

## Homework 2: Bayesian calculations and Conjugate priors

Lecturer: Georgios Karagiannis

georgios.karagiannis@durham.ac.uk

**Exercise 1.** (★★) Let  $x = (x_1, \dots, x_n)$  be observables. Consider a Bayesian model such as

$$\begin{cases} x_i | \lambda & \overset{\text{iid}}{\sim} \text{Pn}(\lambda), \quad \forall i = 1, \dots, n \\ \lambda & \sim \Pi(\lambda) \end{cases}$$

**Hint-1** Poisson distribution  $x \sim \text{Pn}(\lambda)$  has PMF:  $\text{Pn}(x|\lambda) = \frac{1}{x!} \lambda^x \exp(-\lambda) \mathbf{1}_{\mathbb{N}}(x)$ , where  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\lambda > 0$ .

**Hint-2** Gamma distribution  $x \sim \text{Ga}(a, b)$  has PDF:  $\text{Ga}(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) \mathbf{1}_{(0, \infty)}(x)$ , with  $a > 0$  and  $b > 0$ .

**Hint-2** Negative Binomial distribution  $x \sim \text{Nb}(r, \theta)$  has PMF:  $\text{Nb}(x|r, \theta) = \binom{r+x-1}{r-1} \theta^r (1-\theta)^x \mathbf{1}_{\mathbb{N}}(x)$  with  $\theta \in (0, 1)$ ,  $r \in \mathbb{N} - \{0\}$ , and  $\mathbb{N} = \{0, 1, 2, \dots\}$ .

1. Compute the likelihood in the aforesaid Bayesian model.
2. Show that the sampling distribution is a member of the exponential family.
3. Specify the PDF of the conjugate prior distribution  $\Pi(\lambda)$  of  $\lambda$ , and identify the parametric family of distributions as  $\lambda \sim \text{Ga}(a, b)$ , with  $a > 0$ , and  $b > 0$ . While you are deriving the conjugate prior distribution of  $\lambda$ , discuss which of the prior hyper-parameters can be considered as the ‘strength of the prior information and which can be considered as summarizing the prior information.
4. Compute the PDF of the posterior distribution of  $\lambda$ , identify the posterior distribution as a Gamma distribution  $\text{Ga}(\tilde{a}, \tilde{b})$ , and compute the posterior hyper-parameters  $\tilde{a}$ , and  $\tilde{b}$ .
5. Compute the PMF of the predictive distribution of a future outcome  $y = x_{n+1}$ , identify the name of the resulting predictive distribution, and compute its parameters.

**Solution.**

1. The likelihood is

$$f(x|\lambda) = \prod_{i=1}^n \text{Pn}(x_i|\lambda) = \left( \prod_{i=1}^n \frac{1}{x_i!} \right) \lambda^{\sum_{i=1}^n x_i} \exp(-n\lambda) \quad (1)$$

2. The  $k$  parameter exponential family of distributions has the form

$$\text{Ef}_k(x|u, g, h, \phi, \theta, c) = u(x)g(\theta) \exp\left(\sum_{j=1}^k c_j \phi_j(\theta) h_j(x)\right); \quad x \in \mathcal{X}$$

and if sampling space  $\mathcal{X}$  does not depend on  $\theta$  it is also called regular. So I just need to bring the sampling density distribution in this form. It is

$$\text{Pn}(x|\lambda) = \frac{1}{x!} \lambda^x \exp(-\lambda) \mathbf{1}_{\mathbb{N}}(x) = \frac{1}{x!} \exp(-\lambda) \exp(x \log(\lambda)) \mathbf{1}_{\mathbb{N}}(x)$$

So  $\text{Pn}(\lambda)$  is member of the regular 1-parameter exponential family with

$$u(x) = \frac{1}{x!} \mathbf{1}_{\mathbb{N}}(x), \quad g(\lambda) = \exp(-\lambda), \quad h_1(x) = x, \quad \phi_1(\lambda) = \log(\lambda), \quad c_1 = 1.$$

The sampling space  $\mathcal{X}$  does not depend on the uncertain parameter  $\lambda$  and hence it is a regular exponential family of distributions.

3. There are two ways to derive the conjugate prior. I will present both.

**Way-1** (Theorem 20 from the Handout)

The sampling distribution is member of the 1- regular exponential distribution family, as the density of the sampling density distribution  $\text{Pn}(x|\lambda)$  can be written in the form

$$\text{Pn}(x|\lambda) = u(x)g(\lambda) \exp\left(\sum_{j=1}^k c_j \phi_j(\lambda) h_j(x)\right); \quad x \in \mathcal{X}$$

with

$$u(x) = \frac{1}{x!} \mathbf{1}_{\mathbb{N}-\{0\}}(x), \quad g(\lambda) = \exp(-\lambda), \quad h_1(x) = x, \quad \phi_1(\lambda) = \log(\lambda), \quad c_1 = 1.$$

Since the sampling space  $\mathcal{X}$  of the sampling distribution does not depend on the unknown parameter  $\lambda$ , (Theorem 20 from the Handout) the conjugate prior is

$$\begin{aligned} \pi(\lambda) &\propto g(\lambda)^{\tau_0} \exp(c_1 \tau_1 \phi_1(\lambda)) \\ &= \exp(-\lambda \tau_0) \exp(\tau_1 \log(\lambda)) \\ &= \lambda^{\tau_1} \exp(-\lambda \tau_0) \\ &\propto \text{Ga}(\lambda|a, b), \text{ for } a = \tau_1 + 1, \ b = \tau_0 \end{aligned} \quad (2)$$

So the conjugate prior is  $\lambda \sim \text{Ga}(\lambda|a, b)$  with  $a > 0$  and  $b > 0$ .

**Way-2** (Theorem 12 in the Handout)

The likelihood can be written as

$$f(x|\lambda) = \prod_{i=1}^n \text{Pn}(x_i|\lambda) = \underbrace{\lambda^{\sum_{i=1}^n x_i} \exp(-n\lambda)}_{=k(t(x)|\lambda)} \underbrace{\left(\prod_{i=1}^n \frac{1}{x_i!}\right)}_{=\rho(x)} \quad (3)$$

where a kernel of the likelihood is  $k(t(x)|\lambda) = \lambda^{\sum_{i=1}^n x_i} \exp(-n\lambda)$ , with sufficient statistics  $t(x) = (n, \sum_{i=1}^n x_i)$ , and  $\rho(x) = \left(\prod_{i=1}^n \frac{1}{x_i!}\right)$  is the residual term of it. The dimensionality of the sufficient statistic  $t(x)$  does not depend on the sample size  $n$ , and the observables are iid. Hence, (Theorem 12 in the Handout) the conjugate prior results as the aforesaid likelihood kernel from (3) where the sufficient statistics are replaced by a priori hyper-parameters  $\tau = (\tau_0, \tau_1)$ , such as

$$\pi(\lambda) \propto k(\tau|\lambda) = \lambda^{\tau_1} \exp(-\tau_0 \lambda) \propto \text{Ga}(\lambda|a, b), \text{ for } a = \tau_1 + 1, \ b = \tau_0 \quad (4)$$

where I recognize the kernel of the Gamma distribution. So the conjugate prior is  $\lambda \sim \text{Ga}(a, b)$  with  $a > 0$  and  $b > 0$ .

In (2) and (4), as strength of the prior information can be considered the parameter  $\tau_0$  (and hence  $b$ ) because it substitutes the sample size  $n$  in the likelihood (1). In (2) and (4), as prior information summary can be considered the parameter  $\tau_1$  (and hence  $a$ ) because it substitutes the summary  $\sum_{i=1}^n x_i$  in the likelihood (1).

4. According to the definition, the posterior PDF can be computed via the Bayes theorem

$$\begin{aligned}
 \pi(\lambda|x) &\propto f(x|\lambda)\pi(\lambda) \propto \prod_{i=1}^n \text{Pn}(x_i|\lambda)\text{Ga}(\lambda|a, b) \\
 &\propto \left( \prod_{i=1}^n \frac{1}{x_i!} \right) \lambda^{\sum_{i=1}^n x_i} \exp(-n\lambda) \times \frac{b^a}{\Gamma(a)} \lambda^{a-1} \exp(-\lambda b) \\
 &\propto \lambda^{\sum_{i=1}^n x_i + a - 1} \exp(-\lambda(n+b)) \\
 &\propto \text{Ga}(\lambda | \sum_{i=1}^n x_i + a, n+b)
 \end{aligned}$$

So the posterior distribution is  $\lambda|x \sim \text{Ga}(\tilde{a}, \tilde{b})$ ,  $\tilde{a} = \sum_{i=1}^n x_i + a$ ,  $\tilde{b} = n + b$ .

- Alternatively, we could use the Theorem in the Lecture notes stating the properties of the Conjugate priors... I.e.  $\lambda|x \sim \text{Ga}(\sum_{i=1}^n x_i + (\tau_1 + 1), n + (\tau_0))$  –It is up to you...

5. According to the definition, the predictive PMF is

$$\begin{aligned}
 g(y|x) &= \int_{(0,\infty)} f(y|\lambda)\pi(\lambda|x)d\lambda = \int_{(0,\infty)} \text{Pn}(y|\lambda)\text{Ga}(\lambda|\tilde{a}, \tilde{b})d\lambda \\
 &= \int_{(0,\infty)} \frac{1}{y!} \lambda^y \exp(-\lambda) 1_{\mathbb{N}-\{0\}}(y) \frac{\tilde{b}^{\tilde{a}}}{\Gamma(\tilde{a})} \lambda^{\tilde{a}-1} \exp(-\lambda \tilde{b}) d\lambda \\
 &= \frac{1}{y!} \frac{\tilde{b}^{\tilde{a}}}{\Gamma(\tilde{a})} 1_{\mathbb{N}-\{0\}}(y) \int_{(0,\infty)} \lambda^{y+\tilde{a}-1} \exp(-\lambda(\tilde{b}+1)) d\lambda \\
 &= \frac{1}{y!} \frac{\tilde{b}^{\tilde{a}}}{\Gamma(\tilde{a})} \frac{\Gamma(y+\tilde{a})}{(\tilde{b}+1)^{y+\tilde{a}}} 1_{\mathbb{N}-\{0\}}(y) = \frac{1}{y!} \left(\frac{\tilde{b}}{\tilde{b}+1}\right)^{\tilde{a}} \left(\frac{1}{\tilde{b}+1}\right)^y \frac{\Gamma(y+\tilde{a})}{\Gamma(\tilde{a})} 1_{\mathbb{N}-\{0\}}(y) \\
 &= \frac{1}{y!} \left(\frac{\tilde{b}}{\tilde{b}+1}\right)^{\tilde{a}} \left(\frac{1}{\tilde{b}+1}\right)^y \frac{(y+\tilde{a}-1)(y+\tilde{a}-2)\cdots(\tilde{a})\cancel{\Gamma(\tilde{a})}}{\Gamma(\tilde{a})} 1_{\mathbb{N}-\{0\}}(y) \\
 &= \frac{1}{y!} \left(\frac{\tilde{b}}{\tilde{b}+1}\right)^{\tilde{a}} \left(\frac{1}{\tilde{b}+1}\right)^y (y+\tilde{a}-1)(y+\tilde{a}-2)\cdots(\tilde{a}) 1_{\mathbb{N}-\{0\}}(y) \\
 &= \frac{1}{y!} \left(\frac{\tilde{b}}{\tilde{b}+1}\right)^{\tilde{a}} \left(\frac{1}{\tilde{b}+1}\right)^y \frac{(y+\tilde{a}-1)!}{(\tilde{a}-1)!} 1_{\mathbb{N}-\{0\}}(y) = \frac{(y+\tilde{a}-1)!}{(\tilde{a}-1)!y!} \left(\frac{\tilde{b}}{\tilde{b}+1}\right)^{\tilde{a}} \left(\frac{1}{\tilde{b}+1}\right)^y 1_{\mathbb{N}-\{0\}}(y) \\
 &= \binom{y+\tilde{a}-1}{\tilde{a}-1} \left(\frac{\tilde{b}}{\tilde{b}+1}\right)^{\tilde{a}} \left(1 - \frac{\tilde{b}}{\tilde{b}+1}\right)^y 1_{\mathbb{N}-\{0\}}(y) = \text{Nb}(y|\tilde{a}, \frac{\tilde{b}}{\tilde{b}+1})
 \end{aligned}$$

where  $\tilde{a} = \sum_{i=1}^n x_i + a$ ,  $\tilde{b} = n + b$ .