Math Camp: Problem Set 3

1. Differentiate the following functions with respect to x
   - $\frac{1}{x^6}$
   - $\ln(x)(x^2+1)$
   - $\frac{e^{x^2}-x}{2x+1}$
2. Find the second-order Taylor series expansion for $f(x_1,x_2) = \ln(1+x_1+x_2)$ about $(x_1,x_2) = (1,1)$.
3. Evaluate the following integrals **(Just for practice - Will not be on the exam)**
   - $\int(2x+4x^3+7x^4)dx$
   - $\int_1^e \frac{1+\ln x}{x}dx$ (hint: use substitution)
   - $\int x^2e^x dx$ (hint: integrate by parts)
4. Calculate the following integrals over the described sets : **(Just for practice - Will not be on the exam)**
   - $\int_D 1dA$, where $D = \{(x,y) \in [0,1]^2 | y \geq x\}$.
   - $\int_D x_1x_2dA$, where $D = \{(x_1,x_2) \in \mathbb{R}^2; 0 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2\}$
5. Suppose $f(x)$ is quasiconcave over the interval $[a,b]$, and define M as the set of maximum points of f; $M = \{m \in [a,b] | f(x) \leq f(m) \forall x \in [a,b]\}$. Show that M is convex.
6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $f''(x) > 0$ for all x. We will show that f is strictly convex:
   - Let $x_1 < x_2$, and let $x = \lambda x_1 + (1-\lambda)x_2$ for some $\lambda \in (0,1)$
   - Find a Taylor series expansion for $f(x_1)$, expanding around x. Use the fact that $f'' > 0$ to derive an inequality relating $f(x_1), f(x)$, and $f'(x)$
   - Repeat the above step for $f(x_2)$
   - Combine the two inequalities to show that f is strictly convex
7. Calculate the derivative and the Hessian of the following functions. For critical points, use the Hessian to determine whether they are local maxima, minima, or undetermined.
   - $f(x) = x_1^2 + ax_1x_2 + x_2^2$, $|a| < 2$
   - $f(x) = x_1^2 + ax_1x_2 + x_2^2$, $|a| > 2$
   - $f(x) = x^2 - y^2 - xy - x^3$

8. Find all the candidate maxima/minima of the following functions subject to the given constraints (if using the Lagrange method, after findings all the candidate points, we can determine maxima/minima simply by comparing their values (if maxima/minima exist)).

   - $f(x, y) = xy$ subject to $x^2 + y^2 = 2a^2$ $(a > 0)$
   - $f(x, y) = 1/x + 1/y$ subject to $(1/x)^2 + (1/y)^2 = (1/a)^2$
   - $f(x, y) = x + y$ subject to $xy = 16$

9. Let x be a vector of size n and A a real symmetric matrix of size n. Solve the program

$$\max_x x'Ax \text{ subject to } x'x = 1$$

10. Let y be a vector of $\mathbb{R}^n$ and X an $n \times k$ matrix with rank k (so $n \geq k$). Solve the program

$$\min_{b \in \mathbb{R}^k} \|y - Xb\|$$

   This is nothing but the Ordinary Least Squares estimator you will do in econometrics.

11. Consider a consumer with utility function $u(x_1, ..., x_n) = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$ where $\sum_{i=1}^n \alpha_i = 1$. Each good x_i costs p_i per unit, and the consumer has a budget of M dollars. Find the consumer's optimal mix of consumption. That is, solve the program:

$$\max_{x_1, ..., x_n} u(x) \text{ subject to } p \cdot x = B$$

   where $p \cdot x = \sum p_i x_i$