

# Monte Carlo approximation

An introduction for practical use in R

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[Back to README](#)

## ***Aim***

Students will be able to

- apply Monte Carlo approximation with R
- approximate integrals, expected values, modes, quantiles, etc... with R

## **Briefly ...**

**Monte Carlo approximation** is a stochastic procedure for the evaluation of intractable quantities. It involves:

1. properly generating a large random sample from a suitable distribution, and
2. computing a statistic asymptotically consistent to the intractable quantity of interest.

This statistic is the Monte Carlo approximate of the intractable quantity.

## **Approximating expected values, and integrals...**

### ***Notation***

Assume a random variable  $\theta \in \Theta$  following a distribution  $\Pi(\cdot)$ .

Let  $h(\cdot) : \Theta \rightarrow \mathbb{R}^d$ ,  $d \geq 1$ , be a measurable function.

Assume that there is interest in approximating the expected value

$$\mathbb{E}_{\pi}(h(\theta)) = \int h(\theta) d\Pi(\theta) = \begin{cases} \int h(\theta) \pi(\theta) d\theta, & \text{cont.} \\ \sum h(\theta) \pi(\theta), & \text{discr.} \end{cases}$$

### ***Monte Carlo approximation procedure:***

1. Draw a random sample

$$\theta^{(j)} \sim \Pi(\cdot), \text{ for } j = 1, \dots, N$$

2. Compute

$$\bar{h}_N = \frac{1}{N} \sum_{j=1}^N h(\theta^{(j)})$$

We say that  $\bar{h}_N$  is a **Monte Carlo approximate of**  $\mathbb{E}_{\pi}(h(\theta))$ , where

$$\mathbb{E}_{\pi}(h(\theta)) \approx \bar{h}_N \text{ when } N \text{ is large enough.}$$

which can be justified by law of Large Number (LLN) arguments:  $\bar{h}_N \rightarrow \mathbb{E}_{\pi}(h(\theta))$  as  $N \rightarrow \infty$ .

### Example

Consider the integral

$$I = \int_0^{\infty} x^5 \exp(-3x) dx$$

Compute a Monte Carlo approximate of integral  $I$ .

The Exact value is  $I = 0.1646091$ .

### Solution

It is

$$\begin{aligned} I &= \int_0^{\infty} x^5 \exp(-3x) dx = \int x^5 \frac{1}{3} \exp(-3x) 1(x \in (0, \infty)) dx \\ &= \frac{1}{3} \int x^5 \underbrace{\exp(-3x) 1(x \in (0, \infty))}_{=d\Pi_{\text{Exp}(3)}(x)} dx = \frac{1}{3} \int x^5 d\Pi_{\text{Exp}(3)}(x) = \frac{1}{3} E_{\text{Exp}(3)}(x^5) \\ &\approx \frac{1}{3} \frac{1}{N} \sum_{j=1}^N \left(x^{(j)}\right)^5, \quad \text{where } x^{(j)} \sim \text{Exp}(3) \end{aligned}$$

we compute the Monte Carlo approximate for  $N = 1000$ .

```
x = rexp(n = 10^5, rate = 3)
I_mc = (1/3)*mean(x^5)
print(I_mc)
```

```
## [1] 0.1674372
```

## Approximating other quantities...

Monte Carlo approximation can approximate quantities other than expected values, and the theoretical justification is out of the scope.

Let  $\theta \in \Theta$  be a random variable with distribution  $\Pi(\cdot)$ .

- Assume there is a random sample generated:

$$\theta^{(j)} \sim \Pi(\cdot), \text{ for } j = 1, \dots, N$$

for  $j = 1, \dots, N$

- The characteristic of the distribution  $\Pi(\cdot)$  such as population mode, quantiles, etc... can be approximated by their sample analogues such as sample mode, sample quantiles, etc...