Bayesian Statistics III/IV (MATH3341/4031) Michaelmas term, 2019

Handout 1: Random variables [a]

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Aim

To revise a bit, linear algebra, random variables and probabilities

Linear algebra Cholesky decomposition

Probability theory Random variables, probabilities, expected values, covariance/variance matrices, characteristic function, compound distribution function

Reading list:

- DeGroot, M. H. (1970, or 2005). Optimal statistical decisions (Vol. 82). John Wiley & Sons.
  - Part one: Survey of probability theory. Chapters 1-5

[a] Author: Georgios P. Karagiannis.

1 Linear Algebra

Proposition 1. *[Cholesky decomposition] Every symmetric positive definite matrix $A \in \mathbb{R}^d \times \mathbb{R}^d$ can be decomposed into a product of a unique lower triangular matrix $L \in \mathbb{R}^d \times \mathbb{R}^d$ and its transpose $L^\top$, i.e.*

$$A = LL^T$$

Matrix L is called lower triangular factor of the Cholesky decomposition, and it is often denoted as $A^{1/2} = L$.

2 Probability distributions

Definition 2. A collection $\mathscr{F} = \{A, A_1, A_2, ...\}$ of sets $A, A_1, A_2, ...$, each of which are subsets of set Ω, is called a σ-algebra if and only if

1. $\Omega \in \mathscr{F}$
2. If $A \in \mathscr{F}$, then $A^{\complement} \in \mathscr{F}$
3. If $A_1, A_2, ... \in \mathscr{F}$ is an infinite sequence of sets in $\mathscr{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathscr{F}$

Definition 3. Probability distribution P on $(\Omega, \mathscr{F})$ is called a non-negative function if and only if

1. $P(\Omega) = 1$
2. If $A, B \in \mathscr{F}$ and $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$
3. If $A_1, A_2, ... \in \mathscr{F}$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$ then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Notation 4. We call the triple $(\Omega, \mathscr{F}, P)$ as probability space.

3 Random variables

Definition 5. A d-dimensional random variable y on a probability space $(\Omega, \mathscr{F}, P)$ is a function $y : \Omega \rightarrow \mathbb{R}^d$ such as, for any subset $A \subseteq \mathbb{R}^d$, it is $\{\omega \in \Omega \,:\, y(\omega) \in A\} \in \mathscr{F}$.

Definition 6. A d-dimensional random variable y on a probability space $(\Omega, \mathscr{F}, P)$, induces a probability $P_y(\cdot)$ such that, for all subsets $A \subseteq \mathbb{R}^d$,

$$P_y(y \in A) = P(\{\omega \in \Omega \,:\, y(\omega) \in A\})$$

Essentially, it induces a probability space $(\mathbb{R}^d, \mathfrak{B}, P_y)$, with $\mathfrak{B}$ a σ-algebra[1] containing sub-sets of $\mathbb{R}^d$.

Definition 7. The (cumulative) distribution function (CDF) of a d-dimensional random variable $y \in \mathcal{Y}$ is the function $F_y : \mathbb{R}^d \rightarrow [0, 1]$ such that

$$F_y(y) := F_y(y'_1, ..., y'_d) = P_y(y \in (-\infty, y'_1] \times ... \times (-\infty, y'_d])$$

Notation 8. The distribution function defines the distribution of the random variable. As $y \sim F_y$, we will denote that the random variable y follows a distribution with distribution function F_y.

Definition 9. The d-dimensional random variable $y : \Omega \rightarrow \mathcal{Y}$ with distribution F_y is discrete, if $\mathcal{Y}$ is a countable set and the distribution can be described by its Probability Mass Function (PMF)

$$f_y(y') := f_y(y'_1, ..., y'_d) = P(\{\omega \in \Omega \,:\, y(\omega) = y'\})$$

Definition 10. The d-dimensional random variable $y : \Omega \rightarrow \mathcal{Y}$ with distribution F_y is absolutely continuous, if $\mathcal{Y}$ is a uncountable set and the distribution can be described by its Probability Density Function (PDF) $f_y(y)$ such that

$$P_y(y \in A) = \underbrace{\int \cdots \int}_{A} f_y(y'_1, ..., y'_d) \mathrm{d}y'_1 \cdots \mathrm{d}y'_d, \text{ for any } A \subseteq \mathbb{R}^d.$$

or briefly $P_y(A) = \int_A f_y(y') \mathrm{d}y'$, where $\mathrm{d}y' = \prod_{j=1}^{d} \mathrm{d}y'_j$.

Fact 11. *The PDF of d-dimensional random variable $y : \Omega \rightarrow \mathcal{Y}$ with CDF F_y can be computed by the partial derivative as*

$$f_y(y) = \left. \frac{d}{dt_1 \cdots dt_d} F_y(t_1, ..., t_d) \right|_{t_1 = y_1, \cdots t_d = y_d} \qquad \text{if } F_y \text{ is differential.}$$

4 Transforming

Fact 12. *Let $y \in \mathcal{Y}$ be a d-dimensional random variable with PDF $f_y(\cdot)$. Consider a bijective function $h : \mathcal{Y} \rightarrow \mathcal{Z}$ with $z = h(y)$, and h^{-1} its inverse. The PDF of z is*

$$f_z(z) \;= f_y(y) \left| \det \left(\frac{dy}{dz} \right) \right| \;= f_y(h^{-1}(z)) \left| \det \left(\frac{d}{dz} h^{-1}(z) \right) \right|$$

Example 13. Let $y \sim \text{Ex}(\lambda)$ r.v. with Exponential distribution with rate parameter $\lambda > 0$, and $f_{\text{Ex}(\lambda)}(y) = \lambda \exp(-\lambda y) 1(y \geq 0)$. Let $z = 1 - \exp(-\lambda y)$. Calculate the PDF of z, and recognize its distribution.

Solution. It is $z = 1 - \exp(-\lambda y) \iff y = -\frac{1}{\lambda} \log(1 - z)$, and $z \in [0, 1]$. So $h^{-1}(z) = -\frac{1}{\lambda} \log(1 - z)$. Then

$$\begin{aligned} f_z(z) =& f_{\text{Ex}(\lambda)}(h^{-1}(z)) \times \left| \det \left(\frac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right) \right| = f_{\text{Ex}(\lambda)} \left(-\frac{1}{\lambda} \log(1 - z) \right) \times \left| \det(\frac{\mathrm{d}}{\mathrm{d}z} \frac{-1}{\lambda} \log(1 - z) \right| \\ =& \exp \left(-\lambda \frac{-1}{\lambda} \log(1 - z) \right) 1(-\frac{1}{\lambda} \log(1 - z) \geq 0) \times \left| -\frac{1}{\lambda} \frac{1}{1 - z} \right| = 1(z \in [0, 1]) \end{aligned}$$

[1] ...this is not a rigorous definition; a more rigorous definition is out of the scope of the course.

From the density, we recognize that $z \sim \mathrm{U}(0,1)$ follows a uniform distribution.

5 Marginalizing & Integrating out

Fact 14. *Let $(n+d)$-dimensional random variable $y \in \mathcal{Y}$ with distribution $F_y(\cdot)$. Consider a partition $y = (x, \theta)$ where $x \in \mathcal{X}$ is n-dimensional and $\theta \in \Theta$ is d-dimensional . Then*

1. *the marginal CDF of x results by setting θ as ∞*

$$F_x(x) = \lim_{\theta \to \infty} F_y(x, \theta) \qquad = \lim_{\theta_1 \to \infty, \ldots, \theta_d \to \infty} F_y(x_1, \ldots x_n, \theta_1, \ldots, \theta_d)$$

2. *the marginal PDF/PMF of x results by integrating out θ (the dimensions we marginalize)*

$$f_x(x) = \begin{cases} \int_{\mathbb{R}^d} f_y(x, \theta) d\theta & \textit{if } \theta \textit{ is cont.} \\ \\ \sum_{\forall \theta \in \mathbb{R}^d} f_y(x, \theta) & \textit{if } \theta \textit{ is discr.} \end{cases} = \begin{cases} \int_{\mathbb{R}^d} f_y(x_1, \ldots x_n, \theta_1, \ldots, \theta_d) d\theta_1 \ldots d\theta_d & \textit{if } \theta \textit{ is cont.} \\ \\ \sum_{\forall \theta \in \mathbb{R}^d} f_y(x_1, \ldots x_n, \theta_1, \ldots, \theta_d) & \textit{if } \theta \textit{ is discr.} \end{cases}$$

6 Independence

Definition 15. Given a probability space $(\Omega, \mathfrak{B}, P)$, events $A, B \in \mathfrak{B}$ are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Fact 16. *Let $(n+m)$-dimensional random variable $y \in \mathcal{Y}$. Consider a partition $y = (x, z)$ where $x \in \mathcal{X}$ is n-dimensional and $z \in \mathcal{Z}$ is m- dimensional.*

- *The r.v. x and y are independent if and only if for any $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Z}$*

$$P(\{x \in A\} \cap \{z \in B\}) = P(x \in A)P(z \in B)$$

- *The r.v. x and y are independent if and only if*

$$F(x, z) = F(x)F(z), \ \forall x \in \mathcal{X}, \ \forall z \in \mathcal{Z}$$

  where $F(\cdot)$ denotes the CDF.

- *This implies that r.v. x and z are independent if and only if*

$$f(x, z) = f(x)f(z)$$

  where $f(\cdot)$ denotes the PDF/PMF.

7 Expected value

Definition 17. Expected value of the d-dimensional random variable $y \in \mathcal{Y}$ with distribution F is the d-dimensional quantity

$$\mathrm{E}(y) = \int y \mathrm{d}F(y) = \begin{cases} \int_{y \in \mathcal{Y}} y f(y) \mathrm{d}y, & \text{if } y \text{ is cont.} \\ \\ \sum_{y \in \mathcal{Y}} y f(y), & \text{if } y \text{ is discr.} \end{cases}$$

whose ith element is

$$[\mathrm{E}(y)]_i = \begin{cases} \int y_i f_{y_i}(y_i)\mathrm{d}y_i, & \text{if } y_i \text{ is cont.} \\ \\ \sum_{\forall y_i} y_i f_{y_i}(y_i), & \text{if } y_i \text{ is discr.} \end{cases}$$

for $i = 1, ..., d$. Here $f_{y_i}(y_i) = \int_{y\in\mathcal{Y}} f(y)\mathrm{d}y_1 \cdots \mathrm{d}y_{i-1}\mathrm{d}y_{i+1}\cdots \mathrm{d}y_d$ is the marginal PMF/PDF of y_i.

Fact 18. *If $y \in \mathcal{Y}$ is a d-dimensional random variable with PDF/PMF $f_y(\cdot)$, and $\psi : \mathcal{Y} \to \mathbb{R}^d$ is an integrate function with $\psi(\cdot) := (\psi_1(\cdot), ..., \psi_d(\cdot))$, then*

$$E(\psi(y)) = \begin{cases} \int \psi(y) f_y(y)dy, & \text{if } y \text{ is cont.} \\ \\ \sum_{\forall y} \psi(y) f_y(y), & \text{if } y \text{ is discr.} \end{cases}$$

with elements

$$[E(\psi(y))]_i = \begin{cases} \int \psi_i(y) f_y(y)dy, & \text{if } y \text{ is cont.} \\ \\ \sum_{\forall y} \psi_i(y) f_y(y), & \text{if } y \text{ is discr.} \end{cases}$$

Example 19. If $(q \times k)$-dimensional random variable $y \in \mathcal{Y}$ is a matrix, then its expectation $\mathrm{E}(y) = \int y\mathrm{d}F(y)$ is a matrix too

$$\mathrm{E}(y) = \begin{bmatrix} \mathrm{E}(y_{1,1}) & \cdots & \mathrm{E}(y_{1,j}) & \cdots & \mathrm{E}(y_{1,m}) \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ \mathrm{E}(y_{i,1}) & \cdots & \mathrm{E}(y_{i,j}) & \cdots & \mathrm{E}(y_{i,m}) \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ \mathrm{E}(y_{n,1}) & \cdots & \mathrm{E}(y_{n,j}) & \cdots & \mathrm{E}(y_{n,m}) \end{bmatrix}$$

whose (i, j)-th element is

$$[\mathrm{E}(y)]_{i,j} = \mathrm{E}(y_{i,j}) = \begin{cases} \int y_{i,j} f_{y_{i,j}}(y_{i,j})\mathrm{d}y_{i,j}, & \text{if } y_{i,j} \text{ is cont.} \\ \\ \sum_{\forall y_{i,j}} y_{i,j} f_{y_{i,j}}(y_{i,j}), & \text{if } y_{i,j} \text{ is discr.} \end{cases}$$

for $i = 1, ..., n$, and $j = 1, ..., m$. Here $f_{y_{i,j}}(\cdot)$ is the marginal PMF/PDF of $y_{i,j}$.

Proposition 20. *The following properties are valid*

1. *Let fixed matrix/vectors A, c, and $z = c + Ay$ with suitable dimensions then*

$$E(z) = E(c + Ay) = c + AE(y)$$

2. *Let random variables $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$, and let functions ψ_1 and ψ_2 defined on $\mathcal{Z}$ and $\mathcal{Y}$, then*

$$E(\psi_1(z) + \psi_2(y)) = E(\psi_1(z)) + E(\psi_2(y))$$

3. *Random variables $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$ are independent if and only if*

$$E(\psi_1(z)\psi_2(y)) = E(\psi_1(z))E(\psi_2(y))$$

for any functions ψ_1 and ψ_2 defined on $\mathcal{Z}$ and $\mathcal{Y}$.

Proof. Given as Exercise 3 in the Exercise Sheet. □

8 Covariance matrix

Definition 21. The covariance matrix between random variable $z \in \mathcal{Z} \subseteq \mathbb{R}^d$ and random variable $y \in \mathcal{Y} \subseteq \mathbb{R}^q$ is defined as the $d \times q$ matrix

$$\mathrm{Cov}(z, y) = \mathrm{E}\left((z - \mathrm{E}(z))(y - \mathrm{E}(y))^\top\right)$$

Proposition 22. *The following properties are the direct analogues of the $1D$ cases*

1. *Cov$(z, y) =$ E$(zy^\top) -$ E$(z)\,(\text{E}(y))^\top$*
2. *Cov$(z, y) = (\text{Cov}(y, z))^\top$*
3. *Cov$(c_1 + A_1 z, c_2 + A_2 y) = A_1\text{Cov}(z, y)A_2^\top$, for fixed matrices A_1, A_2, and vectors c_1, c_2 with suitable dimensions.*
4. *If z and y are independent random vectors then Cov$(z, y) = 0$*

Proof. (1)-(3) result from the definition. (4) results from Prop 20, as $\mathrm{E}(zy^\top) = \mathrm{E}(z)\,(\mathrm{E}(y))^\top$. □

Proposition 23. *It can be seen that*

$$[\text{Cov}(z, y)]_{i,j} = \text{Cov}(z_i, y_j)$$

for all $i = 1, ..., d$, and $j = 1, ..., q$. Namely, the (i, j)-th element of the covariance matrix between vector z and y is the covariance between their elements z_i and y_j.

Definition 24. The covariance matrix of random vector $y \in \mathcal{Y} \subseteq \mathbb{R}^d$ is defined as the $d \times d$ matrix $\mathrm{Var}(y)$

$$\mathrm{Var}(y) = \mathrm{Cov}(y, y) = \mathrm{E}\left((y - \mathrm{E}(y))(y - \mathrm{E}(y))^\top\right)$$

Proposition 25. *It can be seen that*

$$[\text{Var}(y)]_{i,j} = \text{Cov}(y_i, y_j) \text{ for all } i, j = 1, ..., d$$

and

$$[\text{Var}(y)]_{i,i} = \text{Var}(y_i)$$

for all $i = 1, ..., d$

Proposition 26. *The following properties are the direct analogues of the $1D$ cases*

1. *Var$(y) =$ E$(yy^\top) -$ E$(y)\,(\text{E}(y))^\top$*
2. *Var$(c + Ay) = A\text{Var}(y)A^\top$, for fixed matrix A, and vectors c with suitable dimensions.*
3. *Var$(y) \geq 0$; (semi-positive definite)*

Proof. Given as Exercise 6 in the Exercise Sheet. □

9 Characteristic function

Characteristic functions (CF) provide an alternative way to the probability function for describing a random variable.

Definition 27. The characteristic function of a d dimensional random variable X is

$$\varphi_x(t) = \mathrm{E}(e^{it^T x}) = \int e^{it^T x} \mathrm{d}F(x)$$

for $t \in \mathbb{R}^d$, where $e^{it^T x} = \cos(t^T x) + i \sin(t^T x)$.

Proposition 28. *Some properties of characteristic functions*

1. $\varphi_x(t)$ *exists for all* $t \in \mathbb{R}^d$ *and is absolutely continuous*
2. $\varphi_x(0) = 1$ *and* $|\varphi_x(t)| \leq 1$ *for all* $t \in \mathbb{R}^d$
3. $\varphi_{A+Bx}(t) = e^{it^T A}\varphi_x(B^T t)$ *if* $A \in \mathbb{R}^d$ *and* $B \in \mathbb{R}^{k\times d}$ *are constants*
4. $\varphi_{x+y}(t) = \varphi_x(t)\varphi_y(t)$ *if and only if* x *and* y *are independent*
5. *if* $M_x(t) = E(e^{t^T x})$ *is the moment generating function, then* $M_x(t) = \varphi_x(-it)$

Proof. Given as Exercise 7 in the exercise sheet. □

Fact 29. *Two random variables correspond have equal characteristic functions if and only if they follow the same distribution. AKA: CF completely determines the probability distribution of the random variable*

Fact 30. *If* $\varphi_x(t)$ *is absolutely integrable, then* x *has PDF*

$$f(x) = \frac{1}{(2\pi)^d}\int_{-\infty}^{+\infty} e^{-it^T x}\varphi_x(t)dt$$

Example 31. Address the following:

1. If $z \sim \mathrm{N}(0,1)$ then $\varphi_z(t) = \exp(-\frac{1}{2}t^2)$
2. If $x \sim \mathrm{N}(\mu,\sigma^2)$ then $\varphi_x(t) = \exp(i\mu - \frac{1}{2}t^2\sigma^2)$
3. If $\varphi_z(t) = \exp(-\frac{1}{2}t^2)$ then $f_{\mathrm{N}(0,1)}(z) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2)$

Solution. It is

1. It is

$$\varphi_z(t) = \mathrm{E}(e^{itz}) = \int e^{itz}\mathrm{d}F_{\mathrm{N}(0,1)}(z) = \int e^{itz} f_{\mathrm{N}(0,1)}(z)\mathrm{d}z = \int \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2 + itz)\mathrm{d}z$$
$$= \int \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2 + \frac{2}{2}itz \pm \frac{1}{2}(it)^2 z)\mathrm{d}z = \overbrace{\int \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}(z-it)^2)\mathrm{d}z}^{=1} \times \exp(\frac{1}{2}(it)^2)$$
$$= \exp(-\frac{1}{2}t^2)$$

2. It is $\varphi_x(t) = \varphi_{\mu+\sigma z}(t) = \exp(i\mu)\varphi_z(\sigma t) = \exp(i\mu - \frac{1}{2}t^2\sigma^2)$.

3. It is

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itz}\varphi_z(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-itz)\exp(-\frac{1}{2}t^2)\mathrm{d}t$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-\frac{1}{2}t^2 + i^2 tz)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}\left(t^2 - i^2 z\right)^2 - z^2\right)\mathrm{d}t$$
$$= \sqrt{\frac{1}{2\pi}}\overbrace{\int_{-\infty}^{+\infty}\sqrt{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}\left(t - i^2 z\right)^2\right)\mathrm{d}t}^{=1}\exp\left(-\frac{1}{2}z^2\right) = \sqrt{\frac{1}{2\pi}}\exp\left(-\frac{1}{2}z^2\right) = f_{\mathrm{N}(0,1)}(z)$$

Theorem 32. *The distribution of a* d*-dimensional random variable* $x \in \mathbb{R}^d$ *is completely determined be the set of all* 1-*-dimensional distributions of of linear combinations* $a^\top x$*, for any* $a \in \mathbb{R}^d$.

Proof. Let $y = a^\top x$, for any $a \in \mathbb{R}^d$. Then for any $s \in \mathbb{R}$

$$\varphi_y(s) = \text{E}(e^{is^T y}) = \text{E}(e^{is^\top a^\top x}) = \text{E}(e^{i(as)^\top x}) = \text{E}(e^{i\tilde{t}^\top x}) = \varphi_x(\tilde{t})$$

where $\tilde{t} = as$ is any d-dimensional vector. □

10 Conditioning

Definition 33. Assume a probability space $(\Omega, \mathscr{F}, P)$. For any sets $A, B \in \mathscr{F}$, the conditional probability of A given B is defined as

$$P(A|B) = \frac{P(B \cap A)}{P(B)} \qquad \text{if } P(B) \neq 0.$$

Definition 34. Let $y \in \mathcal{Y}$ be a random variable. Consider a partition $y = (x, \theta)$ with $x \in \mathcal{X}$ and $\theta \in \Theta$. The expected value of θ conditional that random variable $x \in B \subseteq \mathcal{X}$ is

$$\text{E}(\theta|x \in B) = \frac{\text{E}(\theta 1(x \in B))}{P(x \in B)}, \qquad \text{if } P(x \in B) > 0.$$

Fact 35. *Let a random variable $y \in \mathcal{Y}$ with PDF/PMF $f(\cdot)$. Consider a partition $y = (x, \theta)$ with $x \in \mathcal{X}$and $\theta \in \Theta$.*

1. *The conditional MPF/PDF and CDF of random variable θ given the random variable x*

$$f_{\theta|x}(\theta|x) = \frac{f(\theta, x)}{f(x)}, ; \quad F_{\theta|x}(\theta|x) = \begin{cases} \int_{-\infty}^{\theta_1} \cdots \int_{-\infty}^{\theta_d} f_{\theta|x}(\vartheta|x) d\vartheta, & \theta, \textit{ cont.} \\ \\ \sum_{\vartheta_1=-\infty}^{\theta_1} \cdots \sum_{\vartheta_d=-\infty}^{\theta_d} f_{\theta|x}(\vartheta|x), & \theta, \textit{ discr.} \end{cases}$$

   provided that $f(x) > 0$.

2. *The expected value of θ given the random variable x*

$$E(\theta|x) = \int \theta dF_{\theta|x}(\theta|x) = \begin{cases} \int \theta f_{\theta|x}(\theta|x) d\theta & \textit{, if } \theta \textit{ is cont.} \\ \\ \sum_{\forall \theta} \theta f_{\theta|x}(\theta|x) & \textit{, if } \theta \textit{ is discr.} \end{cases} \qquad \textit{provided that } f(x) > 0$$

Example 36. Let a random variable $y \in \mathcal{Y}$ with distribution $F(\cdot)$. Consider a partition $y = (x, \theta)^\top$ with $x \in \mathcal{X}$ and $\theta \in \Theta$. Then

1. $\text{E}(\theta) = \text{E}\left(\text{E}(\theta|x)\right)$
2. $\text{Var}(\theta) = \text{E}\left(\text{Var}(\theta|x)\right) + \text{Var}\left(\text{E}(\theta|x)\right)$

Solution.

1. It is

$$\begin{aligned} \text{E}\left(\text{E}(\theta|x)\right) &= \int \left(\int \theta \text{d}F(\theta|x) \right) \text{d}F(x) = \int \int \theta \text{d}F(x, \theta) = \int \int \theta \text{d}F(\theta|x) \text{d}F(x) \\ &= \int \theta \left(\int \text{d}F(x|\theta) \right) F(\theta) = \int \theta F(\theta) = \text{E}(\theta) \end{aligned}$$

2. It is

$$
\begin{aligned}
\text{Var}(\theta) &= \text{E}\left(\text{E}(\theta\theta^\top)\right) - \text{E}(\theta)\,\text{E}(\theta)^\top = \text{E}\left(\text{E}(\theta\theta^\top|x)\right) - \text{E}\left(\text{E}(\theta|x)\right)\text{E}\left(\text{E}(\theta|x)\right)^\top \\
&= \text{E}\left(\text{E}(\theta\theta^\top|x)\right) - \text{E}\left(\text{E}(\theta|x)\text{E}(\theta|x)^\top\right) + \text{E}\left(\text{E}(\theta|x)\text{E}(\theta|x)^\top\right) - \text{E}\left(\text{E}(\theta|x)\right)\text{E}\left(\text{E}(\theta|x)\right)^\top \\
&= \text{E}\left(\text{E}(\theta\theta^\top|x) - \text{E}(\theta|x)\text{E}(\theta|x)^\top\right) + \text{E}\left(\text{E}(\theta|x)\text{E}(\theta|x) - \text{E}\left(\text{E}(\theta|x)\right)\text{E}\left(\text{E}(\theta|x)\right)^\top\right) \\
&= \text{E}\left(\text{Var}(\theta|x)\right) + \text{Var}\left(\text{E}(\theta|x)\right)
\end{aligned}
$$

Conditional independence

Definition 37. Given a probability space $(\Omega, \mathfrak{B}, P)$, and events $A, B, C \in \mathfrak{B}$, A and B are conditionally independent given C if and only if

$$P(A \cap B|C) = P(A|C)P(B|C), \text{ for } P(C) > 0.$$

Fact 38. *Let $(n+m+d)$-dimensional random variable $y \in \mathcal{Y} \subseteq \mathbb{R}^{n+m+d}$. Consider a partition $y = (x, z, \theta)$ where $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $z \in \mathcal{Z} \subseteq \mathbb{R}^m$, and $\theta \in \Theta \subseteq \mathbb{R}^d$.*

- *The r.v. x and y are independent given θ if and only if*

$$F(y, z|\theta) = F(y|\theta)F(z|\theta),\ \forall x \in \mathbb{R}^n,\ \forall z \in \mathbb{R}^m$$

  where $F(\cdot|\theta)$ denotes the conditional CDF.

- *This implies that r.v. x and z are independent given θ if and only if*

$$f(y, z|\theta) = f(y|\theta)f(z|\theta)$$

  where $f(\cdot|\theta)$ denotes the conditional PDF/PMF.

11 Inverting / updating

This offers a probabilistic mechanism for (1.) inversion $(B|A) \longmapsto (A|B)$, or (2.) updating $(A) \longmapsto (A|B)$.

Fact 39. *[Bayesian theorem with sets] Assume a probability space $(\Omega, \mathscr{F}, P)$. For any sets $A, B \in \mathscr{F}$, it is*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}, \qquad \textit{provided that } P(B) \neq 0.$$

The extension of the Bayesian theorem to the random variables is not straightforward.

Proposition 40. *[Bayesian theorem with random variables] Let a random variable $y \in \mathcal{Y}$. Consider a partition $y = (x, \theta)$ with $x \in \mathcal{X}$ and $\theta \in \Theta$. Then the PDF/PMF of $\theta|x$ is*

$$f(\theta|x) = \frac{f(x|\theta)f(\theta)}{\int f(x|\theta)dF(\theta)} = \begin{cases} \frac{f(x|\theta)f(\theta)}{\int f(x|\theta)f(\theta)d\theta} & \textit{, if } \theta \textit{ is cont.} \\ \frac{f(x|\theta)f(\theta)}{\sum_{\forall\theta} f(x|\theta)f(\theta)} & \textit{, if } \theta \textit{ is discr.} \end{cases}$$

Proof. Given as Exercise 10, in the Exercise Sheet. □

12 Practice

Question 41. *Try the Exercises 8, 9, 10, from the Exercise sheet.*