuitive and conservative: when faced with a decision that could yield consequences say $\{x, y, z\}$, evaluate it by looking exclusively at the worst possible consequence. If x is worse than y which itself is worse than z, then the value attributed to *any* decision whose outcomes lie in this set, will be $u(x)$. It does not matter that x will be the consequence in state 1 or state 2 or any other event, as long as it is a possible consequence of the decision, this will be the way the decision maker assesses the decision.

Arrow and Hurwicz' generalization consists in incorporating not only the worst but also the best outcome of the decision when evaluating it. In our example, this means that a decision f that yields consequences in $\{x, y, z\}$ will be evaluated by $\alpha u(x) + (1 - \alpha)u(z)$. The coefficient α, whose value lies between 0 and 1, is interpreted as a measure of the decision maker's pessimism.

These criteria can be arguably (yet somewhat unfairly as we will see later) criticized as being too crude. Applied in a probabilistic setting for instance, Wald criterion could be said to be overly

pessimistic and would predict that a decision maker is indifferent between betting on black (i.e. receiving say 100 euros if the ball drawn is black and 0 if it is white) in an urn that has 1 white ball and 99 black balls and in an urn that has 99 white balls and 1 black ball. Introducing a pessimism index does not break this indifference.

However, this argument could, in a sense, be used backward. Cohen and Jaffray (1980) study a model of choice under complete ignorance. Ignorance is defined in a behavioural way, as a situation in which the decision maker is indifferent between betting on any two events as long as one is not included in the other (in which case, a dominance principle would simply say that the subject prefers to bet on the larger event). This indifference captures the fact that the decision maker has no means to assess whether one event is more likely than another. Cohen and Jaffray then show that, as an approximation, the Arrow and Hurwicz' criterion is the only decision criterion that is susceptible to model choice in such situations. Thus, the use of this criterion by a rational decision maker would reveal that he faces a situation of complete ignorance.

Nehring (2000) approach to decision under complete ignorance is different, since he assumes that ignorance creates incompleteness in the preferences. More precisely, non-comparability arises from limited information about the likelihood of uncertain events. The way this incompleteness is 'resolved' can be interpreted as agreement in a bargaining game whose players correspond to the (extremal) 'acceptable priors' among which the decision maker has suspended judgment.

3.2 *Non-additive Beliefs*

In this section we present 'first generation' models, namely Choquet expected utility and the cumulative prospect theory. Both rests on the use of capacities to represent beliefs in the face of uncertain situations.

Choquet expected utility. Historically, the first axiomatically sound model in this vein is the Choquet expected utility model developed by Schmeidler (1989). Weakening Savage's sure thing principle, Schmeidler obtains a representation where beliefs are characterized not by a subjective probability but by a capacity, that is, a non-necessarily additive, increasing set function.[7] In this model, an act f is preferred to an act g if and only if there exists a utility function $u : X \to \mathbb{R}$ and a capacity v such that:

$$\int_{Ch} u(f)\,dv \geq \int_{Ch} u(g)\,dv$$

Due to the non-additivity of the capacity v the previous integral is not a standard Lebesgue integral, but a Choquet integral. The expression and intuition behind 'Choquet expected utility' are easier to understand for decisions with a finite set of outcomes. Let $S = \{s_1, s_2, \ldots, s_n\}$ and f such that $f(s_i) = x_i, i = 1, \ldots, n$ with $x_i \leq x_{i+1}$. Then the Choquet integral writes

$$\int_{Ch} u(f)\,dv = u(x_1) + (u(x_2) - u(x_1))v(\{s_2, s_3, \ldots, s_n\}) + \cdots + (u(x_{i+1})$$
$$- u(x_i))v(\{s_{i+1}, \ldots, s_n\}) + \cdots + (u(x_n) - u(x_{n-1}))v(\{s_n\})$$

A decision maker evaluates a decision by considering first the lowest outcome and then adding to this lowest outcome the successive possible increments, weighted by his personal estimation of the occurrence of these increments. Due to the non-additivity of the capacity, the weight of an outcome will depend on its place in the ranking of all the possible outcomes. Note that, for an additive capacity, we get back to subjective expected utility.

In the general case, for an infinite outcome space, the expression for the Choquet integral is:

$$\int_{Ch} u(f)\,dv = \int_{-\infty}^{0} (v[\{s \in S : u(f(s)) \ge x\}] - 1)\,dx$$
$$+ \int_{0}^{+\infty} v[\{s \in S : u(f(s)) \ge x\}]\,dx$$

where the integrals on the right-hand side are the usual Riemann integrals.

One limit of the previous model is the difficulty to give an intuitive interpretation for the capacity representing beliefs (see however Jaffray and Philippe (1997) for an interpretation as a combination of lower and upper envelope of sets of priors). On the other hand, the Choquet expected utility model allows some flexibility in terms of modelling ambiguity attitudes[8] [see e.g. the neo-capacities introduced by Chateauneuf *et al.* (2007)] contrary to the multiple prior model that we present below.

Example 3. In our insurance example, if the individual's preferences are assumed to be represented by the Choquet expected utility model, he will evaluate the available decisions in the following way:

$$V(f) = u(w - d) + v(\overline{L})[u(w) - u(w - d)] = (1 - v(\overline{L}))u(w - d) + v(\overline{L})u(w)$$
$$V(g) = u(w - \pi)$$
$$V(h) = u(w - d + I - \pi') + v(\overline{L})[u(w - \pi') - u(w - d + I - \pi')]$$
$$= (1 - v(\overline{L}))u(w - d + I - \pi') + v(\overline{L})u(w - \pi')$$

Here $(1 - v(\overline{L}))$ is the weight associated by the individual to the event 'loss occurs'. Due to the non-additivity of v, this weight is not necessarily equal to $v(L)$.

Cumulative prospect theory. Cumulative Prospect Theory of Kahneman and Tversky (1979) (under risk), and Kahneman and Tversky (1992) and Wakker and Tversky (1993) is closely related to the Choquet expected utility model,[9] to which is appended a reference point and an asymmetry in the treatment of gains and losses. It generalizes the Choquet Expected Utility by introducing two different capacities, one for events corresponding to gains, and the other, for events corresponding to losses. Let $S = \{s_1, s_2, \ldots, s_n\}$ and f such that $f(s_i) = x_i, i = 1..n$ with $x_1 \le \cdots \le x_k \le 0 \le x_{k+1} \le \cdots \le x_n$. The criterion associates to f the following value:

$$V(f) = u(x_1) + \sum_{i=2}^{k} v^- \left(\cup_{j=i}^{k} \{s_j\} \right) (u(x_i) - u(x_{i-1}))$$
$$+ \sum_{i=k+1}^{n} v^+ \left(\cup_{j=i}^{n} \{s_j\} \right) (u(x_i) - u(x_{i-1}))$$

where v^- and v^+ are two capacities. Note that if $v^- = v^+$ we get back to the Choquet expected utility model. An issue with this model (widely used) is to know what should be considered as gains and what should be considered as losses. This supposes the existence of a reference point above which outcomes are treated as gains and below which they are treated as losses.

3.2.1 *Multiple Priors*

Choquet expected utility paved the way for a new approach to modelling beliefs. The next generation of models rests on the idea that when information is scarce, it is too demanding to ask for precise subjective beliefs (a probability distribution) but may be asking only for 'imprecise' subjective beliefs (i.e. a set of probability distributions) is more appropriate. They vary in the way they deal with this

set of priors. Denote this set C. For most of what follows, we can assume without loss of generality that this set is compact and convex. What can he do with this object? With a single prior, Savage tells us to compute the expected utility of the various possible decisions and then pick the one that yields the highest expected utility. With multiple priors, one can still compute expected utilities, but we now have one value per prior. How does the decision maker then compare two decisions, each one giving rise to a set of expected utilities?

It should be mentioned at the outset that this is very loose formulation: the set of priors in the models we discuss in this section are part of the representation result, i.e. this set and the way it is 'used' all come together in the representation theorem. The set of priors is not an independent object that has its own 'objective' existence.

Incomplete preferences. One solution would be to say that an act f is preferred to an act g if *all* the expected utilities of f with respect to the distributions in C are higher than those of g. This is what Bewley (1986) achieves. Hence, a is preferred to g if no matter what distributions are used to compute their expected utility, the number associated is always higher for f than for g. This means that not all acts can be compared. It can indeed happen that f is better than g for a particular distribution but that g is better than f for another distribution in the set of prior. In this event, the criterion is not binding and nothing can be said, in the sense that f and g are incomparable. This is an example of incomplete preferences. One cannot predict, on the basis of this type of preferences, the behaviour of the decision maker for all possible choice problems.

Example 4. In our insurance example, assume the set of beliefs C is given by $\{(p, 1 - p)|p \in [1/3, 1/2]\}$ where p is the probability of state L. If we take $w = 3/2$, $d = 1/2$, and $u(x) = x$ we get that the set of expected utilities associated with no insurance is the interval $[5/4, 4/3]$. On the other hand, full insurance with π implies a 'sure' (expected) utility equal to $\frac{3}{2} - \pi$. If $\frac{3}{2} - \pi > \frac{4}{3}$, then full insurance is preferred. If $\frac{3}{2} - \pi < \frac{5}{4}$, then no insurance is preferred to full insurance. But in the case $\frac{4}{3} > \frac{3}{2} - \pi > \frac{5}{4}$, Bewley's criterion is silent as, depending on the prior used, it is sometimes better to get full insurance and sometimes better not to have any insurance.

Maxmin expected utility. Gilboa and Schmeidler (1989)'s way of handling the set C is different. They provide an axiomatization of the following criterion. A decision f is preferred to a decision g if and only if

$$\min_{p \in C} E_p u(f) \geq \min_{p \in C} E_p u(g)$$

The set of priors C is uniquely defined and u is unique up to a positive affine transformation. Thus, as opposed to Bewley's, this decision criterion always gives a definite answer as to which decision is better. The decision maker will make his choice between two decisions by comparing the minimal expected utility that each decision yields.

In example 4 above, the decision criterion would unambiguously favour full insurance since the worst expected utility attached to it is higher than the worst expected utility of not getting any insurance (5/4). The axiomatization of the criterion is very much in the Savage vein, namely putting axioms on a preference relationship and then showing that preferences satisfying these axioms can be presented as a minimum over a set of priors of expected utilities.

The key axiom in Gilboa and Schmeidler (1989) is an 'uncertainty aversion' axiom.[10] It captures the intuition behind the usual explanation of the modal behaviour in Ellsberg two-urn experiment. Gilboa and Schmeidler (1989) axiomatization is done in an Anscombe and Aumann (1963) setup in which consequences are lotteries (see Section 2.1.2). Hence, a decision f yields, in state s, a probability distribution $f(s)$ on some prizes. In this context, $\alpha f + (1 - \alpha)g$ for $\alpha \in [0, 1]$ has the following meaning. If state s occurs, then the prize x will be obtained according to the draw of the 'compound

lottery' $\alpha f(s) + (1 - \alpha)g(s)$. The latter can be interpreted as first choosing with probability α to play the lottery $f(s)$ and with probability $(1 - \alpha)$ the lottery $g(s)$ and second, play whichever lottery has been drawn.

The uncertainty aversion axiom of Gilboa and Schmeidler (1989) reads as follows: take two acts f and g among which the decision maker is indifferent. Then, the decision maker (weakly) prefers $\alpha f + (1 - \alpha)g$ to f (or g). To illustrate this axiom, think back to Ellsberg two-urn experiment. Let f be the act that yields (the degenerate lottery yielding for sure) 1 if the ball drawn from the unknown urn is white and g the act that yields (the degenerate lottery yielding for sure) 1 if the ball drawn from the unknown urn is black. Most people would be indifferent between f and g. Now, consider the decision $\frac{1}{2}f + \frac{1}{2}g$. This act yields in case a white ball is drawn the following lottery: get 1 with probability 1/2 and 0 with probability 1/2. Similarly, this act yields in case a black ball is drawn in the very same lottery. Hence, the decision amounts to a risky bet, whose odds are known. Mixing of the two 'uncertain' acts f and g allows one to come up with a new act which is not uncertain: this amounts to say that by cleverly combining the two acts, one can hedge (fully in the present case) against the uncertainty. Thus this act is actually the same as a 50:50 bet, that is, it is equivalent (in a loose sense) to a bet in the known urn. Hence, the preference $\alpha f + (1 - \alpha)g \succeq f$ captures the essence of Ellsberg type behaviour.

As it were, the decision maker chooses which prior in the set he uses to evaluate the act f. For each act f, there is one (or several) prior(s) that lead to the minimum expected utility. So for a single choice, a maxmin expected utility decision maker is indistinguishable from a plain expected utility decision maker. The distinction can be made only when one observes at least two choices from the decision maker, as in Ellsberg's examples.[11]

The functional form shares a common feature with the Wald criterion, which is the presence of the min operator. However, this should not be misleading: the maxmin expected criterion is not as extreme as the Wald criterion. It takes a minimum with respect to a set of priors which is part of the representation. For instance, if it reduces to a singleton, then the min operator is irrelevant. In this case, we are back to a Savage style representation: the decision maker is a subjective expected utility maximizer. To be sure, the attitude of the decision maker with respect to uncertainty is encapsulated both in the operator min and in the set of priors C. The latter is, in Gilboa and Schmeidler's construction entirely subjective. *It does not have to relate to information the decision maker might have*, in the same way beliefs in Savage's construction need not be related to any available information. Indeed, this information is not part of the description of the decision problem. We will come back to this issue.

Example 5. Let us come back to our insurance example. If the individual's subjective set of beliefs C is given by $\{(p, 1 - p)|p \in [p', p'']\}$ and if he decides according to the *Maxmin Expected utility model*, then he will evaluate the available decisions as follows:

$$V(f) = \min_{p \in [p', p'']} pu(w - d) + (1 - p)u(w) = p''u(w - d) + (1 - p'')u(w)$$

$$V(g) = u(w - \pi)$$

$$V(h) = \min_{p \in [p', p'']} pu(w - d + I - \pi') + (1 - p)u(w - \pi')$$

$$= p''u(w - d + I - \pi') + (1 - p'')u(w - \pi')$$

Ambiguity aversion in this example takes the form of a preference for certain outcomes: the individual evaluates acts only according to the worst possible prior in the set, that is, to the highest loss probability p''. This makes full insurance more attractive.

α **maxmin**. One peculiar aspect of the maxmin expected utility is that it includes as a key axiom an axiom of uncertainty aversion. It is peculiar in the sense that one would find peculiar if von Neumann and Morgenstern's axiomatization of expected utility included risk aversion as a key component.

Ghirardato *et al*. (2004) push this idea further and provide a general analysis of what can be said about preferences that satisfy all of Gilboa and Schmeidler (1989)'s axioms but the uncertainty aversion axiom. They obtain a general representation, which under further assumptions can be specialized to the so-called α maxmin expected utility model, which takes the form: f is preferred to an act g if and only if

$$\alpha \min_{p \in C} E_p u(f) + (1 - \alpha) \max_{p \in C} E_p u(f) \geq \min_{p \in C} E_p u(g) + (1 - \alpha) \max_{p \in C} E_p u(g)$$

The set of priors C and the coefficient α are uniquely defined and u is unique up to a positive affine transformation. Obviously, this resembles the Arrow–Hurwicz criterion, with again the caveat that the minimum and maximum are taken with respect to a set of priors which is part of the representation. When $\alpha = 1$ we are back to the maxmin expected utility model. α is viewed in this representation as an index of uncertainty aversion.[12] This raises an interpretational issue. For a given set of priors, higher values of α correspond to (some notion) of less aversion to uncertainty. But α and C come at the same time and both embed some uncertainty aversion. Another, related, issue that arises in this model is the link with the expected utility benchmark. $\alpha = 1/2$ does not have any particular meaning and is certainly not associated to a decision maker that would be expected utility (unless, trivially, the set of priors is reduced to a singleton, but then any α would go). An extension of this work can be found in Amarante (2009).

Maxmin expected utility and Choquet expected utility. As already mentioned, the Choquet expected utility model of Schmeidler (1989) is related to the maxmin expected utility model. More precisely, *under the assumption of uncertainty aversion*, the Choquet expected utility model is a particular case of the maxmin expected utility model. In that case, the set of priors over which the decision maker takes the minimum has a particular property: it is the core of a convex capacity. Indeed, Schmeidler (1986) proved the following result:

For a convex capacity[13] v,

$$\int_{Ch} u(f) \, dv = \min_{p \in core \, v} E_p u(f)$$

where $core \, v = \{p \in \Delta(S) : p(A) \geq v(A), \ \forall A \in \mathcal{A}\}$.

However, Choquet expected utility is compatible with any capacity because it does not presuppose uncertainty aversion, and is, in that sense, more general than maxmin expected utility. On the other hand, any set of priors cannot necessarily be expressed as the core of a capacity.

Confidence function. Chateauneuf and Faro (2009) provide the following representation:

$$V(f) = \min_{p \in L_\alpha} \frac{1}{\varphi(p)} \int_S u(f) \, dp$$

u is the usual utility function, while L_α can be interpreted as the set of priors held by the decision maker. This set depends also on the function φ, the confidence function, in the following manner: $L_\alpha = \{p : \varphi(p) \geq \alpha\}$. The value of $\varphi(p)$ captures the relevance of prior p for the decision. α is a threshold level of confidence under which priors are not taken into account in the evaluation. This criterion generalizes the maxmin expected utility criterion of Gilboa and Schmeidler (1989) – the latter (with set of prior C) is obtained for $\varphi(p) = 1$ if $p \in C$ and 0 if not and any $\alpha \in [0, 1]$.

The following illustrates this model in the framework of our insurance example.

Example 6. An individual believes that there are three possible values for its loss probability: p', p'' and $\frac{p'+p''}{2}$ where $p' < p''$. He associates a maximal confidence to the average level, and a more limited, but equal confidence to the extremes. This is captured by the following confidence function where to simplify notations we denote $\varphi(p) := \varphi((p, 1 - p))$ and $\frac{p'+p''}{2} := p_0$.

$$\varphi(p_0) = 1, \varphi(p') = \varphi(p'') = c \text{ with } c \in]0, 1[$$
$$\text{and } \varphi(p) = 0 \quad \text{for any} \quad p \notin \{p', p'', p_0\}.$$

For $\alpha > c$, only the loss probability of p_0 is considered as plausible enough and thus is used in the decision evaluations. Then, we obtain:

$$V(f) = p_0 u(w - d) + (1 - p_0)u(w)$$
$$V(g) = u(w - \pi)$$
$$V(h) = p_0 u(w - d + I - \pi') + (1 - p_0)u(w - \pi')$$

which is equivalent to a subjective expected utility evaluation.

For $\alpha \leq c$, the three possible values of the loss probability are considered in the evaluation of the acts and the following evaluations are obtained:

$$V(f) = \text{Min} \left\{ \frac{1}{c}[p'u(w - d) + (1 - p')u(w)], \ p_0 u(w - d) + (1 - p_0)u(w), \right.$$

$$\left. \frac{1}{c}[p''u(w - d) + (1 - p'')u(w)] \right\}$$

$$= \text{Min} \left\{ p_0 u(w - d) + (1 - p_0)u(w), \frac{1}{c}[p''u(w - d) + (1 - p'')u(w)] \right\}$$

$$V(g) = u(w - \pi)$$

$$V(h) = \text{Min} \left\{ p_0 u(w - d + I - \pi') + (1 - p_0)u(w - \pi'), \right.$$

$$\left. \frac{1}{c}[p''u(w - d + I - \pi') + (1 - p'')u(w - \pi)] \right\}$$

Then, the probability used in the decision evaluation will depend on the confidence level c. If it is very low, then p_0 will be used, and if it is close enough to 1, then p'' will be used.

3.2.2 *Model Uncertainty*

We have left the question of how to determine the set of priors in the maxmin expected utility model open. Indeed, it comes from the representation of the decision maker's preferences and has, in general, no particular structure.

One convenient formulation, with a nice interpretation, is the so-called ε-contamination (see e.g. Eichberger and Kelsey 1999). In this formulation, the decision maker has a probabilistic benchmark in the sense that he 'believes' the probability distribution on the state space is a given distribution, say p. But he's not totally confident about this. The way to model this is to say that the set of priors he has in mind will be a combination between the 'probabilistic benchmark' on one hand and 'anything can happen' on the other hand. If one denotes $\Delta(S)$ the set of all distributions on the state space S, the set of priors corresponding to the story above is $(1 - \varepsilon)p + \varepsilon\Delta(S)$. ε is a measure of how confident the decision maker is in his probabilistic assessment of the situation. For $\varepsilon = 0$ he behaves as if he were sure of the distribution while for $\varepsilon = 1$, he behaves as if he had no idea of it. Another way to put it

is that the decision maker has a 'model' in mind, represented by p and is unsure about the adequacy of his model.

Robust control. The idea of model uncertainty has been followed by Hansen and Sargent (e.g. Hansen and Sargent 2001, 2008), building on work done by engineers on 'robust control'. In this robust preferences approach, agents have a model p in mind and rank acts f according to the criterion

$$V(f) = \min_{q \in \Delta(S)} \left[\int u(f)\, dq + \theta R(q||p) \right]$$

where $R(.||p): \Delta(S) \to [0, \infty]$ is the relative entropy (or Kullback-Leibler divergence) with respect to p, $R(q||p) = \int q(x) \log \frac{q(x)}{p(x)}\, dx$. This is a measure of the 'distance' between the two distributions p and q. Hence, in this criterion, the decision maker takes into account the possibility that p is not the right distribution and consider possible other distributions q. The relative likelihood of distribution q, rather than p, governing the occurrence of states of nature is given by the relative entropy of q with respect to p. The parameter θ on the other hand is a measure of how much the decision maker weights the possibility of p not being the correct distribution. The larger θ the more the decision maker attaches importance to p being the correct model (i.e. the cost of considering a model q different from p is given more weight in the functional). Hansen and Sargent have applied this criterion to the study of dynamic macroeconomic problems.

Variational preferences. As mentioned by Hansen and Sargent themselves, their model can be given an interpretation in terms of ambiguity aversion. Wang (2003) and especially Maccheroni *et al*. (2005) have developed the formal setting in which the precise link between the robust control approach and the maxmin expected utility model can be made. The maxmin expected utility criterion can be written in the following manner:

$$V(f) = \min_{p \in \Delta(S)} \left[\int u(f)\, dp + \delta_C(p) \right]$$

where $\delta_C(p) = 0$ if $p \in C$ and $\delta_C(p) = \infty$ if $p \notin C$.

The formal analogy with Hansen and Sargent's approach is thus clear: both are a particular case of a general model in which the decision maker's preferences are represented by

$$V(f) = \min_{p \in \Delta(S)} \left[\int u(f)\, dp + c(p) \right]$$

where $c(p)$ is a 'cost function' defined on $\Delta(S)$. If $c(p) = 0$ for all $p \in \Delta(S)$ then the criterion is simply maxmin expected utility with the set of priors equal to $\Delta(S)$, which is the same as Wald criterion (extreme pessimism). If on the other hand $c(q) = 0$ if $q = p$ and $c(q) = \infty$ otherwise, then the criterion is simply expected utility with respect to the distribution p. Finally, the ε-contamination case with which we started this section can be recovered by specifying $c(q) = \delta_{(1-\varepsilon)p+\varepsilon\Delta(S)}(q)$. An illustration of the application of this criterion is provided using our insurance framework.

Example 7. Assume that, according to his personal characteristics and to insurance company's evaluations, an individual believes that his loss probability is p_0, but he is not completely confident in this evaluation. Applying the variational preferences with cost function $c(q) = \delta_{(1-\varepsilon)p+\varepsilon\Delta(S)}(q)$ where p is such that $p(L) = p_0$ and $p(\overline{L}) = 1 - p_0$, gives the following evaluation for the different available decisions:

$$V(f) = (1 - \varepsilon)[p_0 u(w - d) + (1 - p_0)u(w)] + \varepsilon u(w - d)$$
$$= [(1 - \varepsilon)p_0 + \varepsilon]u(w - d) + (1 - \varepsilon)(1 - p_0)u(w)$$

$$V(g) = u(w - \pi)$$
$$V(h) = (1 - \varepsilon)[p_0 u(w - d + I - \pi') + (1 - p_0)u(w - \pi')]$$
$$+ \varepsilon u(w - d + I - \pi')$$
$$= [(1 - \varepsilon)p_0 + \varepsilon]u(w - d + I - \pi') + (1 - \varepsilon)(1 - p_0)u(w - \pi')$$

$(1 - \varepsilon)$ measures the confidence in the probability p_0. If the individual is not confident at all in this probability ($\varepsilon = 1$), he considers only the worst possible consequence of any decision.

3.2.3 *Second-Order Beliefs*

One can look at the Ellsberg three colour urn experiment in a different manner than the way we presented it. The decision maker might reason as follows: the urn might have different compositions (from all sixty balls being blue to all sixty balls being yellow). Once the composition is fixed, the decision maker faces a situation of risk, with known odds. The process can be represented as a two stage process: first the composition of the urn is drawn (among a set of hypothetical urns) then the subject draws in the urn.

If the decision maker has probabilistic beliefs on the set of all possible urn compositions and perform reduction of compound lotteries (i.e. is indifferent between a two stage lottery and its one stage equivalent), then we are back to a probabilistic setting: the decision maker acts as if he had a probability distribution over the colour of the ball drawn in mind. If for instance we assume a uniform distribution over the possible compositions of the urn, i.e. it is equally likely that there are n or m blue balls ($n, m \in \{0, \dots, 60\}$), then a decision maker performing reduction of compound lotteries would have probabilistic beliefs of 1/3 on blue and 1/3 on yellow (see figure 1).

Segal (1987) and Segal (1990) provides an explanation of the Ellsberg paradox based on this idea. More precisely, this author proposes an extension of the Rank Dependent Utility (RDU) to compound (two stage) lotteries by introducing a compound independence axiom and a time neutrality axiom. Preferences representations based on the idea that decision makers do not perform reduction of compound lotteries have been developed notably by Nau (2006) and Klibanoff *et al*. (2005). The latter has become a rather popular model, in particular because it avoids the technical problem (non-differentiability) introduced by the min operator in the maxmin expected utility model.

In this model, one can still think of the decision maker having in mind a set of prior C over the state space S. The decision maker also comes up with a prior over the set of priors, a so-called second-order belief, denoted μ. To understand the construction, think of each prior $p \in C$ as describing a possible scenario (in the Ellsberg case, a possible composition of the urn) and μ as the (probabilistic) belief of the decision maker over the different scenarii. In the simple case of a finite set of states and a finite set of scenarii, the decision criterion to be maximized takes the following form:

$$\sum_{\theta \in \Theta} \mu(\theta) \Phi \left(\sum_{s \in S} p(s; \theta) u(f(s)) \right)$$

where f is an act yielding payoff $f(s)$ in state s, u is the utility function and Φ is a function from $\mathbb{R}$ to $\mathbb{R}$. Φ will be key to assess the decision maker's attitude toward ambiguity. In a more compact way, using the expectation operator, we can write the formula as $E_\mu \Phi(E_p(u(f)))$.[14]

As the reformulation above makes it clear, the criterion can be read as a two-layer expected utility: first the decision maker evaluates the expected utility of the decision with respect to all the priors in C: each prior p is indexed by θ so in the end, we get a set of expected utilities of f, each being indexed by θ. Then, instead of taking the minimum of these expected utilities, as maxmin expected utility does, take an expectation of distorted expected utilities. The role of Φ is crucial here: if Φ were linear, the criterion would simply reduce to expected utility maximization with respect to the combination
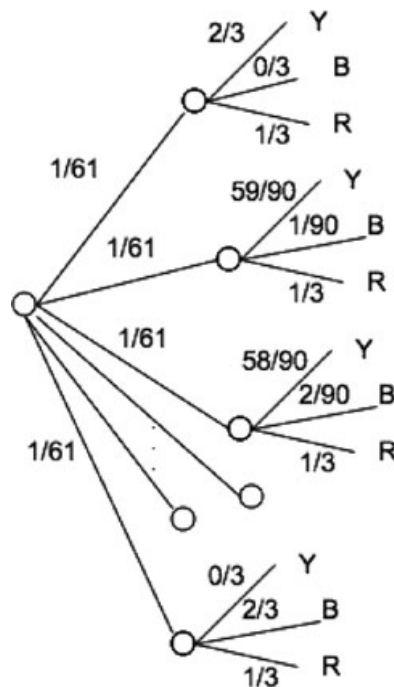
Figure 1. Second-Order Acts.

of μ and possible p's. When Φ is not linear, one cannot combine μ and p's to construct a reduced probability distribution. In this event, the decision maker takes the expected 'Φ-utility' (with respect to μ) of the expected 'u-utility' (with respect to p's).

A concave Φ will reflect ambiguity aversion in the sense that it places a larger weight on bad expected 'u-utility' realizations. In the limit, when Φ is such that it places all the weights on the worst expected 'u-utility', the criterion amounts to Gilboa and Schmeidler's maxmin expected utility. We will come back in Section 4 on the interpretation in terms of ambiguity attitudes of the various constructs presented here. At this stage, we just mention that, as in Gilboa and Schmeidler's analysis, all functions and probability distributions appearing in the formula are subjective constructs, deriving from behavioural axioms. Klibanoff *et al*. (2005)'s axioms are less direct than Gilboa and Schmeidler (1989)'s as they need to introduce 'second-order acts' (that is, in the Ellsberg example, acts whose outcomes depend not on the colour of the ball drawn, but only on the composition of the urn) and relate them with usual acts whose outcomes depend on the realization of a state $s \in S$. Seo (2009) manages to go around this issue and proposes an axiomatization in which the domain on which acts are defined is simply the intuitive state space (i.e. in Ellsberg example, the colour of the ball drawn). Ergin and Gul (2009) also provide a different approach to derive essentially the same decision criterion. They introduce *issue preference*, that is agents may not be indifferent among gambles that yield the same probability distribution if they depend on different issues, and provide a theory of second-order probabilistically sophisticated preferences. Epstein (2009) critically discusses some implications of this model. Based on three Ellsberg urn-type experiments, the author points some counter-intuitive choices appearing with the smooth ambiguity model due for him to the incapacity of this model to

correctly achieve a separation between ambiguity and ambiguity aversion. More precisely, his thought experiment 1 criticizes the assumption of expected utility evaluation of second-order acts. The two other experiments discuss further the behavioural content of this model. A reply to these objections is proposed in Klibanoff *et al.* (2009a).

Example 8. If the probability of loss p in our insurance example can take two values, p' and p'' and if the individual believes that it will be p' with probability q, then, the evaluations of the three available insurance decisions with the previous criterion will be:

$$V(f) = q\Phi[p'u(w-d) + (1-p')u(w)] + (1-q)\Phi[p''u(w-d) + (1-p'')u(w)]$$
$$V(g) = \Phi[u(w-\pi)]$$
$$V(h) = q\Phi[p'u(w-d+I-\pi') + (1-p')u(w-\pi')]$$
$$\qquad + (1-q)\Phi[p''u(w-d+I-\pi') + (1-p'')u(w-\pi')]$$

3.3 *Incorporating Some (Imprecise) Information*

In the purely subjective approaches, preferences are expressed over acts, associating outcomes to any possible state of nature. The available information influences preferences over the acts and thus participates to belief formation. It does so in a very general yet un-modelled way. Thus, the price to pay for this generality is the impossibility to establish an explicit relation between the information possessed on the one hand and the agent's beliefs on the other hand. Indeed, beliefs are the result of a combination between objective information provided for instance by survey data and personal subjective assessment of this information. In subjective models, only the final beliefs appear in the preference representation. Consequently, comparing situations with different available information or evaluating the impact of objective information on belief formation is a difficult task. It is then difficult to have a precise idea of what beliefs associated to a given information situation should be.

For instance, when looking at the Ellsberg urn choice problem, it is natural to consider for the set of probability distributions

$$\mathcal{C} = \left\{ P : P(R) = \frac{1}{3}, \ P(B) = \frac{2}{3} - \beta, \ P(Y) = \beta, \ \beta \in \left[0, \frac{2}{3}\right] \right\}$$

However, nothing excludes the possibility that $\mathcal{C} = \{P : P(R) = P(B) = P(Y) = \frac{1}{3}\}$ or $\mathcal{C} = \{P : P(R) = \frac{1}{3}, \ P(B) = \frac{1}{3} - \beta, \ P(Y) = \frac{1}{3} + \beta, \ \beta \in [0, \frac{1}{6}]\}$.

The previous problems can, at least partially, be solved by having preferences defined over objects explicitly taking into account all the available information.

Linear utility for belief functions. A first step in this direction is taken in Jaffray (1989). His model generalizes the standard expected utility under risk to a framework in which the probabilities of the different states of nature are imperfectly known. Instead, it is only possible to assert that they belong to given intervals. In such a context, the information is compatible with a set of probability distributions, that can be denoted by $\mathcal{P}$. To any set of probability distributions, it is possible to associate its lower envelope v[15], associating to each event its lower probability compatible with the set of distributions. This lower envelope has some mathematical structure associated to it. In particular, it is a capacity, (an increasing, non-necessarily additive set function.) Under some assumptions on this lower envelope, it is possible to associate to any act $f \in D$ not a probability distribution over the outcomes, as under risk, but a capacity v_f over the outcomes. This capacity simply gives the smallest probability of each outcome if this decision is chosen. Applying the usual von Neumann Morgenstern axioms plus a dominance axiom to the preferences over the set of capacities v_f yields a preference representation

generalizing expected utility to situations of objective imprecise information. Individuals preferences are then characterized by a utility function and a pessimism optimism index α.

The value associated to a decision is the following:

$$V(\nu_f) = \sum_{E \in \mathcal{A}} \varphi(E)[\alpha(m_E, M_E)u(m_E) + (1 - \alpha(m_E, M_E))u(M_E)]$$

where

- m_E is the minimal outcome on event E and M_E the maximal outcome on event E,
- φ is called the Möbius transform[16] of the capacity ν. For elementary events, $\varphi(E)$ is the lower probability of the event E, $\varphi(E) = \nu(E)$. For other events, $\varphi(E)$ measures the ambiguity of the event in the sense that it depends on the gap between the lower probability of this event and the sum of the lower probabilities of its sub-events.
- $\alpha(m_E, M_E)$ is a pessimism-optimism index.

If the pessimism-optimism index α is an intrinsic characteristic of the individual and does not depend on the consequences, that is $\alpha(m_E, M_E) = \alpha$ for any m_E and M_E, the previous criterion is equivalent to an α maxmin criterion with objective imprecise information,

$$V(\nu_f) = \alpha \min_{P \in \mathcal{P}} E_P u(f) + (1 - \alpha) \max_{P \in \mathcal{P}} E_P u(f)$$

When the set $\mathcal{P}$ reduces to a single distribution this criterion naturally reduces to expected utility.

Example 9. Let us go back to our insurance example. The individual's information allows him to assert that the probability of a loss is in between p' and p''. This information can be described by the set $\mathcal{P}$ of probability distributions with $\mathcal{P} = \{P : P(L) \in [p', \ p''], P(\overline{L}) \in [1 - p'', 1 - p']\}$. To this set can be associated the following lower envelope

Event	$\emptyset$	L	$\overline{L}$	$L \cup \overline{L}$
ν	0	p'	$1 - p''$	1
φ	0	p'	$1 - p''$	$p'' - p'$

In this simple case of two states of nature, the value of $\varphi(L \cup \overline{L})$ gives a measure of the uncertainty (or ambiguity) in the problem. It corresponds to the probability mass that can not be objectively assigned to one or the other of the elementary events. With the previous criterion, and assuming that α does not depend on wealth levels, the insurance decisions f, g and h are evaluated as follows:

$$V(f) = (\alpha p'' + (1 - \alpha)p')u(w - d) + (1 - \alpha p'' - (1 - \alpha)p')u(w)$$
$$V(g) = u(w - \pi)$$
$$V(h) = (\alpha p'' + (1 - \alpha)p')u(w - d + I - \pi') + (1 - \alpha p'' - (1 - \alpha)p')u(w - \pi')$$

α corresponds here to the weight affected by the individual to the highest probability of the worst outcome. A pessimistic individual ($\alpha = 1$) will act as if the true loss probability was p'', and an optimistic individual ($\alpha = 0$), as if the true probability was p'. Note that, for $\alpha = 1$ we get the same evaluation as with the *Maxmin* Expected Utility criterion. However, the informational assumptions are completely different: the set of possible values for p is completely objective.

The main originality of this criterion is to take explicitly into account the available but partial information, by defining preferences on capacities rather than on distribution functions. It is built on the intuition that individuals use directly (without any subjective transformations) the available (precise

or imprecise) information on the probabilities. However, this model is unable to take into account a subjective treatment of objective imprecise information.

The 'contraction' model. To combine the advantages of Jaffray's model with those of the subjective multiple prior of Gilboa and Schmeidler, Gajdos *et al.* (2008) propose a model (call it for simplicity the 'contraction model') in which it is possible to compare acts under different objective information structures.[17] More precisely, agents are assumed to be able to compare pairs $(\mathcal{P}, f)$ where $\mathcal{P}$ is a set of probability distributions and f is an act in the sense of Savage.

In the spirit of Gilboa & Schmeidler multiple prior model, the authors give axiomatic foundations for the following preference representation:

$$V(f) = \min_{P \in \Phi(\mathcal{P})} E_P u(f)$$

Φ is a mapping that transforms the objective information, given by $\mathcal{P}$, into a subjective set of priors. Under some additional axioms, Φ can be given the specific form of a weighted sum of a kind of 'mean value' of the probability set $\mathcal{P}$ (characterized by the so-called 'Steiner point') and the set itself:

$$\Phi(\mathcal{P}) = \alpha \mathcal{P} + (1 - \alpha) s(\mathcal{P})$$

where α measures imprecision aversion. The corresponding preference representation takes the form:

$$V(f) = \alpha \min_{P \in \mathcal{P}} E_P u(f) + (1 - \alpha) E_{s(\mathcal{P})} u(f)$$

Here, individuals combine a 'Bayesian' criterion (the information is summarized by an objectively calculated single distribution independent of the decisions) with a pessimistic criterion (for any decision, individuals take into account the distribution giving the lowest expected utility).

Example 10. Consider the insurance problem with the informational assumption as described in example 9. $s(\mathcal{P})(L) = \frac{p' + p''}{2}$, $s(\mathcal{P})(\overline{L}) = 1 - \frac{p' + p''}{2}$ and the decisions are evaluated as follows:

$$V(f) = \left(\alpha p'' + (1 - \alpha) \frac{p' + p''}{2} \right) u(w - d) + \left(1 - \alpha p'' - (1 - \alpha) \frac{p' + p''}{2} \right) u(w)$$

$$V(g) = u(w - \pi)$$

$$V(h) = \left(\alpha p'' + (1 - \alpha) \frac{p' + p''}{2} \right) u(w - d + I - \pi')$$

$$+ \left(1 - \alpha p'' - (1 - \alpha) \frac{p' + p''}{2} \right) u(w - \pi')$$

For $\alpha = 1$ individuals choose according to p'' and for $\alpha = 0$, according to the average probability $\frac{p' + p''}{2}$. Note that, in contrast with the previous model, they never act according to p'.

In the previous models, uncertainty concerns the probabilities of the different events. There are however situations where it concerns not only probabilities, but also consequences (i.e. given a state, consequences are not fully determined but are only known to belong to some set.) At least two approaches have been proposed to model decisions in such 'unforeseen contingencies' situations due to Ghirardato (2001) and Jaffray and Jeleva (2009).

3.4 *Ambiguity, Probabilistic Sophistication and Source Dependence*

We finally mention another line of research, that takes a somewhat different approach to explaining ambiguity related phenomena. As we mentioned in several occasions, Ellsberg examples constitute

actually more than a violation of expected utility. It violates 'probabilistic sophistication' a term coined by Machina and Schmeidler (1992) to describe the behaviour of decision makers that behave in a way consistent with a probability distribution, but not necessarily by treating it linearly. A decision maker that would 'distort' a probability distribution, like in Prospect Theory or Rank Dependent Utility theories would qualify as being probabilistically sophisticated although he is not an expected utility decision maker.

Recent theoretical research has given foundations for an idea that was already present in the experimental and psychological literature, namely, the idea of 'source dependence'. The aim here is to identify domains or, to use Savage's terminology, 'small worlds' in which the decision maker has well-formed probabilistic beliefs. Thus, within each small world, the decision maker does not perceive ambiguity. He will however possibly reveal source-dependent risk attitudes: across domains, he might not hold probabilistic beliefs. For a general treatment, we refer the reader to Chew and Sagi (2006) and Chew and Sagi (2008). There, the idea that ambiguity aversion might be interpreted in terms of comparative risk aversion across small worlds is given an axiomatic treatment based on de Finetti's idea of exchangeability.

This can be easily explained in the Ellsberg two urn examples. Imagine that behaviour reveals that the decision maker is indifferent between betting on a black ball being drawn from the unknown urn and betting on white being drawn from the same urn. We could be entitled to say that the decision maker acts as if he had equiprobable beliefs on these two events. But then, how can we take into account the fact that the decision maker strictly prefers betting (on any colour) in the known urn rather than in the unknown urn?

The source dependent story would go as follows: the decision maker probabilistic assessment of the unknown urn (to simplify, putting a probability 0.5 on each colour) is valid only in this 'small world'. Note they are the same as the beliefs in the other domain made of the known urn. However, the way the decision maker acts on these beliefs is different across the two domains. Hence, identifying the two urns as two different sources of uncertainty and recognizing that risk aversion might be different across domains would explain the behaviour usually observed in these experiments. As we will discuss in Section 6, one can for instance assume that the decision maker distorts probability in a different manner for the two urns. Interestingly, Ergin and Gul (2009) link the 'source dependent' (or issue dependence as they name it) approach with the second-order belief approach to ambiguity aversion.

4. Ambiguity Attitudes

At this stage, we are equipped with various decision criteria that are based on the fact that the underlying preferences are 'ambiguity sensitive'. However, we have not attempted so far to define properly what ambiguity really is. Nor have we discussed ways of representing differences in ambiguity attitudes. We tackle these issues in this section.

4.1 Statement of the Problem

What is the source of the difficulty faced when attempting to define notions and measures of ambiguity attitudes? In a nutshell, it is difficult to come up with a definition of what it means for an act to be ambiguous or unambiguous, or what it means for an act to be more ambiguous than another act. As a matter of fact, the very definition of an ambiguous event and thus more generally of 'ambiguity' is not that straightforward.

In contrast, take the situation of risk, where objects of choice are lotteries. Then, it is possible to use stochastic orders to rank lotteries according to their riskiness. The usual notion of 'more risky than' is

the one developed in Rothschild and Stiglitz (1970), of a mean preserving spread. One can then say that a decision maker is risk averse if he always prefers a lottery ℓ to a mean preserving spread of ℓ.[18] One can also measure the strength of one's risk aversion: it is measured by the risk premium (the premium the decision maker is willing to pay to replace the lottery by its expectations).

When we move to the Savagian framework in which objects of choices are not lotteries but acts, these tools are no longer available. A way around this issue has been proposed by Yaari (1969): 'It seems quite natural to say that Mr. A is more risk averse than Mr. B (...) if every gamble which is acceptable to A is also acceptable to B.' This idea has been recently exploited by Epstein (1999) and Ghirardato and Marinacci (2001). Before describing this approach, we first present approaches of uncertainty aversion that do not directly tackle the problem of defining ambiguity *per se*.

4.2 *Ambiguity Aversion without Ambiguity*

The literature we refer to in this section is one in which the foundational issue of what it means for an act or an event to be ambiguous is not central.

The first definition has been proposed by Schmeidler. An individual is ambiguity averse if he has a preference for diversification. A preference relation $\succsim$ exhibits ambiguity aversion if for any act f and g:

$$f \succsim g \Rightarrow \alpha f + (1 - \alpha)g \succsim g$$

Note that this definition requires that mixing acts is possible (which is the case for instance if consequences are lotteries, or real numbers). It implies that it is possible for an ambiguity averse individual to observe both $f \sim g$ and $\alpha f + (1 - \alpha)g \succ g$. The underlying idea is that a convex combination of acts either preserves ambiguity, or decreases it. We already illustrated this axiom in Section 3.2.1.

Chateauneuf and Tallon (2002) also base a definition of ambiguity aversion on the idea of diversification.[19]

An individual is said to be (weakly) ambiguity averse (or has a preference for sure diversification) if, for any collection of acts f_i, $i = 1, \ldots, n$, such that $\exists \alpha_i \in \mathbb{R}_+, i = 1, \ldots, n$, $\sum_{i=1}^{n} \alpha_i = 1$ and $\sum_{i=1}^{n} \alpha_i f_i = x$, (abusing notation x denote here the constant act giving x in all states).

$$f_1 \sim f_2 \sim \cdots \sim f_n \Rightarrow x \succ f_i, \quad i = 1, \ldots, n$$

Thus, sure diversification means that if the decision maker can attain certainty by a convex combination of equally desirable random variables, then he prefers certainty to any of these random variables

These definitions do not allow to identify ambiguous acts and have thus been criticized. For instance, it has been observed (see Epstein (1999)) that for individuals with Choquet expected utility preferences, this type of ambiguity aversion is neither necessary, nor sufficient, to explain the choices observed in the Ellsberg three-colours urn experiment.

4.3 *Comparative Ambiguity Aversion*

Epstein (1999) and Ghirardato and Marinacci (2002) propose other definitions inspired from the one proposed by Yaari (1969) to define risk aversion.

Comparative ambiguity aversion is defined in Epstein using unambiguous acts. An unambiguous act is an act that is measurable with respect to an exogenously defined set of unambiguous events. If whenever Mr. *A* prefers an unambiguous act to some other act, Mr. *B* exhibits the same preference,

then B is deemed more ambiguity averse than A. Ghirardato and Marinacci differ in the set of acts used as benchmarks. They will say that B is more ambiguity averse than A is whenever A prefers a constant act to some other act (since they use an Anscombe–Aumann setting, constant acts are lotteries), then so does B.

The other difference lies in the notion of ambiguity neutrality retained. In Epstein, a decision maker that is probabilistically sophisticated (i.e. assigns subjective probabilities to events but does not necessarily treat them in a linear fashion) is said to be ambiguity neutral. In Ghirardato and Marinacci ambiguity neutral preferences are subjective expected utility preferences. Thus, a rank dependent expected utility maximizer will be said to be ambiguity neutral according to Epstein, but ambiguity averse according to Ghirardato and Marinacci.

Note also that in the comparative notions proposed, the ambiguity attitudes of two individuals can be compared *only if some assumptions are made on their preferences under risk*. In Ghirardato and Marinacci (2002), individuals can be compared if they have the same preferences over constant acts, that is, if they have the same attitude towards risk. In Epstein (1999), individuals' preferences have to coincide on the set of unambiguous acts. Thus, these definitions do not allow to separate completely attitudes towards risk from attitudes towards uncertainty in the sense that one cannot say that A is more ambiguity averse than B if A and B do not have the same preferences under risk.

Epstein and Zhang (2001), Nehring (2001) and Ghirardato *et al.* (2004) tackle the issue of defining an unambiguous event. Amarante and Filiz (2007) characterize these notions in the maxmin expected utility model. Grant and Quiggin (2005) goes one step further by providing a definition of 'increasing uncertainty'.

4.4 *Imprecision Attitude*

Gajdos *et al.* (2008) give a definition of comparative imprecision aversion, in a setting in which objective information is available, in the form of sets of probability distributions. B is more imprecision averse than A if whenever A prefers a bet defined on a precise event (one for which the probabilities in the information set all coincide) to an act so does B. Using these bets as reference acts has the feature of providing a separation between attitudes towards risk and uncertainty (see Gajdos *et al.* 2008).

In this setting, given that objective information exists, one could think that a natural notion of increasing imprecision comes with this extra structure. This is not the case, at least not so trivially. Imagine that we used a definition saying that imprecision aversion means that whenever the information available in situation J corresponds to a subset of the information available in situation K then the decision maker prefers any act in J to the same act in K. Then, we would be led to assume that an imprecision averse decision maker prefers to bet on black when he knows there is only one black ball (out of say 1000 balls) to bet on black when he has no information whatsoever on the composition of this 1000 balls urn. This is clearly a very extreme form of imprecision aversion and makes this definition rather unappealing.

4.5 *Characterization in Different Models*

Due to the different settings in which they are defined, it is not possible to give the characterizations of all types of ambiguity attitudes in all the decision models under uncertainty that were presented before.

The first point to note is that subjective expected utility preferences are unable to take into account ambiguity aversion, whichever definition is used. There, the utility function characterizes only attitudes towards risk and wealth. Thus, individuals with subjective expected utility preferences are ambiguity neutral according to all definitions.

4.5.1 *Ambiguity Aversion in the Choquet Expected Utility Model*

In the Choquet expected utility model, ambiguity aversion is mainly captured by the properties of the capacity characterizing individuals' beliefs. Two types of capacities seem relevant for ambiguity aversion:

- convex capacities;
- capacities with non-empty cores (convex capacities are a subset of these capacities).

An individual with Choquet expected utility preferences is Ambiguity Averse in the sense of Schmeidler if his capacity ν is convex and if his utility function is concave or linear. The characterization of Epstein's notion is similar under the assumption of the existence of a sufficiently rich set of unambiguous events (events to which a unique probability can be associated).

An individual with Choquet expected utility preferences is weakly ambiguity averse in the sense of Chateauneuf and Tallon (2002) if his capacity has non-empty core and if his utility function is concave or linear. Ambiguity aversion in the sense of Ghirardato and Marinacci is also characterized by a capacity with non-empty core.

4.5.2 *Ambiguity Aversion in Maxmin Models*

The Wald maxmin model is a special case of the Choquet expected utility model. It is easy to see that it does correspond to ambiguity aversion. Indeed, for a capacity ν such that $\nu(A) = 0$, $\forall A \in \mathcal{A} \backslash \{\Omega\}$, it follows directly from the expression of Choquet expected utility that the evaluation of an act equals the utility of the worst possible outcome, as in the Wald criterion. This capacity being convex, this criterion corresponds to ambiguity aversion in the sense of Schmeidler.

The Maxmin expected utility model of Gilboa and Schmeidler (1989), as mentioned in Section 3.2.1., has ambiguity aversion built in, since it is one of the axioms characterizing the decision criterion. Note that in this model, it is not possible to measure the intensity of ambiguity aversion, due to the lack of an 'objective' reference for the set of prior.

In the α max min model, ambiguity aversion is captured by the value of the parameter α. This model allows, in some sense, for a separation between the ambiguity revealed (perceived) by individuals (given by the size of the set of priors $\mathcal{C}$) and the ambiguity attitude. More precisely, an individual 1 reveals more ambiguity than 2 if the utility function u_1 is a positive affine transformation of u_2 and $\mathcal{C}_1 \supseteq \mathcal{C}_2$. For two individuals revealing the same ambiguity, 1 is more ambiguity averse than 2 if $\alpha_1 > \alpha_2$. One limit of this model is that it is impossible to compare the ambiguity attitudes of individuals who do not reveal the same ambiguity and, in particular, whose utility functions are not identical (up to a positive affine transformation).

4.5.3 *Ambiguity Aversion and Variational Preferences*

By construction, variational preferences are ambiguity averse, because they verify the same ambiguity aversion axiom as Gilboa and Schmeidler (1989) maxmin expected utility preferences. The intensity of ambiguity aversion is measured by the function $c(p)$. More precisely, applying Ghirardato and Marinacci (2002) definition of comparative ambiguity aversion, for two individuals with the same utility function u, individual 1 is more ambiguity averse than individual 2 if $c_1 \leq c_2$.

4.5.4 *Ambiguity and Risk Aversion in the Smooth Ambiguity Model*

In Klibanoff *et al*. (2005), individual's tastes are characterized by two utility functions, a standard utility function u transforming outcomes and a function Φ, transforming expected utility for any

possible scenario. Risk aversion is characterized by the shape of the utility function u and ambiguity aversion, in the sense of Ghirardato and Marinacci (2002), by the shape of Φ. More precisely, an individual with concave functions u and Φ is both risk and ambiguity averse. It is possible to compare ambiguity attitudes only for individuals who have the same attitude towards risk (that is the same utility function u). For two individuals with the same utility function u, individual 1 is more ambiguity averse than individual 2 if and only if, for any x, $-\frac{\Phi_1''(x)}{\Phi_1'(x)} \geq -\frac{\Phi_2''(x)}{\Phi_2'(x)}$.

4.5.5 *Imprecision and Attitudes Towards Imprecise Information in the Contraction Model*

In Gajdos *et al.* (2008), attitude towards imprecision is characterized by the function $\Phi(\mathcal{P})$ which determines the selected probability-possibility set of an individual facing objective imprecision given by the set $\mathcal{P}$ of probability distributions.

An individual 1 is more averse to imprecision than 2 if, for all $P \in \mathcal{P}$, $\Phi_1(\mathcal{P}) \subset \Phi_2(\mathcal{P})$. In the special case of $\Phi(\mathcal{P}) = \alpha \mathcal{P} + (1 - \alpha)s(\mathcal{P})$ this corresponds to $\alpha_1 > \alpha_2$. Note that no restriction is necessary on the utility functions of the two individuals. Thus, it is not necessary to ensure that two individuals have the same risk preferences to compare their attitude towards imprecision.

5. Ambiguity Aversion and Dynamics

Since ambiguity leads to violations of the Sure-Thing Principle, defining updating and ensuring dynamic consistency for maxmin expected utility or other similar models seems a rather complicated task, and indeed, has been subject to different attempts and debates.

5.1 *A Dynamic Version of the Ellsberg Paradox*

To understand the potential problems created by ambiguity sensitive preferences in dynamic setting, it is instructive to start with a simple dynamic version of Ellsberg paradox (see figure 2). Consider the three colours experiment, an urn containing 30 red balls and 60 other balls that are either blue or yellow. The first step consists in obtaining the individual's preference over the two gambles: 'win 100 if a red ball is drawn' versus 'win 100 if it is a blue ball'.

The second step of the analysis consists in presenting the individual with the dynamic choice problems illustrated below.

Consider the left-hand tree: the choice of the *upper* branch at the choice node corresponds to the gamble 'win 100 if a red ball is drawn'. The choice of the *lower* branch corresponds to the gamble 'win 100 if a blue ball is drawn'. Thus an individual with the typical static Ellsberg preferences, would plan to choose up at the choice node in the left-hand tree.
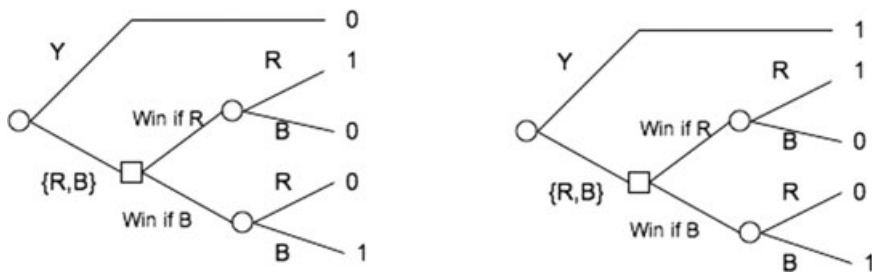

Figure 2. Dynamic Version of Ellsberg Three-Colour Urn.

Consider now the right-hand tree, where the choice of the *upper* branch at the choice node corresponds to the gamble 'win 100 if a red or a yellow ball is drawn'. The choice of the *lower* branch corresponds to the gamble 'win 100 if a blue or a yellow ball is drawn'. Thus, an individual with the typical Ellsberg preferences would plan to choose down at the choice node in the right-hand tree.

At this point, goes the argument, an individual with Ellsberg-type preferences will be in trouble. Suppose his preference in the first step had been for the gamble 'win 100 if a red ball is drawn'. If nature were to choose down at the initial chance node in the right-hand tree, the individual would reverse his original plan of choosing down in favour of choosing up. In short, his behaviour will be dynamically inconsistent, as he will fail to follow the path he decided ex ante.

Similarly to what happens under framework, the inconsistency can be explained by the non-relevance of consequentialism. Following Hammond (1988), consequentialism means that the individual would act, at any point in time, as if he had started out with the rest of the tree (choice node in our illustration): all what has not been realized can be safely ignored as it does not affect the choice of a consequentialist agent. Consequentialism is characterized by the following two properties: (i) the preference relation conditional on the realization of an event E depends only on E, and (ii) when evaluating an act f conditional upon the realization of an event E, outcomes in states outside of E do not matter.

In the context of risk, Machina (1989) and McClennen (1990) propose 'to avoid consequentialism'. Applying this argument to situations of uncertainty, it would be odd to impose that dynamic preferences satisfy some form of dynamic separability (i.e. consequentialism) while ambiguity aversion rests precisely on the fact that preferences need not be separable across events (as exemplified by Ellsberg example).

Recently, some authors have proposed different representations of preferences under uncertainty which do not allow some dynamic inconsistency. We can distinguish two categories of models. First, there are models on updating rules for ambiguity sensitive preferences. In these models, behaviour is naively inconsistent, in the sense that an ex ante choice of what to do contingent on the event $\{B, R\}$ may differ from what will actually be chosen when and if that event occurs. Second, some papers propose characterizations of recursive versions of the maxmin expected utility model and of the second-order belief model.

5.2 *Updating Multiple Priors*

As we saw previously, maxmin expected utility is a leading model of ambiguity aversion. Many papers have examined updating rules for these preferences. Gilboa and Schmeidler (1993) axiomatize a large class of rules for updating sets of priors (see also Jaffray 1994; Siniscalchi 2009). This class includes both maximum likelihood and full Bayesian updating. The logic behind these two rules are simple to express. The maximum likelihood updating rule (called also Dempster–Shaffer rule) singles out the distributions according to which the realized (conditioning) event is the most likely and drops all the other distributions from the set of priors. The updated set of priors thus consists of the former distributions, updated according to Bayes rule. The full Bayesian updating rule simply consists in taking as the updated set of priors the set of all priors, updated according to Bayes rule. These updating rules may lead to 'dynamic inconsistencies' in the presence of ambiguity (note that consequentialism is automatically satisfied with these two updating rules). Indeed, as shown in Ghirardato (2002), violations of the sure thing principle imply dynamic inconsistencies, when consequentialism is satisfied.

Machina (1989) argues that for preferences that are 'non-separable', one should abandon consequentialism and let conditional preferences depend on the optimal plan. Epstein and LeBreton (1993) precisely do that. However, their result is somewhat negative since they show that if conditional preferences – that could now depend on the optimal plan – are based on beliefs, then dynamic consistency implies probabilistic sophistication (and hence, ambiguity neutrality according to Epstein's definition).

This has led Hanany and Klibanoff (2009) to propose a different class of updating rules. Following Machina (1989) advice (for models under risk), their analysis drops consequentialism (actually, drops only the property (i) mentioned above) but in a more 'extreme way' than Epstein and LeBreton (1993). They then achieve dynamic consistency, by having conditional preferences at each node depend on the event, on the (*ex ante*) optimal decision taken as well as the choice set for which this decision is optimal. Then dynamic consistency is ensured for any multiple prior model.

5.3 *Backward Induction*

Skiadas (1998) first axiomatized a subjective version of recursive utility by considering a multi-period consumption-information profiles. Epstein and Schneider (2003b) consider a model of recursive maxmin expected utility preferences and explicitly impose dynamic consistency with respect to a pre-specified filtration. The essential axioms are that conditional orderings (i) satisfy the Gilboa–Schmeidler axioms, (ii) are connected by dynamic consistency, and (iii) do not depend on unrealized parts of the decision tree and depend only on consumption in states of the world that can still occur. They thus retain consequentialism and restrict dynamic consistency to a fixed, pre-specified collection of events. They show that this structure implies that the set of priors have a property they label 'rectangularity'.

Hayashi (2005) axiomatizes a form of recursive utility on intertemporal consumption that permits a distinction between three concepts: intertemporal substitution, risk aversion and ambiguity aversion. This model has two special cases: (i) the recursive model of risk preference by Kreps and Porteus (1978), Epstein and Zin (1989); (ii) the intertemporal model of multiple-priors utility by Epstein and Schneider (2003b). Recently, Klibanoff *et al.* (2009b) proposed a model of recursive preferences over intertemporal plans, extending the smooth ambiguity model developed in Klibanoff *et al.* (2005) to a setting involving dynamic decision making. Finally, we should mention the work of Siniscalchi (2009) which does not fall under the heading of recursive models. He analyses sophisticated dynamic choice for ambiguity-sensitive decision makers. Sophistication can be formulated as an assumption on preferences over decision trees that does not restrict the decision maker's preferences over acts.

5.4 *Learning*

Somewhat related to the issue raised in this section, one could wonder whether ambiguity should not disappear in the long run. If ambiguity takes the form of uncertainty about the data generating process, isn't it the case that the mere repetition of draws from that data generating process should be enough to fully learn it?

Dow and Werlang (1994a) addresses the issue of the law of large number under ambiguity and show in what sense learning can or cannot occur. Marinacci (1999) provides a thorough analysis of this. In a series of recent contributions Epstein and co-authors study further this issue and show that the Bayesian model of learning is but one extreme of a more general class of learning models in complex environment (see in particular, Epstein and Schneider 2003a, 2007; Epstein and Seo 2009a,b).

6. Experimental Evidence

In this section, we focus on the empirical (mostly experimental) literature on individuals' behaviour in presence of uncertainty. Many factors can explain uncertainty attitudes (personality traits, psychological environment, decision context,...). We pay more attention to the measure of ambiguity aversion than to its determinants. We also limit attention to 'traditional' experimental work and do not attempt to discuss the recent advances in neuroeconomics directly dealing with ambiguity (see e.g. Hsu *et al.*, 2005).

6.1 *Attitude Toward Ambiguity*

Since Ellsberg's paper, a host of papers have studied attitudes regarding ambiguity. The evidence on ambiguity aversion has been mixed. Raiffa (1961) argues that a rational decision maker should not be ambiguity averse. Heath and Tversky (1991) establish that when they feel competent in the domain under consideration, subjects tend to be ambiguity seeking, not averse. Fox and Tversky (1995) find that ambiguity aversion disappears when individuals evaluate a single gamble: the subjects asked about their willingness to pay for a bet in the unknown urn did not provide an answer different from the subjects who were asked about their willingness to pay for a bet in the known urn. Only when the subjects were asked to evaluate both bets did they make significantly different assessments of the bets. See also Chow and Sarin (2002). Viscusi and Chesson (1999) show that ambiguity aversion depends on some benchmark level of probability. For high probability of gains, subjects are averse to ambiguity while for low probability of gains, they can prefer higher levels of ambiguity. It is the converse in case the loss domain.

Yates and Zukowski (1976), Curley and Yates (1985) on the other hand report significant ambiguity aversion, through the elicitation of willingness to bet in various (informational) circumstances. Dealing with complete ignorance (no information given on the composition of the experimental urn) Cohen *et al*. (1985) find a variety of behaviour (including ambiguity seeking). They also find an absence of correlation between risk and ambiguity attitudes. Camerer and Weber (1992) reviewed the many studies that have examined how individuals react to ambiguity. These different studies among others show that ambiguity aversion is more pertinent in gains than in losses. Cohen *et al*. (1987) showed that for a majority of subjects, attitude in the gain domain is not correlated to the one in loss domain. More recently, Di Mauro and Maffioletti (2004) confirmed the existence of a 'reflection effect' in both risky and ambiguous tasks. Du and Budescu (2005) found strong ambiguity aversion in the gain domain but close to ambiguity neutrality in the loss domain. Lauriola and Levin (2001) report the results of an experiment in which they ask subjects to choose between a completely unknown urn and a known urn, with the odds of winning changing from one question to another. They find a positive correlation between risk and imprecision attitudes in the loss domain but none in the gain domain. Chakravarty and Roy (2009) find a positive correlation between risk and ambiguity aversion in the domain of gains but not in the loss one. Cohen *et al*. (2009) found significant ambiguity aversion in both a student and a 'general' population. They find no correlation among measures of risk and ambiguity aversion. Borghans *et al*. (2009) find that men and women act similarly in face of low ambiguity, but that women are more ambiguity averse than men in situations of high ambiguity. They also find that psychological variables explain none of the differences in ambiguity aversion among subjects.

Other studies have looked at non-student population. Guiso and Jappelli (2008) conducted a survey on some Italian bank's clients. They find a positive correlation between answers to questions about risk and imprecision attitude. They relate this to modes of decision making (intuitive versus reasoned). Cabantous (2007) surveyed insurance professionals and found that imprecision aversion was pervasive in this population. She also finds that sources of ambiguity (conflict of expert opinion or imprecision) matter. Burks *et al*. (2008) use data collected among truck drivers and show that there is a positive and strong correlation between risk and ambiguity aversion. They show that a common factor, cognitive ability, explain many features of these subjects. Potamites and Zhang (2007) present a field experiment on ambiguity aversion among investors in China. Their data shows a substantial heterogeneity in ambiguity aversion among this population, ranging from high ambiguity aversion to ambiguity seeking. Akay *et al*. (2009) ran experiment on two populations: the usual western students population and Ethiopian peasants. They find similar ambiguity aversion in the two populations (while Ethiopian peasants are much more risk averse). Poor health increases both ambiguity and risk aversion. Keller *et al*. (2007) examine willingness to pay for gambles involving risk and ambiguity made by individuals

and dyads (marriage partners, business partners) who exhibit more complex attitudes toward risk and ambiguity. They find that dyads display risk aversion and ambiguity aversion.

Maffioletti and Schmidt (2008) find that the elicitation method matters and observe quite a few 'preference reversals' depending on which method is used. Trautmann *et al*. (2009) also find preference reversal for ambiguity averse preferences.

Little has been done in dynamics. Liu and Colman (2009) study ambiguity aversion in a repeated game approach. It appears that participants chose ambiguous options significantly more frequently in repeated-choice than in single-choice. This suggests that repetition diminishes the effect of ambiguity aversion. Dominiak *et al*. (2009) run a dynamic Ellsberg urn experiment and find that subjects act more often in line with consequentialism than with dynamic consistency.

6.2 *Model Comparison*

A by now fairly robust result in the domain of experimental economics is that behaviour is very heterogeneous among individuals. Not only are ambiguity attitudes diverse; the various implications of the different models presented in Section 3 can be used to estimate which model best fits which individual behaviour.

Halevy (2007) proposed an experiment that permits to distinguish between four models of ambiguity aversion (subjective expected utility, maxmin expected utility, recursive non-expected utility, and recursive expected utility). Subjects are asked their reservation values for four different urns, representing different types of uncertainty. As he puts it, 'the findings indicate that currently there is no unique theoretical model that universally captures ambiguity preferences'. However, 'the results suggest that failure to reduce compound (objective) lotteries is the underlying factor of the Ellsberg paradox, and call upon decision theory to uncover the theoretical relationship between ambiguity aversion and different forms in which reduction may fail.' Ahn *et al*. (2009) estimate the parameters of four models of choice under ambiguity (subjective expected utility, maxmin expected utility, recursive expected utility (of the smooth ambiguity type) and α-maxmin expected utility). They use some portfolio-choice problem to analyse behaviour. They find that although individual preferences are heterogeneous, about two-thirds of the subjects have a positive degree of ambiguity aversion. They also find that a significant fraction of subjects behave according to the α-maxmin expected utility model, and another significant fraction of subjects' behaviour is best explained by the second-order prior model of Klibanoff *et al*. (2005). Hayashi and Wada (2010) find that subjects are sensitive to the dimension and the shape of the sets of probability distributions given to them and not only to the worst- and best-case, thus violating the α-maxmin models. They also show that the presence of non-extreme points matter, in contradiction to what the contraction model assumes. They find mixed support for the second-order prior model. Andersen *et al*. (2009) estimate a second-order prior model in which they jointly elicit risk and ambiguity attitude as well as subjective beliefs. They show that ambiguity aversion is quantitatively significant. They also show that attitude towards risk and uncertainty can be different, quantitatively and qualitatively. Hey *et al*. (2008) find results that are rather discouraging for the recent theories and favour simple rules of thumb. They use a new device to generate ambiguity (a Bingo blower) that does not induce second-order probability thinking nor fear of manipulation.

Using variants of the Ellsberg urn problems to estimate parameter values or functional forms for individual subjects, Abdellaoui *et al*. (2009) capture attitudes towards uncertainty and ambiguity by using a choice-based probabilities approach. They find heterogeneity in subjects' preferences both in an Ellsberg urn experiment and in experiments using naturally occurring uncertainties. They also implement the 'source dependent' approach and manage to estimate the source dependent weighting function that explain apparently ambiguous averse choices.

6.3 *Ambiguity in Economic Situations*

We finally mention some work that has been done beyond simple experimental choices that often consist in artificial draws from bizarre urns. Indeed, eventually, all the models discussed so far aim at giving some new insights into 'real' economic issues. The papers we mention here give interesting results in some simple, controlled, examples.

Camerer and Kunreuther (1989) consider an insurance market and use bidding between insurees and insurers. Aversion toward ambiguity plays an important role for explaining the observed behaviour. Ambiguity has an impact on prices. This result is in line with Hogarth and Kunreuther (1985) who observe that insurance premia are higher in presence of ambiguity. They also find that ambiguity aversion is a decreasing function of the probability of loss; see also Kunreuther *et al*. (1993). Bossaerts *et al*. (2010) study the impact of ambiguity and ambiguity aversion on portfolio holdings and asset prices in a financial market experiment. They obtain that ambiguity aversion can be observed in competitive markets and that ambiguity aversion matters for portfolio choices and for prices. They also observe that there is substantial heterogeneity in ambiguity preferences and that there is a positive correlation between risk aversion and ambiguity aversion. Kozhan and Salmon (2009) explain behaviour in foreign exchange markets by using the multiple prior approach. Their empirical model based on daily GBP/USD data over a 10 year period, test the investors' ambiguity-aversion. They find significant evidence of uncertainty aversion.

There exist a few empirical analyses on ambiguity aversion when ambiguity concerns health risks. For example, Viscusi *et al*. (1991) or Viscusi and Magat (1992) consider a potential risk of illness (such as cancer) from environmental exposure. The lack of information on the impact of environment on health risk involves ambiguity. In particular, they find evidence of individuals' ambiguity aversion. Viscusi *et al*. (1999) highlight that, in presence of ambiguity on the impact of smoking on health status, smokers and non-smokers present different levels of ambiguity aversion.

7. Imports into Economics

There is by now quite a sizable literature that applies the decision criteria described in this survey to various economic issues. For instance, Treich (2010) uses the smooth ambiguity model to show that ambiguity aversion increases the value of a statistical life. Based on the same model Snow (2010) shows that the value of information increases with greater ambiguity and with greater ambiguity aversion.

Below, we mention some further work in the domain of finance and general equilibrium on the one hand, and auctions and strategic interactions on the other hand.

7.1 *Finance and General Equilibrium*

Portfolio choice has been a first instance of an application of the Choquet expected utility model by Dow and Werlang (1992). See Asano (2006) for an elaboration on this. The effect of ambiguity and uncertainty aversion on equilibria, in representative agent models has been discussed by Epstein and Wang (1994) in a dynamic setting. See also Chen and Epstein (2002). Epstein and Miao (2003) consider heterogeneous agents. Chateauneuf *et al*. (2000) studies static optimal risk sharing in an economy populated by Choquet expected utility maximizers. Dana (2004) characterizes equilibria in the Choquet Expected Utility model and analyses the determinacy of equilibria. Rigotti and Shannon (2005) studies this issue in the Bewley model while Rigotti *et al*. (2008) adopt a more general approach to the problem. Billot *et al*. (2000) show that in the multiple prior model, the absence of betting is equivalent to the sets of priors of the different agents to be non-empty. Mukerji and Tallon (2001)

builds a model in which incompleteness of financial markets is due to ambiguity aversion, using the Choquet expected utility model. See Rinaldi (2009) for a similar construct using the smooth ambiguity model.

Cao *et al*. (2005) find that the presence of model uncertainty leads to higher equity premium than that obtained in standard expected utility framework without model uncertainty. Epstein and Schneider (2008) show in particular that expected excess returns are higher when information quality is more uncertain. Leippold *et al*. (2008) extend this model in a continuous-time framework. Condie (2008) analyses the impact of ambiguity aversion on asset prices when investors are heterogeneous with respect to their beliefs, their degree of risk aversion, and their degree of ambiguity aversion. He shows that ambiguity aversion is not sufficient to explain long run market prices. Ludwig and Zimper (2006) analyses the decision problem of a Choquet Expected Utility decision maker who can choose between a liquid and an illiquid uncertain investment project without knowing her liquidity type. For two standard Bayesian update rules – the optimistic and the pessimistic update rule – they show that the decision maker may strictly prefer investing in the illiquid project since it serves as an intrapersonal commitment device that guarantees an ex ante favourable outcome. In Routledge and Zin (2009), an aversion to uncertainty increases the market-makers bid ask spreads and reduces liquidity.

7.2 *Auctions and Strategic Interactions*

There is an emerging interest in the literature in robustness of auction market mechanisms under ambiguity. Ambiguity is introduced in the distribution of types, or of players' beliefs about the distribution of types. Implications of ambiguity in the value distribution have been investigated theoretically by Salo and Weber (1995) within a Choquet expected utility framework, Lo (1998) or more recently Turocy (2008) using the multiple prior framework. Ambiguity in strategic games can be applied to many economic problems. Dow and Werlang (1994b) first show existence of an equilibrium under uncertainty, in the framework of two player games. See also among others, Marinacci (2000), Haller (2000) and Kelsey and Eichberger (2000). In a model of voluntary contributions to public goods, Eichberger and Kelsey (2002) show that ambiguity increases the provision of public goods. Recently, Eichberger *et al*. (2009) apply equilibrium concepts under uncertainty to oligopoly models and show that ambiguity influences equilibrium prices.

8. Conclusion

As we hope we conveyed to the reader, the field of decision under uncertainty is very rich and expanding rapidly. The basis of most advances – the natural fact that acting in a situation in which one has very little knowledge of the odds of different scenarios is very different from acting when the odds are known – is so intuitive that it even made it to a best seller like Robert Ludlum's The Ambler Warning (2005), where Ellsberg two colour urn experiment is accurately described. The field nevertheless definitively needs more applications in the realm of concrete economics phenomena. We hope that this survey, by providing an introduction to the literature will encourage some readers to tackle this important enterprise.

Acknowledgements

Notes

1. To summarize in very broad terms, Al Najjar and Weinstein (2009) question the relevance of this literature and provide examples in which decisions made by ambiguity or uncertainty averse decision makers appear to be silly. They also point to the problem of identifying beliefs in models of ambiguity aversion due to the lack of natural updating rules. The replies in turn take issues with the examples and discuss ways of dealing with arrival of new information.
2. For an excellent textbook, see Gilboa (2009).
3. We however will deal with some uncertainty about possible consequences.
4. Recently, Vierø (2009) has argued in favour of a generalization of the Anscombe–Aumann framework, where the outcome space is the set of subsets of lotteries.
5. The interested reader might want also to look at some discussion among researchers in the field on the forum http://groups.google.fr/group/decision_theory_forum?lnk=srg.
6. See nevertheless Section 3.4 for a discussion of how rank dependent utility has been advocated, together with the notion of source dependence, to explain this paradox.
7. A capacity ν is a set function from the algebra $\mathcal{A}$ to [0, 1] such that $\nu(\emptyset) = 0$ and $\nu(S) = 1$, and if $E \subset F$, $E, F \in \mathcal{A}$, $\nu(E) \leq \nu(F)$. Note that the equality defining a probability distribution, that is $\nu(E \cup F) = \nu(E) + \nu(F)$ for any disjoint events E, F is not necessarily satisfied here. See Jeleva and Rossignol (2009) for further references.
8. See Section 4.
9. The rank dependent aspect in prospect theory was problematic in Kahneman and Tversky (1979) formulation, which failed to satisfy basic dominance principles. Quiggin (1982) is the first correct formulation of a model in which probabilities are used in a non-linear manner, while still satisfying dominance.
10. We will deal more thoroughly with the notion of ambiguity aversion in Section 4. Since the axiom of uncertainty/ambiguity aversion is built in the (original) multiple prior model, we nevertheless need to present it at this stage.
11. This last statement is true of all other theories presented here.
12. See Section 4 for more on this issue.
13. The relation between uncertainty aversion and convex capacity in the Choquet Expected Utility model will be established in Section 4.
14. An equivalent formulation has Φ^{-1} in front of this formula: $\Phi^{-1}(E_\mu \Phi(E_p(u(f))))$. Since Φ^- is increasing, this is an equivalent representation of the preferences. It makes more explicit the fact that under no ambiguity, the criterion is an expected utility criterion.
15. The lower envelope of a set of probability distributions being a capacity, we use the same notation ν as for the capacity in the Choquet expected utility model.
16. φ is obtained from a capacity ν using the following formula: for any event E, $\varphi(E) = \nu(E) - \sum_{B \subset E} (-1)^{|E \setminus B| - 1} \nu(B)$.
17. For a related approach, that mixes 'objective' and 'subjective' rationality, see, Gilboa *et al.* (2008a).
18. See Cohen (1995) for alternative definitions. One can also use other stochastic orders, e.g. the monotone mean preserving increase in risk, the left monotone mean preserving increase in risk etc; a presentation of these notions and of the associated characterizations in different decision models under risk can be found in Chateauneuf *et al.* (2004) .
19. See also Wakker (1990) and Chateauneuf (1994).

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