Columbia MA Math Camp

Real Analysis

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Motivation

- In real analysis (and also in micro) we often need to use concepts related to limits and convergence
- By saying that a sequence of objects converges to a limiting object, we mean, roughly speaking, that the sequence will get "as close as we want" to the limit.
- To be able to talk about how close 2 objects are, we need the concept of distance.
- Metric spaces are the general framework that capture the concept of distance, but we will focus on Euclidean metric spaces
  - Pretty much the only thing economists work with
  - Easier to visualize

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Defining Metric Spaces

Definition 1.1
Let X be a set, and $d : X \times X \to \mathbb{R}$ a function. We call d a **metric** *on X if:*

- **Positive Definiteness :** $d(x, y) \geq 0$ *for all* $x, y \in X$*, and* $d(x, y) = 0$ *iff* $x = y$
- **Symmetry :** $d(x, y) = d(y, x)$
- **Triangle Inequality :** $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$

A **metric space** (X, d) *is a set X with a metric d defined on X*

Example : A trivial example is the discrete metric :

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Does this satisfy the properties of a metric? We will however focus on Euclidean metric spaces.

Discrete Metric is actually a metric

Discrete metric : $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

Let us check that the discrete metric satisfies the 3 properties of a metric :

(a) By definition $d(x, y) = 0$ iff $x = y$

(b) Symmetric is also trivial

(c) Take $x, y, z \in X$. If $x = y$, then $d(x, y) = 0 \leq d(x, z) + d(z, y)$. If $x \neq y$, then we must have $d(x, z) = 1$ OR $d(z, y) = 1$. In either case we have :

$$d(x, y) = 1 \leq d(x, z) + d(z, y)$$

Definition 1.2
In $\mathbb{R}^n$, the **Euclidean distance** *is the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$:*

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

Euclidean distance satisfies the 3 properties that we mentioned earlier:

- $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

Let us prove the first two properties. Triangle Inequality is hard to show - requires the Cauchy Schwarz inequality.

Euclidean metric proof

Fact 1
Prove that the Euclidean metric satisfies the first two properties of a metric

Proof.
(a) WTS that $d(x, y) = 0$ iff $x = y$ for any $x, y \in \mathbb{R}^n$.

$(\implies)$: If $d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0$, this implies $\sum_{i=1}^{n} (x_i - y_i)^2 = 0$. Since $(x_i - y_i)^2 \geq 0$ for each i which means that $(x_i - y_i)^2 = 0$ $\forall i$ and thus $x_i = y_i$ $\forall i$. Thus $x = y$

$(\impliedby)$: If $x = y$, we have $x_i = y_i$ for each i. Therefore

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0$$

(b) WTS that $d(x, y) = d(y, x)$. This follows from the fact that :

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$$

Definition 1.3
A subset S of $\mathbb{R}^n$ is **bounded** *if there exists $M \in \mathbb{R}$ such that for all $x \in S$, $d(0,x) \leq M$.*

Note: We could have chosen any $a \in \mathbb{R}^n$ in the place of 0. Suppose S is bounded with respect to 0. Then for any $x \in S$ the triangle inequality tells us

$$d(a,x) \leq d(a,0) + d(0,x) \leq d(a,0) + M$$

Thus S is bounded with respect to a as well.

Least Upper Bounds

Definition 1.4
Let $S \subseteq \mathbb{R}$. A number $M \in \mathbb{R}$ is an **upper bound** *of S if $s \leq M$ for every $s \in S$.*

If no $M' < M$ is an upper bound of S, then M is called the **least upper bound** *or* **supremum** *of S.*

We make one important assumption about the real numbers: every bounded set of real numbers has a least upper bound. This is called the **least upper bound property**.

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Sequences: Definition

Formally, a **sequence** in a set X is a function from $\mathbb{N}$ to X . We denote x_n the image of n, and (x_n) the sequence.

Less formally, a sequence is an **ordered collection** of elements $(x_0, x_1, x_2, ...)$. Many problems in math boil down to understanding the long-term behavior of some sequence.

We typically write sequences as a formula or by enumerating the first few terms.

- $(x_n) = (n)_{n=0}^{\infty}$: $(0, 1, 2, 3, \dots)$
- $(x_n) = (1)_{n=0}^{\infty}$: $(1, 1, 1, 1, \dots)$
- $(x_n) = \left(\frac{1+(-1)^n}{2}\right)_{n=0}^{\infty}$: $(1, 0, 1, 0, \dots)$
- $(x_n) = \left(\frac{1}{n+1}\right)_{n=0}^{\infty}$: $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$

You'll also see people write things like $x_n = n$.

Bounded Sequences

In $\mathbb{R}^n$ we can define properties of sequences that rely on the notion of distance.

Definition 2.1
A sequence (x_n) of $S \subseteq \mathbb{R}^n$ is **bounded** *iff $\{x_0, ..., x_n, ...\}$ is a bounded subset of $\mathbb{R}^n$.*

Are the sequences $x_n = n$ and $x_n = 1/n$ bounded or unbounded?

- The sequence $x_n = n$ is not bounded. For any $M \in \mathbb{R}$, $d(x_n, 0) > M$ for $n > M$.
- The sequence $x_n = \frac{1}{n}$ is bounded: $d(x_n, 0) \leq 1$ for all n

Definition 2.2
A sequence (x_n) of $S \subseteq \mathbb{R}^n$ **converges to a limit** *$\ell \in S$ iff for all $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that for all $n \geq N(\epsilon)$*

$$d(x_n, \ell) < \epsilon$$

We write $x_n \to \ell$ or $\lim_{n\to\infty} x_n = \ell$. If (x_n) does not converge, we say it **diverges***.*

This definition is important, so let's unpack it a little:

- The sequence must eventually **get and remain** arbitrarily close to ℓ
- N can be different for each ϵ.
- We require $\ell \in S$. Consider $(x_n) = \left(1, \frac{1}{2}, \frac{1}{3}, ...\right)$. What happens if $S = \mathbb{R}$? If $S = (0, 2)$?

Proposition 2.1
If a sequence converges, then it is bounded.

Proof.
Assume $x_n \to \ell$.

- Take $\epsilon = 1$
- Since $x_n \to \ell$, by definition there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, \ell) < 1$.

This proves that the sequence starting at N is bounded, but we need to deal with the first $N - 1$ terms.

- Define $M \equiv \max\{d(x_1, \ell), ..., d(x_{N-1}, \ell), 1\}$
- For all n, $d(x_n, l) \leq M$

$\square$

Limit (if exists) must be unique

Proposition 2.2
The limit of a sequence (x_n) *is unique provided it exists i.e. if* $x_n \to x$ *and* $x_n \to x'$*, then* $x = x'$

Proof.
We prove this by contradiction. Suppose $x \neq x'$. Thus it must be that $d(x, x') > 0$.

Consider $\varepsilon = \frac{d\left(x, x'\right)}{2}$.

Because $x_n \to x$, there exists N such that $d(x_n, x) < \varepsilon$ for any $n > N$. Similarly, because $x_n \to x'$, there exists N' such that $d(x_n, x') < \varepsilon$ for any $n > N'$. Take $\hat{n} = \max\{N, N'\} + 1$ so that $\hat{n} > N$ and $\hat{n} > N'$. There $d(x_{\hat{n}}, x) < \varepsilon$ and $d(x_{\hat{n}}, x') < \varepsilon$. Thus it must be that :

$$d(x, x_{\hat{n}}) + d\left(x_{\hat{n}}, x'\right) < 2\varepsilon = d\left(x, x'\right)$$

which contradicts the triangle inequality of d. Thus we have reached a logical contradiction and therefore $x = x'$. □

Proposition 2.3
In the Euclidean metric space $(\mathbb{R}, d)$, suppose there are 2 convergent sequences $x_n \to x$ and $y_n \to y$. If $x_n \leq y_n \ \forall n$, then prove that $x \leq y$

Proof.
We do this by contradiction. Suppose $x > y$ and set $\varepsilon = \frac{x-y}{2}$. Since $x_n \to x$ and $y_n \to y$, then by definition, $\exists N_x$ and N_y such that $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ $\forall n > N_x$ and $n > N_y$. Take $\hat{n} > \max\{N_x, N_y\}$. Then we must have $|x_{\hat{n}} - x| < \varepsilon$ and $|y_{\hat{n}} - y| < \varepsilon$. Since $\varepsilon = \frac{x-y}{2}$, this implies,$x - \varepsilon = y + \varepsilon$

$$x_{\hat{n}} > x - \varepsilon = y + \varepsilon > y_{\hat{n}}$$

which contradicts $x_n \leq y_n \ \forall n$ $\square$

Here is a result you'll show on your problem set:

Proposition 2.4

A sequence $(x^k) = (x_1^k, ..., x_n^k)$ of $\mathbb{R}^n$ converges to a limit x iff each component converges to the corresponding component of x in $\mathbb{R}$.

The result boils down to the fact that for all $j \in \{1, ..., n\}$:

$$|x_j - x_j^k| \leq \left(\sum_{i=1}^{n} (x_j - x_j^i)^2 \right)^{\frac{1}{2}} \leq n \max_i |x_i - x_i^k|$$

In general, working with the definition of convergence is cumbersome. There are some important results we'll use frequently

Proposition 2.5
Let (x_n) and (y_n) be sequences of $\mathbb{R}$. If $(x_n) \to x$ and $(y_n) \to y$:

(a) $x_n + y_n \to x + y$

(b) $x_n y_n \to xy$

(c) $1/x_n \to 1/x$ *if* $x \neq 0$

Proof.
We'll show the second one and the rest will probably be on your problem set. □

Proof of Property (b)

Proposition 2.6
Let (x_n) and (y_n) be sequences of $\mathbb{R}$. If $(x_n) \to x$ and $(y_n) \to y$ then $x_n y_n \to xy$

Proof.
Take any $\varepsilon > 0$. I want to find N such that $|x_n y_n - xy| < \varepsilon$ for any $n > N$.

Because (y_n) is convergent, it is bounded i.e. there exists M such that $|y_n| < M\ \forall n$. Because $x_n \to x$, there exists N_x s.t $|x_n - x| < \frac{\varepsilon}{2M}$. Again since $y_n \to y$, there exists N_y such that $|y_n - y| < \frac{\varepsilon}{2(|x|+1)}$.

Let $N = \max\{N_x, N_y\}$ and I claim this is the N we need to find. This is because for any $n > N$,

$$\begin{aligned}
|x_n y_n - xy| &= |(x_n - x)y_n + (y_n - y)\,x| \\
&\leq |x_n - x|\,|y_n| + |y_n - y|\,|x| \\
&< \frac{\varepsilon}{2M}.M + \frac{\varepsilon}{2\,(|x| + 1)}.|x| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

$\square$

Definition 2.3
Let (x_n) be a sequence. A **subsequence** *of (x_n) is a sequence (x_{n_k}) where $n_1 < n_2 < \dots$ is an increasing sequence of indices.*

- $(x_2, x_3, x_5, \dots)$ is a subsequence of (x_n)
- $(x_4, x_3, x_2, \dots)$ is not (the terms are out of order).

Proposition 2.7
A sequence (x_n) converges to a limit ℓ iff all its subsequences converge to the same limit ℓ.

Proof.
($\Rightarrow$) Assume $(x_n) \to \ell$ and consider a subsequence (x_{n_k}) of (x_n). Fix $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, \ell) < \epsilon$. Take K such that $n_K \geq N$. Then for all $k \geq K$, $n_k \geq n_K \geq N$, so $d(x_{n_k}, \ell) < \epsilon$.

($\Leftarrow$) Since (x_n) is a subsequence of itself, this implication is immediate. $\square$

Subsequences (cont.)

Proposition 2.8

Every bounded sequence of real numbers has a convergent subsequence.

Proof.
Let (x_n) be a bounded sequence of real numbers.

- Since (x_n) is bounded, some integer part D occurs infinitely many times; consider only terms whose integer part is D.
- Among these terms, some first digit d_1 must occur infinitely many times.
- Continuing this process we can construct some $\ell = D.d_1 d_2 \ldots$.

Construct the subsequence as follows:

- Let x_{n_1} be an element that begins with $D.d_1$.
- Take x_{n_2} to be a term after x_{n_1} that begins with $D.d_1 d_2$ (we can take $n_2 > n_1$ because there are infinitely many such elements).
- Continue this process for n_3 and so on. We see $d(\ell, x_{n_k}) < 10^{-k}$, so the subsequence clearly converges to ℓ.

□