Homework 1: Manipulation of multivariate probability distributions

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Exercise 1. ($\star\star$)

Let $x \sim \mathrm{T}_d(\mu, \Sigma, \nu)$. Recall that $x \sim \mathrm{T}_d(\mu, \Sigma, \nu)$ is the marginal distribution $f_x(x) = \int f_{x|\xi}(x|\xi) f_\xi(\xi) \mathrm{d}\xi$ of (x, ξ) where

$$x|\xi \sim \mathrm{N}_d(\mu, \Sigma\xi v)$$
$$\xi \sim \mathrm{IG}(\frac{v}{2}, \frac{1}{2})$$

Consider partition such that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}; \qquad \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{21}^\top \\ \Sigma_{21} & \Sigma_2 \end{bmatrix},$$

where $x_1 \in \mathbb{R}^{d_1}$ and $x_2 \in \mathbb{R}^{d_2}$.

Address the following:

1. Show that the marginal distribution of x_1 is such that

$$x_1 \sim \mathrm{T}_{d_1}(\mu_1, \Sigma_1, \nu)$$

Hint: Try to use the form $f_x(x) = \int f_{x|\xi}(x|\xi) f_\xi(\xi) \mathrm{d}\xi$.

2. Show that

$$\xi|x_1 \sim \mathrm{IG}(\frac{1}{2}(d_1 + v), \frac{1}{2}\frac{Q + v}{v})$$

where $Q = (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)$.

Hint: The PDF of $y \sim \mathrm{N}_d(\mu, \Sigma)$ is

$$f(y) = (2\pi)^{-\frac{d}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(y - \mu)^\top \Sigma^{-1}(y - \mu)\right)$$

Hint: The PDF of $y \sim \mathrm{IG}(a, b)$ is

$$f_{\mathrm{IG}(a,b)}(y) = \frac{b^a}{\Gamma(a)} y^{-a-1} \exp(-\frac{b}{y}) 1_{(0,+\infty)}(y)$$

3. Let $\xi' = \xi\frac{v}{Q+v}$, with $Q = (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)$, show that

$$\xi'|x_1 \sim \mathrm{IG}(\frac{v + d_1}{2}, \frac{1}{2})$$

4. Show that the conditional distribution of $x_2|x_1$ is such that

$$x_2|x_1 \sim \mathrm{T}_{d_2}(\mu_{2|1}, \dot{\Sigma}_{2|1}, \nu_{2|1})$$

where

$$\begin{aligned}
\mu_{2|1} =& \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) \\
\dot{\Sigma}_{2|1} =& \frac{\nu + (\mu_1 - x_1)^\top \Sigma_1^{-1}(\mu_1 - x_1)}{\nu + d_1}\Sigma_{2|1} \\
\Sigma_{2|1} =& \Sigma_{22} - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top \\
\nu_{2|1} =& \nu + d_1
\end{aligned}$$

Hint: You can use the Example [Marginalization & conditioning] from the Lecture Handout