Bayesian Statistics III/IV (MATH3361/4071)

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## Homework 3: Jeffreys' priors and Maximum entropy priors

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Exercise 1. (\*\*)Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \Pr(\theta), \ \forall i = 1, ..., n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $Pn(\theta)$  is the Poisson distribution with expected value  $\theta$ . Specify a Jeffreys' prior for  $\theta$ .

**Hint:** Poisson distribution:  $x \sim Pn(\theta)$  has PMF

$$Pn(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

Solution.

It is

$$\operatorname{Pn}(x|\theta) = \exp(-\theta) \frac{\theta^x}{x!} \Longrightarrow \\ \log(\operatorname{Pn}(x|\theta)) = -\theta + x \log(\theta) - \log(x!) \Longrightarrow \\ \frac{\mathrm{d}}{\mathrm{d}\theta} \log(\operatorname{Pn}(x|\theta)) = -1 + x \frac{1}{\theta} \Longrightarrow \\ \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log(\operatorname{Pn}(x|\theta)) = -x \frac{1}{\theta^2} \Longrightarrow \\ -\underline{\mathrm{E}(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log(\operatorname{Pn}(x|\theta)))} = \frac{1}{\theta} \Longrightarrow$$

so  $\pi(\theta) \propto \sqrt{I(\theta)} \propto \frac{1}{\theta^{1/2}}$ .

Exercise 2.  $(\star\star)$ Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \Pr(\theta), \ \forall i = 1, ..., n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $Pn(\theta)$  is the Poisson distribution with expected value  $\theta$ . Specify a Maximum entropy prior under the constrain  $E(\theta)=2$  and reference measure such as  $\pi_0(\theta)=\frac{1}{\sqrt{\theta}}$ . In particular, you also have to state the name of the derived Maximum entropy prior distribution and report the values of its parameters.

**Hint-1:** Poisson distribution:  $x \sim Pn(\theta)$  has PMF

$$Pn(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

**Hint-2:** Gamma distribution:  $x \sim \text{Ga}(a, b)$  has PDF

$$Ga(x|a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-\beta x) \mathbb{1}(x>0)$$

Solution.

It is  $\int_{\mathbb{R}_+} \pi_0(\theta) \mathrm{d}\theta = +\infty.$  The maximum entropy prior is such that

$$\pi(\theta|\lambda) \propto \frac{1}{\sqrt{\theta}} \exp(\lambda\theta) = \theta^{-\frac{1}{2}} \exp(\lambda\theta)$$

where I can recognize  $\theta | \lambda \sim \text{Ga}(\frac{1}{2}, -\lambda)$ .

The expected value of the Gamma distribution Ga(a, b) is

$$E_{Ga(a,b)}(x) = \int x \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac{b^{a+1}}{\Gamma(a+1)} x^{a+1-1} \exp(-\beta x) 1(x > 0) = \frac{a}{b} \int \frac$$

Because  $2=\mathrm{E}(\theta|\lambda)=-\frac{1}{2\lambda}$  , it is  $\lambda=-\frac{1}{4}.$  Hence,

$$\theta | \lambda \sim \operatorname{Ga}(\frac{1}{2}, \frac{1}{4})$$

**Exercise 3.**  $(\star\star)$ Let x be an observation. Consider the Bayesian model

$$\begin{cases} x | \theta & \sim \Pr(\theta) \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $Pn(\theta)$  is the Poisson distribution with expected value  $\theta$ . Consider a prior  $\Pi(\theta)$  with density such as  $\pi(\theta) \propto \frac{1}{\theta}$ . Show that the posterior distribution is not always defined.

**Hint-1:** It suffices to show that the posterior is not defined in the case that you collect only one observation x = 0.

**Hint-2:** Poisson distribution:  $x \sim Pn(\theta)$  has PMF

$$Pn(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

**Solution.** The prior with  $\pi(\theta) \propto \frac{1}{\theta}$  is improper because

$$\int \pi(\theta) d\theta \propto \int \frac{1}{\theta} d\theta = \infty$$

So I need to check the proneness condition,

$$\underbrace{\int_{\mathbb{R}_+} \Pr(x|\theta) \pi(\theta) d\theta}_{\propto f(x)} \quad \begin{cases} < \infty & \text{posterior distribution is defined} \\ = \infty & \text{posterior distribution is not defined} \end{cases}$$

I will show that the posterior distribution is not defined given that I have collected a single observation x = 0. So I need to show that

$$\underbrace{\int_{\mathbb{R}_{+}} \operatorname{Pn}(x=0|\theta)\pi(\theta) d\theta}_{\propto f(x=0)} = \infty$$

It is

$$\begin{split} f(x) &\propto \int_{\mathbb{R}_+} \text{Pn}(x|\theta) \frac{1}{\theta} \text{d}\theta = \int_0^\infty \exp(-\theta) \frac{\theta^x}{x!} \frac{1}{\theta} \text{d}\theta \\ f(x=0) &\propto \int_{\mathbb{R}_+} \exp(-\theta) \frac{\theta^0}{0!} \frac{1}{\theta} \text{d}\theta = \int_0^\infty \exp(-\theta) \frac{1}{\theta} \text{d}\theta \end{split}$$

We will use a convergence criteria in order to check if  $\int_0^\infty \exp(-\theta) \frac{1}{\theta} \mathrm{d}\theta = \infty$ .

Consider  $h(\theta) = \exp(-\theta)\frac{1}{\theta}$ . The function  $h(\theta)$  has an improper behavior at 0, as it is not bounded there. Let  $g(\theta) = \frac{1}{\theta}$ . According to the Limit Comparison Test, it is

$$\lim_{\theta \to 0^+} \frac{h(\theta)}{g(\theta)} = \lim_{\theta \to 0^+} \frac{\frac{1}{\theta} \exp(-\theta)}{\frac{1}{\theta}} = 1 \neq 0$$

and

$$\int_0^\infty g(\theta)\mathrm{d}\theta = \int_0^\infty \frac{1}{\theta}\mathrm{d}\theta = \infty.$$

Therefore, it will be

$$\underbrace{\int_0^\infty h(\theta) \mathrm{d}\theta}_{=f(x=0)} = \infty$$

as well.

## The Limit Comparison Theorem for Improper Integrals

• Brand, L. (2006; Chapter 7). Advanced calculus: an introduction to classical analysis.

**General:** Let integrable functions f(x), and g(x) for  $x \ge a$ .

Let

$$0 \le f(x) \le g(x)$$
, for  $x \ge a$ 

Then

$$\int_{a}^{\infty}g(x)\mathrm{d}x<\infty\implies\int_{a}^{\infty}f(x)\mathrm{d}x<\infty$$
 
$$\int_{a}^{\infty}f(x)\mathrm{d}x=\infty\implies\int_{a}^{\infty}g(x)\mathrm{d}x=\infty$$

**Type I:** Let integrable functions f(x), and g(x) for  $x \ge a$ , and let g(x) be positive.

Let

$$\lim_{n \to \infty} \frac{f(x)}{g(x)} = c$$

Then

- If  $c\in(0,\infty)$  :  $\int_a^\infty g(x)\mathrm{d}x<\infty\Longleftrightarrow\int_a^\infty f(x)\mathrm{d}x<\infty$
- If c=0 :  $\int_a^\infty g(x)\mathrm{d}x < \infty \implies \int_a^\infty f(x)\mathrm{d}x < \infty$
- If  $c=\infty$ :  $\int_a^\infty f(x)\mathrm{d}x = \infty \implies \int_a^\infty g(x)\mathrm{d}x = \infty$

**Type II:** Let integrable functions f(x), and g(x) for  $a < x \le b$ , and let g(x) be positive.

Let

$$\lim_{n \to a^+} \frac{f(x)}{g(x)} = c$$

Then

- If  $c\in(0,\infty)$ :  $\int_{a}^{\infty}g(x)\mathrm{d}x<\infty\Longleftrightarrow\int_{a}^{\infty}f(x)\mathrm{d}x<\infty$
- If c=0 :  $\int_a^\infty g(x)\mathrm{d}x < \infty \implies \int_a^\infty f(x)\mathrm{d}x < \infty$
- If  $c=\infty$ :  $\int_a^\infty f(x)\mathrm{d}x = \infty \implies \int_a^\infty g(x)\mathrm{d}x = \infty$

**Note:** A useful test function is

$$\int_0^\infty \left(\frac{1}{x}\right)^p \mathrm{d}x \quad \begin{cases} <\infty &, \text{ when } p>1\\ =\infty &, \text{ when } p\leq 1 \end{cases}$$