Monte Carlo approximation

An intoduction for practical use in R

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Aim

Students will be able to

- apply Monte Carlo approximation with R
- approximate integrals, expected values, modes, quantiles, etc... with R

Briefly ...

Monte Carlo approximation is a stochastic procedure for the evaluation of intractable quantities. It involves:
1. properly generating a large random sample from a suitable distribution, and
2. computing a statistic asymptotically consistent to the intractable quantity of interest.

This statistic is the Monte Carlo approximate of the intractable quantity.

Approximating expected values, and integrals...

Notation

Assume a random variable $\theta \in \Theta$ following a distribution $\Pi(\cdot)$.

Let $h(\cdot) : \Theta \to \mathbb{R}^d$, $d \ge 1$, be a measurable function.

Assume that there is interest in approximating the expected value

$$\mathrm{E}_{\pi}(h(\theta)) = \int h(\theta)\mathrm{d}\Pi(\theta) = \begin{cases} \int h(\theta)\pi(\theta)\mathrm{d}\theta, & \text{cont.} \\ \\ \sum h(\theta)\pi(\theta), & \text{discr.} \end{cases}$$

Monte Carlo approximation procedure:

1. Draw a random sample
$$\theta^{(j)} \sim \Pi(\cdot), \text{ for } j = 1, ..., N$$
2. Compute
$$\bar{h}_N = \frac{1}{N}\sum_{j=1}^{N} h(\theta^{(j)})$$

We say that $\bar{h}_N$ **is a Monte Carlo approximate of $\mathbf{E}_{\pi}(h(\theta))$**, where

$$\mathrm{E}_{\pi}(h(\theta)) \approx \bar{h}_N \quad \text{when } N \text{ is large enough.}$$

which can be justified by aw of Large Number (LLN) arguments: $\bar{h}_N \to \mathrm{E}_{\pi}(h(\theta)) \quad \text{as} \quad N \to \infty$.

Example

Consider the integral

$$I = \int_0^\infty x^5 \exp(-3x) \mathrm{d}x$$

Compute a Monte Carlo approximate of integral I.

The Exact value is $I = 0.1646091$.

Solution

It is

$$\begin{aligned} I = \int_0^\infty x^5 \exp(-3x) \mathrm{d}x &= \int x^5 \frac{1}{3} 3 \exp(-3x) 1(x \in (0, \infty)) \mathrm{d}x \\ = \frac{1}{3} \int x^5 \underbrace{3 \exp(-3x) 1(x \in (0, \infty)) \mathrm{d}x}_{= \mathrm{d}\Pi_{\mathrm{Exp}(3)}(x)} &= \frac{1}{3} \int x^5 \mathrm{d}\Pi_{\mathrm{Exp}(3)}(x) = \frac{1}{3} \mathrm{E}_{\mathrm{Exp}(3)}(x^5) \\ \approx \frac{1}{3} \frac{1}{N} \sum_{j=1}^{N} \left(x^{(j)} \right)^5, &\quad \text{where} \quad x^{(j)} \sim \mathrm{Exp}(3) \end{aligned}$$

we compute the Monte Carlo approximate for $N = 1000$.

```
x = rexp(n = 10^5, rate = 3)
I_mc = (1/3)*mean(x^5)
print(I_mc)
```

```
## [1] 0.1734107
```

Approximating other quantities...

Monte Carlo approximation can approximate quantities other than expected values, and the theoretical justification is out of the scope.

Let $\theta \in \Theta$ be a random variable with distribution $\Pi(\cdot)$.

- Assume there is a random sample generated:

$$\theta^{(j)} \sim \Pi(\cdot), \text{ for } j = 1, ..., N$$

  for $j = 1, ..., N$

- The characteristic of the distribution $\Pi(\cdot)$ such as population mode, quantiles , etc... can be approximated by their sample analogues such as sample mode, sample quantiles, etc...