Bayesian Statistics III/IV (MATH3361/4071) Michaelmas term 2019

Homework 3: Jeffreys' priors and Maximum entropy priors

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Exercise 1. ($\star\star$)Consider the Bayesian model

$$\begin{cases} x_i|\theta & \overset{\text{iid}}{\sim} \text{Pn}(\theta),\ \forall i=1,...,n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $\text{Pn}(\theta)$ is the Poisson distribution with expected value θ. Specify a Jeffreys' prior for θ.

Hint: Poisson distribution: $x \sim \text{Pn}(\theta)$ has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

Solution.

It is

$$\begin{aligned} \text{Pn}(x|\theta) &= \exp(-\theta)\frac{\theta^x}{x!} \Longrightarrow \\ \log(\text{Pn}(x|\theta)) &= -\theta + x\log(\theta) - \log(x!) \Longrightarrow \\ \frac{\text{d}}{\text{d}\theta}\log(\text{Pn}(x|\theta)) &= -1 + x\frac{1}{\theta} \Longrightarrow \\ \frac{\text{d}^2}{\text{d}\theta^2}\log(\text{Pn}(x|\theta)) &= -x\frac{1}{\theta^2} \Longrightarrow \\ \underbrace{-\text{E}(\frac{\text{d}^2}{\text{d}\theta^2}\log(\text{Pn}(x|\theta)))}_{=I(\theta)} &= \frac{1}{\theta} \Longrightarrow \end{aligned}$$

so $\pi(\theta) \propto \sqrt{I(\theta)} \propto \frac{1}{\theta^{1/2}}$.

Exercise 2. ($\star\star$)Consider the Bayesian model

$$\begin{cases} x_i|\theta & \overset{\text{iid}}{\sim} \text{Pn}(\theta),\ \forall i=1,...,n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $\text{Pn}(\theta)$ is the Poisson distribution with expected value θ. Specify a Maximum entropy prior under the constrain $\text{E}(\theta) = 2$ and reference measure such as $\pi_0(\theta) = \frac{1}{\sqrt{\theta}}$. In particular, you also have to state the name of the derived Maximum entropy prior distribution and report the values of its parameters.

Hint-1: Poisson distribution: $x \sim \text{Pn}(\theta)$ has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

Hint-2: Gamma distribution: $x \sim \text{Ga}(a, b)$ has PDF

$$\text{Ga}(x|a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-\beta x) 1(x > 0)$$

Solution.

It is $\int_{\mathbb{R}_+} \pi_0(\theta)\mathrm{d}\theta = +\infty$. The maximum entropy prior is such that

$$\pi(\theta|\lambda) \propto \frac{1}{\sqrt{\theta}} \exp(\lambda\theta) = \theta^{-\frac{1}{2}} \exp(\lambda\theta)$$

where I can recognize $\theta|\lambda \sim \mathrm{Ga}(\frac{1}{2}, -\lambda)$.

The expected value of the Gamma distribution $\mathrm{Ga}(a, b)$ is

$$E_{\mathrm{Ga}(a,b)}(x) = \int x \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b}$$

Because $2 = \mathrm{E}(\theta|\lambda) = -\frac{1}{2\lambda}$, it is $\lambda = -\frac{1}{4}$. Hence,

$$\theta|\lambda \sim \mathrm{Ga}(\frac{1}{2}, \frac{1}{4})$$

Exercise 3. ($\star\star$)Let x be an observation. Consider the Bayesian model

$$\begin{cases} x|\theta & \sim \mathrm{Pn}(\theta) \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $\mathrm{Pn}(\theta)$ is the Poisson distribution with expected value θ. Consider a prior $\Pi(\theta)$ with density such as $\pi(\theta) \propto \frac{1}{\theta}$. Show that the posterior distribution is not always defined.

Hint-1: It suffices to show that the posterior is not defined in the case that you collect only one observation $x = 0$.

Hint-2: Poisson distribution: $x \sim \mathrm{Pn}(\theta)$ has PMF

$$\mathrm{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

Solution. The prior with $\pi(\theta) \propto \frac{1}{\theta}$ is improper because

$$\int \pi(\theta)\mathrm{d}\theta \propto \int \frac{1}{\theta}\mathrm{d}\theta = \infty$$

So I need to check the proneness condition,

$$\underbrace{\int_{\mathbb{R}_+} \mathrm{Pn}(x|\theta)\pi(\theta)\mathrm{d}\theta}_{\propto f(x)} \quad \begin{cases} < \infty & \text{posterior distribution is defined} \\ = \infty & \text{posterior distribution is not defined} \end{cases}$$

I will show that the posterior distribution is not defined given that I have collected a single observation $x = 0$. So I need to show that

$$\underbrace{\int_{\mathbb{R}_+} \mathrm{Pn}(x = 0|\theta)\pi(\theta)\mathrm{d}\theta}_{\propto f(x=0)} = \infty$$

It is

$$f(x) \propto \int_{\mathbb{R}_+} \mathrm{Pn}(x|\theta)\frac{1}{\theta}\mathrm{d}\theta = \int_0^\infty \exp(-\theta)\frac{\theta^x}{x!}\frac{1}{\theta}\mathrm{d}\theta$$
$$f(x = 0) \propto \int_{\mathbb{R}_+} \exp(-\theta)\frac{\theta^0}{0!}\frac{1}{\theta}\mathrm{d}\theta = \int_0^\infty \exp(-\theta)\frac{1}{\theta}\mathrm{d}\theta$$

We will use a convergence criteria in order to check if $\int_0^\infty \exp(-\theta)\frac{1}{\theta}\mathrm{d}\theta = \infty$.

Consider $h(\theta) = \exp(-\theta)\frac{1}{\theta}$. The function $h(\theta)$ has an improper behavior at 0, as it is not bounded there. Let $g(\theta) = \frac{1}{\theta}$. According to the Limit Comparison Test, it is

$$\lim_{\theta\to 0^+} \frac{h(\theta)}{g(\theta)} = \lim_{\theta\to 0^+} \frac{\frac{1}{\theta}\exp(-\theta)}{\frac{1}{\theta}} = 1 \neq 0$$

and

$$\int_0^\infty g(\theta)\mathrm{d}\theta = \int_0^\infty \frac{1}{\theta}\mathrm{d}\theta = \infty.$$

Therefore, it will be

$$\underbrace{\int_0^\infty h(\theta)\mathrm{d}\theta}_{=f(x=0)} = \infty$$

as well.

The Limit Comparison Theorem for Improper Integrals

- Brand, L. (2006; Chapter 7). Advanced calculus: an introduction to classical analysis.

General: Let integrable functions $f(x)$, and $g(x)$ for $x \geq a$.

Let

$$0 \leq f(x) \leq g(x), \quad \text{for } x \geq a$$

Then

$$\int_a^\infty g(x)\mathrm{d}x < \infty \implies \int_a^\infty f(x)\mathrm{d}x < \infty$$
$$\int_a^\infty f(x)\mathrm{d}x = \infty \implies \int_a^\infty g(x)\mathrm{d}x = \infty$$

Type I: Let integrable functions $f(x)$, and $g(x)$ for $x \geq a$, and let $g(x)$ be positive.

Let

$$\lim_{n\to\infty} \frac{f(x)}{g(x)} = c$$

Then

- If $c \in (0, \infty)$:
$$\int_a^\infty g(x)\mathrm{d}x < \infty \iff \int_a^\infty f(x)\mathrm{d}x < \infty$$
- If $c = 0$:
$$\int_a^\infty g(x)\mathrm{d}x < \infty \implies \int_a^\infty f(x)\mathrm{d}x < \infty$$
- If $c = \infty$:
$$\int_a^\infty f(x)\mathrm{d}x = \infty \implies \int_a^\infty g(x)\mathrm{d}x = \infty$$

Type II: Let integrable functions $f(x)$, and $g(x)$ for $a < x \leq b$, and let $g(x)$ be positive.

Let

$$\lim_{n\to a^+} \frac{f(x)}{g(x)} = c$$

Then

- If $c \in (0, \infty)$:
$$\int_a^\infty g(x)\mathrm{d}x < \infty \iff \int_a^\infty f(x)\mathrm{d}x < \infty$$
- If $c = 0$:
$$\int_a^\infty g(x)\mathrm{d}x < \infty \implies \int_a^\infty f(x)\mathrm{d}x < \infty$$
- If $c = \infty$:
$$\int_a^\infty f(x)\mathrm{d}x = \infty \implies \int_a^\infty g(x)\mathrm{d}x = \infty$$

Note: A useful test function is

$$\int_0^\infty \left(\frac{1}{x}\right)^p \mathrm{d}x \quad \begin{cases} < \infty & \text{, when } p > 1 \\ = \infty & \text{, when } p \leq 1 \end{cases}$$