Handout 13: Hypothesis tests

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Aim: To explain, design, and use hypothesis tests in the Bayesian framework

References:

- Berger, J. O. (2013; Section 4.3.3). Statistical decision theory and Bayesian analysis. Springer Science & Business Media.
- DeGroot, M. H. (2005, Sections 11.5-11.13). Optimal statistical decisions (Vol. 82). John Wiley & Sons
- Robert, C. (2007; Section 5.2(exclude 5.2.6)). The Bayesian choice: from decision-theoretic foundations to computational implementation. Springer Science & Business Media.

1 Set-up of a hypothesis test

Aim: Let $y = (y_1, ..., y_n)$ generated from the real unknown data-generating process $y \sim R(y)$. Statistician approximates/parametrizes $R(\cdot)$ by a statistical model $F(y|\theta)$ with unknown $\theta \in \Theta$. Then you wish to find useful statements about θ: E.g. is there a smaller $\Theta_\star \subseteq \Theta$ where You can restrict the possible values of unknown θ?

Notation 1. Let $y = (y_1, ..., y_n)$ be a sequence of observables modeled to have been generated from the sampling distribution $F(y|\theta)$ labeled by an unknown parameter $\theta \in \Theta$ following a priori distribution $\Pi(\theta)$; namely

$$\begin{cases} y|\theta & \sim F(y|\theta) \\ \theta & \sim \Pi(\theta) \end{cases} \tag{1}$$

Assume there is interest to test/compare the following hypotheses/statements

$$\mathrm{H}_0 : \theta \in \Theta_0; \text{ vs. } \mathrm{H}_1 : \theta \in \Theta_1 \tag{2}$$

where $\Theta = \Theta_0 \cup \Theta_1$, under the Bayesian model (1).

Note 2. The pair of hypotheses (2) partitions the overall prior $\Pi(\theta)$ (representing overall prior believes about θ) as

$$\mathrm{d}\Pi(\theta) = \pi_0 \times \mathrm{d}\Pi_0(\theta) + \pi_1 \times \mathrm{d}\Pi_1(\theta) \tag{3}$$

where π_0, and π_1 describe the prior probabilities of hypotheses H_0 and H_1

$$\pi_0 = \underbrace{\mathsf{P}_\Pi(\theta \in \Theta_0)}_{=\mathsf{P}_\Pi(\mathrm{H}_0)} = \int 1(\theta \in \Theta_0)\, \mathrm{d}\Pi(\theta), \qquad \pi_1 = \underbrace{\mathsf{P}_\Pi(\theta \in \Theta_1)}_{=\mathsf{P}_\Pi(\mathrm{H}_1)} = \int 1(\theta \in \Theta_1)\, \mathrm{d}\Pi(\theta),$$

respectively while $\Pi_0(\theta) := \Pi(\theta|\theta \in \Theta_0)$ and $\Pi_1(\theta) := \Pi(\theta|\theta \in \Theta_1)$ are prior distributions with pdf/pmf

$$\pi_0(\theta) := \underbrace{\pi(\theta|\theta \in \Theta_0)}_{=\pi(\theta|\mathrm{H}_0)} = \frac{\pi(\theta) 1(\theta \in \Theta_0)}{\int 1(\theta \in \Theta_0)\, \mathrm{d}\Pi_0(\theta)}; \quad \text{and} \quad \pi_1(\theta) := \underbrace{\pi(\theta|\theta \in \Theta_1)}_{=\pi(\theta|\mathrm{H}_1)} = \frac{\pi(\theta) 1(\theta \in \Theta_1)}{\int 1(\theta \in \Theta_1)\, \mathrm{d}\Pi_1(\theta)},$$

describing how the prior mass of θ is spread out over the hypotheses H_0 and H_1 respectively. Then the Bayesian hypothesis test can also be expressed as

$$\mathrm{H}_0 : \begin{cases} y|\theta & \sim F(y|\theta) \\ \theta & \sim \Pi_0(\theta),\ \theta \in \Theta_0 \end{cases} \qquad \text{vs} \qquad \mathrm{H}_1 : \begin{cases} y|\theta & \sim F(y|\theta) \\ \theta & \sim \Pi_1(\theta),\ \theta \in \Theta_1 \end{cases} \tag{4}$$

with prior $\pi_0 = \mathsf{P}_\Pi\left(\theta \in \Theta_0\right)$ and $\pi_1 = \mathsf{P}_\Pi\left(\theta \in \Theta_1\right)$.

Question 3. *Which Bayesian model (H_0 or H_1) describes 'better' the real data generating process?*

Note 4. In Bayesian framework, hypothesis testing is rather straightforward. All You need to do is to calculate the corresponding posterior probabilities $\mathsf{P}_\Pi(\theta \in \Theta_0|y)$, and $\mathsf{P}_\Pi(\theta \in \Theta_1|y)$, and decide between H_0 and H_1.

2 Decision theory prespective

Note 5. Bayes hypothesis test (4) can be addressed as a Bayesian statistical decision problem with decision space $\mathcal{D} = \{\text{accept } \mathrm{H}_0, \text{ accept } \mathrm{H}_1\}$ or simpler $\mathcal{D} = \{0, 1\}$, and under Bayesian model (1). It can be seen as a parametric point estimation about the indicator function

$$1_{\Theta_1}(\theta) = \begin{cases} 0 & , \theta \in \Theta_0 \\ 1 & , \theta \in \Theta_1 \end{cases} \tag{5}$$

under Bayesian model (1), prior (3), and a loss function $\ell(\theta, \delta)$, with $\theta \in \Theta$, $\delta \in \mathcal{D}$; E.g., the $0 - 1$ loss function.

Theorem 6. *The Bayes estimator of $1_{\Theta_1}(\theta)$ in (5), under the prior $\Pi(\theta)$ in (3) and the $c_I - c_{II}$ loss function*

$$\ell(\theta, \delta) = \begin{cases} 0 & , \textit{if } \theta \in \Theta_0,\ \delta = 0 \\ 0 & , \textit{if } \theta \notin \Theta_0,\ \delta = 1 \\ c_{II} & , \textit{if } \theta \notin \Theta_0,\ \delta = 0 \\ c_I & , \textit{if } \theta \in \Theta_0,\ \delta = 1 \end{cases} \tag{6}$$

where $c_I > 0$ and $c_{II} > 0$ are specified by the researcher is

$$\delta(y) = \begin{cases} 0 & , \mathsf{P}_\Pi\left(\theta \in \Theta_0|y\right) > \frac{c_{II}}{c_{II}+c_I} \\ 1 & , \textit{otherwise} \end{cases} \tag{7}$$

where $\{\Theta_0, \Theta_1\}$ constitute a partition for Θ, and $\mathsf{P}_\Pi\left(\theta \in \Theta_0|y\right) = \int 1\left(\theta \in \Theta_0\right) d\Pi(\theta|y)$.

Proof. The posterior expected loss is[1]

$$\begin{aligned} \varrho(\pi, \delta|y) &= \mathrm{E}_\Pi(\ell(\theta, \delta)|y) = \int \ell(\theta, \delta)\mathrm{d}\Pi(\theta|y) = \int_{\Theta_0} \ell(\theta, \delta)\mathrm{d}\Pi(\theta|y) + \int_{\Theta_1} \ell(\theta, \delta)\mathrm{d}\Pi(\theta|y) \\ &= \begin{cases} \cancelto{=0}{\int_{\Theta_0} 0\mathrm{d}\Pi(\theta|y)} + \int_{\Theta_1} c_{\mathrm{II}}\mathrm{d}\Pi(\theta|y) & , \quad \text{if } \delta = 0 \\ \int_{\Theta_0} c_{\mathrm{I}}\mathrm{d}\Pi(\theta|y) + \cancelto{=0}{\int_{\Theta_1} 0\mathrm{d}\Pi(\theta|y)} & , \quad \text{if } \delta = 1 \end{cases} = \begin{cases} c_{\mathrm{II}} \int 1\left(\theta \in \Theta_1\right) \mathrm{d}\Pi(\theta|y) & , \quad \text{if } \delta = 0 \\ c_{\mathrm{I}} \int 1\left(\theta \in \Theta_0\right) \mathrm{d}\Pi(\theta|y) & , \quad \text{if } \delta = 1 \end{cases} \\ &= c_{\mathrm{II}}\mathsf{P}_\Pi\left(\theta \notin \Theta_0|y\right) 1\left(\delta \in \{0\}\right) + c_{\mathrm{I}}\mathsf{P}_\Pi\left(\theta \in \Theta_0|y\right) \end{aligned}$$

[1]Notation: $\int_{\Theta_j} \mathrm{d}\Pi(\theta|y) = \int 1\left(\theta \in \Theta_j\right) \mathrm{d}\Pi(\theta|y)$

The Bayes rule (estimator) of (5) is $\delta(y) = 0$ when

$$\varrho(\pi, \delta = 0|y) < \varrho(\pi, \delta = 1|y) \iff c_{\mathrm{II}} \mathsf{P}_{\Pi} \left(\theta \notin \Theta_0|y\right) < c_{\mathrm{I}} \mathsf{P}_{\Pi} \left(\theta \in \Theta_0|y\right) \iff \mathsf{P}_{\Pi} \left(\theta \in \Theta_0|y\right) > \frac{c_{\mathrm{II}}}{c_{\mathrm{II}} + c_{\mathrm{I}}}$$

The Bayes rule (estimator) of (5) is $\delta(y) = 1$ when

$$\varrho(\pi, \delta = 0|y) > \varrho(\pi, \delta = 1|y) \iff c_{\mathrm{II}} \mathsf{P}_{\Pi} \left(\theta \notin \Theta_0|y\right) > c_{\mathrm{I}} \mathsf{P}_{\Pi} \left(\theta \in \Theta_0|y\right) \iff \mathsf{P}_{\Pi} \left(\theta \in \Theta_0|y\right) < \frac{c_{\mathrm{II}}}{c_{\mathrm{II}} + c_{\mathrm{I}}}$$

So $\varrho(\pi, \delta|y)$ is minimised for (7). □

3 Bayes factors perspective

Note 7. Hypothesis tests in Bayesian statistics can be addressed by using Bayes factors.

Definition 8. The Bayes factor $\mathrm{B}_{01}(y)$ is the ratio of the posterior probabilities of H_0 and H_1 over the ratio of the prior probabilities of H_0 and H_1.

$$\mathrm{B}_{01}(y) = \frac{\mathsf{P}_{\Pi} \left(\theta \in \Theta_0|y\right) / \mathsf{P}_{\Pi} \left(\theta \in \Theta_0\right)}{\mathsf{P}_{\Pi} \left(\theta \in \Theta_1|y\right) / \mathsf{P}_{\Pi} \left(\theta \in \Theta_1\right)} \tag{8}$$

where

$$\mathsf{P}_{\Pi} \left(\theta \in \Theta_j\right) = \int 1 \left(\theta \in \Theta_j\right) \mathrm{d}\Pi(\theta); \quad \text{and} \quad \mathsf{P}_{\Pi} \left(\theta \in \Theta_j|y\right) = \int 1 \left(\theta \in \Theta_j\right) \mathrm{d}\Pi(\theta|y); \quad \text{for } j = 0, 1.$$

Proposition 9. *For Hypothesis pair (2) and Bayes model (1), where the prior is formed as in (3), the Bayes factor in (8) can be written as*

$$B_{01}(y) = \frac{\int_{\Theta_0} f(y|\theta) d\Pi_0(\theta)}{\int_{\Theta_1} f(y|\theta) d\Pi_1(\theta)} = \frac{f_0(y)}{f_1(y)}$$

where $f_j(y) = \int_{\Theta_j} f(y|\theta) d\Pi_j(\theta)$ is the conditional marginal likelihood (or prior predictive pdf/pmf) given H_j, for $j = 0, 1$.

Proof. It results by showing that for $j = 0, 1$, it is

$$\mathsf{P}_{\Pi}(\theta \in \Theta_j|y) = \int_{\Theta_j} \mathrm{d}\Pi(\theta|y) \quad = \int_{\Theta_j} \frac{f(y|\theta)\mathrm{d}\Pi(\theta)}{\int_{\Theta} f(y|\theta)\mathrm{d}\Pi(\theta)} \quad = \int_{\Theta_j} \frac{f(y|\theta) \left(\pi_0 \times \mathrm{d}\Pi_0(\theta) + \pi_1 \times \mathrm{d}\Pi_1(\theta)\right)}{\underbrace{\int_{\Theta} f(y|\theta)\mathrm{d}\Pi(\theta)}_{=f(y)}}$$

$$= \frac{\pi_0}{f(y)} \int_{\Theta_j} f(y|\theta)\mathrm{d}\Pi_0(\theta) + \frac{\pi_0}{f(y)} \int_{\Theta_j} \pi_1 f(y|\theta)\mathrm{d}\Pi_1(\theta) = \begin{cases} \frac{\pi_0}{f(y)} \int_{\Theta_0} f(y|\theta)\mathrm{d}\Pi_0(\theta) & \text{,if } j = 0 \\ \frac{\pi_1}{f(y)} \int_{\Theta_1} f(y|\theta)\mathrm{d}\Pi_1(\theta) & \text{,if } j = 1 \end{cases}$$

□

Remark 10. Obviously, $\mathrm{B}_{10}(y) = 1/\mathrm{B}_{01}(y)$

Remark 11. Bayes factor $\mathrm{B}_{01}(y)$:

- ...is the 'odds in favour of H_0 against H_1 that are given by the data' y.
- ...evaluate the modification of the odds of Θ_0 against Θ_1 due to the observations y.
- ...is the ratio of the likelihoods, weighted by the conditional priors $\mathrm{d}\Pi_0(\theta)$ and $\mathrm{d}\Pi_1(\theta)$.

Proposition 12. *One can write*

$$\mathsf{P}_{\Pi}(\theta\in\Theta_0|y) \quad =\left[1+\frac{1-\mathsf{P}_{\Pi}(\theta\in\Theta_0)}{\mathsf{P}_{\Pi}(\theta\in\Theta_0)}B_{01}(y)^{-1}\right]^{-1} \quad =\left[1+\frac{\pi_1}{\pi_0}B_{01}(y)^{-1}\right]^{-1}$$

where $\pi_j=\mathsf{P}_{\Pi}\left(\theta\in\Theta_j\right)$, for $j=0,1$, by rearranging (8) (please check).

Criterion 13. *Consider a hypothesis test $H_0:\theta\in\Theta_0$ versus $H_1:\theta\in\Theta_1$ as described in (4) with loss function (6), and given a Bayesian model (1). The hypothesis H_0 is accepted when*

$$B_{01}(y)>\frac{c_{II}}{c_I}\frac{\pi_1}{\pi_0} \tag{9}$$

where $\pi_j=\mathsf{P}_{\Pi}\left(\theta\in\Theta_j\right)$, for $j=0,1$.

Proof. Straightforward result from Definition 8 and Theorem 6. □

Remark 14. Eq. 9 shows the duality between loss function and the prior distribution. Different combinations of priors and loss functions may lead to the same result. For instance, for $c'_{\text{II}}=c'_{\text{I}}=1$, $\pi'_0=\frac{c_{\text{I}}\pi_0}{c_{\text{I}}\pi_0+c_{\text{II}}\pi_1}$, and $\pi'_1=\frac{c_{\text{II}}\pi_1}{c_{\text{I}}\pi_0+c_{\text{II}}\pi_1}$, we get again (9) !!!

Criterion 15. *Jeffreys developed a scale to judge the strength of evidence in favor of H_0 or against H_0 brought by the data, outside a true decision-theoretic setting (aka; without the need to specify c_I and c_{II} in (9)).*

B_{01}	$\log_{10}(B_{01})$	**Strength of evidence**
$(1,+\infty)$	$(0,+\infty)$	**H_0 is supported**
$(10^{-1/2},1)$	$(-1/2,0)$	**Evidence against H_0: not worth more than a bare**
$(10^{-1},10^{-1/2})$	$(-1,-1/2)$	**Evidence against H_0: substantial**
$(10^{-3/2},10^{-1})$	$(-3/2,-1)$	**Evidence against H_0: strong**
$(10^{-2},10^{-3/2})$	$(-2,-3/2)$	**Evidence against H_0: very strong**
$(0,10^{-2})$	$(-\infty,-2)$	**Evidence against H_0: decisive**

The precise bounds separating one strength from another are a matter of convention. Note that similar criticism exists in frequentist hypothesis tests with the choice of the significance level $a=\{0.01,0.05,0.1,...\}$.

4 Special cases in hypotheses tests

Definition 16. Traditionally, hypotheses, H_j, are categorized as:

- **Single (or point) hypothesis** for θ is called the hypothesis $\mathrm{H}_j:\theta\in\Theta_j$ where $\Theta_j=\{\theta_j\}$ contains a single element, namely when $\Pi_j(\theta)$ assigns probability one to a specific value for θ.
- **Composite hypothesis** for θ is called the hypothesis $\mathrm{H}_j:\theta\in\Theta_j$ where $\Theta_j\subseteq\Theta$ contains many elements. Namely when $\Pi_j(\theta)$ defines a non-degenerate density $\pi_j(\theta)$ over $\Theta_j\subseteq\Theta$.
- **General alternative hypothesis** for θ is called the composite hypothesis $\mathrm{H}_1:\theta\in\Theta_1$ where $\Theta_1=\Theta-\{\theta_0\}$ and θ_0 a single value. It is often denoted as $\mathrm{H}_1:\theta\neq\theta_0$ and compared against a single null hypothesis $\mathrm{H}_0:\theta=\theta_0$.

We present some special cases of hypothesis tests.

Case 1. **Composite vs Composite** is the hypothesis test:

$$\mathrm{H}_0:\theta\in\Theta_0 \qquad \text{vs} \qquad \mathrm{H}_1:\theta\in\Theta_1$$

where both $\Theta_0 \subseteq \Theta$ and $\Theta_1 \subseteq \Theta$ contain more than one elements. Overall prior can be partitioned as

$$\mathrm{d}\Pi(\theta) = \pi_0 \times \mathrm{d}\Pi_0(\theta) + \pi_1 \times \mathrm{d}\Pi_1(\theta)$$

Then the conditional marginal likelihoods are

$$f_0(y) = \int_{\Theta_0} f(y|\theta)\mathrm{d}\Pi_0(\theta); \qquad\qquad f_1(y) = \int_{\Theta_1} f(y|\theta)\mathrm{d}\Pi_1(\theta).$$

Example. A composite vs. composite hypothesis is:

$$\mathrm{H}_0 : \begin{cases} y_i|\mu,\sigma^2 & \stackrel{\mathrm{IID}}{\sim} \mathrm{N}(\mu,\sigma^2),\ i=1,...,n \\ \mu|\sigma^2 \sim; & \sim \mathrm{N}(\mu_0, \sigma^2\frac{1}{\lambda_0}) \\ \sigma^2 & \sim \mathrm{Ga}(a_0,b_0) \end{cases} \qquad \text{vs} \qquad \mathrm{H}_1 : \begin{cases} y_i|\mu & \stackrel{\mathrm{IID}}{\sim} \mathrm{T}(\mu,1,k_0),\ i=1,...,n \\ \mu & \sim \mathrm{N}(\xi_0, v_0) \\ & \end{cases}$$

In H_0: I consider a sampling model $y_i \stackrel{\mathrm{IID}}{\sim} \mathrm{N}(\mu,\sigma^2)$ with prior $(\mu,\sigma^2) \sim \mathrm{N}(\mu_0,\sigma^2/\lambda_0)\mathrm{IG}(a_0,b_0)$, and $\Theta_0 = \{\mathrm{N}\} \cup \mathbb{R} \cup (0,\infty)$. Here $\mu_0, \lambda_0, a_0, b_0$ are fixed.
In H_1: I consider a sampling model $y_i \stackrel{\mathrm{IID}}{\sim} \mathrm{T}(\mu,1,k_0)$ with prior $(\mu,\sigma^2) \sim \mathrm{N}(\mu_0,\sigma^2/\lambda_0)\mathrm{IG}(a_0,b_0)$, and $\Theta_1 = \{\mathrm{T}\} \cup \mathbb{R}$. Here μ_0, k_0, ξ_0, v_0 are fixed.

Case 2. **Single vs. General alternative** is the pair of hypotheses

$$\mathrm{H}_0 : \theta = \theta_0 \qquad \text{vs} \qquad \mathrm{H}_1 : \theta \neq \theta_0.$$

If θ is continuous, the difficulty is that we cannot use a continuous prior for $\Pi_0(\theta)$ to conduct a test with point null hypothesis $\mathrm{H}_0 : \theta = \theta_0$ because it would give a prior probability zero for $\theta = \theta_0$. To overcome this, we specify the conditional distribution $\Pi_0(\theta)$ as a Dirac prior distribution with concentration point at θ_0 ; namely $\mathrm{d}\Pi_0(\theta) = 1\left(\theta \in \{\theta_0\}\right)\mathrm{d}\theta$. The conditional distribution $\Pi_1(\theta)$ can be any reasonable distribution $\mathrm{d}\Pi_1(\theta) = \mathrm{d}\Pi_1(\theta|\theta \in \Theta_1)$. Then the overall prior is

$$\mathrm{d}\Pi(\theta) = \pi_0 \times 1\left(\theta \in \{\theta_0\}\right)\mathrm{d}\theta + \pi_1 \times \mathrm{d}\Pi_1(\theta|\theta \in \Theta_1) \tag{10}$$

and it is called spike-and-slab. Then the conditional marginal likelihoods are

$$f_0(y) = \int_{\Theta_0} f(y|\theta)\mathrm{d}\Pi_0(\theta) = \int_{\{\theta=\theta_0\}} f(y|\theta)1\left(\theta \in \{\theta_0\}\right)\mathrm{d}\theta = f(y|\theta_0)$$
$$f_1(y) = \int_{\Theta_1} f(y|\theta)\mathrm{d}\Pi_1(\theta) = \int_{\{\theta\neq\theta_0\}} f(y|\theta)\mathrm{d}\Pi_1(\theta)$$

Example. The standard two side test $\mathrm{H}_0 : \mu = \theta_0$ vs. $\mathrm{H}_1 : \mu \neq \theta_0$, where the sampling distribution is assumed to be $y_i \stackrel{\mathrm{IID}}{\sim} \mathrm{N}(\mu,\sigma^2)$ with known variance σ^2 for $i = 1,...,n$, is a simple vs. general alternative hypothesis test and can also be formulated as:

$$\mathrm{H}_0 : y_i|\theta_0,\sigma_0^2 \stackrel{\mathrm{IID}}{\sim} \mathrm{N}\left(\theta_0,\sigma_0^2\right),\ i=1,...,n \qquad \text{vs} \qquad \mathrm{H}_1 : \begin{cases} y_i|\mu,\sigma^2 \stackrel{\mathrm{IID}}{\sim} \mathrm{N}\left(\mu,\sigma_0^2\right), & i=1,...,n \\ \mu \sim \mathrm{N}(\mu_0,\sigma_0^2) \end{cases}$$

Here it is $\Theta_0 = \{\theta_0\}$, $\Theta_1 = \{\theta \in \mathbb{R} : \theta \neq \theta_0\}$, while $\theta_0, \mu_0, \sigma_0^2$ are fixed values.

Case 3. **Single vs. Single** is the pair of hypothesis

$$\mathrm{H}_0 : \theta = \theta_0 \qquad \text{vs} \qquad \mathrm{H}_1 : \theta = \theta_1$$

where $\Theta_0 = \{\theta_0\}$ and $\Theta_1 = \{\theta_1\}$ for some values of θ_0 and θ_1. The hypotheses H_0 and H_1 are single, and hence the corresponding priors can be considered as having a point mass around θ_0 and θ_1. Mathematically, we assign Dirac prior distributions $\mathrm{d}\Pi_0(\theta) = 1\left(\theta \in \{\theta_0\}\right)\mathrm{d}\theta$ and $\mathrm{d}\Pi_1(\theta) = 1(\theta \in \{\theta_1\})\mathrm{d}\theta$, which imply

$$\mathrm{d}\Pi(\theta) = \Big(\quad \pi_0 \times 1\left(\theta \in \{\theta_0\}\right) + \pi_1 \times 1(\theta \in \{\theta_1\})\Big)\,\mathrm{d}\theta$$

Then the conditional marginal likelihoods are

$$f_0(y) = \int_{\Theta_0} f(y|\theta)\mathrm{d}\Pi_0(\theta) = \int_{\{\theta=\theta_0\}} f(y|\theta)1\left(\theta \in \{\theta_0\}\right)\mathrm{d}\theta = f(y|\theta_0)$$
$$f_1(y) = \int_{\Theta_1} f(y|\theta)\mathrm{d}\Pi_1(\theta) = \int_{\{\theta=\theta_1\}} f(y|\theta)1\left(\theta \in \{\theta_1\}\right)\mathrm{d}\theta = f(y|\theta_1)$$

Note 17. In this case, Bayes factor is the likelihood ratio of H_0 against H_1 which most statisticians (whether Bayesian or not) view as the odds in favor of H_0 against H_1 that are given by the data.

Example. Given the statistical model $y_i \overset{\text{IID}}{\sim} \mathrm{N}(\mu, \sigma^2)$ the comparison $\mathrm{H}_0 : \mu = \theta_0$ vs. $\mathrm{H}_1 : \mu = \theta_1$, is a simple vs. simple hypothesis, where $\Theta_0 = \{\theta_0\}$, $\Theta_1 = \{\theta_1\}$ are sets with a single elements $\theta_0 \neq \theta_1$.

Example. The model comparison

$$\mathrm{H}_0 : y_i|\phi_0 \overset{\text{IID}}{\sim} \mathrm{Nb}(\phi_0, 1) \text{ vs.} \quad \mathrm{H}_1 : y_i|\lambda_0 \overset{\text{IID}}{\sim} \mathrm{Pn}(\lambda_0)$$

where $\phi_0 > 0, \lambda_0 > 0$ are known, is a simple vs. simple hypothesis. Here it is $\Theta_0 = \{\mathrm{Nb}\}$, $\Theta_1 = \{\mathrm{Pn}\}$.

Example 18. Let $y = (y_1, ..., y_n)$ a sequence of observables, and assume that $n = 5$, and $y_* = \sum_{i=1}^{5} y_i = 3$. Assume a sampling distribution $y_i|\theta \overset{\text{iid}}{\sim} \mathrm{Br}(\theta)$, with unknown parameter $\theta \in [0, 1]$, a priori following a uniform distribution.

1. By using Jeffreys' scaling rule, perform the following hypothesis test for $\theta_0 = 1/2$

$$\mathrm{H}_0 : \theta = \theta_0 \qquad \text{vs} \qquad \mathrm{H}_1 : \theta \neq \theta_0$$

2. Compute the posterior probability of the NULL hypothesis.

Solution. This is a simple vs. general alternative hypothesis. I specify the overall prior with pdf

$$\pi(\theta) = \pi_0 1(\theta = \theta_0) + (1 - \pi_0)\mathrm{U}(\theta|0, 1)$$

for some $\pi_0 > 0$. I leave π_0 abstract, however the usual choice (but maybe not the best) is $\pi_0 = 1/2$.

1. The Bayes factor is

$$\mathrm{B}_{01}(y) = \frac{\prod_{i=1}^{n} \mathrm{Br}(y_i|\theta_0)}{\int_{(0,1)} \prod_{i=1}^{n} \mathrm{Br}(y_i|\theta)\mathrm{U}(\theta|0, 1)\mathrm{d}\theta} = \frac{\theta_0^{y_*}(1 - \theta_0)^{n-y_*}}{\int_{(0,1)} \theta^{y_*}(1 - \theta)^{n-y_*}\mathrm{d}\theta} = \frac{\theta_0^{y_*}(1 - \theta_0)^{n-y_*}}{\mathrm{B}(y_* + 1, n - y_* + 1)} = \frac{(1/2)^5}{\mathrm{B}(4, 3)} = \frac{15}{8}$$

Then $B_{01}(y) = \frac{15}{8} \approx 2$, and $\log_{10}(\mathrm{B}_{01}(y)) \approx 0.27$. According to Jeffreys' scaling rule, H_0 is supported. We can accept the null hypothesis.

2. The posterior probability of H_0 is

$$\mathrm{P}_{\Pi}(\theta = \theta_0|y) = \mathrm{P}_{\Pi}(\theta \in \Theta_0|y) = \left[1 + \frac{1 - \pi_0}{\pi_0}\mathrm{B}_{01}(y)^{-1}\right]^{-1} = \left[1 + \frac{1/2}{1 - 1/2}(\frac{15}{8})^{-1}\right]^{-1} = \frac{15}{23} \approx 0.65$$

and hence the posterior distribution tends to support H_0.

Example 19. Let $y = (y_1, ..., y_n)$ a sequence of observables. There is interest in performing the following hypothesis test

$$\mathrm{H}_0 : \begin{cases} y_i|\phi \sim \mathrm{Nb}(1, \phi); & \phi > 0 \\ \phi \sim \mathrm{Be}(a_0, b_0); & a_0 = 2,\, b_0 = 2 \end{cases} \qquad \text{vs} \qquad \mathrm{H}_1 : \begin{cases} y_i|\lambda \sim \mathrm{Pn}(\lambda); & \lambda > 0 \\ \lambda \sim \mathrm{Ga}(a_1, b_1); & a_1 = 2,\, b_1 = 1 \end{cases}$$

1. Perform the test for $n = 2$, and $y_1 = y_2 = 0$, by using Jeffrey's scaling.
2. Perform the test for $n = 2$, and $y_1 = y_2 = 2$, by using Jeffrey's scaling.

Hint-1 Poisson distribution $x \sim \mathrm{Pn}(\lambda)$ has PMF: $\mathrm{Pn}(x|\lambda) = \frac{1}{x!}\lambda^x \exp(-\lambda)1_{\mathbb{N}}(x)$, where $\mathbb{N} = \{0, 1, 2, ...\}$ and $\lambda > 0$.

Hint-2 Negative Binomial distribution $x \sim \mathrm{Nb}(r, \theta)$ has PMF: $\mathrm{Nb}(x|r, \theta) = \binom{r+x-1}{r-1}\theta^r(1-\theta)^x 1_{\mathbb{N}}(x)$ with $\theta \in (0, 1)$, $r \in \mathbb{N} - \{0\}$, and $\mathbb{N} = \{0, 1, 2, ...\}$.

Hint-3 Gamma distribution $x \sim \mathrm{Ga}(a, b)$ has PDF: $\mathrm{Ga}(x|a, b) = \frac{b^a}{\Gamma(a)}x^{a-1}\exp(-bx)1_{(0,\infty)}(x)$, with $a > 0$ and $b > 0$.

Hint-4 Beta distribution $x \sim \mathrm{Be}(a, b)$ has PDF: $\mathrm{Be}(x|a, b) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}1_{(0,1)}(x)$, with $a > 0$ and $b > 0$.

Solution. This is a Composite vs composite hypotheses. The overall a priori distribution $\mathrm{d}\Pi(\theta)$ with $\theta \in \Theta$ and $\Theta = \{\mathrm{Nb}\} \times (0, 1) \cup \{\mathrm{Pn}\} \times (0, \infty)$ has density

$$\pi(\theta) = \pi_0 \mathrm{Be}(\phi|a_0, b_0) + \pi_1 \mathrm{Ga}(\lambda|a_1, b_1);$$

where $\pi_0 = \pi_1 = 0.5$. Let's compute the Bayes factor

$$\begin{aligned} f_0(y) &= \int \prod_{i=1}^{n} \mathrm{Nb}(y_i|\phi, 1)\mathrm{Be}(\phi|a_0, b_0)\mathrm{d}\phi = \frac{1}{B(a_0, b_0)}\int_0^1 \phi^{n+a_0-1}(1-\phi)^{n\bar{y}+b_0-1}\mathrm{d}\phi = \frac{B(n + a_0, n\bar{y} + b_0)}{B(a_0, b_0)} \\ f_1(y) &= \int \prod_{i=1}^{n} \mathrm{Pn}(y_i|\lambda)\mathrm{Ga}(\lambda|a_1, b_1)\mathrm{d}\lambda = \frac{1}{\prod_{i=1}^{n} y_i!}\frac{b_1^{a_1}}{\Gamma(a_1)}\int_0^\infty \lambda^{n\bar{y}+a_1-1}\exp(-(n + b_1)\lambda)\mathrm{d}\lambda \\ &= \frac{\Gamma(n\bar{y} + a_1)}{\Gamma(a_1)}\frac{b_1^{a_1}}{(n + b_1)^{n\bar{y}+a_1}}\frac{1}{\prod_{i=1}^{n} y_i!} \end{aligned}$$

So the Bayes Factor is

$$B_{01}(y) = \frac{B(n + a_0, n\bar{y} + b_0)}{B(a_0, b_0)}\frac{\Gamma(a_1)}{\Gamma(n\bar{y} + a_1)}\frac{(n + b_1)^{n\bar{y}+a_1}}{b_1^{a_1}}\prod_{i=1}^{n} y_i!$$

1. Then $B_{01}(y) = 2.70$, and $\log_{10}(\mathrm{B}_{01}(y)) \approx 0.43$. According to Jeffrey's scaling rule, H_0 is supported.
2. Then $B_{01}(y) = 0.29$, and $\log_{10}(\mathrm{B}_{01}(y)) \approx -0.53$. According to Jeffrey's scaling rule, the evidence against H_0 is substantial.

Example 20. Let $y = (y_1, ..., y_n)$ a sequence of observables. There is interest in performing the following hypothesis test

$$\mathrm{H}_0 : y_i|\phi \sim \mathrm{Nb}(\phi, 1); \text{ with } \phi = 1/3 \qquad \text{vs} \qquad \mathrm{H}_1 : y_i|\lambda \sim \mathrm{Pn}(\lambda); \text{ with } \lambda = 2$$

1. Perform the test for $n = 2$, and $y_1 = y_2 = 0$, by using Jeffreys' scaling.
2. Perform the test for $n = 2$, and $y_1 = y_2 = 2$, by using Jeffreys' scaling.

Hint-1 Poisson distribution $x \sim \mathrm{Pn}(\lambda)$ has PMF: $\mathrm{Pn}(x|\lambda) = \frac{1}{x!}\lambda^x \exp(-\lambda) 1_{\mathbb{N}}(x)$, where $\mathbb{N} = \{0, 1, 2, ...\}$ and $\lambda > 0$.

Hint-2 Negative Binomial distribution $x \sim \mathrm{Nb}(r, \theta)$ has PMF: $\mathrm{Nb}(x|r, \theta) = \binom{r+x-1}{r-1}\theta^r(1-\theta)^x 1_{\mathbb{N}}(x)$ with $\theta \in (0, 1)$, $r \in \mathbb{N} - \{0\}$, and $\mathbb{N} = \{0, 1, 2, ...\}$.

Solution. This is a simple vs simple hypothesis test. I specify priors $\pi(\mathrm{Nb}) = \pi(\mathrm{Pn}) = 1/2$, due to the a priori ignorance about the parametric statistical model, however, we do not really need it now .... The Bayes factor is

$$B_{01}(y) = \frac{f_0(y)}{f_1(y)} = \frac{\prod_{i=1}^n \mathrm{Nb}(y_i|\phi, 1)}{\prod_{i=1}^n \mathrm{Pn}(y_i|\lambda)} = \frac{\phi^n(1-\phi)^{n\bar{y}}}{\lambda^{n\bar{y}} \exp(-n\lambda)/\prod_{i=1}^n y_i!}$$

1. Then $B_{01}(y) = \exp(4)/9 \approx 6.07$, and $\log_{10}(\mathrm{B}_{01}(y)) \approx 0.78$. According to Jeffrey's scaling rule, H_0 is supported.
2. Then $B_{01}(y) = 4\exp(4)/729 \approx 0.30$, and $\log_{10}(\mathrm{B}_{01}(y)) \approx -0.54$. According to Jeffrey's scaling rule, the evidence against H_0 is substantial.

Practice

Question 21. *To practice try to work on the Exercise 70 from the Exercise sheet.*