

# Columbia MA Math Camp

## Introduction to Propositional Logic

---

Vinayak Iyer

August 12, 2019

## Outline of Today's Class

- Introduction to Propositional Logic
- Introduction to Set Theory
- Overview of how to write proofs

But first, more important stuff :

- How to approach classes
- Rubinstein Interview 1
- Rubinstein Interview 2

## Before we start...

- This is meant to prepare a basic foundation of mathematics that will help you in your classes
  - Not meant to teach you advanced mathematics
- Your goal should be to understand the concepts well. The more you use this time to really grasp concepts, the easier it will be in your classes
- I will try and go slow, but please let me know if I am going too fast.
- Please interrupt me and ask questions! You will benefit and so will others!
- Don't forget to enjoy the wonderful NY summer!

# Why do we need to learn Propositional Logic?

- In economics, we rely on a lot of proofs. You will encounter proofs in most of your classes.
- So it is useful to go through some foundations of logic which will help you in your class

# Propositions

## Definition

A **proposition** is a statement that can be either **True (T)** or **False (F)**

## Examples :

- *“The earth is flat”* - **False**
- *“March has 31 days”* - **True**
- *“You should maintain a work-life balance”* - **Not a proposition**

**Notation** : Lower case letters are often used to represent propositions.

## Example :

- $p$  : *“The earth is flat”*
- $q$  : *“March has 31 days”*

# Logical Operators (Connectives)

## Definition

A **connective** is a symbol that combine propositions. Propositions separated by connectives make a **compound proposition**.

There are 3 basic connectives :

1. **Conjunction** : “ $p$  and  $q$ ” and is denoted by  $\wedge$

- “*The Earth is flat and March has 31 days*”

2. **Disjunction** : “ $p$  or  $q$ ” and denoted by  $\vee$

- “*The Earth is flat or March has 31 days*”

**Note** : The meaning of  $\vee$  is *inclusive* in the sense that the either one statement is true or both can be true

3. **Negation** : “ $\neg p$ ” is the negation of the statement  $p$

# Truth Values

- We now must determine the **truth values** of compound statements.  
(What were compound statements?)
- Note that the truth or falsity of a compound statement depends on the truth or falsity of the individual statements.
- We do this through a **truth table** which lists all possible combinations of the truth value of a compound statement based on the truth value of the individual components.

$p \wedge q$			$p \vee q$			$\neg p$	
$p$	$q$	$p \wedge q$	$p$	$q$	$p \vee q$	$p$	$\neg p$
T	T	T	T	T	T	T	F
T	F	F	T	F	T	F	T
F	T	F	F	T	T		
F	F	F	F	F	F		

# Examples of Evaluating Compound Propositions

## Example

Suppose  $p, q, r$  are 3 statements. Suppose  $p$  and  $r$  are true while  $q$  is false. Evaluate whether or not the following statements are true.

- $\neg(p \wedge q)$  :- Since  $p$  is True and  $q$  is False  $\implies p \wedge q$  is False  
 $\implies \neg(p \wedge q)$  is **True**
- $(\neg p) \wedge (\neg q)$  :-  $(\neg p)$  is False and  $(\neg q)$  is True  $\implies (\neg p) \wedge (\neg q)$  is **False**
- $p \vee ((\neg q) \wedge r)$  :- (?)



# Filling out a Truth Table

In the previous slide I specified the truth values of the individual statements. If I had not specified the truth values, then we would have to fill out the truth table based on all possible combinations of  $p$ ,  $q$  and  $r$ . So let's fill it out for  $p \vee ((\neg q) \wedge r)$ .

$p$	$q$	$r$	$p \vee \neg q \wedge r$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

# Conditional Propositions

## Definition

The conditional operator **if** is used in a compound statement of the form “*if  $p$  then  $q$* ” and is denoted by  $p \rightarrow q$ . It is also called the logical implication sometimes.

- $p$  is called the **antecedent or hypothesis**
- $q$  is called the **consequent or conclusion**
- $p \rightarrow q$  is also read as “ *$p$  implies  $q$* ”

## Example :

- $p$  : “*You got an A*”
- $q$  : “*I give you a dollar*”
- $p \rightarrow q$  : “*If you get an A, then I will give you a dollar*”

# Truth Table for $\rightarrow$

Like the other logical operators, the **logical implication** ( $\rightarrow$ ) also has a truth table.

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- The last two are by definition. To understand why, let us go back to the previous example.
- $p \rightarrow q$  :- *"If you get an A, then I will give you a dollar"*
  - Statement is true if I keep my promise and false if I don't...
  - If you don't get an A, then irrespective of whether or not I give you a dollar, I haven't broken my promise!

# Example

## Example

Suppose  $p, q$  and  $r$  are statements such that  $p, r$  are true and  $q$  is false.

Evaluate the following compound statements.

1.  $\neg(p \rightarrow q)$  :- From the truth table, since  $p$  is true and  $q$  is false  $\implies p \rightarrow q$  is False and the negation of a False statement is True. Hence  $\neg(p \rightarrow q)$  is **True**
2.  $(p \wedge q) \rightarrow r$  :-  $(p \wedge q)$  is False. This implies that  $(p \wedge q) \rightarrow r$  is a **True** statement.
3.  $(p \rightarrow q) \rightarrow r$  :- (?)

# Necessary and Sufficient Conditions

- You will come across these terms a LOT in economics!
- Consider the logical statement  $p \rightarrow q$ 
  - $p$  is said to be a **sufficient condition** for  $q$ 
    - If the statement *“If you get an A, then I will give you a dollar”* is true, then knowing that you got an A is sufficient to know that you also received a dollar.
  - $q$  is said to be a **necessary condition** for  $p$ 
    - Similarly, for the statement above to be true, then getting a dollar is necessarily true if you get an A.
- Of course  $p$  being sufficient for  $q$  does not mean it is necessary and vice versa.

# More examples of Necessary vs Sufficient Conditions

## 1. Consider the statement *“If it rains, then it is cloudy”*

- Then knowing that it is raining is **sufficient** to know that it is cloudy. (But being cloudy is **not sufficient** for rain!)
- It is **necessary** for it to be cloudy for rain to occur. In other words, if it is not cloudy, then it cannot rain! (It is **not necessary** for it to rain for it be cloudy!)

## 2. How does this relate to economics? Most of you must have heard the phrases **“First Order Condition”** and **“Second Order Condition”** right?

- The FOC when maximizing a function  $f(x)$  is given by  $f'(x) = 0$ . For  $x$  to be an interior maximum, it is **necessary** that  $f'(x) = 0$  (but not sufficient!)
- The SOC sufficient condition is generally given by the statement that if  $x$  is such that  $f'(x) = 0$  and  $f''(x) < 0$ , then  $x$  is a maximizer. But this is not necessary! - Find the maxima of  $f(x) = -x^2$

# Some statements can be both necessary and sufficient

- Consider the conditional statement given by :
  - “A number being 2,4,6 or 8  $\rightarrow$  it is an even number”
    - As per the discussion previously, knowing a number being 2,4,6 or 8 is **sufficient** to know that it is even. But it is **not necessary** that number be 2,4,6 or 8 for it to be even
- Now consider the alternate statement :
  - “A number being 2,4,6 or 8  $\rightarrow$  it is a positive even number  $<10$ ”
    - Now a number being 2,4,6 or 8 is **both necessary and sufficient** for it to be a positive even number  $<10$
    - In this case, we can actually write “A number being 2,4,6 or 8  $\leftrightarrow$  it is an even number  $<10$ ”

## Definition

We write  $p \leftrightarrow q$  or  $p$  if and only if  $q$  when  $p$  is **both necessary and sufficient** for each other.

What does the truth table look like? When is  $p \leftrightarrow q$  true?

# Converse and Contrapositive

## Definition

For a conditional statement  $p \rightarrow q$

- The **converse** is defined as  $q \rightarrow p$
- The **contrapositive** is defined as  $(\neg q) \rightarrow (\neg p)$

## Properties :

1. If  $p \rightarrow q$  is true, that **does not** mean that  $q \rightarrow p$  is true!
  - 1.1 When  $p$  is False and  $q$  is True, then  $p \rightarrow q$  is True but  $q \rightarrow p$  is False
2. The contrapositive  $(\neg q) \rightarrow (\neg p)$  is **equivalent** to  $p \rightarrow q$ . (Why?)

**Question :** Does  $p \rightarrow q$  imply  $\neg p \rightarrow \neg q$  ?



# Example by Rubinstein!

## Q6. My paper has just been rejected. What should I do?

I have a lot of experience with the mental state you must be in, so I have three pieces of advice:

- (a) Don't read the referee reports. They are likely to depress you. Even if they are potentially useful, you are not in a state of mind to benefit from them.
- (b) Find comfort in my motto: "A paper that has not been rejected should not be published." But beware of the faulty logic in assuming that "every paper that has been rejected should be published."

**Figure 1:** Words of Wisdom from Rubinstein

- What is the faulty logic?  $p \rightarrow q$  DOES NOT IMPLY  $\neg A \implies \neg B$
- Contrapositive is that "*A paper that has been published must have been rejected*"

# Examples of Converse and Contrapositive Statements

Consider the statement *“If a paper that has been published, then it must have been rejected”*

- **Converse** : *If a paper has been rejected, then it must be published*
- **Contrapositive** : *If a paper has not been rejected, then it must not be published*

**Note** : The contrapositive has the same truth value as the original statement.

# Logical Equivalence

## Definition

Two statements  $p$  and  $q$  are **logically equivalent** denoted by  $p \leftrightarrow q$  when the truth values in **all** rows in the truth table are the same.

For example, show that the statements  $(p \rightarrow q)$  and  $(\neg p \vee q)$  are logically equivalent

$p$	$q$	$p \rightarrow q$	$\neg p \vee q$
T	T		
T	F		
F	T		
F	F		

# DeMorgan's Law

## Fact

Consider 2 propositions  $p$  and  $q$ . Then the statements  $\neg(p \vee q)$  and  $(\neg p \wedge \neg q)$  are logically equivalent i.e.

$$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$$

## Proof.

$p$	$q$	$\neg(p \vee q)$	$(\neg p \wedge \neg q)$
$T$	$T$	$F$	$F$
$T$	$F$	$F$	$F$
$F$	$T$	$F$	$F$
$F$	$F$	$T$	$T$



# Tautology

## Definition

A **tautology** is a formula that is always true i.e. for every combination of truth values of its components, the compound statement is true.

## Example

Show that  $(p \rightarrow q) \vee (q \rightarrow p)$  is a tautology.

$P$	$Q$	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

# Final Example

## Example

Suppose  $x$  is a real number. Consider the statement :

$$\text{If } x^2 = 4, \text{ then } x = 2$$

Construct the **converse** and **contrapositive** of the statement. Determine the truth or falsity of the original, converse and the contrapositive using your knowledge of algebra. In the converse statement, identify the **necessary and sufficient statements**.

## Proof.

The converse of the statement is “If  $x = 2$ , then  $x^2 = 4$ ”. The contrapositive is “If  $x \neq 2$ , then  $x^2 \neq 4$ ”.

**Truth value** : The original statement is false since  $(-2)^2 = 4$ . The contrapositive has the same truth value as the original, hence is also false, while the converse is true i.e. If  $x = 2$ , then  $x^2 = 4$  .

Knowing that  $x = 2$  is sufficient to know that  $x^2 = 4$ , while  $x^2$  being equal to 4 is necessary for  $x$  being equal to 2. □