

## Homework 3: Jeffreys' priors and Maximum entropy priors

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**Exercise 1.** (\*\*) Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Pn}(\theta), \forall i = 1, \dots, n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $\text{Pn}(\theta)$  is the Poisson distribution with expected value  $\theta$ . Specify a Jeffreys' prior for  $\theta$ .

**Hint:** Poisson distribution:  $x \sim \text{Pn}(\theta)$  has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

**Solution.**

It is

$$\begin{aligned} \text{Pn}(x|\theta) &= \exp(-\theta) \frac{\theta^x}{x!} \implies \\ \log(\text{Pn}(x|\theta)) &= -\theta + x \log(\theta) - \log(x!) \implies \\ \frac{d}{d\theta} \log(\text{Pn}(x|\theta)) &= -1 + x \frac{1}{\theta} \implies \\ \frac{d^2}{d\theta^2} \log(\text{Pn}(x|\theta)) &= -x \frac{1}{\theta^2} \implies \\ \underbrace{-E\left(\frac{d^2}{d\theta^2} \log(\text{Pn}(x|\theta))\right)}_{=I(\theta)} &= \frac{1}{\theta} \implies \end{aligned}$$

so  $\pi(\theta) \propto \sqrt{I(\theta)} \propto \frac{1}{\theta^{1/2}}$ .

**Exercise 2.** (\*\*) Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Pn}(\theta), \forall i = 1, \dots, n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $\text{Pn}(\theta)$  is the Poisson distribution with expected value  $\theta$ . Specify a Maximum entropy prior under the constraint  $E(\theta) = 2$  and reference measure such as  $\pi_0(\theta) = \frac{1}{\sqrt{\theta}}$ . In particular, you also have to state the name of the derived Maximum entropy prior distribution and report the values of its parameters.

**Hint-1:** Poisson distribution:  $x \sim \text{Pn}(\theta)$  has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

**Hint-2:** Gamma distribution:  $x \sim \text{Ga}(a, b)$  has PDF

$$\text{Ga}(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-\beta x) 1(x > 0)$$

**Solution.**

It is  $\int_{\mathbb{R}_+} \pi_0(\theta) d\theta = +\infty$ . The maximum entropy prior is such that

$$\pi(\theta|\lambda) \propto \frac{1}{\sqrt{\theta}} \exp(\lambda\theta) = \theta^{-\frac{1}{2}} \exp(\lambda\theta)$$

where I can recognize  $\theta|\lambda \sim \text{Ga}(\frac{1}{2}, -\lambda)$ .

The expected value of the Gamma distribution  $\text{Ga}(a, b)$  is

$$E_{\text{Ga}(a,b)}(x) = \int x \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b}$$

Because  $2 = E(\theta|\lambda) = -\frac{1}{2\lambda}$ , it is  $\lambda = -\frac{1}{4}$ . Hence,

$$\theta|\lambda \sim \text{Ga}(\frac{1}{2}, \frac{1}{4})$$

**Exercise 3.** (★★) Let  $x$  be an observation. Consider the Bayesian model

$$\begin{cases} x|\theta & \sim \text{Pn}(\theta) \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $\text{Pn}(\theta)$  is the Poisson distribution with expected value  $\theta$ . Consider a prior  $\Pi(\theta)$  with density such as  $\pi(\theta) \propto \frac{1}{\theta}$ . Show that the posterior distribution is not always defined.

**Hint-1:** It suffices to show that the posterior is not defined in the case that you collect only one observation  $x = 0$ .

**Hint-2:** Poisson distribution:  $x \sim \text{Pn}(\theta)$  has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

**Solution.** The prior with  $\pi(\theta) \propto \frac{1}{\theta}$  is improper because

$$\int \pi(\theta) d\theta \propto \int \frac{1}{\theta} d\theta = \infty$$

So I need to check the proneness condition,

$$\underbrace{\int_{\mathbb{R}_+} \text{Pn}(x|\theta) \pi(\theta) d\theta}_{\propto f(x)} \begin{cases} < \infty & \text{posterior distribution is defined} \\ = \infty & \text{posterior distribution is not defined} \end{cases}$$

I will show that the posterior distribution is not defined given that I have collected a single observation  $x = 0$ . So I need to show that

$$\underbrace{\int_{\mathbb{R}_+} \text{Pn}(x = 0|\theta) \pi(\theta) d\theta}_{\propto f(x=0)} = \infty$$

It is

$$\begin{aligned} f(x) &\propto \int_{\mathbb{R}_+} \text{Pn}(x|\theta) \frac{1}{\theta} d\theta = \int_0^\infty \exp(-\theta) \frac{\theta^x}{x!} \frac{1}{\theta} d\theta \\ f(x = 0) &\propto \int_{\mathbb{R}_+} \exp(-\theta) \frac{\theta^0}{0!} \frac{1}{\theta} d\theta = \int_0^\infty \exp(-\theta) \frac{1}{\theta} d\theta \end{aligned}$$

We will use a convergence criteria in order to check if  $\int_0^\infty \exp(-\theta)^{\frac{1}{\theta}} d\theta = \infty$ .

Consider  $h(\theta) = \exp(-\theta)^{\frac{1}{\theta}}$ . The function  $h(\theta)$  has an improper behavior at 0, as it is not bounded there. Let  $g(\theta) = \frac{1}{\theta}$ . According to the Limit Comparison Test, it is

$$\lim_{\theta \rightarrow 0^+} \frac{h(\theta)}{g(\theta)} = \lim_{\theta \rightarrow 0^+} \frac{\frac{1}{\theta} \exp(-\theta)}{\frac{1}{\theta}} = 1 \neq 0$$

and

$$\int_0^\infty g(\theta) d\theta = \int_0^\infty \frac{1}{\theta} d\theta = \infty.$$

Therefore, it will be

$$\underbrace{\int_0^\infty h(\theta) d\theta}_{=f(x=0)} = \infty$$

as well.

## The Limit Comparison Theorem for Improper Integrals

- Brand, L. (2006; Chapter 7). Advanced calculus: an introduction to classical analysis.

**General:** Let integrable functions  $f(x)$ , and  $g(x)$  for  $x \geq a$ .

Let

$$0 \leq f(x) \leq g(x), \quad \text{for } x \geq a$$

Then

$$\begin{aligned} \int_a^\infty g(x)dx < \infty &\implies \int_a^\infty f(x)dx < \infty \\ \int_a^\infty f(x)dx = \infty &\implies \int_a^\infty g(x)dx = \infty \end{aligned}$$

**Type I:** Let integrable functions  $f(x)$ , and  $g(x)$  for  $x \geq a$ , and let  $g(x)$  be positive.

Let

$$\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)} = c$$

Then

- If  $c \in (0, \infty)$  :

$$\int_a^\infty g(x)dx < \infty \iff \int_a^\infty f(x)dx < \infty$$

- If  $c = 0$  :

$$\int_a^\infty g(x)dx < \infty \implies \int_a^\infty f(x)dx < \infty$$

- If  $c = \infty$  :

$$\int_a^\infty f(x)dx = \infty \implies \int_a^\infty g(x)dx = \infty$$

**Type II:** Let integrable functions  $f(x)$ , and  $g(x)$  for  $a < x \leq b$ , and let  $g(x)$  be positive.

Let

$$\lim_{n \rightarrow a^+} \frac{f(x)}{g(x)} = c$$

Then

- If  $c \in (0, \infty)$  :

$$\int_a^\infty g(x)dx < \infty \iff \int_a^\infty f(x)dx < \infty$$

- If  $c = 0$  :

$$\int_a^\infty g(x)dx < \infty \implies \int_a^\infty f(x)dx < \infty$$

- If  $c = \infty$  :

$$\int_a^\infty f(x)dx = \infty \implies \int_a^\infty g(x)dx = \infty$$

**Note:** A useful test function is

$$\int_0^\infty \left(\frac{1}{x}\right)^p dx \begin{cases} < \infty & , \text{ when } p > 1 \\ = \infty & , \text{ when } p \leq 1 \end{cases}$$