Bayesian Statistics

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Welcome!

- This is a 60-hour, PhD-level course on Bayesian inference.
- We have 11 planned weeks. Reading material is posted at `https://github.com/maxbiostat/BayesianStatisticsCourse/`
- Assessment will be done via a written exam (70%) and an assignment (30%);
- Tenets:
    - Respect the instructor and your classmates;
    - Read before class;
    - Engage in the discussion;
    - Don't be afraid to ask/disagree.
- Books are
    - Robert (2007);
    - Hoff (2009);
    - Bernardo and Smith (2000).

Bayes's Theorem

What do

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}, \tag{1}$$

and

$$\Pr(A_i \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\sum_{i=1}^{n}\Pr(B \mid A_i)\Pr(A_i)}, \tag{2}$$

and

$$p(\theta \mid \boldsymbol{y}) = \frac{l(\boldsymbol{y} \mid \theta)\pi(\theta)}{\int_{\Theta} l(\boldsymbol{y} \mid t)\pi(t)\,dt}, \tag{3}$$

and

$$p(\theta \mid \boldsymbol{y}) = \frac{l(\boldsymbol{y} \mid \theta)\pi(\theta)}{m(\boldsymbol{y})}, \tag{4}$$

all have in common? In this course, we will find out how to use Bayes's rule in order to draw statistical inferences in a coherent and mathematically sound way.

Bayesian Statistics is a complete approach

Our whole paradigm revolves around the posterior:

$$p(\theta \mid \boldsymbol{x}) \propto l(\theta \mid \boldsymbol{x})\pi(\theta).$$

Within the Bayesian paradigm, you are able to

- Perform point and interval inference about unknown quantities;

$$\delta(\boldsymbol{x}) = E_p[\theta] := \int_\Theta t p(t \mid \boldsymbol{x})\, dt,$$
$$\Pr(a \leq \theta \leq b) = 0.95 = \int_a^b p(t \mid \boldsymbol{x})\, dt;$$

- Compare models:

$$\mathrm{BF}_{12} = \frac{\Pr(M_1 \mid \boldsymbol{x})}{\Pr(M_2 \mid \boldsymbol{x})} = \frac{\Pr(\boldsymbol{x} \mid M_1)\Pr(M_1)}{\Pr(\boldsymbol{x} \mid M_2)\Pr(M_2)};$$

- Make predictions: $g(\tilde{x} \mid \boldsymbol{x}) := \int_\Theta f(\tilde{x} \mid t) p(t \mid \boldsymbol{x})\, dt$;
- Make decisions: $E_p[U(r)]$.

Statistical model: informal definition

Stuff you say at the bar:

Definition 1 (Statistical model: informal)

DeGroot, def 7.1.1, pp. 377 A statistical model consists in identifying the random variables of interest (observable and potentially observable), the specification of the joint distribution of these variables and the identification of parameters (θ) that index this joint distribution. Sometimes it is also convenient to assum that the parameters are themselves random variables, but then one needs to specify a joint distribution for θ also.

Stuff you say in a Lecture:

Definition 2 (Statistical model: formal)

McCullagh, 2002. Let $\mathcal{X}$ be an arbitrary sample space, Θ a non-empty set and $\mathcal{P}(\mathcal{X})$ the set of all probability distributions on $\mathcal{X}$, i.e. $P : \Theta \to [0, \infty)$, $P \in \mathcal{P}$. A parametric statistical model is a function $P : \Theta \to \mathcal{P}(\mathcal{X})$, that associates each point $\theta \in \Theta$ to a probability distribution P_θ over $\mathcal{X}$.

Examples:

- Put $\mathcal{X} = \mathbb{R}$ and $\Theta = (-\infty, \infty) \times (0, \infty)$. We say P is a *normal* (or *Gaussian*) statistical model[1] if for every $\theta = \{\mu, \sigma^2\} \in \Theta$,

$$P_\theta(x) \equiv \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \; x \in \mathbb{R}.$$

- Put $\mathcal{X} = \mathbb{N} \cup \{0\}$ and $\Theta = (0, \infty)$. P is a Poisson statistical model if, for $\lambda \in \Theta$,

$$P_\lambda(k) \equiv \frac{e^{-\lambda}\lambda^k}{k!}, \; k = 0, 1, \ldots$$

[1] Note the abuse of notation: striclty speaking, P_θ is a probability **measure** and not a *density* as we have presented it here.

Principle I: the sufficiency principle

Sufficiency plays a central role in all of Statistics.

Definition 3 (Sufficient statistic)

Let $x \sim f(x \mid \theta)$. We say $T : \mathcal{X} \to \mathbb{R}$ is a ***sufficient statistic*** *for the parameter θ if $\operatorname{Pr}(X = x \mid T(x), \theta)$ is independent of θ.*

This is the basis for a cornerstone of Statistics,

Theorem 1 (Factorisation theorem)

Under mild regularity conditions, we can write:

$$f(x \mid \theta) = g(T(x) \mid \theta)h(x \mid T(x)).$$

We can now state

Idea 1 (Sufficiency principle (SP))

For $x, y \in \mathcal{X}$, if T is sufficient for θ and $T(x) = T(y)$, then x and y should lead to the same inferences about θ.

Principle II: the Likelihood principle

The Likelihood Principle (LP) is a key concept in Statistics, of particular Bayesian Statistics.

Idea 2 (Likelihood Principle)

*The information brought by an observation $x \in \mathcal{X}$ about a parameter $\theta \in \Theta$ is **completely** contained in the likelihood function $l(\theta \mid x) \propto f(x \mid \theta)$.*

Example 1 (Uma vez Flamengo...)

Principle II: the Likelihood principle

Suppose a pollster is interested in estimating the fraction θ of football fans that cheer for Clube de Regatas do Flamengo (CRF). They survey $n = 12$ people and get $x = 9$ supporters and $y = 3$ "antis". Consider the following two designs:

i) Survey 12 people and record the number of supporters;

ii) Survey until they get $y = 3$.

The likelihoods for both surveys are, respectively,

$$x \sim \text{Binomial}(n, \theta) \implies l_1(\theta \mid x, n) = \binom{n}{x}\theta^x(1-\theta)^{n-x},$$

$$n \sim \text{NegativeBinomial}(y, 1-\theta) \implies l2(\theta \mid n, y) = \binom{n-1}{y-1}y(1-\theta)^{n-y}\theta^y,$$

hence

$$l_1(\theta) \propto l_2(\theta) \propto \theta^3(1-\theta)^9.$$

Therefore, we say that these two experiments bring exactly the same information about θ.

A generalised version of the LP can be stated as follows:

Principle II: the Likelihood principle

Theorem 2 (Likelihood Proportionality Theorem (Gonçalves and Franklin, 2019))

Let Θ *be a nonempty set and* $\mathcal{P} = \{P_\theta; \theta \in \Theta\}$ *be a family of probability measures on* $(\Omega, \mathcal{A})$ *and* ν_1 *and* ν_2 *be* σ*-finite measures on* $(\Omega, \mathcal{A})$*. Suppose* $P \ll \nu_1$ *and* $P \ll \nu_2$ *for all* $P \in \mathcal{P}$*. Then there exists a measurable set* $A \in \mathcal{A}$ *such that* $P_\theta(A) = 1$ *for all* $\theta \in \Theta$ *and there exist* $f_{1,\theta} \in \left[\frac{dP_\theta}{d\nu_1}\right]$ *and* $f_{2,\theta} \in \left[\frac{dP_\theta}{d\nu_2}\right]$ *and a measurable function h such that*

$$f_{1,\theta}(\omega) = h(\omega) f_{2,\theta}(\omega), \forall\, \theta \in \Theta\, \forall\, \omega \in A.$$

Principle III: stopping rule principle

A subject of contention between inference paradigms is the role of stopping rules in the inferences drawn.

Idea 3 (Stopping rule principle (SRP))

Let τ be a stopping rule directing a series of experiments $\mathcal{E}_1, \mathcal{E}_2, \ldots$, which generates data $\boldsymbol{x} = (x_1, x_2, \ldots)$. Inferences about θ should depend on τ only through $\boldsymbol{x}$.

Example 3 (Finite stopping rules)

Suppose experiment $\mathcal{E}_i$ leads to the observation of $x_i \sim f(x_i \mid \theta)$ and let $\mathcal{A}_i \subset \mathcal{X}_1 \times \ldots \times \mathcal{X}_i$ be a sequence of events. Define

$$\tau := \inf \{ n : (x_1, \ldots, x_n) \in \mathcal{A}_n \} .$$

It can be shown that $\Pr(\tau < \infty) = 1$ (exercise 1.20 BC).

Principle IV: the conditionality principle

We will now state one of the main ingredients of the derivation of the LP. The Conditionality Principle (CP) is a statement about the permissible inferences from randomised experiments.

Idea 4 (Conditionality Principle)

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two experiments about θ. Let $Z \sim \text{Bernoulli}(p)$ and

- *If $Z = 1$, perform $\mathcal{E}_1$ to generate $x_1 \sim f_1(x_1 \mid \theta)$;*
- *If $Z = 0$ perform $\mathcal{E}_2$ to generate $x_2 \sim f_2(x_2 \mid \theta)$.*

Inferences about θ should depend ***only*** *on the selected experiment, $\mathcal{E}_i$.*

Deriving the Likelihood Principle

Birnbaum (1962) showed that the simpler and mostly uncontroversial Sufficiency and Conditionality principles lead to the Likelihood Principle.

Theorem 2 (Birnbaum's theorem (Birnbaum, 1962))

$$\mathrm{SP} + \mathrm{CP} \implies \mathrm{LP}\,. \tag{5}$$

Proof.

Sketch:

- Define a function $\mathrm{EV}(\mathcal{E}, x)$ to quantify the evidence about θ brought by data x from experiment $\mathcal{E}$ and consider a randomised experiment $\mathcal{E}^*$ in which $\mathcal{E}_1$ and $\mathcal{E}_2$ are performed with probability p;
- Show that CP implies $\mathrm{EV}(\mathcal{E}^*, (j, x_j)) = \mathrm{EV}(\mathcal{E}_j, x_j), j = 1, 2$;
- Show that SP implies $\mathrm{EV}(\mathcal{E}^*, (1, x_1)) = \mathrm{EV}(\mathcal{E}^*, (2, x_2))$ when

$$l(\theta \mid x_1) = cl(\theta \mid x_2).$$

$\square$

See Robert (2007), pg.18 for a complete proof.

Recommended reading

- Robert (2007) Ch. 1;
- Next lecture: Robert (2007) Ch. 2 and * Schervish (2012) Ch.3;

Belief functions

Let F, G and $H \in \mathcal{S}$ be three (possibly overlapping) statements about the world. For example, consider the following statements about a person:

$$F = \{\text{votes for a left-wing candidate}\};$$

$$G = \{\text{is in the 10\% lower income bracket}\};$$

$$H = \{\text{lives in a large}\};$$

Definition 4 (Belief function)

For $A, B \in \mathcal{S}$, a belief function $\operatorname{Be} : \mathcal{S} \to \mathbb{R}$ *assigns numbers to statements such that* $\operatorname{Be}(A) < \operatorname{Be}(B)$ *implies one is more confident in B than in A.*

Belief functions: properties

It is useful to think of Be as **preferences over bets**:

- $\mathrm{Be}(F) > \mathrm{Be}(G)$ means we would bet on F being true over G being true;
- $\mathrm{Be}(F \mid H) > \mathrm{Be}(G \mid H)$ means that, **conditional** on knowing H to be true, we would bet on F over G;
- $\mathrm{Be}(F \mid G) > \mathrm{Be}(F \mid H)$ means that if we were forced to be on F, we would be prefer doing so if G were true than H.

Belief functions: axioms

In order for Be to be **coherent**, it must adhere to a certain set of properties/axioms. A self-sufficient collection is:

A1 (boundedness of complete [dis]belief):

$$\mathrm{Be}(\neg H \mid H) \leq \mathrm{Be}(F \mid H) \leq \mathrm{Be}(H \mid H),\ \forall\, F \in \mathcal{S};$$

A2 (monotonicity):

$$\mathrm{Be}(F \text{ or } G \mid H) \leq \max\left\{\mathrm{Be}(F \mid H), \mathrm{Be}(G \mid H)\right\};$$

A3 (sequentiality): There exists $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\mathrm{Be}(F \text{ and } G \mid H) = f\left(\mathrm{Be}(G \mid H), \mathrm{Be}(F \mid G \text{ and } H)\right).$$

Probabilities can be beliefs!

Exercise 1 (Probabilities and beliefs)

Show that the axioms of belief functions map one-to-one to the axioms of probability:

P1. $0 \leq \Pr(E), \forall E \in \mathcal{S}$;

P2. $\Pr(\mathcal{S}) = 1$;

P3. *For any countable sequence of disjoint statements* $E_1, E_2, \ldots \in \mathcal{S}$ *we have*

$$\Pr\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \Pr(E_i).$$

Hint: derive the consequences (e.g. monotonicity) of these axioms and compare them with the axioms of belief functions.

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Definition 5 (Partition)

If $H = \{H_1, H_2, \ldots, H_k\}$, $H_i \in \mathcal{S}$, such that $H_i \cap H_j = \emptyset$ for all $i \neq j$ and $\bigcup_{k=1}^{K} = \mathcal{S}$, we say H is a partition of $\mathcal{S}$.

For any $H \in \mathcal{D}(\mathcal{S})$:

- **Total probability**: $\sum_{k=1}^{K} \operatorname{Pr}(H_k) = 1$;
- **Marginal probability**:

$$\operatorname{Pr}(E) = \sum_{k=1}^{K} = \operatorname{Pr}(E \cap H_k) = \sum_{k=1}^{K} \operatorname{Pr}(E \mid H_k) \operatorname{Pr}(H_k),$$

  for all $E \in \mathcal{S}$;
- Consequence $\implies$ Bayes's rule:

$$\operatorname{Pr}(H_j \mid E) = \frac{\operatorname{Pr}(E \mid H_j) \operatorname{Pr}(H_j)}{\sum_{k=1}^{K} \operatorname{Pr}(E \mid H_k) \operatorname{Pr}(H_k)}.$$

Independence

We will now state a central concept in probability theory and Statistics.

Definition 6 ((Conditional) Independence)

For any $F, G \in \mathcal{S}$, we say F and G are ***conditionally independent*** *given A if*

$$\Pr(F \cap G \mid A) = \Pr(F \mid A) \Pr(G \mid A).$$

Remark 1

If F and G are conditionally independent given A, then

$$\Pr(F \mid A \cap G) = \Pr(F \mid A).$$

Proof.

First, notice that the axioms P1-P3 imply $\Pr(F \cap G \mid A) = \Pr(G \mid A) \Pr(F \mid A \cap G)$. Now use conditional independence to write

$$\Pr(G \mid A) \Pr(F \mid A \cap G) = \Pr(F \cap G \mid A) = \Pr(F \mid A) \Pr(G \mid A),$$
$$\Pr(G \mid A) \Pr(F \mid A \cap G) = \Pr(F \mid A) \Pr(G \mid A).$$

Definition 7 (Exchangeable)

We say a sequence of random variables $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_n\}$ *are* ***exchangeable*** *if*

$$\Pr(Y_1, Y_2, \ldots Y_n) = \Pr(Y_{\xi_1}, Y_{\xi_2}, \ldots Y_{\xi_n}),$$

for all ***permutations*** ξ *of the labels of* $\mathbf{Y}$.

Example 4 (Uma vez Flamengo... continued)

Suppose we survey 12 people and record whether they cheer for Flamengo $Y_i = 1$ or not $Y_i = 0$, $i = 1, 2, \ldots, 12$. What value shoud we assign to :

- $p_1 := \Pr(1, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1)$;
- $p_2 := \Pr(1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1)$;
- $p_3 := \Pr(1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0)$?

If your answer is $p_1 = p_2 = p_3$ then you are saying the Y_i are (at least partially) exchangeable!

An application of conditional independence

For $\theta \in (0, 1)$, consider the following sequence of probability statements:

$$
\begin{aligned}
\Pr(Y_{12} = 1 \mid \theta) &= \theta, \\
\Pr(Y_{12} = 1 \mid Y_1, \ldots Y_{11}, \theta) &= \theta, \\
\Pr(Y_{11} = 1 \mid Y_1, \ldots Y_{10}, Y_{12}, \theta) &= \theta.
\end{aligned}
$$

These imply that the Y_i are conditionally independent and identically distributed (iid), and in particular:

$$
\begin{aligned}
\Pr(Y_1 = y_1, \ldots, Y_{12} = y_{12} \mid \theta) &= \prod_{i=1}^{12} \theta^{y_i}(1-\theta)^{1-y_i}, \\
&= \theta^{S}(1-\theta)^{12-S},
\end{aligned}
$$

with $S := \sum_{i=1}^{12} y_i$. Also, under a uniform prior,

$$
\Pr(Y_1, \ldots Y_{12}) = \int_0^1 t^S(1-t)^{12-S}\pi(t)\,dt = \frac{(S+1)!(12-S+1)!}{13!} = \binom{13}{S+1}^{-1}.
$$

Relaxing exchangeability (a bit)

Sometimes total symmetry can be a burden. We can relax this slightly by introducing the concept of **partial exchangeability**:

Definition 8 (Partially exchangeable)

Let $\boldsymbol{X} = \{X_1, \ldots, X_n\}$ *and* $\boldsymbol{X} = \{Y_1, \ldots, Y_m\}$ *be two sets of random variables. We say* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are* ***partially*** *exchangeable if*

$$\Pr(X_1, \ldots, X_n; Y_1, \ldots, Y_m) = \Pr\left(X_{\xi_1}, \ldots, X_{\xi_n}; Y_{\sigma_1}, \ldots, Y_{\sigma_m}\right),$$

for any two permutations $\boldsymbol{\xi}$ and $\boldsymbol{\sigma}$ of $1, \ldots, n$ and $1, \ldots, m$, respectively.

Example 5 (Uma vez Flamengo...continued)

To see how exchangeability can be relaxed into partial exchangeability, consider $\boldsymbol{X}$ and $\boldsymbol{Y}$ as observations coming from populations from Rio de Janeiro and Ceará, respectively. If the covariate "state" were deemed to not matter, then we would have complete exchangeability.

A statistically useful remark

Remark 2 (Exchangeability from conditional independence)

Take $\theta \sim \pi(\theta)$, i.e., represent uncertainty about θ using a probability distribution. If $\Pr(Y_1 = y_1, \ldots, Y_n = y_n \mid \theta) = \prod_{i=1}^{n} \Pr(Y_i = y_i \mid \theta)$, then $Y_1, \ldots, Y_n$ are exchangeable.

Proof.

Sketch: Use

- Marginalisation;
- Conditional independence;
- Commutativity of products in $\mathbb{R}$;
- Definition of exchangeability.

$\square$

A fabulous theorem!

Theorem 3 (De Finetti's theorem[2])

If $\Pr(Y_1, \ldots, Y_n) = \Pr\left(Y_{\xi_1}, \ldots, Y_{\xi_n}\right)$ *for all permutations* ξ *of* $1, \ldots, n$*, then*

$$\Pr(Y_1, \ldots, Y_n) = \Pr\left(Y_{\xi_1}, \ldots, Y_{\xi_n}\right) = \int_{\Theta} \Pr(Y_1, \ldots, Y_n \mid t)\, \pi(t)\, dt, \tag{6}$$

for some choice of triplet $\{\theta, \pi(\theta), f(y_i \mid \theta)\}$*, i.e., a parameter, a prior and a sampling model.*

See Proposition 4.3 in Bernardo and Smith (2000) for a proof outline. Here we shall prove the version from De Finetti (1931).

[2]Technically, the theorem stated here is more general than the representation theorem proven by De Finetti in his seminal memoir, which concerned binary variables only.

Consequences

This theorem has a few important implications, namely:

- $\pi(\theta)$ represents our beliefs about $\lim_{n\to\infty} \sum_i (Y_i \leq c)/n$ for all $c \in \mathcal{Y}$;
- $\{ Y_1, \ldots, Y_n \mid \theta \text{ are i.i.d} \} + \{ \theta \sim \pi(\theta) \} \iff \{ Y_1, \ldots, Y_n \text{ are exchangeable for all } n \}$;
- If $Y_i \in \{0, 1\}$, we can also claim that:
  - If the Y_i are assumed to be independent, then they are distributed Bernoulli conditional on a random quantity θ;
  - θ has a prior measure $\Pi \in \mathcal{P}((0, 1))$;
  - By the strong law of large numbers (SLLN), $\theta = \lim_{n\to\infty}(\frac{1}{n}\sum_{i=1}^{n} Y_i)$, so Π can be interpreted as a "belief about the limiting relative frequency of 1's".

As the exchangeability results above clearly demonstrate, being able to use conditional independence is a handy tool. More specifically, knowing on what to condition so as to make things exchangeable is key to statistical analysis.

Idea 5 (Conditioning is the soul of Statistics[3])

Knowing on what to condition can be the difference between an unsolvable problem and a trivial one. When confronted with a statistical problem, always ask yourself "What do I know for sure?" and then "How can I create a conditional structure to include this information?".

[3] This idea is due to Joe Blitzstein, who did his PhD under no other than the great Persi Diaconis.

Recommended reading

- Hoff (2009) Ch. 2 and *Schervish (2012) Ch.1;
  - *Paper: Diaconis and Freedman (1980) explains why if n samples are taken from an exchangeable population of size $N \gg n$ without replacement, then the sample $Y_1, \ldots Y_n$ can be modelled as approximately exchangeable;
- Next lecture: Robert (2007) Ch. 3.

Priors: a curse and a blessing

- Priors are the main point of contention between Bayesians and non-Bayesians;
- As we shall see, there is usually no unique way of constructing a prior measure;
- Moreover, in many situations the choice of prior is not inconsequential.
- There is always a question of when to stop adding uncertainty...

Determination of priors: existence

It is usually quite hard to determine a (unique) prior even when substantial knowledge. Why? One reason is that a prior measure is guaranteed to exist only when there is a **coherent ordering** of the Borel sigma-algebra $\mathcal{B}(\Theta)$. This entails that the following axioms hold:

(A1) Total ordering: For all measurable $A, B \in \mathcal{B}(\Theta)$ one and <u>only one</u> of these can hold:

$$A \prec B, B \prec A \text{ or } A \sim B.$$

(A2) Transitivity: For measurable $A_1, A_2, B_1, B_2 \in \mathcal{B}(\Theta)$ such that $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$ and $A_i \preceq B_i, i = 1, 2$ then the following holds:

- $A_1 \cup A_2 \preceq B_1 \cup B_2$;
- If $A_1 \prec B_1$ then $A_1 \cup A_2 \prec B_1 \cup B_2$;

(A3) For any measurable A, $\emptyset \preceq A$ and also $\emptyset \prec \Theta$;

(A4) Continuity: If $E_1 \supset E_2 \ldots$ is a decreasing sequence of measurable sets and B is such that $B \preceq E_i$ for all i, then

$$B \preceq \bigcap_{i=1}^{\infty} E_i.$$

Approximation I: marginalisation

One way to approach the problem of determining a prior measure is to consider the marginal distribution of the data:

$$m(x) = \int_{\Theta} f(x \mid \theta)\pi(\theta)\, d\theta. \tag{7}$$

In other words we are trying to solve an inverse problem in the form of an integral equation by placing restrictions on $m(x)$ and calibrating π to satisfy them.

Approximation II: moments

Another variation on the integral-equation-inverse-problem theme is to consider expectations of measurable functions. Suppose

$$E_\pi[g_k] := \int_\Theta g_k(t)\pi(t)\,dt = w_k. \tag{8}$$

For instance, if the analyst knows that $E_\pi[\theta] = \mu$ and $\operatorname{Var}_\pi(\theta) = \sigma^2$, then this restricts the class of functions in $\mathcal{L}_1(\Theta)$ that can be considered as prior density[4]. One can also consider *order statistics* by taking $g_k(x) = \mathbb{I}_{(-\infty, a_k]}(x)$.

[4]As we shall see in the coming lectures, π needs not be in $\mathcal{L}_1(\Theta)$, i.e., needs not be **proper**. But this "method-of-moments" approach is then complicated by lack of integrability.

Maximum entropy priors

The moments-based approach is not complete in the sense that it does not lead to a unique prior measure π.

Definition 9 (Entropy)

The entropy of a probability distribution P is defined as

$$H(P) := E_p[-\log p] = -\int_{\mathcal{X}} \log p(x)\, dP(x). \tag{9}$$

When θ has finite support, we get the familiar

$$H(P) = -\sum_i p(\theta_i) \log(p(\theta_i)).$$

We can leverage this concept in order to pick π.

Definition 10 (Maximum entropy prior)

Let $\mathcal{P}_r$ be a class of probability measures on $\mathcal{B}(\Theta)$. A maximum entropy prior in $\mathcal{P}_r$ is a distribution that satisfies

$$\arg\max_{P \in \mathcal{P}_r} H(P).$$

Finding a maxent prior

When Θ is finite, we can write

$$\pi^*(\theta_i) = \frac{\exp\left\{\sum_{k=1} \lambda_k g_k(\theta_i)\right\}}{\sum_j \exp\left\{\sum_{k=1} \lambda_k g_k(\theta_j)\right\}},$$

where the λ_k are Lagrange multipliers. In the uncountable case things are significantly more delicate, but under regularity conditions there exists a reference measure Π_0 such that

$$\begin{aligned} H_\Pi &= E_{\pi_0}\left[\log\left(\frac{\pi(\theta)}{\pi_0(\theta)}\right)\right], \\ &= \int_\Theta \log\left(\frac{\pi(\theta)}{\pi_0(\theta)}\right) \Pi_0(d\theta). \end{aligned}$$

Exercise 2 (Maximum entropy Beta prior)

Find the maximum entropy Beta distribution under the following constraints:

- $E[\theta] = 1/2$;
- $E[\theta] = 9/10$.

Hint: If P is a Beta distribution with parameters α and β, then

$$H_P = \log B(\alpha, \beta) - (\alpha - 1)\psi(\alpha) - (\beta - 1)\psi(\beta) + (\alpha + \beta - 2)\psi(\alpha + \beta),$$

where $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function and $\psi(x) = \frac{d}{dx}\log(\Gamma(x))$ is the digamma function.

Parametric approximations: easy-peasy

In some situations, the “right” parametric family presents itself naturally.

Example 6 (Eliciting Beta distributions)

Let $x_i \sim \operatorname{Binomial}(n_i, p_i)$ be the number of Flamengo supporters out of n_i people surveyed. Over the years, the average of p_i has been 0.70 with variance 0.1. If we assume $p_i \sim \operatorname{Beta}(\alpha, \beta)$ we can elicit an informative distribution based on historical data by solving the system of equations

$$
\begin{aligned}
E[\theta] &= \frac{\alpha}{\alpha + \beta} = 0.7, \\
\operatorname{Var}(\theta) &= \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} = 0.1.
\end{aligned}
$$

Parametric approximations: difficulties

Other times we may have a hard time narrowing down the prior to a specific parametric family. Consider the following example.

Example 7 (Normal or Cauchy?)

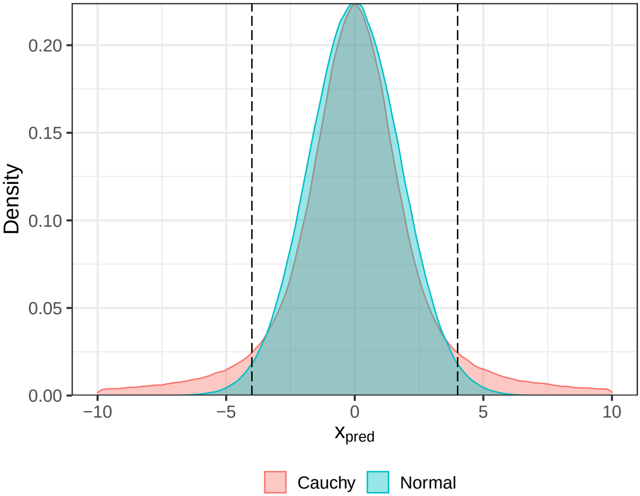
Suppose $x_i \sim \text{Normal}(\theta, 1)$ and we are informed that $\Pr(\theta \leq -1) = 1/4$, $\Pr(\theta \leq 0) = 1/2$ and $\Pr(\theta \leq 1) = 3/4$. Seems like plenty of information. It can be shown that

$$\pi_1(\theta) = \frac{1}{\sqrt{2\pi 2.19}} \exp\left(-\frac{\theta^2}{2 \times 2.19}\right) \text{ (Normal)},$$
$$\pi_2(\theta) = \frac{1}{\pi(1 + x^2)} \text{ (Cauchy)},$$

both satisfy the requirements. Unfortunately, under quadratic loss we get $\delta_1(4) = 2.75$ and $\delta_2(4) = 3.76$ and differences are exacerbated for $|x| \geq 4$.

Why, though?

Remember the marginal approach? It is illuminating in this case. Heres $m(x)$:



Prior predictive distributions of x under Normal and Cauchy priors.

Conjugacy is a central concept in Bayesian statistics. It provides a functional view of the prior-posterior mechanic that emphasises tractability over coherence.

Definition 11 (Conjugate)

A family $\mathcal{F}$ of distributions on Θ is called ***conjugate*** *or closed under sampling for a likelihood $f(x \mid \theta)$ if, for every $\pi \in \mathcal{F}$, $p(\theta \mid x) \in \mathcal{F}$.*

Arguments for using conjugate priors

- "Form-preservation": in a limited-information setting it makes sense that $p(\theta \mid x)$ and $\pi(\theta)$ lie on the same family, since the information in x might not be enough to change the structure of the model, just its parameters;
- Simplicity: when you do not know a whole lot, it makes sense to KISS[5];
- Sequential learning: since $\mathcal{F}$ is closed under sampling, one can update a sequence of posteriors $p_i(\theta \mid x_1, \ldots, x_i)$ as data comes in.

[5] Keep it simple, stupid!

Exponential families

The exponential family of distributions is a cornerstone of statistical practice, underlying many often-used models. Here are a few useful definitions.

Definition 12 ((Natural) Exponential family)

Let μ be a σ-finite measure on $\mathcal{X}$ and let Θ be a non-empty set serving as the parameter space. Let $C : \Theta \to (0, \infty)$ and $h : \Theta \to (0, \infty)$ and let $R : \Theta \times \mathcal{X} \to \mathbb{R}^k$ and $T : \Theta \times \mathcal{X} \to \mathbb{R}^k$. The family of distributions with density

$$f(x \mid \theta) = C(\theta)h(x) \exp\left(R(\theta) \cdot T(x)\right)$$

*w.r.t. μ is called an **exponential family**. Moreover, if $R(\theta) = \theta$, the family is said to be **natural**.*

Definition 13 (Regular exponential family)

*We say a natural exponential family $f(x \mid \theta)$ is **regular** if the natural parameter space*

$$N := \left\{\theta : \int_{\mathcal{X}} \exp(\theta \cdot x)h(x)\, d\mu(x) < \infty\right\}, \tag{10}$$

is an open set of the same dimension as the closure of the convex hull of $\operatorname{supp}(\mu)$.

Conjugacy and sufficiency

There is an intimate link between sufficiency (i.e. the existence of sufficient statistics) and conjugacy. The following is a staple of Bayesian theory.

Theorem 4 (Pitman-Koopman-Darmois)

If a family of distributions $f(\cdot \mid \theta)$ whose support does not depend on θ is such that, for a sample size large enough, there exists a sufficient statistic of fixed dimension, then $f(\cdot \mid \theta)$ is an exponential family.

The support condition is not a complete deal breaker, however:

Remark 3 (Quasi-exponential)

The $\mathrm{Uniform}(-\theta, \theta)$ and $\mathrm{Pareto}(\theta, \alpha)$ families are called quasi-exponential due to the fact that there do exist sufficient statistics of fixed dimension for these families, even though their supports depend on θ.

I hope you are convinced of the utility of the exponential family by now. It would be nice to have an automated way to deduce a conjugate prior for $f(x \mid \theta)$ when it is in the exponential family. This is exactly what the next result gives us.

Remark 4 (Conjugate prior for the exponential family)

A conjugate family for $f(x \mid \theta)$ is given by

$$\pi(\theta \mid \mu, \lambda) = K(\mu, \lambda) \exp\left(\theta \cdot \mu - \lambda g(\theta)\right), \tag{11}$$

such that the posterior is given by $p(\theta \mid \mu + x, \lambda + 1)$.

Please do note that (11) is only a valid density when $\lambda > 0$ and μ/λ belongs to the interior of the natural space parameter. Then, it is a σ-finite measure. See Diaconis and Ylvisaker (1979) for more details.

Conjugacy: common families

Table 3.3.1. *Natural conjugate priors for some common exponential families*

$f(x\|\theta)$	$\pi(\theta)$	$\pi(\theta\|x)$
Normal $\mathcal{N}(\theta, \sigma^2)$	Normal $\mathcal{N}(\mu, \tau^2)$	$\mathcal{N}(\varrho(\sigma^2\mu + \tau^2 x), \varrho\sigma^2\tau^2)$ $\varrho^{-1} = \sigma^2 + \tau^2$
Poisson $\mathcal{P}(\theta)$	Gamma $\mathcal{G}(\alpha, \beta)$	$\mathcal{G}(\alpha + x, \beta + 1)$
Gamma $\mathcal{G}(\nu, \theta)$	Gamma $\mathcal{G}(\alpha, \beta)$	$\mathcal{G}(\alpha + \nu, \beta + x)$
Binomial $\mathcal{B}(n, \theta)$	Beta $\mathcal{B}e(\alpha, \beta)$	$\mathcal{B}e(\alpha + x, \beta + n - x)$
Negative Binomial $\mathcal{N}eg(m, \theta)$	Beta $\mathcal{B}e(\alpha, \beta)$	$\mathcal{B}e(\alpha + m, \beta + x)$
Multinomial $\mathcal{M}_k(\theta_1, \ldots, \theta_k)$	Dirichlet $\mathcal{D}(\alpha_1, \ldots, \alpha_k)$	$\mathcal{D}(\alpha_1 + x_1, \ldots, \alpha_k + x_k)$
Normal $\mathcal{N}(\mu, 1/\theta)$	Gamma $\mathcal{G}a(\alpha, \beta)$	$\mathcal{G}(\alpha + 0.5, \beta + (\mu - x)^2/2)$

Taken from Robert (2007), page 121.

Conjugacy: drawbacks

Conjugate modelling is certainly useful, but has its fair share of pitfalls.

Arguments against using conjugate priors

- Conjugate priors are restrictive *a priori*: in many settings, specially in high dimensions, the set of conjugate priors that retain tractability is so limited so as to not be able to encode all prior information available;
- Conjugate priors are not truly subjective: they limit the analyst's input to picking values for the hyperparameters;
- Conjugate priors are restrictive *a posteriori*: you are stuck with a given structure forever, no matter how much data you run into.

Recommended reading

- Robert (2007) Ch. 3;
- Next lecture: Robert (2007) Ch. 3.6, Seaman III et al. (2012), Gelman et al. (2017) and Simpson et al. (2017).

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