

## Handout 8: Non-informative priors<sup>a</sup>

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**Aim:** Explain the non-informative priors. Explain, theorize, and construct Laplace and Jeffreys' priors.

### References:

- Robert, C. (2007; Sections 3.5.1 - 3.5.3.). The Bayesian choice: from decision-theoretic foundations to computational implementation. Springer Science & Business Media.
- Berger, J. O. (2013; Sections 3.3, & 4.2.3). Statistical decision theory and Bayesian analysis. Springer Science & Business Media.

**Web applets:** [https://georgios-stats-1.shinyapps.io/demo\\_conjugatejeffreyslplacepriors/](https://georgios-stats-1.shinyapps.io/demo_conjugatejeffreyslplacepriors/)

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## 1 Non-informative priors

*Note 1.* Non-informative priors (or objective priors) are often specified (in practice) when no prior information is available.

*Note 2.* Non-informative prior distributions can be hardly justified in the subjective Bayesian stats because probabilities are considered to be subjective, and hence every researcher has some personal believe about the unknowns. Non-informative priors are used in the Subjective Bayes framework as a last resort when no prior information exist or when the specific application requires them.

*Note 3.* Objective Bayes and objective probability are variants of the subjective Bayesian stats and probability, where the probability is assumed to represent degree of believe about a proposition (similar to Subjective framework) but this is not a matter of an individual's personal degree of believe (in contrast to Subjective framework). There is no room for personal belief, hence everyone should assign the same prior probabilities, hence the posterior probability should be the same for different researchers. This Bayesian philosophical variation can be hardly justified, no?

*Note 4.* Objective Bayes requires the specification of non-informative priors as a mean to eliminate subjective/individual believes to the priors, and assign priors generally accepted by everybody. So these non-informative priors are also called objective priors.

*Note 5.* Objective Bayes is an arguable variation of the Bayesian framework. Different people may have different prior degrees of believe and hence the use of a prior accepted by everybody is arguable. Also, prior degree of believe of a group or several groups of researchers can still be expressed by subjective priors and be justified in the Subjective Bayesian framework.

*Note 6.* Some tools used to specify non-informative (objective) priors often break the Bayesian paradigm, and may produce unreasonable results, e.g. violation of the likelihood principle.

*Note 7.* In most applications it is almost impossible to specify non-informative priors representing exactly total ignorance about the problem at hand. They should rather be taken as reference or default priors, upon which everyone could fall back when the prior information is missing. More realistically, non-informative priors about specific aspects

of the problem or features of the statistical model (e.g., transformations, rotations) can be derived, and justified in the Subjective and Objective Bayes framework.

## 2 Laplace non-informative priors

**Definition 8.** The principle of insufficient prior states that we do not have any reason to think that one value of the unknown quantity is more likely than any other, so we should use a Uniform prior.

**Definition 9.** Laplace prior  $\Pi(\theta)$  for  $\theta \in \Theta$  is specified such as

$$d\Pi(\theta) \propto \underbrace{1}_{\propto \pi(\theta)} d\theta \quad \text{with pdf/pmf} \quad \pi(\theta) \propto 1 \quad (1)$$

and builds upon the principle of insufficient prior.

*Remark 10.* Laplace prior (1) places the same degree of believe at each value of  $\theta$  if  $\theta$  discrete, and at each interval  $d\theta$  of the same length if  $\theta$  is continuous.

*Remark 11.* Laplace priors (1) can be improper. In such cases the properness condition has to be checked. Laplace priors are improper when  $\Theta$  is unbounded parametric space but are proper when  $\Theta$  is bounded.

**Example 12.** Consider the Bayesian model

$$\begin{cases} y_i | \mu & \sim N(\mu, 1), \quad \forall i = 1, \dots, n \\ \mu & \sim \Pi(\mu) \end{cases}$$

Laplace prior  $d\Pi(\mu) \propto 1d\mu$  is an improper prior since  $\int_{\Theta} d\Pi(\mu) = \int_{\mathbb{R}} \pi(\mu)d\mu = \int_{\mathbb{R}} 1d\mu = +\infty$ . However, it can be used as a prior because it satisfies the properness condition

$$\int_{\mathbb{R}} \prod_{i=1}^n N(y_i | \mu, 1) 1d\mu \propto 2^{-\frac{n}{2}} (\pi)^{-\frac{n-1}{2}} \exp\left(-\frac{1}{2} \left(\sum x_i^2\right) + \frac{1}{2} \left(\sum x_i\right)^2\right) < \infty$$

and hence it leads to a well defined posterior probability distribution.

*Remark 13.* Laplace prior (1) are not invariant under transformations. Assume Laplace (non-informative) prior for  $\theta$ , with pdf  $\pi(\theta) \propto 1$ . Consider a random quantity  $\psi$  where  $\psi = g(\theta)$ , such that  $g : \Theta \rightarrow \Psi$  is an 1-1 transformation. Then it is

$$\pi_{\psi}(\psi) \propto \pi_{\theta}(g^{-1}(\psi)) \left| \frac{d}{d\psi} g^{-1}(\psi) \right|$$

which is not necessarily flat, and hence not necessarily non-informative. This is strange because it means that we a priori know nothing about  $\theta$  but we a priori know something about  $\psi = g(\theta)$ ...

**Example 14.** Consider an experiment with sampling distribution

$$y_i | \theta \sim \text{Br}(\theta), \quad \forall i = 1, \dots, n$$

The Laplace (non-informative) prior for success frequency  $\theta \in [0, 1]$  is  $d\Pi(\theta) \propto 1d\theta$  and hence  $\theta \sim \text{Un}(0, 1)$  which is a proper prior. This implies that odds  $\psi = \frac{\theta}{1-\theta}$  have prior

$$\pi_{\psi}(\psi) \propto \pi_{\theta}\left(\frac{\psi}{1+\psi}\right) \left| \frac{d}{d\psi} \frac{\psi}{1+\psi} \right| \propto \frac{1}{(1+\psi)^2}$$

which is informative. So I a priori know nothing about the frequency  $\theta$  but I know something about the odds  $\psi$ ...

*Note 15.* Those using Laplace priors argue that You should parametrize the likelihood  $f(y|\theta)$  according to a desired parameterization (e.g., success frequency  $\theta$ ), assign a Laplace prior, and stick with it ignoring reparametrizations...

### 3 Jeffreys' priors

**Note 16.** Let  $y = (y_1, \dots, y_n)$  observables drawn from a sampling distribution  $F(y|\theta)$  with density  $f(y|\theta)$ . I use this notation hereafter. We aim to specify a prior  $\Pi(\theta)$  with density  $\pi(\theta)$  in the Bayesian model

$$\begin{cases} y|\theta & \sim F(y|\theta) \\ \theta & \sim \Pi(\theta) \end{cases} \quad (2)$$

so that  $\Pi(\theta)$  can be invariant to 1 – 1 transformations. Precisely, this is an invariance to a 1-1 transformations, and it is reasonable in certain type of applications (mentioned in the classroom).

**Definition 17.** The Jeffreys' prior distribution  $\Pi(\theta)$  of the unknown parameter  $\theta \in \Theta = \mathbb{R}^k$   $k \geq 1$  is defined such as

$$d\Pi(\theta) \propto \underbrace{\sqrt{\det(\mathcal{J}(\theta))}}_{\propto \pi(\theta)} d\theta \quad \text{with pdf/pmf} \quad \pi(\theta) \propto \sqrt{\mathcal{J}(\theta)},$$

where  $\mathcal{J}(\theta)$  is the Fisher Information.

**Definition 18.** Let  $y = (y_1, \dots, y_n)$  observables drawn from a sampling distribution  $F(y|\theta)$  with density  $f(y|\theta)$  labeled by parameter  $\theta \in \Theta \subseteq \mathbb{R}^k$ . Let  $f(y|\theta)$  denote the likelihood.

Fisher Information  $\mathcal{J}(\theta)$  is a  $k \times k$  matrix defined as

$$\mathcal{J}(\theta) = E_{F(\cdot|\theta)} \left( \left[ \frac{d}{d\theta} \log(f(y|\theta)) \right]^\top \left[ \frac{d}{d\theta} \log(f(y|\theta)) \right] \right)$$

where

$$\frac{d}{d\theta} \log f(y|\theta) = \left[ \frac{d}{d\theta_1} \log(f(y|\theta)), \dots, \frac{d}{d\theta_j} \log(f(y|\theta)), \dots, \frac{d}{d\theta_d} \log(f(y|\theta)) \right] \in \mathbb{R}^k$$

So the  $(i, j)$  element of Fisher Information  $\mathcal{J}(\theta)$  is

$$[\mathcal{J}(\theta)]_{i,j} = E_{F(\cdot|\theta)} \left( \left[ \frac{d}{d\theta_i} \log(f(y|\theta)) \right] \left[ \frac{d}{d\theta_j} \log(f(y|\theta)) \right] \right)$$

In the univariate case  $\theta \in \Theta \subseteq \mathbb{R}$ , it is

$$\mathcal{J}(\theta) = E_{F(\cdot|\theta)} \left( \left( \frac{d}{d\theta} \log(f(y|\theta)) \right)^2 \right)$$

**Fact 19.** Some properties of Fisher information  $\mathcal{J}(\theta)$ :

1. Under regularity conditions, when  $\theta \in \Theta = \mathbb{R}^k$ ,  $\mathcal{J}(\theta)$  simplifies to

$$[\mathcal{J}(\theta)]_{i,j} = -E_{F(\cdot|\theta)} \left( \frac{d^2}{d\theta_i d\theta_j} \log(f(y|\theta)) \mid \theta \right) = - \int_{\mathcal{X}} \frac{d^2}{d\theta_i d\theta_j} \log(f(y|\theta)) dF(y|\theta).$$

and when  $\theta \in \Theta = \mathbb{R}$  (univariate),  $\mathcal{J}(\theta)$  simplifies to

$$\mathcal{J}(\theta) = -E_{F(\cdot|\theta)} \left( \frac{d^2}{d\theta^2} \log(f(y|\theta)) \mid \theta \right) = - \int_{\mathcal{X}} \frac{d^2}{d\theta^2} \log(f(y|\theta)) dF(y|\theta),$$

2. Let  $y = (y_1, \dots, y_n)$  be observables frown iid from distribution  $F(y|\theta)$ , and let  $\mathcal{J}_n(\theta)$  be the associated Fisher information based on  $n$  observables. Then  $\mathcal{J}_n(\theta) = n\mathcal{J}_1(\theta)$ .
3. Let  $g : \Theta \rightarrow \Psi$  with  $\psi = g(\theta)$  be a 1 – 1 transformation. Then the Fisher information is

$$\mathcal{J}(\psi) = J^\top \mathcal{J}(\theta) J, \quad \text{and} \quad \det(\mathcal{J}(\psi)) = \det(\mathcal{J}(\theta)) \det(J)^2$$

where  $J = \frac{d\theta}{d\psi}$  is the Jacobian of the transformation  $\psi = g(\theta)$  whose  $(i, j)$  element is  $[J]_{i,j} = \frac{\partial \theta_i}{\partial \psi_j}$ .

In the univariate case where  $\theta \in \Theta \subseteq \mathbb{R}$   $\psi \in \Psi \subseteq \mathbb{R}$  it is

$$\mathcal{I}(\psi) = \mathcal{I}(\theta) \left( \frac{d\theta}{d\psi} \right)^2$$

*Note 20.* The rationale of Jeffreys' prior is that Fisher information is widely accepted as an indicator of the amount of information brought by the statistics model (or the observation) about  $\theta$ , and hence the values of  $\theta$  for which  $\mathcal{I}(\theta)$  is larger should be more likely for the prior distribution. E.g, in Fact 19(2),  $\mathcal{I}_n(\theta)$  is understood as  $n$  unites of observation information if  $\mathcal{I}_1(\theta)$  is understood as 1 unite of observational information. Fisher information  $\mathcal{I}(\theta)$  can evaluate the ability of the model to discriminate between  $\theta$  and  $\theta + d\theta$  through the expected slope of  $\log(f(y|\theta))$ .

- Therefore, to favor the values of  $\theta$  for which  $\mathcal{I}(\theta)$  is large is equivalent to minimizing the influence of the prior distribution and is therefore as noninformative as possible.

**Theorem 21.** *Jeffreys' priors are invariant under 1-1 transformations. Let Jeffreys' prior  $\Pi_\theta(\theta)$  on  $\theta$  with density  $\pi_\theta(\theta) \propto \sqrt{\mathcal{I}(\theta)}$  associated to sampling pdf/pmf  $f(y|\theta)$ , and Jeffreys' prior  $\Pi_\psi(\psi)$  on  $\psi$  with density  $\pi_\psi(\psi) \propto \sqrt{\mathcal{I}(\psi)}$  associated to sampling pdf/pmf  $f(y|\psi)$ , where  $\psi = g(\theta)$  and  $g : \Theta \rightarrow \Psi$  is a 1-1 transformation. Then  $\pi_\psi(\psi) \propto \pi_\theta(\theta) \left| \frac{\partial \theta}{\partial \psi} \right|$ .*

*Proof.* We prove it for the case where  $\theta \in \mathbb{R}$ , as the extension to the multivariate setting is straightforward. It is  $\mathcal{I}(\psi) = \mathcal{I}(\theta) \left( \frac{\partial \theta}{\partial \psi} \right)^2$ , because

$$\begin{aligned} \mathcal{I}(\psi) &= \mathbb{E}_{y \sim F(\cdot|\psi)} \left( \frac{\partial}{\partial \psi} \log(f(y|\psi)) \right)^2 = \int_{\mathcal{Y}} \left( \frac{\partial}{\partial \psi} \log(f(y|\psi)) \right)^2 dF(y|\psi) \\ &= \int_{\mathcal{Y}} \left( \frac{\partial}{\partial \psi} \log(f(y|\theta)) \right)^2 dF(y|\theta) = \int_{\mathcal{Y}} \left( \frac{\partial \theta}{\partial \psi} \frac{\partial}{\partial \theta} \log(f(y|\theta)) \right)^2 dF(y|\theta) \\ &= \int_{\mathcal{X}} \left( \frac{\partial}{\partial \theta} \log(f(y|\theta)) \right)^2 dF(y|\theta) \left( \frac{\partial \theta}{\partial \psi} \right)^2 = \mathbb{E}_{y \sim F(\cdot|\theta)} \left( \frac{\partial}{\partial \theta} \log(f(y|\theta)) \right)^2 \left( \frac{\partial \theta}{\partial \psi} \right)^2 \\ &= \mathcal{I}(\theta) \left( \frac{\partial \theta}{\partial \psi} \right)^2 \end{aligned}$$

Then

$$\pi_\psi(\psi) \propto \sqrt{\mathcal{I}(\psi)} = \sqrt{\mathcal{I}(\theta) \left( \frac{\partial \theta}{\partial \psi} \right)^2} \propto \sqrt{\mathcal{I}(\theta)} \left| \frac{\partial \theta}{\partial \psi} \right| \propto \pi_\theta(\theta) \left| \frac{\partial \theta}{\partial \psi} \right|$$

□

**Example 22.** Consider observable  $r$  drawn from a Binomial sampling distribution  $r|\theta \sim \text{Bn}(n, \theta)$  with pdf  $\text{Br}(r|\theta) = \binom{n}{r} \theta^r (1-\theta)^{n-r}$  and mean  $\mathbb{E}_{\text{Bn}(n, \theta)}(r) = n\theta$ . Find the Jeffreys prior for  $\theta$ . Compute the posterior of  $\theta$  given  $r$ .

**Solution.** The likelihood is  $f(r|\theta) = \text{Br}(r|\theta) = \binom{n}{r}\theta^r(1-\theta)^{n-r}$  So

$$\begin{aligned}\log(f(r|\theta)) &= \log\left(\binom{n}{r}\right) + r\log(\theta) + (n-r)\log(1-\theta) \implies \\ \frac{d^2}{d\theta^2} \log(f(r|\theta)) &= -\frac{r}{\theta^2} - \frac{n-r}{(1-\theta)^2} \implies \\ \mathcal{J}(\theta) &= -E_{\text{Bn}(n,\theta)}\left(-\frac{r}{\theta^2} - \frac{n-r}{(1-\theta)^2}\right) = \frac{n}{\theta(1-\theta)} \implies \\ \text{Jeffreys' prior is } \pi^{(\text{Jef;Bn})}(\theta) &\propto \frac{1}{\theta^{1/2}(1-\theta)^{1/2}} \propto \text{Be}(\theta|0.5, 0.5)\end{aligned}\quad (3)$$

The posterior is

$$\pi^{(\text{Jef;Bn})}(\theta|r) \propto \text{Bn}(r|n, \theta)\pi^{(\text{Jef;Bn})}(\theta) \propto \theta^{r-1/2}(1-\theta)^{n-r-1/2}$$

**Example 23.** Consider observable  $n$  drawn from a Negative binomial  $n|\theta \sim \text{Nb}(r, \theta)$  with pdf  $\text{Nb}(n|r, \theta) = \binom{n-1}{r-1}\theta^r(1-\theta)^{n-r}$  and mean  $E_{\text{Nb}(r,\theta)}(n) = r/\theta$ . Find the Jeffreys prior for  $\theta$ . Compute posterior of  $\theta$  given  $n$ .

**Solution.** So

$$\begin{aligned}\log(f(r|\theta)) &= \log\left(\binom{n-1}{r-1}\right) + r\log(\theta) + (n-r)\log(1-\theta) \implies \\ \frac{d^2}{d\theta^2} \log(f(r|\theta)) &= -\frac{r}{\theta^2} - \frac{n-r}{(1-\theta)^2} \implies \\ \mathcal{J}(\theta) &= -E_{\text{Nb}(r,\theta)}\left(-\frac{r}{\theta^2} - \frac{n-r}{(1-\theta)^2}\right) = \frac{r}{\theta^2(1-\theta)} \implies \\ \text{Jeffreys' prior is } \pi^{(\text{Jef;Nb})}(\theta) &\propto \frac{1}{\theta^1(1-\theta)^{1/2}}\end{aligned}\quad (4)$$

The posterior is  $\pi^{(\text{Jef;Nb})}(\theta|n) \propto f(r|\theta)\pi^{(\text{Jef;Nb})}(\theta) \propto \theta^{r-1}(1-\theta)^{n-r-1/2}$ .

*Remark 24.* Jeffreys' prior can violate the Likelihood Principle. This is because its derivation depends on the form of the specific experiment via Fisher information which can differ for two different experiments even though they have proportional likelihoods. E.g.; For two experiments with propositional likelihoods  $f_1(\theta|y_1) \propto f_2(\theta|y_2) \propto L(\theta)$ , Jeffreys' priors are  $\pi^{(\text{ip},1)}(\theta) \propto \mathcal{J}_1(\theta)$ , and  $\pi^{(\text{ip},2)}(\theta) \propto \mathcal{J}_2(\theta)$ , so

$$\pi^{(\text{exp } 1)}(\theta|y_1) = \frac{L(\theta)\sqrt{\mathcal{J}_1(\theta)}}{\int L(\theta)\sqrt{\mathcal{J}_1(\theta)}d\theta}; \quad \pi^{(\text{exp } 2)}(\theta|y_2) = \frac{L(\theta)\sqrt{\mathcal{J}_2(\theta)}}{\int L(\theta)\sqrt{\mathcal{J}_2(\theta)}d\theta}; \quad \mathcal{J}_1(\theta) \not\propto \mathcal{J}_2(\theta) \implies \pi^{(\text{exp } 1)}(\theta|y_1) \neq \pi^{(\text{exp } 2)}(\theta|y_2)$$

In the Examples 22 and 23, even though the two likelihoods are equal up to a multiplicative constant, i.e.

$$\text{Bn}(r|n, \theta) \propto \text{Nb}(n|r, \theta) \propto \theta^r(1-\theta)^{n-r} \quad (5)$$

Jeffreys' priors led to different posteriors  $\pi^{(\text{Jef;Br})}(\theta|r) \neq \pi^{(\text{Jef;Nb})}(\theta|n)$ .

**Example 25.** Let  $y \in \mathbb{R}$  be an observable. Consider the statistical model

$$y|\mu, \sigma^2 \stackrel{\text{iid}}{\sim} \text{N}(\mu, \sigma^2) \quad \text{where} \quad (\mu, \sigma^2) \in \mathbb{R} \times (0, \infty).$$

1. Specify the Jeffreys' prior for  $\theta = \mu$ , when  $\sigma$  is known.
2. Specify the Jeffreys' prior for  $\theta = \sigma$ , when  $\mu$  is known.
3. Specify the Jeffreys' prior for  $\theta = (\mu, \sigma)$ .

**Solution.** It is

$$\log f(y|\theta) = \log(N(y|\mu, \sigma^2)) = -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log(\sigma^2) - \frac{1}{2} \frac{(y-\mu)^2}{\sigma^2}$$

1. It is

$$\begin{aligned} \frac{d}{d\mu} \log(N(y|\mu, \sigma^2)) &= \frac{(y-\mu)}{\sigma^2} \\ \frac{d^2}{d\mu^2} \log(N(y|\mu, \sigma^2)) &= \frac{\partial}{\partial \mu} \frac{(y-\mu)}{\sigma^2} = -\frac{1}{\sigma^2} \\ \mathcal{J}(\mu) &= -E_{y \sim N(\mu, \sigma^2)} \left( \frac{d^2}{d\mu^2} \log(N(y|\mu, \sigma^2)) \right) = \frac{1}{\sigma^2} \end{aligned}$$

and hence  $\pi^{(JP)}(\mu) \propto \sqrt{\mathcal{J}(\mu)} \propto 1$

2. It is

$$\begin{aligned} \frac{d}{d\sigma} \log(N(y|\mu, \sigma^2)) &= -\frac{1}{\sigma} + \frac{(y-\mu)^2}{\sigma^3} \\ \frac{d^2}{d\sigma^2} \log(N(y|\mu, \sigma^2)) &= \frac{1}{\sigma^2} - 3 \frac{(y-\mu)^2}{\sigma^4} = \frac{1}{\sigma^2} - 3 \frac{1}{\sigma^2} \left( \frac{y-\mu}{\sigma^2} \right)^2 \end{aligned}$$

So

$$\mathcal{J}(\sigma) = -E_{y \sim N(\mu, \sigma^2)} \left( \frac{\partial^2}{\partial \theta^2} \log(N(y|\mu, \sigma^2)) \right) = -E_{y \sim N(\mu, \sigma^2)} \left( \frac{1}{\sigma^2} - 3 \frac{1}{\sigma^2} \left( \frac{y-\mu}{\sigma^2} \right)^2 \right) \propto \frac{1}{\sigma^2}$$

and hence  $\pi^{(JP)}(\sigma) \propto \sqrt{\mathcal{J}(\sigma)} \propto \frac{1}{\sigma}$

3. it is

$$\frac{d^2}{d\mu d\sigma} \log f(y|\theta) = \frac{d^2}{d\mu d\sigma} \log(N(y|\mu, \sigma^2)) = -2 \frac{(y-\mu)}{\sigma^3}$$

So

$$\mathcal{J}(\mu, \sigma) = -E_{y \sim N(\mu, \sigma^2)} \left( \frac{d^2}{d\theta^2} \log(N(y|\mu, \sigma^2)) \right) = -E_{y \sim N(\mu, \sigma^2)} \begin{bmatrix} -\frac{1}{\sigma^2} & -2 \frac{(y-\mu)}{\sigma^3} \\ -2 \frac{(y-\mu)}{\sigma^3} & \frac{1}{\sigma^2} - 3 \frac{(y-\mu)^2}{\sigma^4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 2 \frac{1}{\sigma^2} \end{bmatrix}$$

and  $\pi^{(JP)}(\mu, \sigma) \propto \sqrt{\det(\mathcal{J}(\mu, \sigma))} \propto \frac{1}{\sigma^2}$ .

**Example 26.** Consider the model of Normal linear regression where the observables are pairs  $(\phi_i, y_i)$  for  $i = 1, \dots, n$ , assumed to be modeled according to the sampling distribution  $y_i|\beta, \sigma^2 \stackrel{\text{ind}}{\sim} N(\phi_i^\top \beta, \sigma^2)$  for  $i = 1, \dots, n$  with unknown  $(\beta, \sigma^2) \in \mathbb{R}^d \times \mathbb{R}_+$ . Namely, the sampling distribution in vector form is

$$y|\beta, \sigma^2 \sim N_n(\Phi\beta, I\sigma^2)$$

where  $y = (y_1, \dots, y_n)$ , and  $\Phi$  is the design matrix. Here  $\beta$  is  $d$ -dimensional. Find the Jeffreys' priors for  $(\beta, \sigma^2)$ .

**Hint:** Recall your AMV:  $\frac{d}{dx} x^\top A x = 2Ax$ ,  $\frac{d}{dx} (c + Ax) = A$ , and  $\frac{d}{dx} (A(x))^\top = \left( \frac{d}{dx} A(x) \right)^\top$ .

**Hint:** If  $y|\beta, \sigma^2 \sim N_n(\Phi\beta, I\sigma^2)$ , then  $E_{y|\beta, \sigma^2 \sim N_n(\Phi\beta, I\sigma^2)} \left( (y - \Phi\beta)^\top (y - \Phi\beta) \right) = n\sigma^2$ .

**Solution.** Let's set  $\xi = \sigma^2$  to simplify notation... The log likelihood is

$$\log(f(y|\beta, \xi)) = -\frac{n}{2} \log(\xi) - \frac{1}{2\xi} (y - \Phi\beta)^\top (y - \Phi\beta)$$

Let's compute the derivatives

$$\begin{aligned}
\frac{d}{d\xi} \log(f(y|\beta, \xi)) &= -\frac{n}{2} \frac{1}{\xi} + \frac{1}{2} \frac{1}{\xi^2} (y - \Phi\beta)^\top (y - \Phi\beta) \\
\frac{d^2}{d\xi^2} \log(f(y|\beta, \xi)) &= \frac{d}{d\xi} \left( -\frac{n}{2} \frac{1}{\xi} + \frac{1}{2} \frac{1}{\xi^2} (y - \Phi\beta)^\top (y - \Phi\beta) \right) = \frac{n}{2} \frac{1}{\xi^2} - \frac{1}{\xi^3} (y - \Phi\beta)^\top (y - \Phi\beta) \\
\frac{d}{d\beta} \log(f(y|\beta, \xi)) &= -\frac{1}{\xi} \Phi^\top (y - \Phi\beta) \\
\frac{d^2}{d\beta^2} \log(f(y|\beta, \xi)) &= \frac{d}{d\beta} \left( -\frac{1}{\xi} \Phi^\top (y - \Phi\beta) \right) = -\frac{1}{\xi} \Phi^\top \Phi \\
\frac{d}{d\xi} \frac{d}{d\beta} \log(f(y|\beta, \xi)) &= \frac{d}{d\xi} \left( -\frac{1}{\xi} \Phi^\top (y - \Phi\beta) \right) = \frac{1}{\xi^2} \Phi^\top (y - \Phi\beta)
\end{aligned}$$

Lets compute the components of the Fisher information. The sampling distribution of  $y$  is  $y|\beta, \xi \sim \mathbf{N}(\Phi\beta, I\sigma^2)$  with  $E(y|\beta, \xi) = \Phi\beta$  and  $\text{Var}(y|\beta, \xi) = I\sigma^2$ . So

$$\begin{aligned}
E_{\mathbf{N}(\Phi\beta, I\sigma^2)} \left( \frac{d^2}{d\xi^2} \log(f(y|\beta, \xi)) \right) &= E_{\mathbf{N}(\Phi\beta, I\sigma^2)} \left( \frac{n}{2} \frac{1}{\xi^2} - \frac{1}{\xi^3} (y - \Phi\beta)^\top (y - \Phi\beta) \right) = +\frac{n}{2} \frac{1}{\xi^2} - \frac{1}{\xi^3} n\xi = -\frac{n}{2} \frac{1}{\xi^2} \\
E_{\mathbf{N}(\Phi\beta, I\sigma^2)} \left( \frac{d^2}{d\beta^2} \log(f(y|\beta, \xi)) \right) &= -\frac{1}{\xi} \Phi^\top \Phi \\
E_{\mathbf{N}(\Phi\beta, I\sigma^2)} \left( \frac{d}{d\xi} \frac{d}{d\beta} \log(f(y|\beta, \xi)) \right) &= E_{\mathbf{N}(\Phi\beta, I\sigma^2)} \left( \frac{1}{\xi^2} \Phi^\top (y - \Phi\beta) \right) = \frac{1}{\xi^2} \Phi^\top (E_{\mathbf{N}(\Phi\beta, I\sigma^2)}(y) - \Phi\beta) = 0
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{J} &= -E_{\mathbf{N}(\Phi\beta, I\sigma^2)} \left( \frac{d^2}{d\theta^2} \log(f(y|\theta)) \right) = -E_{\mathbf{N}(\Phi\beta, I\sigma^2)} \left( \begin{bmatrix} \frac{d^2}{d\beta^2} \log(f(y|\beta, \xi)) & \frac{d}{d\xi} \frac{d}{d\beta} \log(f(y|\beta, \xi)) \\ \frac{d}{d\xi} \frac{d}{d\beta} \log(f(y|\beta, \xi)) & \frac{d^2}{d\xi^2} \log(f(y|\beta, \xi)) \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{1}{\xi} \Phi^\top \Phi & 0 \\ 0 & \frac{n}{2} \frac{1}{\xi^2} \end{bmatrix}
\end{aligned}$$

So the Jeffreys prior for  $(\beta, \xi)$  has density such as

$$\pi^{(\text{JP})}(\beta, \xi) \propto \sqrt{\det(\mathcal{J})} = \sqrt{\det \left( \frac{1}{\xi} \Phi^\top \Phi \right) \det \left( \frac{n}{2} \frac{1}{\xi^2} \right)} = \left( \frac{1}{\xi} \right)^{\frac{d}{2}+1} \sqrt{\det(\Phi^\top \Phi) \det \left( \frac{n}{2} \right)} \propto \left( \frac{1}{\xi} \right)^{\frac{d}{2}+1}$$

Namely  $\pi^{(\text{JP})}(\beta, \sigma^2) = (\sigma^2)^{-\frac{d}{2}-1}$ .

## 4 Limiting posterior distributions

*Note 27.* One way to derive a posterior in the absence of prior information, is to specify a non-informative (possibly improper) prior, check the properness condition, and compute the posterior by the Bayes theorem; i.e.

$$\pi(\theta|y) = \frac{f(y|\theta) \pi^{(\text{impr.})}(\theta)}{\int f(y|\theta) \pi^{(\text{impr.})}(\theta) d\theta}$$

*Note 28.* An alternative way to derive a posterior distribution in the absence of prior information, is to compute it as a limit of a posterior updated from a proper prior. Namely:

1. specify a proper prior with a specific parametric form  $\pi(\theta|\tau) \propto \tilde{\pi}(\theta|\tau)$  (e.g., conjugate prior with hyperparameter  $\tau$ ), and compute the kernel  $\tilde{\pi}(\theta|\tau)$  of its pdf/pmf as  $\pi(\theta|\tau) \propto \tilde{\pi}(\theta|\tau)$

2. find values  $\tau'$  for the prior hyper parameters such that the limit

$$\tilde{\pi}'(\theta) = \lim_{\tau \rightarrow \tau'} \tilde{\pi}(\theta|\tau), \quad \text{when } \tau \rightarrow \tau'$$

can be considered as the kernel of a non-informative prior. E.g.,  $\tilde{\pi}'(\theta)$  can be a Jeffreys prior.

3. compute the posterior distribution  $\pi(\theta|y, \tau)$  updated from the proper prior  $\pi(\theta|\tau)$ , by Bayes theorem

$$\pi(\theta|y, \tau) = \frac{f(y|\theta)\tilde{\pi}(\theta|\tau)}{\int f(y|\theta)\tilde{\pi}(\theta|\tau)d\theta}$$

4. compute the limiting posterior as

$$\pi'(\theta|y) = \lim_{\tau \rightarrow \tau'} \pi(\theta|y, \tau)$$

The limiting posterior  $\pi'(\theta|y)$  is derived based on no prior information,

*Note 29.* A convenient way is to specify a conjugate prior  $\pi(\theta|\tau)$  and let the  $\tau$  approach values that reduce the strength of the prior information.

**Example 30.** Let  $\pi^{(CP)}(\theta|\tau)$  denote the conjugate prior,  $\pi^{(JP)}(\theta|\tau)$  denote the Jeffreys' prior, and  $\pi^{(LP)}(\theta|\tau)$  denote the Laplace prior. For Bernoulli statistical model  $y_i \stackrel{\text{iid}}{\sim} \text{Br}(\theta)$ ,  $\forall i = 1, \dots, n$ , it is

**Conjugate prior:**  $\theta|\tau = (a, b) \sim \text{Be}(a, b)$  has pdf kernel  $\pi^{(CP)}(\theta|a, b) \propto \tilde{\pi}^{(CP)}(\theta|a, b) = \theta^{a-1}(1-\theta)^{b-1}$  is updated By Bayes theorem to

$$\pi^{(CP)}(\theta|y, a, b) = \text{Be}\left(\sum_{i=1}^n y_i + a, n - \sum_{i=1}^n y_i + b\right)$$

**Laplace prior:**  $\theta \sim \text{U}(0, 1) \equiv \text{Be}(1, 1)$  has pdf kernel  $\pi^{(LP)}(\theta) \propto \tilde{\pi}^{(LP)}(\theta) = 1$ . We find point  $(a', b') = (1, 1)$  such that

$$\lim_{(a,b) \rightarrow (1,1)} \tilde{\pi}^{(CP)}(\theta|a, b) = 1 = \tilde{\pi}^{(LP)}(\theta)$$

so

$$\pi^{(LP)}(\theta|y) = \lim_{(a,b) \rightarrow (1,1)} \pi^{(CP)}(\theta|y, a, b) = \text{Be}\left(\sum_{i=1}^n y_i + 1, n - \sum_{i=1}^n y_i + 1\right)$$

**Jeffreys prior:**  $\theta \sim \text{Be}(\theta|0.5, 0.5)$  has pdf kernel  $\pi^{(JP)}(\theta) \propto \tilde{\pi}^{(JP)}(\theta) = \theta^{-0.5}(1-\theta)^{-0.5}$ . We find point  $(a', b') = (0.5, 0.5)$  such that

$$\lim_{(a,b) \rightarrow (0.5,0.5)} \tilde{\pi}^{(JP)}(\theta|a, b) = 1 = \tilde{\pi}^{(JP)}(\theta)$$

so

$$\pi^{(JP)}(\theta|y) = \lim_{(a,b) \rightarrow (0.5,0.5)} \pi^{(CP)}(\theta|y, a, b) = \text{Be}\left(\sum_{i=1}^n y_i + 0.5, n - \sum_{i=1}^n y_i + 0.5\right)$$

## 5 Practice

**Question 31.** For practice try to address the Exercises 47, 48, and 52, from the Exercise sheet. You can try the Exercise 53 from the Exercise sheet. which is related to Regression.