

Monte Carlo approximation

An introduction for practical use in R

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Aim

Students will be able to

- apply Monte Carlo approximation with R
- approximate integrals, expected values, modes, quantiles, etc... with R

Briefly ...

Monte Carlo approximation is a stochastic procedure for the evaluation of intractable quantities. It involves:

1. properly generating a large random sample from a suitable distribution, and
2. computing a statistic asymptotically consistent to the intractable quantity of interest.

This statistic is the Monte Carlo approximate of the intractable quantity.

Approximating expected values, and integrals...

Notation

Assume a random variable $\theta \in \Theta$ following a distribution $\Pi(\cdot)$.

Let $h(\cdot) : \Theta \rightarrow \mathbb{R}^d$, $d \geq 1$, be a measurable function.

Assume that there is interest in approximating the expected value

$$\mathbb{E}_{\pi}(h(\theta)) = \int h(\theta) d\Pi(\theta) = \begin{cases} \int h(\theta) \pi(\theta) d\theta, & \text{cont.} \\ \sum h(\theta) \pi(\theta), & \text{discr.} \end{cases}$$

Monte Carlo approximation procedure:

1. Draw a random sample

$$\theta^{(j)} \sim \Pi(\cdot), \text{ for } j = 1, \dots, N$$

2. Compute

$$\bar{h}_N = \frac{1}{N} \sum_{j=1}^N h(\theta^{(j)})$$

We say that \bar{h}_N is a **Monte Carlo approximate of** $\mathbb{E}_{\pi}(h(\theta))$, where

$$\mathbb{E}_{\pi}(h(\theta)) \approx \bar{h}_N \text{ when } N \text{ is large enough.}$$

which can be justified by law of Large Number (LLN) arguments: $\bar{h}_N \rightarrow E_\pi(h(\theta))$ as $N \rightarrow \infty$.

Example

Consider the integral

$$I = \int_0^\infty x^5 \exp(-3x) dx$$

Compute a Monte Carlo approximate of integral I .

The Exact value is $I = 0.1646091$.

Solution

It is

$$\begin{aligned} I &= \int_0^\infty x^5 \exp(-3x) dx = \int x^5 \frac{1}{3} \exp(-3x) 1(x \in (0, \infty)) dx \\ &= \frac{1}{3} \int x^5 \underbrace{3 \exp(-3x) 1(x \in (0, \infty))}_{=d\Pi_{\text{Exp}(3)}(x)} dx = \frac{1}{3} \int x^5 d\Pi_{\text{Exp}(3)}(x) = \frac{1}{3} E_{\text{Exp}(3)}(x^5) \\ &\approx \frac{1}{3} \frac{1}{N} \sum_{j=1}^N \left(x^{(j)}\right)^5, \quad \text{where } x^{(j)} \sim \text{Exp}(3) \end{aligned}$$

we compute the Monte Carlo approximate for $N = 1000$.

```
x = rexp(n = 10^5, rate = 3)
I_mc = (1/3)*mean(x^5)
print(I_mc)
```

```
## [1] 0.1708962
```

Approximating other quantities...

Monte Carlo approximation can approximate quantities other than expected values, and the theoretical justification is out of the scope.

Let $\theta \in \Theta$ be a random variable with distribution $\Pi(\cdot)$.

- Assume there is a random sample generated:

$$\theta^{(j)} \sim \Pi(\cdot), \text{ for } j = 1, \dots, N$$

for $j = 1, \dots, N$

- The characteristic of the distribution $\Pi(\cdot)$ such as population mode, quantiles, etc... can be approximated by their sample analogues such as sample mode, sample quantiles, etc...