Bayesian Statistics III/IV (MATH3361/4071)

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Homework 3: Jeffreys' priors and Maximum entropy priors

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Exercise 1. $(\star\star)$ Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Pn}(\theta), \ \forall i = 1, ..., n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $Pn(\theta)$ is the Poisson distribution with expected value θ . Specify a Jeffreys' prior for θ .

Hint: Poisson distribution: $x \sim Pn(\theta)$ has PMF

$$Pn(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

Exercise 2. (**)Consider the Bayesian model

$$\begin{cases} x_i | \theta & \stackrel{\text{iid}}{\sim} \text{Pn}(\theta), \ \forall i = 1, ..., n \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $Pn(\theta)$ is the Poisson distribution with expected value θ . Specify a Maximum entropy prior under the constrain $E(\theta)=2$ and reference measure such as $\pi_0(\theta)=\frac{1}{\sqrt{\theta}}$. In particular, you also have to state the name of the derived Maximum entropy prior distribution and report the values of its parameters.

Hint-1: Poisson distribution: $x \sim Pn(\theta)$ has PMF

$$Pn(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

Hint-2: Gamma distribution: $x \sim Ga(a, b)$ has PDF

$$Ga(x|a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-\beta x) \mathbb{1}(x>0)$$

Exercise 3. $(\star\star)$ Let x be an observation. Consider the Bayesian model

$$\begin{cases} x|\theta & \sim \operatorname{Pn}(\theta) \\ \theta & \sim \Pi(\theta) \end{cases}$$

where $Pn(\theta)$ is the Poisson distribution with expected value θ . Consider a prior $\Pi(\theta)$ with density such as $\pi(\theta) \propto \frac{1}{\theta}$. Show that the posterior distribution is not always defined.

Hint-1: It suffices to show that the posterior is not defined in the case that you collect only one observation x = 0.

Hint-2: Poisson distribution: $x \sim Pn(\theta)$ has PMF

$$Pn(x|\theta) = \frac{\theta^x \exp(-\theta)}{x!} 1(x \in \mathbb{N})$$

The Limit Comparison Theorem for Improper Integrals

• Brand, L. (2006; Chapter 7). Advanced calculus: an introduction to classical analysis.

General: Let integrable functions f(x), and g(x) for $x \ge a$.

Let

$$0 \le f(x) \le g(x)$$
, for $x \ge a$

Then

$$\int_{a}^{\infty} g(x) dx < \infty \implies \int_{a}^{\infty} f(x) dx < \infty$$
$$\int_{a}^{\infty} f(x) dx = \infty \implies \int_{a}^{\infty} g(x) dx = \infty$$

Type I: Let integrable functions f(x), and g(x) for $x \ge a$, and let g(x) be positive.

Let

$$\lim_{n \to \infty} \frac{f(x)}{g(x)} = c$$

Then

- If $c\in(0,\infty)$: $\int_a^\infty g(x)\mathrm{d}x<\infty\Longleftrightarrow\int_a^\infty f(x)\mathrm{d}x<\infty$
- If c=0 : $\int_a^\infty g(x)\mathrm{d}x <\infty \implies \int_a^\infty f(x)\mathrm{d}x <\infty$
- If $c=\infty$: $\int_a^\infty f(x)\mathrm{d}x = \infty \implies \int_a^\infty g(x)\mathrm{d}x = \infty$

Type II: Let integrable functions f(x), and g(x) for $a < x \le b$, and let g(x) be positive.

Let

$$\lim_{n \to a^+} \frac{f(x)}{g(x)} = c$$

Then

- If $c\in(0,\infty)$: $\int_{a}^{\infty}g(x)\mathrm{d}x<\infty\Longleftrightarrow\int_{a}^{\infty}f(x)\mathrm{d}x<\infty$
- If c=0 : $\int_a^\infty g(x)\mathrm{d}x <\infty \implies \int_a^\infty f(x)\mathrm{d}x <\infty$
- If $c = \infty$: $\int_a^\infty f(x) \mathrm{d}x = \infty \implies \int_a^\infty g(x) \mathrm{d}x = \infty$

Note: A useful test function is

$$\int_0^\infty \left(\frac{1}{x}\right)^p \mathrm{d}x \quad \begin{cases} <\infty &, \text{ when } p>1\\ =\infty &, \text{ when } p\leq 1 \end{cases}$$