

**Handout 14: Jeffreys-Lindley ‘Paradox’ (?) <sup>a</sup>**

Lecturer: Georgios P. Karagiannis

georgios.karagiannis@durham.ac.uk

**Aim:** To explain, and theorize Jeffreys-Lindley Paradox**References:**

- Berger, J. O. (2013; Section 4.3.3). Statistical decision theory and Bayesian analysis. Springer Science & Business Media.
- Robert, C. (2007; Section 5.2(exclude 5.2.6)). The Bayesian choice: from decision-theoretic foundations to computational implementation. Springer Science & Business Media.

<sup>a</sup>Author: Georgios P. Karagiannis.**1 Improper priors situations**

Jeffreys-Lindley Paradox describes the overwhelming statistical evidence in favor a hypothesis when the priors of the alternative hypotheses diverge faster. It occurs in Bayesian hypothesis test and model selection with improper priors.

**Example 1.** [Single-vs-General-alternative]Let  $y = (y_1, \dots, y_n)$  observables and consider the Bayesian hypothesis test

$$H_0 : y_i | \theta_0 \stackrel{\text{iid}}{\sim} N(\theta_0, \sigma^2), i = 1, \dots, n \quad \text{vs} \quad H_1 : \begin{cases} y_i | \mu & \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2), i = 1, \dots, n \\ \mu & \sim N(\mu_0, \sigma_0^2) \end{cases} \quad (1)$$

with  $\pi_j = P_{\Pi}(\theta \in \Theta_j)$  for  $j = 0, 1$ . Let  $\theta_0, \sigma^2, \mu_0, \sigma_0^2$  be fixed values. It can be computed (Appendix A) that

$$B_{01}(y) = \frac{\left(\frac{\sigma^2}{n}\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(\bar{y}-\theta_0)^2}{\frac{\sigma^2}{n}}\right)}{\left(\frac{\sigma^2}{n} + \sigma_0^2\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(\bar{y}-\mu_0)^2}{\frac{\sigma^2}{n} + \sigma_0^2}\right)}; \quad P_{\Pi}(H_0|y) = \left(1 + \frac{1 - \pi_0}{\pi_0} \frac{\left(\frac{\sigma^2}{n} + \sigma_0^2\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(\bar{y}-\mu_0)^2}{\frac{\sigma^2}{n} + \sigma_0^2}\right)}{\left(\frac{\sigma^2}{n}\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(\bar{y}-\theta_0)^2}{\frac{\sigma^2}{n}}\right)}\right)^{-1}$$

1. The Jeffreys’ (and Laplace) prior of  $\mu$  is  $\pi^{(j)}(\mu) \propto 1(\mu \in \mathbb{R})$  (see Handout 7). Find values  $\mu_*, \sigma_*^2$  such that

$$\tilde{\pi}(\mu|H_1) \rightarrow \tilde{\pi}^{(j)}(\mu); \text{ as } (\mu_0, \sigma_0^2) \rightarrow (\mu_*, \sigma_*^2) \quad (2)$$

where  $\tilde{\pi}(\mu|H_1)$  denotes the kernel of the pdf of the conditional prior  $\mu \sim N(\mu_0, \sigma_0^2)$  of  $\mu$  under  $H_1$  in (1), and  $\tilde{\pi}^{(j)}(\mu)$  denotes that of the the Jeffreys prior  $\pi^{(j)}(\mu) \propto 1(\mu \in \mathbb{R})$ .

2. [Lindley’s Paradox] Let  $\sigma_0^2 \rightarrow \infty$ . Investigate, how, and why

- the conditional prior  $\pi(\mu|H_1)$  given the hypothesis  $H_1$  behaves?
- the Bayes Factor  $B_{01}(y)$  and the posterior  $P_{\Pi}(H_0|y)$  behave?

**Solution.** The calculation of  $B_{01}(y)$  and of  $P_{\Pi}(H_0|y)$  is presented in the Appendix A.

1. The kernel  $\tilde{\pi}(\mu|H_1)$  of the prior  $\pi(\mu|H_1)$  is

$$\pi(\mu|H_1) = N(\mu|\mu_0, \sigma_0^2) \propto \exp\left(-\frac{1}{2\sigma_0^2}(\mu - \mu_0)^2\right) 1(\mu \in \mathbb{R}) = \tilde{\pi}(\mu|H_1)$$

So for  $\sigma_0^2 = \infty$  and any  $|\mu_*| < \infty$ , I get the (2).

2. If  $\sigma_0^2 \rightarrow \infty$ , the prior  $N(\mu_0, \sigma_0^2)$  of  $H_1$  becomes flat (the mass spreads uniformly around  $\mathbb{R}$ ) and improper:

$$\pi(\mu|H_1) = N(\mu|\mu_0, \sigma_0^2) \propto \exp\left(-\frac{1}{2}\frac{(\mu - \mu_0)^2}{\sigma_0^2}\right) 1(\mu \in \mathbb{R}) \xrightarrow{\sigma_0^2 \rightarrow \infty} 1(\mu \in \mathbb{R})$$

In fact  $\pi(\mu|H_1)$  meets Jeffreys' (or Laplace) prior of  $\mu$  in the limit  $\sigma_0^2 \rightarrow \infty$ .

Then in the limit the Bayes factor  $B_{01}(y)$  and the posterior  $P_{\Pi}(H_0|y)$  approach the values

$$B_{01}(y) \rightarrow \infty, \text{ and } P_{\Pi}(H_0|y) \rightarrow 1 \text{ as } \sigma_0^2 \rightarrow \infty$$

We observe that regardless the number of the observables  $n$ , when the prior variance  $\sigma_0^2$  of  $\mu$  given  $H_1$  becomes huge ( $\sigma_0^2 \rightarrow \infty$ ), the prior of  $\mu$  in  $H_1$  becomes more diverge

$$\pi(\mu|H_1) \xrightarrow{\sigma_0^2 \rightarrow \infty} 1(\mu \in \mathbb{R}), \text{ as } \sigma_0^2 \rightarrow \infty$$

spreading the prior mass uniformly in  $\mathbb{R}$  while the evidence in favor  $H_0 : \mu = \theta_0$  that  $\mu$  is equal to single value  $\theta_0$  becomes overwhelming.

$$B_{01}(y) \xrightarrow{\sigma_0^2 \rightarrow \infty} \infty \text{ and } P_{\Pi}(H_0|y) \xrightarrow{\sigma_0^2 \rightarrow \infty} 1$$

Paradoxical behavior: One would not expect for  $\sigma_0^2 \rightarrow \infty$  favor  $H_0 : \mu = \theta_0$  because  $\sigma_0^2 \rightarrow \infty$  gives increases ignorance and give uniformly positive mass to more values. Also, we wouldn't expect increasing the sample size  $n$  to have no effect.

Essentially, the above paradoxical behavior says that: When  $\sigma_0^2 \rightarrow \infty$ , it means that a priori, I know  $\mu = \theta_0$  with probability  $\pi_0$ , but I know nothing about  $\mu$  probability  $1 - \pi_0$ ; however after I get any observation  $y$  I am a posteriori certain that  $\mu = \theta_0$ .

Observe that the overall posterior is

$$\pi(\mu) = \pi_0 1(\mu \in \{\theta_0\}) + (1 - \pi_0) \left[ \left(\frac{1}{2\pi\sigma_0^2}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\frac{(\mu - \mu_0)^2}{\sigma_0^2}\right) 1(\mu \in \mathbb{R}) \right]$$

When  $\sigma_0^2 \rightarrow \infty$ , it involves one conditional component that concentrates the mass  $\pi(\mu|H_0) = 1_{\{\theta_0\}}(\mu)$  and one component with infinite normalizing constant. So we cannot apply the Bayes theorem by using the kernels of the priors and in the sense of canceling normalizing constants as  $\pi(\mu|y) \propto f(y|\mu)\tilde{\pi}(\mu)$  where  $\pi(\mu) \propto \tilde{\pi}(\mu)$ .

Lindley's paradox also appears in composite-vs-composite tests with improper priors, when the one prior diverges faster than the other in the limit. See the following realistic example in Bayesian regression Variable selection problem.

#### Example 2. [Composite-vs-Composite]

Consider a Normal linear regression model with dependent variable  $y$  and a set of regressors  $\{\Phi_j\}_{j \in \mathcal{M}}$  where  $\mathcal{M}$  is the set of size  $d$  that includes the labels of the available regressors; e.g.

$$y_i|\beta, \sigma^2 \sim N\left(\sum_{j \in \mathcal{M}} \Phi_{i,j} \beta_j, I\sigma^2\right), \text{ for } i = 1, \dots, n$$

where the regression coefficients  $\{\beta_j\}_{j \in \mathcal{M}}$  and the noise variance  $\sigma^2$  are unknown.

Let  $\mathcal{M}_0$  and  $\mathcal{M}_1$  denote two sets of regressors (nested or not) with  $\dim(\mathcal{M}_j) = d_j$ . We are interested in learning whether the linear model with  $\mathcal{M}_0$  set of regressors or that with  $\mathcal{M}_1$  set of regressors models the data generating processes ‘better’. I.e. we may test

$$H_0 : \begin{cases} y|\beta_{\mathcal{M}_0}, \sigma^2 & \sim N(\Phi_{\mathcal{M}_0}\beta_{\mathcal{M}_0}, I\sigma^2) \\ \beta_{\mathcal{M}_0}|\sigma^2 & \sim N(\mu_{\mathcal{M}_0}, V_{\mathcal{M}_0}\sigma^2) \\ \sigma^2 & \sim \text{IG}(a, k) \end{cases} \quad \text{v.s.} \quad H_1 : \begin{cases} y|\beta_{\mathcal{M}_1}, \sigma^2 & \sim N(\Phi_{\mathcal{M}_1}\beta_{\mathcal{M}_1}, I\sigma^2) \\ \beta_{\mathcal{M}_1}|\sigma^2 & \sim N(\mu_{\mathcal{M}_1}, V_{\mathcal{M}_1}\sigma^2) \\ \sigma^2 & \sim \text{IG}(a, k) \end{cases}$$

The Bayes factor  $B_{01}(y)$  and the posterior marginal probability  $P_{\Pi}(H_0|y)$  are (See the Appendix B)

$$B_{01}(y) = \sqrt{\frac{|V_1|}{|V_0|}} \sqrt{\frac{|V_0^*|}{|V_1^*|}} \left( \frac{k_0^*}{k_1^*} \right)^{-\frac{n}{2}-a}; \quad P_{\Pi}(H_0|y) = \left( 1 + \frac{1 - \pi_0}{\pi_0} B_{01}^{-1}(y) \right)^{-1}$$

where for  $j = 0, 1$

$$k_j^* = k + \frac{1}{2} \mu_j^\top V_j^{-1} \mu_j - \frac{1}{2} (\mu_j^*)^\top (V_j^*)^{-1} \mu_j^* + \frac{1}{2} y^\top y$$

$$V_j^* = (V_j^{-1} + \Phi_j^\top \Phi_j)^{-1}; \quad \mu_j^* = V_j^* (V_j^{-1} \mu_j + \Phi_j^\top y)$$

Assume that  $V_{\mathcal{M}_0} = vI_{d_0}$  and  $V_{\mathcal{M}_1} = vI_{d_1}$ . Let  $v \rightarrow \infty$ , how  $B_{01}(y)$  and  $P_{\Pi}(H_0|y)$  behave when (1.)  $d_0 < d_1$ , (2.)  $d_0 > d_1$ , and (3.)  $d_0 = d_1$

**Solution.** For  $V_0 = vI_{d_0}$  and  $V_1 = vI_{d_1}$  it is

$$\lim_{v \rightarrow \infty} B_{01}(y) = \lim_{v \rightarrow \infty} (v)^{\frac{d_1 - d_0}{2}} \times \sqrt{\frac{|\Phi_0^\top \Phi_0|}{|\Phi_1^\top \Phi_1|}} \left( \frac{k - \frac{1}{2} y^\top \Phi_0 (\Phi_0^\top \Phi_0)^{-1} \Phi_0^\top y_0 + y^\top y}{k - \frac{1}{2} y^\top \Phi_1 (\Phi_1^\top \Phi_1)^{-1} \Phi_1^\top y_1 + y^\top y} \right)^{-\frac{n}{2}-a}$$

$$\text{So } B_{01}(y) \xrightarrow{v \rightarrow \infty} \begin{cases} +\infty, & d_0 < d_1 \\ 0, & d_0 > d_1 \\ < \infty, & d_0 = d_1 \end{cases} \quad \text{and} \quad P_{\Pi}(H_0|y) \xrightarrow{v \rightarrow \infty} \begin{cases} 1, & d_0 < d_1 \\ 0, & d_0 > d_1 \\ \in (0, 1), & d_0 = d_1 \end{cases}$$

As  $v \rightarrow \infty$ , both conditional priors  $\beta_0|\sigma^2 \sim N(\mu_0, \sigma^2 I v)$  and  $\beta_1|\sigma^2 \sim N(\mu_1, \sigma^2 I v)$  under hypothesis  $H_0$  and  $H_1$  become more and more diverge, while the evidence becomes more overwhelming in favor of the hypothesis that the prior diverges slower. In our example, when  $d_0 < d_1$  the conditional prior  $\pi(\beta|\sigma^2, H_0)$  given  $H_0$  diverges slower.

## 2 Informal but intuitive investigation of the phenomenon

Consider

$$H_0 : \theta \in \Theta_0, \text{ vs } H_1 : \theta \in \Theta_1$$

where  $\{\Theta_0, \Theta_1\}$  unbounded sets partitioning  $\Theta \subseteq \mathbb{R}^d$ . Consider overall prior with pdf

$$\pi(\theta) = \pi_0 \pi_0(\theta) + \pi_1 \pi_1(\theta)$$

Let conditional priors  $\pi_0(\theta) = \frac{1}{C_0} \tilde{\pi}_0(\theta)$  and  $\pi_1(\theta) = \frac{1}{C_1} \tilde{\pi}_1(\theta)$  where  $\tilde{\pi}_0(\theta)$  and  $\tilde{\pi}_1(\theta)$  are pdf kernels.

The posterior probability  $P_{\Pi}(H_0|y)$  of the hypothesis  $H_0$  is

$$P_{\Pi}(\mu \in \Theta_0|y) = \frac{\pi_0 C_0^{-1} \int_{\Theta_0} \tilde{\pi}_0(\theta) f(y|\theta) d\theta}{\pi_0 C_0^{-1} \int_{\Theta_0} \tilde{\pi}_0(\theta) f(y|\theta) d\theta + \pi_1 C_1^{-1} \tilde{\pi}_1(\theta) \int_{\Theta_1} f(y|\theta) d\theta} = \left[ 1 + \frac{C_0 \int_{\Theta_1} \tilde{\pi}_1(\theta) f(y|\theta) d\theta}{C_1 \int_{\Theta_0} \tilde{\pi}_0(\theta) f(y|\theta) d\theta} \right]^{-1}$$

The Bayes factor is

$$B_{01}(y) = \frac{C_1 \int_{\Theta_0} \tilde{\pi}_0(\theta) f(y|\theta) d\theta}{C_0 \int_{\Theta_1} \tilde{\pi}_1(\theta) f(y|\theta) d\theta}.$$

When  $\tilde{\pi}_0(\theta) \rightarrow \text{const}$  and  $\tilde{\pi}_1(\theta) \rightarrow \text{const}$ , the normalizing constants of  $\pi_0(\theta)$  and  $\pi_1(\theta)$  become infinite  $C_0 \rightarrow \infty$  and  $C_1 \rightarrow \infty$ . Then whether  $B_{01}(y) \rightarrow 0$  or  $B_{01}(y) \rightarrow \infty$  depends on which prior  $\pi_0(\theta)$  or  $\pi_1(\theta)$  becomes improper faster than the other, namely

$$\frac{C_1}{C_0} \xrightarrow[\tilde{\pi}_1(\theta) \rightarrow \text{const}]{\tilde{\pi}_0(\theta) \rightarrow \text{const}} \begin{cases} \infty & , \text{ if } C_0 \ll C_1 \\ 0 & , \text{ if } C_1 \ll C_0 \end{cases}$$

In Example 2,

$$C_j = (2\pi)^{\frac{n}{2}} |V_j|^{\frac{1}{2}} \stackrel{V_j = vI}{=} (2\pi)^{\frac{n}{2}} v^{\frac{d_j}{2}}; \implies \frac{C_1}{C_0} = (v)^{\frac{d_1 - d_0}{2}} \xrightarrow{v \rightarrow \infty} \begin{cases} \infty & , \text{ if } d_0 < d_1 \\ 0 & , \text{ if } d_0 > d_1 \end{cases} . \text{as}$$

**Summary 3.** Improper priors do not really work in hypothesis tests, or model comparison. Be cautious when You use improper priors.

### 3 Large samples situation

It is not a secret<sup>1</sup> that in Frequentist hypothesis tests you can reject  $H_0$  if you get a large enough sample. P-values is a misleading description of the evidence against  $H_0$ . What is the case in Bayesian hypothesis tests?

**Example 4.** (Cont. of Example 1) Set  $\mu_0 = \theta_0$  and let  $z_n = \frac{\bar{y} - \mu_0}{\sigma} \sqrt{n}$ . Then, it is

$$B_{01}(y) \stackrel{\text{calc.}}{=} \frac{(1 + n\sigma_0^2/\sigma^2)^{\frac{1}{2}}}{\exp\left(\frac{1}{2}z_n^2 (1 + \sigma^2/(\sigma_0^2 n))^{-1}\right)} \geq \frac{(1 + n\sigma_0^2/\sigma^2)^{\frac{1}{2}}}{\exp\left(\frac{1}{2}z_n^2\right)},$$

Assume that  $|z_n| \geq z_{1-\frac{\alpha}{2}}^*$ ; compare the result of the Bayesian hypothesis test against the  $\alpha$ -sig. level Frequentist test.

**Solution.** Consider  $z_n$  as fixed. For the Bayesian hypothesis test, I get

$$B_{01}(y) \geq \frac{(1 + n\sigma_0^2/\sigma^2)^{\frac{1}{2}}}{\exp\left(\frac{1}{2}z_n^2\right)}, \text{ and } P_{\Pi}(H_0|y) \geq \left(1 + \frac{1 - \pi_0}{\pi_0} \frac{\exp\left(\frac{1}{2}z_n^2\right)}{(1 + n\sigma_0^2/\sigma^2)^{\frac{1}{2}}}\right)^{-1}$$

As  $n \rightarrow \infty$   $|z_n| < \infty$  is bounded due to the CLT and hence

$$B_{01}(y) \rightarrow \infty, \text{ and } P_{\Pi}(H_0|y) \rightarrow 1$$

so  $H_0$  is accepted for sure. In the frequentist hypothesis test, I reject  $H_0$  at  $\alpha$ -sig. level, because  $|z_n| > z_{1-\frac{\alpha}{2}}^*$ .

The Bayesian behavior is the reasonable one: It has to be  $\bar{y} \approx \mu_0$  for the observed  $z_n = \frac{\bar{y} - \mu_0}{\sigma} \sqrt{n}$  to be a fixed finite number and not an infinite as  $n \rightarrow \infty$

**Note 5.** It is not difficult to see that for Single-vs-General-alternative tests  $H_0 : \mu = \theta_0$  vs  $H_1 : \mu \neq \theta_0$

$$B_{01}(y) = \frac{f_0(y)}{f_1(y)} \geq \frac{f(y|\theta_0)}{\sup_{\mu \neq \theta_0} f(y|\mu)} = \frac{f(y|\theta_0)}{f(y|\hat{\mu}_{MLE})} \equiv \text{Max. Likl. Ratio}$$

implying that the  $B_{01}(y)$  has the maximum likelihood ratio as a lower bound.

**Question.** Practice with Exercises 66 and 67 in the Exercise sheet.

<sup>1</sup>...maybe it is kept as a secret from students attending only frequentist courses in any university ...

# Appendix

The following calculations are given for completeness. They are part of Exercises 66 and 67 in the Exercise sheet.

## A Calculations

**Hint-1:** It is

$$-\frac{1}{2} \sum_{i=1}^n \frac{(x - \mu_i)^2}{\sigma_i^2} = -\frac{1}{2} \frac{(x - \hat{\mu})^2}{\hat{\sigma}^2} + C(\hat{\mu}, \hat{\sigma}^2)$$

$$\hat{\sigma}^2 = \left( \sum_{i=1}^n \frac{1}{\sigma_i^2} \right)^{-1}; \quad \hat{\mu} = \hat{\sigma}^2 \left( \sum_{i=1}^n \frac{\mu_i}{\sigma_i^2} \right); \quad C(\hat{\mu}, \hat{\sigma}^2) = \underbrace{\frac{1}{2} \frac{(\sum_{i=1}^n \frac{\mu_i}{\sigma_i^2})^2}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} - \frac{1}{2} \sum_{i=1}^n \frac{\mu_i^2}{\sigma_i^2}}_{=\text{independent of } x}$$

**Hint-2:** It is  $\sum_{i=1}^n (x_i - \theta)^2 = \sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \theta)^2$

**Solution.** The overall prior is

$$\pi(\mu) = \pi_0 1_{\{\theta_0\}}(\mu) + (1 - \pi_0) N(\mu | \mu_0, \sigma_0^2)$$

with  $\pi_0 = 1/2$  (although the value does not play any role here), and known  $\mu_0$  and  $\sigma_0^2$ .

The Bayes factor is

$$B_{01}(y) = \frac{f_0(y)}{f_1(y)}$$

So

$$\begin{aligned} f_0(y) &= f(y | \theta_0) = \prod_{i=1}^n N(y_i | \theta_0, \sigma^2) = \left( \frac{1}{2\pi\sigma^2} \right)^{\frac{n}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(y_i - \theta_0)^2}{\sigma^2}\right) \\ &= (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2} \frac{1}{\sigma^2} \underbrace{\left( \sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{y} - \theta_0)^2 \right)}_{=ns^2}\right) \\ &= (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2} \frac{1}{\sigma^2/n} s^2\right) \exp\left(-\frac{1}{2} \frac{1}{\sigma^2/n} (\bar{y} - \theta_0)^2\right) \\ f_1(y) &= \int_{\mathbb{R}} f(y | \theta) d\Pi_1(\theta) = \int_{\mathbb{R}} \prod_{i=1}^n N(y_i | \mu, \sigma^2) N(\mu | \mu_0, \sigma_0^2) d\mu \\ &= \int_{\mathbb{R}} (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2} \frac{1}{\sigma^2} \underbrace{\left( \sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2 \right)}_{=ns^2}\right) \times \\ &\quad \times (2\pi)^{-\frac{1}{2}} (\sigma_0^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{1}{\sigma_0^2} (\mu - \mu_0)^2\right) d\mu \\ &= (2\pi)^{-\frac{n}{2} - \frac{1}{2}} (\sigma^2)^{-\frac{n}{2}} (\sigma_0^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{1}{\sigma^2/n} s^2\right) \\ &\quad \times \int_{\mathbb{R}} \underbrace{\exp\left(-\frac{1}{2} \frac{1}{\sigma^2/n} (\bar{x} - \mu)^2 - \frac{1}{2} \frac{1}{\sigma_0^2} (\mu - \mu_0)^2\right)}_{=A(\mu)} d\mu \end{aligned}$$

$$\begin{aligned}
A(\mu) &= -\frac{1}{2} \frac{1}{\sigma^2/n} (\bar{y} - \mu)^2 - \frac{1}{2} \frac{1}{\sigma_0^2} (\mu - \mu_0)^2 = -\frac{1}{2} \frac{(\mu - \hat{\mu})^2}{\hat{\sigma}^2} - \frac{1}{2} \left( \frac{\bar{y}^2}{\sigma^2/n} + \frac{\mu_0^2}{\sigma_0^2} \right) + \frac{1}{2} \frac{(\frac{\bar{y}}{\sigma^2/n} + \frac{\mu_0}{\sigma_0^2})^2}{\frac{1}{\sigma^2/n} + \frac{1}{\sigma_0^2}} \\
&= -\frac{1}{2} \frac{(\mu - \hat{\mu})^2}{\hat{\sigma}^2} - \frac{1}{2} \frac{(\bar{y} - \mu_0)^2}{\frac{\sigma^2}{n} + \sigma_0^2}
\end{aligned}$$

where

$$\hat{\sigma}^2 = \left( \frac{1}{\sigma^2/n} + \frac{1}{\sigma_0^2} \right)^{-1}$$

and

$$\int_{\mathbb{R}} \exp(A(\mu)) d\mu = \exp\left(-\frac{1}{2} \frac{(\bar{y} - \mu_0)^2}{\frac{\sigma^2}{n} + \sigma_0^2}\right) \int_{\mathbb{R}} \exp\left(-\frac{1}{2} \frac{(\mu - \hat{\mu})^2}{\hat{\sigma}^2}\right) d\mu = \exp\left(-\frac{1}{2} \frac{(\bar{y} - \mu_0)^2}{\frac{\sigma^2}{n} + \sigma_0^2}\right) (2\pi)^{\frac{1}{2}} (\hat{\sigma}^2)^{\frac{1}{2}}$$

So

$$\begin{aligned}
f_1(y) &= (2\pi)^{-\frac{n}{2}-\frac{1}{2}} (\sigma^2)^{-\frac{n}{2}} (\sigma_0^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{1}{\sigma^2/n} s^2\right) \int_{\mathbb{R}} \exp(A(\mu)) d\mu \\
&= (2\pi)^{-\frac{n}{2}-\frac{1}{2}} (\sigma^2)^{-\frac{n}{2}} (\sigma_0^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{1}{\sigma^2/n} s^2\right) (2\pi \hat{\sigma}^2)^{1/2} \exp\left(-\frac{1}{2} \frac{(\bar{y} - \mu_0)^2}{\frac{\sigma^2}{n} + \sigma_0^2}\right)
\end{aligned}$$

It is

$$B_{01}(y) = \frac{f_0(y)}{f_1(y)} = \frac{\left(\frac{\sigma^2}{n}\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(\bar{y} - \theta_0)^2}{\frac{\sigma^2}{n}}\right)}{\left(\frac{\sigma^2}{n} + \sigma_0^2\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(\bar{y} - \mu_0)^2}{\frac{\sigma^2}{n} + \sigma_0^2}\right)}$$

It is

$$P_{\Pi}(H_0|y) = \left(1 + \frac{1 - \pi_0}{\pi_0} B_{01}(y)^{-1}\right)^{-1} = \left(1 + \frac{1 - \pi_0}{\pi_0} \frac{\left(\frac{\sigma^2}{n} + \sigma_0^2\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(y - \mu_0)^2}{\frac{\sigma^2}{n} + \sigma_0^2}\right)}{\left(\frac{\sigma^2}{n}\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(\bar{y} - \theta_0)^2}{\frac{\sigma^2}{n}}\right)}\right)^{-1}$$

## B Calculations

You may use the following identity:

$$\begin{aligned}
(y - \Phi\beta)^\top (y - \Phi\beta) + (\beta - \mu)^\top V^{-1}(\beta - \mu) &= (\beta - \mu^*)^\top (V^*)^{-1}(\beta - \mu^*) + S^*; \\
S^* &= \mu^\top V^{-1}\mu - (\mu^*)^\top (V^*)^{-1}(\mu^*) + y^\top y; \quad V^* = (V^{-1} + \Phi^\top \Phi)^{-1}; \quad \mu^* = V^* (V^{-1}\mu + \Phi^\top y)
\end{aligned}$$

**Solution.** For simplicity, we suppress the indexing denoting the sub-set of the regressors. It is

$$\begin{aligned}
f(y) &= \int N(y|\Phi\beta, I\sigma^2) N(\beta|\mu, V\sigma^2) \text{IG}(\sigma^2|a, k) d\beta d\sigma^2 \\
&= \int \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}(y - \Phi\beta)^\top (y - \Phi\beta)\right) \times \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \left(\frac{1}{|\sigma^2 V|}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(\beta - \mu)^\top V^{-1}(\beta - \mu)\right) \\
&\quad \times \frac{k^a}{\Gamma(a)} \left(\frac{1}{\sigma^2}\right)^{a+1} \exp\left(-\frac{k}{\sigma^2}\right) d\beta d\sigma^2 = \dots
\end{aligned}$$

$$\begin{aligned}
f(y) &= \left(\frac{1}{2\pi}\right)^{\frac{n+d}{2}} \left(\frac{1}{|V|}\right)^{\frac{1}{2}} \frac{k^a}{\Gamma(a)} \\
&\times \int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2} + \frac{d}{2} + a + 1} \exp\left(-\frac{1}{2\sigma^2}(y - \Phi\beta)^\top(y - \Phi\beta) - \frac{1}{2\sigma^2}(\beta - \mu)^\top V^{-1}(\beta - \mu) - \frac{k}{\sigma^2}\right) d\beta d\sigma^2 \\
&= \left(\frac{1}{2\pi}\right)^{\frac{n+d}{2}} \left(\frac{1}{|V|}\right)^{\frac{1}{2}} \frac{k^a}{\Gamma(a)} \\
&\times \int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2} + \frac{d}{2} + a + 1} \exp\left(-\frac{1}{\sigma^2} \left[\frac{(y - \Phi\beta)^\top(y - \Phi\beta) + (\beta - \mu)^\top V^{-1}(\beta - \mu)}{2} + k\right]\right) d\beta d\sigma^2 \\
&= \left(\frac{1}{2\pi}\right)^{\frac{n+d}{2}} \left(\frac{1}{|V|}\right)^{\frac{1}{2}} \frac{k^a}{\Gamma(a)} \\
&\times \int \left[\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2} + \frac{d}{2} + a + 1} \exp\left(-\frac{1}{\sigma^2} \left(\frac{1}{2}S + k\right)\right) \left[\int \exp\left(-\frac{1}{2} \frac{1}{\sigma^2}(\beta - v)^\top (V^*)^{-1}(\beta - \mu^*)\right) d\beta\right]\right] d\sigma^2 \\
&= \left(\frac{1}{2\pi}\right)^{\frac{n}{2} + \frac{d}{2}} \left(\frac{1}{|V|}\right)^{\frac{1}{2}} \frac{k^a}{\Gamma(a)} \int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2} + \frac{d}{2} + a + 1} \exp\left(-\frac{1}{\sigma^2} \left(\frac{1}{2}S + k\right)\right) \times \left[(2\pi)^{\frac{d}{2}} (\sigma^2)^{\frac{d}{2}} |V^*|^{\frac{1}{2}}\right] d\sigma^2 \\
&= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left(\frac{|V^*|}{|V|}\right)^{\frac{1}{2}} \frac{k^a}{\Gamma(a)} \frac{\Gamma\left(\frac{n}{2} + a\right)}{\left(\frac{1}{2}S + k\right)^{\frac{n}{2} + a}} = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \left(\frac{|V^*|}{|V|}\right)^{\frac{1}{2}} \frac{k^a}{\Gamma(a)} \frac{\Gamma\left(\frac{n}{2} + a\right)}{(k^*)^{\frac{n}{2} + a}}
\end{aligned}$$

Where

$$\begin{aligned}
S &= \mu^\top V^{-1} \mu - (\mu^*)^\top (V^*)^{-1} (\mu^*) + y^\top y \\
k^* &= k + \frac{1}{2}S; \quad V^* = (V^{-1} + \Phi^\top \Phi)^{-1}; \quad \mu^* = V^* (V^{-1} \mu + \Phi^\top y)
\end{aligned}$$

For simplicity, we use the indexing  $\cdot_0$  and  $\cdot_1$  instead of  $\cdot_{\mathcal{M}_0}$  and  $\cdot_{\mathcal{M}_1}$  in what follows. So the Bayes factor is

$$B_{01}(y) = \frac{f_0(y)}{f_1(y)} = \sqrt{\frac{|V_1|}{|V_0|}} \sqrt{\frac{|V_0^*|}{|V_1^*|}} \left(\frac{k_0^*}{k_1^*}\right)^{-\frac{n}{2} - a}$$

and

$$P_{\Pi}(H_0|y) = \left(1 + \frac{1 - \pi_0}{\pi_0} \sqrt{\frac{|V_0|}{|V_1|}} \sqrt{\frac{|V_1^*|}{|V_0^*|}} \left(\frac{k_1^*}{k_0^*}\right)^{-\frac{n}{2} - a}\right)^{-1}$$