

## Homework 3: Jeffreys' priors and Maximum entropy priors

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**Exercise 1.** (\*\*) Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Pn}(\theta), \forall i = 1, \dots, n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $\text{Pn}(\theta)$  is the Poisson distribution with expected value  $\theta$ . Specify a Jeffreys' prior for  $\theta$ .

**Hint:** Poisson distribution:  $x \sim \text{Pn}(\theta)$  has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

**Exercise 2.** (\*\*) Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Pn}(\theta), \forall i = 1, \dots, n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $\text{Pn}(\theta)$  is the Poisson distribution with expected value  $\theta$ . Specify a Maximum entropy prior under the constraint  $E(\theta) = 2$  and reference measure such as  $\pi_0(\theta) = \frac{1}{\sqrt{\theta}}$ . In particular, you also have to state the name of the derived Maximum entropy prior distribution and report the values of its parameters.

**Hint-1:** Poisson distribution:  $x \sim \text{Pn}(\theta)$  has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

**Hint-2:** Gamma distribution:  $x \sim \text{Ga}(a, b)$  has PDF

$$\text{Ga}(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-\beta x) 1(x > 0)$$

**Exercise 3.** (\*\*) Let  $x$  be an observation. Consider the Bayesian model

$$\begin{cases} x | \theta & \sim \text{Pn}(\theta) \\ \theta & \sim \Pi(\theta) \end{cases}$$

where  $\text{Pn}(\theta)$  is the Poisson distribution with expected value  $\theta$ . Consider a prior  $\Pi(\theta)$  with density such as  $\pi(\theta) \propto \frac{1}{\theta}$ . Show that the posterior distribution is not always defined.

**Hint-1:** It suffices to show that the posterior is not defined in the case that you collect only one observation  $x = 0$ .**Hint-2:** Poisson distribution:  $x \sim \text{Pn}(\theta)$  has PMF

$$\text{Pn}(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

## The Limit Comparison Theorem for Improper Integrals

- Brand, L. (2006; Chapter 7). Advanced calculus: an introduction to classical analysis.

**General:** Let integrable functions  $f(x)$ , and  $g(x)$  for  $x \geq a$ .

Let

$$0 \leq f(x) \leq g(x), \quad \text{for } x \geq a$$

Then

$$\begin{aligned}\int_a^\infty g(x)dx < \infty &\implies \int_a^\infty f(x)dx < \infty \\ \int_a^\infty f(x)dx = \infty &\implies \int_a^\infty g(x)dx = \infty\end{aligned}$$

**Type I:** Let integrable functions  $f(x)$ , and  $g(x)$  for  $x \geq a$ , and let  $g(x)$  be positive.

Let

$$\lim_{n \rightarrow \infty} \frac{f(x)}{g(x)} = c$$

Then

- If  $c \in (0, \infty)$  :

$$\int_a^\infty g(x)dx < \infty \iff \int_a^\infty f(x)dx < \infty$$

- If  $c = 0$  :

$$\int_a^\infty g(x)dx < \infty \implies \int_a^\infty f(x)dx < \infty$$

- If  $c = \infty$  :

$$\int_a^\infty f(x)dx = \infty \implies \int_a^\infty g(x)dx = \infty$$

**Type II:** Let integrable functions  $f(x)$ , and  $g(x)$  for  $a < x \leq b$ , and let  $g(x)$  be positive.

Let

$$\lim_{n \rightarrow a^+} \frac{f(x)}{g(x)} = c$$

Then

- If  $c \in (0, \infty)$  :

$$\int_a^\infty g(x)dx < \infty \iff \int_a^\infty f(x)dx < \infty$$

- If  $c = 0$  :

$$\int_a^\infty g(x)dx < \infty \implies \int_a^\infty f(x)dx < \infty$$

- If  $c = \infty$  :

$$\int_a^\infty f(x)dx = \infty \implies \int_a^\infty g(x)dx = \infty$$

**Note:** A useful test function is

$$\int_0^\infty \left(\frac{1}{x}\right)^p dx \begin{cases} < \infty & , \text{ when } p > 1 \\ = \infty & , \text{ when } p \leq 1 \end{cases}$$