Homework 2: Bayesian calculations and Conjugate priors

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Exercise 1. ($\star\star$)Let $x = (x_1, ..., x_n)$ be observables. Consider a Bayesian model such as

$$\begin{cases} x_i|\lambda & \overset{\text{iid}}{\sim} \text{Pn}(\lambda),\ \forall i = 1, ..., n \\ \lambda & \sim \Pi(\lambda) \end{cases}$$

Hint-1 Poisson distribution $x \sim \text{Pn}(\lambda)$ has PMF: $\text{Pn}(x|\lambda) = \frac{1}{x!}\lambda^x \exp(-\lambda)1_{\mathbb{N}}(x)$, where $\mathbb{N} = \{0, 1, 2, ...\}$ and $\lambda > 0$.

Hint-2 Gamma distribution $x \sim \text{Ga}(a, b)$ has PDF: $\text{Ga}(x|a, b) = \frac{b^a}{\Gamma(a)}x^{a-1}\exp(-bx)1_{(0,\infty)}(x)$, with $a > 0$ and $b > 0$.

Hint-2 Negative Binomial distribution $x \sim \text{Nb}(r, \theta)$ has PMF: $\text{Nb}(x|r, \theta) = \binom{r+x-1}{r-1}\theta^r(1-\theta)^x 1_{\mathbb{N}}(x)$ with $\theta \in (0, 1)$, $r \in \mathbb{N} - \{0\}$, and $\mathbb{N} = \{0, 1, 2, ...\}$.

1. Compute the likelihood in the aforesaid Bayesian model.
2. Show that the sampling distribution is a member of the exponential family.
3. Specify the PDF of the conjugate prior distribution $\Pi(\lambda)$ of λ, and identify the parametric family of distributions as $\lambda \sim \text{Ga}(a, b)$, with $a > 0$, and $b > 0$. While you are deriving the conjugate prior distribution of λ, discuss which of the prior hyper-parameters can be considered as the ‘strength of the prior information and which can be considered as summarizing the prior information.
4. Compute the PDF of the posterior distribution of λ, identify the posterior distribution as a Gamma distribution $\text{Ga}(\tilde{a}, \tilde{b})$, and compute the posterior hyper-parameters $\tilde{a}$, and $\tilde{b}$.
5. Compute the PMF of the predictive distribution of a future outcome $y = x_{n+1}$, identify the name of the resulting predictive distribution, and compute its parameters.