

Lecture handout 1: Random variables ^a

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Aim

To revise a bit, linear algebra, random variables and probabilities

Linear algebra Cholesky decomposition**Probability theory** Random variables, probabilities, expected values, covariance/variance matrices, characteristic function, compound distribution function

Reading list:

- DeGroot, M. H. (1970, or 2005). Optimal statistical decisions (Vol. 82). John Wiley & Sons.
 - Part one: Survey of probability theory. Chapters 1-5

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1 Linear Algebra

Proposition 1. [Cholesky decomposition] Every symmetric positive definite matrix $A \in \mathbb{R}^d \times \mathbb{R}^d$ can be decomposed into a product of a unique lower triangular matrix $L \in \mathbb{R}^d \times \mathbb{R}^d$ and its transpose L^\top , i.e.

$$A = LL^\top$$

Matrix L is called lower triangular factor of the Cholesky decomposition, and it is often denoted as $A^{1/2} = L$.

2 Probability distributions

Definition 2. A collection $\mathcal{F} = \{A, A_1, A_2, \dots\}$ of sets A, A_1, A_2, \dots , each of which are subsets of set Ω , is called a σ -algebra if and only if

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. If $A_1, A_2, \dots \in \mathcal{F}$ is an infinite sequence of sets in \mathcal{F} then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Definition 3. Probability distribution P on (Ω, \mathcal{F}) is called a non-negative function if and only if

1. $P(\Omega) = 1$
2. If $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$
3. If $A_1, A_2, \dots \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$ then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Notation 4. We call the triple (Ω, \mathcal{F}, P) as probability space.

3 Random variables

Definition 5. A d -dimensional random variable y on a probability space (Ω, \mathcal{F}, P) is a function $y : \Omega \rightarrow \mathbb{R}^d$ such as, for any subset $A \subseteq \mathbb{R}^d$, it is $\{\omega \in \Omega : y(\omega) \in A\} \in \mathcal{F}$.

Definition 6. A d -dimensional random variable y on a probability space (Ω, \mathcal{F}, P) , induces a probability $P_y(\cdot)$ such that, for all subsets $A \subseteq \mathbb{R}^d$,

$$P_y(y \in A) = P(\{\omega \in \Omega : y(\omega) \in A\})$$

Essentially, it induces a probability space $(\mathbb{R}^d, \mathfrak{B}, P_y)$, with \mathfrak{B} a σ -algebra¹ containing sub-sets of \mathbb{R}^d .

Definition 7. The (cumulative) distribution function (CDF) of a d -dimensional random variable $y \in \mathcal{Y}$ is the function $F_y : \mathbb{R}^d \rightarrow [0, 1]$ such that

$$F_y(y) := F_y(y'_1, \dots, y'_d) = P_y(y \in (-\infty, y'_1] \times \dots \times (-\infty, y'_d])$$

Notation 8. The distribution function defines the distribution of the random variable. As $y \sim F_y$, we will denote that the random variable y follows a distribution with distribution function F_y .

Definition 9. The d -dimensional random variable $y : \Omega \rightarrow \mathcal{Y}$ with distribution F_y is discrete, if \mathcal{Y} is a countable set and the distribution can be described by its Probability Mass Function (PMF)

$$f_y(y') := f_y(y'_1, \dots, y'_d) = P(\{\omega \in \Omega : y(\omega) = y'\})$$

Definition 10. The d -dimensional random variable $y : \Omega \rightarrow \mathcal{Y}$ with distribution F_y is absolutely continuous, if \mathcal{Y} is a uncountable set and the distribution can be described by its Probability Density Function (PDF) $f_y(y)$ such that

$$P_y(y \in A) = \underbrace{\int \dots \int_A f_y(y'_1, \dots, y'_d) dy'_1 \dots dy'_d}_{A}, \text{ for any } A \subseteq \mathbb{R}^d.$$

or briefly $P_y(A) = \int_A f_y(y') dy'$, where $dy' = \prod_{j=1}^d dy'_j$.

Fact 11. The PDF of d -dimensional random variable $y : \Omega \rightarrow \mathcal{Y}$ with CDF F_y can be computed by the partial derivative as

$$f_y(y) = \frac{d}{dt_1 \dots dt_d} F_y(t_1, \dots, t_d) \Big|_{t_1=y_1, \dots, t_d=y_d} \quad \text{if } F_y \text{ is differential.}$$

4 Transforming

Fact 12. Let $y \in \mathcal{Y}$ be a d -dimensional random variable with PDF $f_y(\cdot)$. Consider a bijective function $h : \mathcal{Y} \rightarrow \mathcal{Z}$ with $z = h(y)$, and h^{-1} its inverse. The PDF of z is

$$f_z(z) = f_y(y) \left| \det \left(\frac{dy}{dz} \right) \right| = f_y(h^{-1}(z)) \left| \det \left(\frac{d}{dz} h^{-1}(z) \right) \right|$$

Example 13. Let $y \sim \text{Ex}(\lambda)$ r.v. with Exponential distribution with rate parameter $\lambda > 0$, and $f_{\text{Ex}(\lambda)}(y) = \lambda \exp(-\lambda y) 1(y \geq 0)$. Let $z = 1 - \exp(-\lambda y)$. Calculate the PDF of z , and recognize its distribution.

Solution. It is $z = 1 - \exp(-\lambda y) \iff y = -\frac{1}{\lambda} \log(1 - z)$, and $z \in [0, 1]$. So $h^{-1}(z) = -\frac{1}{\lambda} \log(1 - z)$. Then

$$\begin{aligned} f_z(z) &= f_{\text{Ex}(\lambda)}(h^{-1}(z)) \times \left| \det \left(\frac{d}{dz} h^{-1}(z) \right) \right| = f_{\text{Ex}(\lambda)} \left(-\frac{1}{\lambda} \log(1 - z) \right) \times \left| \det \left(\frac{d}{dz} -\frac{1}{\lambda} \log(1 - z) \right) \right| \\ &= \exp \left(-\lambda \frac{-1}{\lambda} \log(1 - z) \right) 1 \left(-\frac{1}{\lambda} \log(1 - z) \geq 0 \right) \times \left| -\frac{1}{\lambda} \frac{1}{1 - z} \right| = 1(z \in [0, 1]) \end{aligned}$$

¹...this is not a rigorous definition; a more rigorous definition is out of the scope of the course.

From the density, we recognize that $z \sim U(0, 1)$ follows a uniform distribution.

5 Marginalizing & Integrating out

Fact 14. Let $(n + d)$ -dimensional random variable $y \in \mathcal{Y}$ with distribution $F_y(\cdot)$. Consider a partition $y = (x, \theta)$ where $x \in \mathcal{X}$ is n -dimensional and $\theta \in \Theta$ is d -dimensional. Then

1. the marginal CDF of x results by setting θ as ∞

$$F_x(x) = \lim_{\theta \rightarrow \infty} F_y(x, \theta) = \lim_{\theta_1 \rightarrow \infty, \dots, \theta_d \rightarrow \infty} F_y(x_1, \dots, x_n, \theta_1, \dots, \theta_d)$$

2. the marginal PDF/PMF of x results by integrating out θ (the dimensions we marginalize)

$$f_x(x) = \begin{cases} \int_{\mathbb{R}^d} f_y(x, \theta) d\theta & \text{if } \theta \text{ is cont.} \\ \sum_{\forall \theta \in \mathbb{R}^d} f_y(x, \theta) & \text{if } \theta \text{ is discr.} \end{cases} = \begin{cases} \int_{\mathbb{R}^d} f_y(x_1, \dots, x_n, \theta_1, \dots, \theta_d) d\theta_1 \dots d\theta_d & \text{if } \theta \text{ is cont.} \\ \sum_{\forall \theta \in \mathbb{R}^d} f_y(x_1, \dots, x_n, \theta_1, \dots, \theta_d) & \text{if } \theta \text{ is discr.} \end{cases}$$

6 Independence

Definition 15. Given a probability space $(\Omega, \mathfrak{B}, P)$, events $A, B \in \mathfrak{B}$ are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Fact 16. Let $(n + m)$ -dimensional random variable $y \in \mathcal{Y}$. Consider a partition $y = (x, z)$ where $x \in \mathcal{X}$ is n -dimensional and $z \in \mathcal{Z}$ is m -dimensional.

- The r.v. x and y are independent if and only if for any $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Z}$

$$P(\{x \in A\} \cap \{z \in B\}) = P(x \in A)P(z \in B)$$

- The r.v. x and y are independent if and only if

$$F(x, z) = F(x)F(z), \quad \forall x \in \mathcal{X}, \forall z \in \mathcal{Z}$$

where $F(\cdot)$ denotes the CDF.

- This implies that r.v. x and z are independent if and only if

$$f(x, z) = f(x)f(z)$$

where $f(\cdot)$ denotes the PDF/PMF.

7 Expected value

Definition 17. Expected value of the d -dimensional random variable $y \in \mathcal{Y}$ with distribution F is the d -dimensional quantity

$$E(y) = \int y dF(y) = \begin{cases} \int_{y \in \mathcal{Y}} y f(y) dy, & \text{if } y \text{ is cont.} \\ \sum_{y \in \mathcal{Y}} y f(y), & \text{if } y \text{ is discr.} \end{cases}$$

whose i th element is

$$[E(y)]_i = \begin{cases} \int y_i f_{y_i}(y_i) dy_i, & \text{if } y_i \text{ is cont.} \\ \sum_{\forall y_i} y_i f_{y_i}(y_i), & \text{if } y_i \text{ is discr.} \end{cases}$$

for $i = 1, \dots, d$. Here $f_{y_i}(y_i) = \int_{y \in \mathcal{Y}} f(y) dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_d$ is the marginal PMF/PDF of y_i .

Fact 18. If $y \in \mathcal{Y}$ is a d -dimensional random variable with PDF/PMF $f_y(\cdot)$, and $\psi : \mathcal{Y} \rightarrow \mathbb{R}^d$ is an integrate function with $\psi(\cdot) := (\psi_1(\cdot), \dots, \psi_d(\cdot))$, then

$$E(\psi(y)) = \begin{cases} \int \psi(y) f_y(y) dy, & \text{if } y \text{ is cont.} \\ \sum_{\forall y} \psi(y) f_y(y), & \text{if } y \text{ is discr.} \end{cases}$$

with elements

$$[E(\psi(y))]_i = \begin{cases} \int \psi_i(y) f_y(y) dy, & \text{if } y \text{ is cont.} \\ \sum_{\forall y} \psi_i(y) f_y(y), & \text{if } y \text{ is discr.} \end{cases}$$

Example 19. If $(q \times k)$ -dimensional random variable $y \in \mathcal{Y}$ is a matrix, then its expectation $E(y) = \int y dF(y)$ is a matrix too

$$E(y) = \begin{bmatrix} E(y_{1,1}) & \cdots & E(y_{1,j}) & \cdots & E(y_{1,m}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ E(y_{i,1}) & \cdots & E(y_{i,j}) & \cdots & E(y_{i,m}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ E(y_{n,1}) & \cdots & E(y_{n,j}) & \cdots & E(y_{n,m}) \end{bmatrix}$$

whose (i, j) -th element is

$$[E(y)]_{i,j} = E(y_{i,j}) = \begin{cases} \int y_{i,j} f_{y_{i,j}}(y_{i,j}) dy_{i,j}, & \text{if } y_{i,j} \text{ is cont.} \\ \sum_{\forall y_{i,j}} y_{i,j} f_{y_{i,j}}(y_{i,j}), & \text{if } y_{i,j} \text{ is discr.} \end{cases}$$

for $i = 1, \dots, n$, and $j = 1, \dots, m$. Here $f_{y_{i,j}}(\cdot)$ is the marginal PMF/PDF of $y_{i,j}$.

Proposition 20. The following properties are valid

1. Let fixed matrix/vectors A , c , and $z = c + Ay$ with suitable dimensions then

$$E(z) = E(c + Ay) = c + AE(y)$$

2. Let random variables $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$, and let functions ψ_1 and ψ_2 defined on \mathcal{Z} and \mathcal{Y} , then

$$E(\psi_1(z) + \psi_2(y)) = E(\psi_1(z)) + E(\psi_2(y))$$

3. Random variables $z \in \mathcal{Z}$ and $y \in \mathcal{Y}$ are independent if and only if

$$E(\psi_1(z)\psi_2(y)) = E(\psi_1(z))E(\psi_2(y))$$

for any functions ψ_1 and ψ_2 defined on \mathcal{Z} and \mathcal{Y} .

Proof. Given as Exercise 3 in the Exercise Sheet. □

8 Covariance matrix

Definition 21. The covariance matrix between random variable $z \in \mathcal{Z} \subseteq \mathbb{R}^d$ and random variable $y \in \mathcal{Y} \subseteq \mathbb{R}^q$ is defined as the $d \times q$ matrix

$$\text{Cov}(z, y) = E((z - E(z))(y - E(y))^\top)$$

Proposition 22. The following properties are the direct analogues of the 1D cases

1. $\text{Cov}(z, y) = E(zy^\top) - E(z) (E(y))^\top$
2. $\text{Cov}(z, y) = (\text{Cov}(y, z))^\top$
3. $\text{Cov}(c_1 + A_1 z, c_2 + A_2 y) = A_1 \text{Cov}(z, y) A_2^\top$, for fixed matrices A_1, A_2 , and vectors c_1, c_2 with suitable dimensions.
4. If z and y are independent random vectors then $\text{Cov}(z, y) = 0$

Proof. (1)-(3) result from the definition. (4) results from Prop 20, as $E(zy^\top) = E(z) (E(y))^\top$. □

Proposition 23. It can be seen that

$$[\text{Cov}(z, y)]_{i,j} = \text{Cov}(z_i, y_j)$$

for all $i = 1, \dots, d$, and $j = 1, \dots, q$. Namely, the (i, j) -th element of the covariance matrix between vector z and y is the covariance between their elements z_i and y_j .

Definition 24. The covariance matrix of random vector $y \in \mathcal{Y} \subseteq \mathbb{R}^d$ is defined as the $d \times d$ matrix $\text{Var}(y)$

$$\text{Var}(y) = \text{Cov}(y, y) = E((y - E(y))(y - E(y))^\top)$$

Proposition 25. It can be seen that

$$[\text{Var}(y)]_{i,j} = \text{Cov}(y_i, y_j) \text{ for all } i, j = 1, \dots, d$$

and

$$[\text{Var}(y)]_{i,i} = \text{Var}(y_i)$$

for all $i = 1, \dots, d$

Proposition 26. The following properties are the direct analogues of the 1D cases

1. $\text{Var}(y) = E(yy^\top) - E(y) (E(y))^\top$
2. $\text{Var}(c + Ay) = A \text{Var}(y) A^\top$, for fixed matrix A , and vectors c with suitable dimensions.
3. $\text{Var}(y) \geq 0$; (semi-positive definite)

Proof. Given as Exercise 6 in the Exercise Sheet. □

9 Characteristic function

Characteristic functions (CF) provide an alternative way to the probability function for describing a random variable.

Definition 27. The characteristic function of a d dimensional random variable X is

$$\varphi_x(t) = E(e^{it^T x}) = \int e^{it^T x} dF(x)$$

for $t \in \mathbb{R}^d$, where $e^{it^T x} = \cos(t^T x) + i \sin(t^T x)$.

Proposition 28. Some properties of characteristic functions

1. $\varphi_x(t)$ exists for all $t \in \mathbb{R}^d$ and is absolutely continuous
2. $\varphi_x(0) = 1$ and $|\varphi_x(t)| \leq 1$ for all $t \in \mathbb{R}^d$
3. $\varphi_{A+Bx}(t) = e^{it^T A} \varphi_x(B^T t)$ if $A \in \mathbb{R}^d$ and $B \in \mathbb{R}^{k \times d}$ are constants
4. $\varphi_{x+y}(t) = \varphi_x(t) \varphi_y(t)$ if and only if x and y are independent
5. if $M_x(t) = E(e^{t^T x})$ is the moment generating function, then $M_x(t) = \varphi_x(-it)$

Proof. Given as Exercise 7 in the exercise sheet. □

Fact 29. Two random variables correspond have equal characteristic functions if and only if they follow the same distribution. AKA: CF completely determines the probability distribution of the random variable

Fact 30. If $\varphi_x(t)$ is absolutely integrable, then x has PDF

$$f(x) = \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} e^{-it^T x} \varphi_x(t) dt$$

Example 31. Address the following:

1. If $z \sim N(0, 1)$ then $\varphi_z(t) = \exp(-\frac{1}{2}t^2)$
2. If $x \sim N(\mu, \sigma^2)$ then $\varphi_x(t) = \exp(i\mu t - \frac{1}{2}t^2 \sigma^2)$
3. If $\varphi_z(t) = \exp(-\frac{1}{2}t^2)$ then $f_{N(0,1)}(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2)$

Solution. It is

1. It is

$$\begin{aligned} \varphi_z(t) &= E(e^{itz}) = \int e^{itz} dF_{N(0,1)}(z) = \int e^{itz} f_{N(0,1)}(z) dz = \int \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2 + itz) dz \\ &= \int \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2 + \frac{2}{2}itz \pm \frac{1}{2}(it)^2 z^2) dz = \int \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(z - it)^2) dz \times \exp(\frac{1}{2}(it)^2) \\ &= \exp(-\frac{1}{2}t^2) \end{aligned}$$

2. It is $\varphi_x(t) = \varphi_{\mu+\sigma z}(t) = \exp(i\mu t) \varphi_z(\sigma t) = \exp(i\mu t - \frac{1}{2}t^2 \sigma^2)$.

3. It is

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itz} \varphi_z(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-itz) \exp(-\frac{1}{2}t^2) dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-\frac{1}{2}t^2 + i^2 tz) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}(t^2 - i^2 z^2) - z^2\right) dt \\ &= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} \sqrt{\frac{1}{2\pi}} \exp\left(-\frac{1}{2}(t - i^2 z)^2\right) dt \exp\left(-\frac{1}{2}z^2\right) = \sqrt{\frac{1}{2\pi}} \exp\left(-\frac{1}{2}z^2\right) = f_{N(0,1)}(z) \end{aligned}$$

Theorem 32. The distribution of a d -dimensional random variable $x \in \mathbb{R}^d$ is completely determined by the set of all 1-dimensional distributions of linear combinations $a^\top x$, for any $a \in \mathbb{R}^d$.

157 *Proof.* Let $y = a^\top x$, for any $a \in \mathbb{R}^d$. Then for any $s \in \mathbb{R}$

$$158 \quad \varphi_y(s) = \mathbb{E}(e^{is^\top y}) = \mathbb{E}(e^{is^\top a^\top x}) = \mathbb{E}(e^{i(as)^\top x}) = \mathbb{E}(e^{i\tilde{t}^\top x}) = \varphi_x(\tilde{t})$$

159 where $\tilde{t} = as$ is any d -dimensional vector. □

160 10 Conditioning

161 **Definition 33.** Assume a probability space (Ω, \mathcal{F}, P) . For any sets $A, B \in \mathcal{F}$, the conditional probability of A given
162 B is defined as

$$163 \quad P(A|B) = \frac{P(B \cap A)}{P(B)} \quad \text{if } P(B) \neq 0.$$

164 **Definition 34.** Let $y \in \mathcal{Y}$ be a random variable. Consider a partition $y = (x, \theta)$ with $x \in \mathcal{X}$ and $\theta \in \Theta$. The expected
165 value of θ conditional that random variable $x \in B \subseteq \mathcal{X}$ is

$$166 \quad \mathbb{E}(\theta|x \in B) = \frac{\mathbb{E}(\theta 1(x \in B))}{P(x \in B)}, \quad \text{if } P(x \in B) > 0.$$

167 **Fact 35.** Let a random variable $y \in \mathcal{Y}$ with PDF/PMF $f(\cdot)$. Consider a partition $y = (x, \theta)$ with $x \in \mathcal{X}$ and $\theta \in \Theta$.

168 1. The conditional MPF/PDF and CDF of random variable θ given the random variable x

$$169 \quad f_{\theta|x}(\theta|x) = \frac{f(\theta, x)}{f(x)}, \quad ; \quad F_{\theta|x}(\theta|x) = \begin{cases} \int_{-\infty}^{\theta_1} \cdots \int_{-\infty}^{\theta_d} f_{\theta|x}(\vartheta|x) d\vartheta, & \theta, \text{ cont.} \\ \sum_{\vartheta_1=-\infty}^{\theta_1} \cdots \sum_{\vartheta_d=-\infty}^{\theta_d} f_{\theta|x}(\vartheta|x), & \theta, \text{ discr.} \end{cases}$$

170 provided that $f(x) > 0$.

171 2. The expected value of θ given the random variable x

$$172 \quad E(\theta|x) = \int \theta dF_{\theta|x}(\theta|x) = \begin{cases} \int \theta f_{\theta|x}(\theta|x) d\theta & , \text{ if } \theta \text{ is cont.} \\ \sum_{\vartheta \in \Theta} \vartheta f_{\vartheta|x}(\vartheta|x) & , \text{ if } \theta \text{ is discr.} \end{cases} \quad \text{provided that } f(x) > 0$$

173 **Example 36.** Let a random variable $y \in \mathcal{Y}$ with distribution $F(\cdot)$. Consider a partition $y = (x, \theta)^\top$ with $x \in \mathcal{X}$ and
174 $\theta \in \Theta$. Then

$$175 \quad 1. \quad \mathbb{E}(\theta) = \mathbb{E}(\mathbb{E}(\theta|x))$$

$$176 \quad 2. \quad \text{Var}(\theta) = \mathbb{E}(\text{Var}(\theta|x)) + \text{Var}(\mathbb{E}(\theta|x))$$

177 **Solution.**

178 1. It is

$$179 \quad \begin{aligned} \mathbb{E}(\mathbb{E}(\theta|x)) &= \int \left(\int \theta dF(\theta|x) \right) dF(x) = \int \int \theta dF(x, \theta) = \int \int \theta dF(\theta|x) dF(x) \\ 180 \quad &= \int \theta \left(\int dF(x|\theta) \right) F(\theta) = \int \theta F(\theta) = \mathbb{E}(\theta) \end{aligned}$$

2. It is

$$\begin{aligned}
 \text{Var}(\theta) &= \mathbb{E}(\mathbb{E}(\theta\theta^\top)) - \mathbb{E}(\theta)\mathbb{E}(\theta)^\top = \mathbb{E}(\mathbb{E}(\theta\theta^\top|x)) - \mathbb{E}(\mathbb{E}(\theta|x))\mathbb{E}(\mathbb{E}(\theta|x))^\top \\
 &= \mathbb{E}(\mathbb{E}(\theta\theta^\top|x)) - \mathbb{E}(\mathbb{E}(\theta|x)\mathbb{E}(\theta|x)^\top) + \mathbb{E}(\mathbb{E}(\theta|x)\mathbb{E}(\theta|x)^\top) - \mathbb{E}(\mathbb{E}(\theta|x))\mathbb{E}(\mathbb{E}(\theta|x))^\top \\
 &= \mathbb{E}(\mathbb{E}(\theta\theta^\top|x) - \mathbb{E}(\theta|x)\mathbb{E}(\theta|x)^\top) + \mathbb{E}(\mathbb{E}(\theta|x)\mathbb{E}(\theta|x) - \mathbb{E}(\mathbb{E}(\theta|x))\mathbb{E}(\mathbb{E}(\theta|x))^\top) \\
 &= \mathbb{E}(\text{Var}(\theta|x)) + \text{Var}(\mathbb{E}(\theta|x))
 \end{aligned}$$

Conditional independence

Definition 37. Given a probability space $(\Omega, \mathfrak{B}, P)$, and events $A, B, C \in \mathfrak{B}$, A and B are conditionally independent given C if and only if

$$P(A \cap B|C) = P(A|C)P(B|C), \text{ for } P(C) > 0.$$

Fact 38. Let $(n + m + d)$ -dimensional random variable $y \in \mathcal{Y} \subseteq \mathbb{R}^{n+m+d}$. Consider a partition $y = (x, z, \theta)$ where $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $z \in \mathcal{Z} \subseteq \mathbb{R}^m$, and $\theta \in \Theta \subseteq \mathbb{R}^d$.

- The r.v. x and y are independent given θ if and only if

$$F(y, z|\theta) = F(y|\theta)F(z|\theta), \quad \forall x \in \mathbb{R}^n, \forall z \in \mathbb{R}^m$$

where $F(\cdot|\theta)$ denotes the conditional CDF.

- This implies that r.v. x and z are independent given θ if and only if

$$f(y, z|\theta) = f(y|\theta)f(z|\theta)$$

where $f(\cdot|\theta)$ denotes the conditional PDF/PMF.

11 Inverting / updating

This offers a probabilistic mechanism for (1.) inversion $(B|A) \mapsto (A|B)$, or (2.) updating $(A) \mapsto (A|B)$.

Fact 39. [Bayesian theorem with sets] Assume a probability space (Ω, \mathcal{F}, P) . For any sets $A, B \in \mathcal{F}$, it is

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}, \quad \text{provided that } P(B) \neq 0.$$

The extension of the Bayesian theorem to the random variables is not straightforward.

Proposition 40. [Bayesian theorem with random variables] Let a random variable $y \in \mathcal{Y}$. Consider a partition $y = (x, \theta)$ with $x \in \mathcal{X}$ and $\theta \in \Theta$. Then the PDF/PMF of $\theta|x$ is

$$f(\theta|x) = \frac{f(x|\theta)f(\theta)}{\int f(x|\theta)f(\theta)dF(\theta)} = \begin{cases} \frac{f(x|\theta)f(\theta)}{\int f(x|\theta)f(\theta)d\theta} & , \text{ if } \theta \text{ is cont.} \\ \frac{f(x|\theta)f(\theta)}{\sum_{\forall \theta} f(x|\theta)f(\theta)} & , \text{ if } \theta \text{ is discr.} \end{cases}$$

Proof. Given as Exercise 10, in the Exercise Sheet. □

12 Practice

Question 41. Try the Exercises 8, 9, 10, from the Exercise sheet.