

Handout 7: Mixture priors ^a

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Aim: Explain the mixture distribution. Explain, theorize, and construct conjugate mixture prior distribution.

References:

- Berger, J. O. (2013; Sections 4.2.2). Statistical decision theory and Bayesian analysis. Springer Science & Business Media.
- Robert, C. (2007; Sections 3, pp. 105-123, & pp. 127-141). The Bayesian choice: from decision-theoretic foundations to computational implementation. Springer Science & Business Media.

Web applets: https://georgios-stats-1.shinyapps.io/demo_mixturepriors/

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1 Finite mixture distributions

Definition 1. Let $\mathcal{P}_m = \{\Pi_l(\theta|\chi_l); l = 1, \dots, m\}$ be a collection of probability distributions where $\{\chi_l\}_{l=1}^m$ are parameters of the l -th component $\Pi_l(\theta|\chi_l)$. Let $\{\varpi_l\}_{l=1}^m$ be a set of weights where $\varpi_l > 0$ and $\sum_{l=1}^m \varpi_l = 1$. The mixture distribution derived from the aforementioned collections is

$$\Pi(\theta|\varpi, \chi) = \sum_{l=1}^m \varpi_l \Pi_l(\theta|\chi_l), \quad \theta \in \Theta \quad (1)$$

where $\chi := (\chi_l, l = 1 : m)$ and $\varpi := (\varpi_l, l = 1, \dots, m)$. $\Pi_l(\theta|\chi_l)$ is called l -th mixture component with mixture weight ϖ_l .

Example 2. Let $h(\cdot)$ be a function defined on Θ . The expectation of $h(\cdot)$ with respect to (1) is

$$E_{\Pi}(h(\theta)|\varpi, \chi) = \int \sum_{l=1}^m \varpi_l h(\theta) d\Pi_l(\theta|\chi_l) = \sum_{l=1}^m \varpi_l \int h(\theta) d\Pi_l(\theta|\chi_l) = \sum_{l=1}^m \varpi_l E_{\Pi_l}(h(\theta)|\chi_l)$$

where $E_{\Pi_l}(h(\theta)|\chi_l) = \int h(\theta) d\Pi_l(\theta|\chi_l)$.

Example 3. A mixture of probability distributions (1) is a probability distribution; i.e.

$$\int_{\Theta} \pi(\theta|\varpi, \chi) d\theta = \sum_{l=1}^m \varpi_l \underbrace{\int_{\Theta} d\Pi_l(\theta|\chi_l)}_{=1} d\theta = \sum_{l=1}^m \varpi_l = 1$$

Definition 4. The mixture is called finite mixture if $m < \infty$, and countably infinite mixture if $m \rightarrow \infty$. Here, we focus on finite mixtures.

Definition 5. A mixture model is called parametric mixture model if its components are members of the same parametric family of distributions eg. $\mathcal{P}_m = \{\Pi(\theta|\chi_l); l = 1, \dots, m\}$, and hence

$$\Pi(\theta|\varpi, \chi) = \sum_{l=1}^m \varpi_l \Pi(\theta|\chi_l)$$

Example 6. An example of a parametric mixture model is the Normal mixture model

$$\pi(\theta|\varpi, \mu, \sigma^2) = \sum_{l=1}^m \varpi_l \mathcal{N}(\theta|\mu_l, \sigma_l^2),$$

where all components belong to the Normal distribution family.

Note 7. Mixture distributions are useful because (among others):

- they can approximate complicate other distributions by using a combination of simpler distributions $\{\Pi_l(\theta|\chi_l)\}$ which lead to more convenient computations.
- they can naturally model heterogeneity. E.g. consider a population which is heterogeneous in the sense that there are multiple sub-groups labeled by $\ell \in \{1, \dots, m\}$, each group represented in the population with proportion ϖ_ℓ , and distributed as $\Pi_l(\theta|\chi_l)$. Then $y \sim \Pi(\theta|\chi)$ can be realized by drawing ℓ with probability $P(\ell = l) = \varpi_l$ and drawing θ from $\Pi_\ell(\theta|\chi_\ell)$ given ℓ , aka

$$\begin{cases} \theta|\ell & \sim \Pi_\ell(\theta|\chi_\ell) \\ \ell & \sim P(\ell) \end{cases} \quad \text{which implies } \theta \sim \Pi(\theta|\chi) \quad \text{since} \quad \Pi(\theta|\chi) = \sum_{\ell=1}^m \Pi(\theta|\chi_\ell)P(\ell) = \sum_{l=1}^m \varpi_l \Pi(\theta|\chi_l)$$

See Section 1 in Handout 2.

Example 8. A bimodal, or a right skewed (non-symmetric) distribution can be approximated by a Mixture of (unimodal and symmetric) Normal distributions with different parameter values. Also it describes a population with two groups Normally distributed with different parameters.



(a) Bimodal PDF:
 $\pi(\theta) = 0.7\mathcal{N}(\theta|1, 0.9^2) + 0.3\mathcal{N}(\theta|6, 1.9^2)$



(b) Right skewed PDF:
 $\pi(\theta) = 0.7\mathcal{N}(\theta|1, 0.9^2) + 0.3\mathcal{N}(\theta|4, 1.9^2)$

Figure 1: Normal mixture models. $\mathcal{N}(x|\mu, \sigma^2)$ denotes the Normal distribution density at value x with mean μ and variance σ^2 .

2 Mixture prior distributions

Note 9. Mixture models can be used to specify prior distributions, either as a mean to approximate Your actual prior distribution with simpler & tractable distributions, or as a mean to represent heterogeneous prior believes.

Theorem 10. Let $y := (y_1, \dots, y_n)$ be observables generated from the sampling distribution $F(y|\theta)$. Prior mixture distribution $\Pi(\theta|\varpi)$ is called the prior with pdf/pmf

$$\pi(\theta|\varpi) = \sum_{l=1}^m \varpi_l \pi_l(\theta), \quad (2)$$

where, $\mathcal{P}_m = \{\pi_l(\theta), \theta \in \Theta\}_{l=1}^m$ is a collection of distributions, and $\{\varpi_l\}$ are weights such that $\sum_{l=1}^m \varpi_l = 1$ and $\varpi_l > 0$. Then:

1. the posterior distribution $\Pi(\theta|y, \varpi)$ has pdf/pmf

$$\pi(\theta|y, \varpi) = \sum_{l=1}^m \varpi_l^* \pi_l(\theta|y), \quad (3)$$

2. the predictive distribution of a future outcome z has pdf/pmf

$$g(z|y, \varpi) = \sum_{l=1}^m \varpi_l^* g_l(z|y)$$

where

$$\begin{aligned} \varpi_l^* &= \frac{\varpi_l f_l(y)}{\sum_{l=1}^m \varpi_l f_l(y)} \propto \varpi_l f_l(y) \\ f_l(y) &= \int_{\Theta} f(y|\theta) d\Pi_l(\theta) \\ \pi_l(\theta|y) &= \frac{f(y|\theta) \pi_l(\theta)}{\int_{\Theta} f(y|\vartheta) d\Pi_l(\vartheta)} \propto f(y|\theta) \pi_l(\theta) \\ g_l(z|y) &= \int_{\Theta} f(z|\theta) \pi_l(\theta|y) d\theta \end{aligned}$$

Proof. From Bayes theorem, we have:

$$\begin{aligned} d\Pi(\theta|y) &= \frac{f(y|\theta) d\Pi(\theta)}{\int_{\Theta} f(y|\vartheta) d\Pi(\vartheta)} = \frac{f(y|\theta) \sum_{l=1}^m \varpi_l d\Pi_l(\theta)}{\int_{\Theta} f(y|\vartheta) \sum_{l'=1}^m \varpi_{l'} d\Pi_{l'}(\vartheta)} = \frac{\sum_{l=1}^m \varpi_l f(y|\theta) d\Pi_l(\theta)}{\sum_{l'=1}^m \int_{\Theta} \varpi_{l'} f(y|\vartheta) d\Pi_{l'}(\vartheta)} \\ &= \frac{\sum_{l=1}^m \varpi_l f_l(y) \overbrace{\frac{f(y|\theta) d\Pi_l(\theta)}{f_l(y)}}^{\substack{=d\Pi_l(\theta|y)}}}{\sum_{l'=1}^m \int_{\Theta} \varpi_{l'} f_{l'}(y) \underbrace{\frac{f(y|\vartheta) d\Pi_{l'}(\vartheta)}{f_{l'}(y)}}_{=d\Pi_{l'}(\vartheta|y)}} = \frac{\sum_{l=1}^m \varpi_l f_l(y) d\Pi_l(\theta|y)}{\sum_{l'=1}^m \varpi_{l'} f_{l'}(y) \underbrace{\int_{\Theta} d\Pi_{l'}(\vartheta|y)}_{=1}} \\ &= \sum_{l=1}^m \underbrace{\frac{\varpi_l f_l(y)}{\sum_{l'=1}^m \varpi_{l'} f_{l'}(y)}}_{=\varpi_l^*} d\Pi_l(\theta|y) = \sum_{l=1}^m \varpi_l^* d\Pi_l(\theta|y). \end{aligned}$$

Also

$$g(z|y, \varpi) = \int_{\Theta} f(z|\theta) d\Pi_l(\theta|y) = \int_{\Theta} f(y|\theta) \sum_{l=1}^m \varpi_l^* d\Pi_l(\theta|y) = \sum_{l=1}^m \varpi_l^* g_l(z|y)$$

□

Remark 11. Theorem 10 shows that the posterior distribution $\Pi(\theta|y)$ (derived by a mixture prior) is a mixture of 'individual posterior distributions' $\Pi_l(\theta|y)$ weighted by ϖ_l^* . It is determined not only by the observables but also by the weights of the individual distributions. Note that, for $l = 1, \dots, m$, the prior weights ϖ_l and posterior weights ϖ_l^* may differ a lot, however they can be close each other.

Remark 12. Mixture priors $\Pi(\theta)$ whose components $\{\Pi_l(\theta)\}$ are conjugate to the likelihood $f(y|\theta)$ can facilitate tractable Bayesian inference. In Theorem 10, if each $\Pi_l(\theta)$ is conjugate to the likelihood $f(y|\theta)$, then obviously each $\Pi_l(\theta|y)$ will belong to the same distribution family as the corresponding $\Pi_l(\theta)$ because $\pi_l(\theta|y) \propto f(y|\theta)\pi_l(\theta)$. Then, provided that components $\pi_l(\theta)$ are tractable, the components $\pi_l(\theta|y)$ will be tractable too (as they belong to the same distr. family). Likewise, the posterior weights can be calculated in closed form i.e., $\varpi_l^* \propto \varpi_l f_l(y)$ since the integral $f_l(y) = \int_{\Theta} f(y|\theta) d\Pi_l(\theta)$ will be tractable. Hence, the produced posterior mixture pdf/pmf will be tractable.

Remark 13. Mixtures of conjugate priors are as easy to manipulate as regular conjugate distributions, while leading to a greater freedom in the modeling of the prior information.

Example 14. Let $y = (y_1, \dots, y_n)$ observable quantities, generated iid from a Bernoulli sampling distribution with unknown parameter θ ; aka $y_i|\theta \stackrel{\text{iid}}{\sim} \text{Br}(\theta)$, $i = 1, \dots, n$.

1. Find the likelihood function
2. Find the PDF of the conjugate prior mixture prior $\pi(\theta) = \sum_{l=1}^m \varpi_l \pi_l(\theta)$, with m components $\{\pi_l(\theta)\}_{l=1}^m$.
3. Compute the pdf of the posterior distribution, and recognize it.
4. Compute the predictive pdf for the next outcome $z = (y_{n+1}, \dots, y_{n+m})$ given we have observed y ? What do you observe?

Hint: Beta distribution: $x \sim \text{Be}(a, b)$ has pdf

$$f(x) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} \mathbf{1}(x \in [0, 1]); \quad \text{where} \quad B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a > 0, b > 0$$

Solution.

1. The likelihood function is

$$f(y|\theta) = \prod_{i=1}^n f(y_i|\theta) = \prod_{i=1}^n \text{Br}(x_i|\theta) = \theta^{\sum_{i=1}^n x_i} (1-\theta)^{n-\sum_{i=1}^n x_i}$$

2. In previous examples, we found that

- the sampling distribution $F(\cdot|\theta)$ is a Bernoulli distribution which is a member of the Exponential family.
- the conjugate prior is Beta distribution, aka $\theta \sim \text{Be}(a, b)$ $a > 0$ and $b > 0$.

Therefore, the components $\{\pi_l(\theta)\}_{l=1}^m$ in the prior mixture distribution will be from the family of Beta distributions $\mathcal{P}_m = \{\text{Be}(\theta|a_l, b_l); l = 1, \dots, m\}$.

Therefore, the conjugate mixture prior has pdf

$$\pi(\theta) = \sum_{l=1}^m \varpi_l \text{Be}(\theta|a_l, b_l)$$

where $\text{Be}(\theta|a_l, b_l)$ is the pdf of $\text{Be}(a_l, b_l)$, and $\{(a_l, b_l)\}$ are fixed prior hyper-parameters.

3. The posterior mixture posterior has PDF

$$\pi(\theta|y) = \sum_{l=1}^m \varpi_l^* \pi_l(\theta|y)$$

According to Theorem 10, the l -th components of the mixture posterior is

$$\begin{aligned} \pi_l(\theta|y) &\propto f(y|\theta)\pi_l(\theta) = \prod_{i=1}^n \text{Br}(y_i|\theta) \times \text{Be}(\theta|a_l, b_l) \propto \prod_{i=1}^n [\theta^{y_i}(1-\theta)^{1-y_i}] \times \theta^{a_l-1}(1-\theta)^{b_l-1} \\ &\quad \theta^{r_n+\sum_{i=1}^n y_i} \theta^{r_n+a_l-1}(1-\theta)^{n-r_n+b_l-1} \propto \text{Be}(\theta|a_l^*, b_l^*) \end{aligned}$$

with $a_l^* = r_n + a_l$, $b_l^* = n - r_n + b_l$, and $r_n = \sum_{i=1}^n y_i$.

According to Theorem 10, the posterior weights can be calculated as

$$\begin{aligned} \varpi_l^* &\propto \varpi_l f_l(y) \propto \varpi_l \int_{(0,\infty)} \prod_{i=1}^n f(x_i|\theta)\pi_l(\theta)d\theta = \varpi_l \int_{(0,\infty)} \prod_{i=1}^n \text{Br}(y_i|\theta)\text{Be}(\theta|a_l, b_l)d\theta \\ &= \varpi_l \int_{(0,\infty)} \theta^{r_n}(1-\theta)^{n-r_n} \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \theta^{a_l-1}(1-\theta)^{b_l-1} d\theta \\ &= \varpi_l \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \int_{(0,1)} \theta^{r_n+a_l-1}(1-\theta)^{n-r_n+b_l-1} d\theta = \varpi_l \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \frac{\Gamma(a_l^*)\Gamma(b_l^*)}{\Gamma(a_l^*+b_l^*)} \end{aligned} \quad (4)$$

namely,

$$\varpi_l^* = \frac{\varpi_l \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \frac{\Gamma(a_l^*)\Gamma(b_l^*)}{\Gamma(a_l^*+b_l^*)}}{\sum_{l=1}^m \varpi_l \frac{\Gamma(a_l+b_l)}{\Gamma(a_l)\Gamma(b_l)} \frac{\Gamma(a_l^*)\Gamma(b_l^*)}{\Gamma(a_l^*+b_l^*)}}$$

4. It is ...

$$g(z|y) = \int_{\Theta} f(z|\theta)d\Pi(\theta|y) = \sum_{i=1}^m \varpi_i^*(y) \underbrace{\int_{\Theta} f(z|\theta)d\Pi_i(\theta|y)}_{=g_i(z|y)}$$

where the following is just copy-paste from Handout 3...

$$\begin{aligned} g_l(z|y) &= \int_{\Theta} f(z|\theta)\pi_l(\theta|y)d\theta = \int_{\Theta} \prod_{i=1}^m f(z_i|\theta)\pi_l(\theta|y)d\theta = \int_{(0,\infty)} \prod_{i=1}^m \text{Br}(z_i|\theta)\text{Be}(\theta|a_l^*, b_l^*)d\theta \\ &= \int_0^1 \left[\theta^{\sum_{i=1}^m z_i} (1-\theta)^{m-\sum_{i=1}^m z_i} \right] \left[\frac{\theta^{a_l^*-1}(1-\theta)^{b_l^*-1}}{B(a_l^*, b_l^*)} \right] d\theta \mathbf{1}(z \in \{0, 1\}^m) \\ &= \frac{1}{B(a_l^*, b_l^*)} \int_0^1 \theta^{\sum_{i=1}^m z_i + a_l^* - 1} (1-\theta)^{m - \sum_{i=1}^m z_i + b_l^* - 1} d\theta \mathbf{1}(z \in \{0, 1\}^m) \\ &= \frac{B(\sum_{i=1}^m z_i + a_l^*, m - \sum_{i=1}^m z_i + b_l^*)}{B(a_l^*, b_l^*)} \mathbf{1}(z \in \{0, 1\}^m) \end{aligned}$$

3 Practice

Question 15. Try the Exercise 43 from the Exercise sheet.