Math Camp: Problem Set 3

1. Differentiate the following functions with respect to x

   - $\frac{1}{x^6}$
     Solution: Use the power rule: $\frac{d}{dx}x^n = nx^{n-1}$. In this case, $\frac{d}{dx}x^{-6} = -6x^{-7}$.

   - $\ln(x)(x^2+1)$
     Solution: Use the product rule:

     $$\begin{aligned}\frac{d}{dx}\ln(x)(x^2+1) &= \left(\frac{d}{dx}\ln(x)\right)(x^2+1) + \ln(x)\left(\frac{d}{dx}(x^2+1)\right) \\ &= \frac{x^2+1}{x} + 2x\ln(x)\end{aligned}$$

   - $\frac{e^{x^2}-x}{2x+1}$
     Solution: Quotient rule and chain rule:

     $$\begin{aligned}\frac{d}{dx}\frac{e^{x^2}-x}{2x+1} &= \frac{\frac{d}{dx}\left(e^{x^2}-x\right)}{2x+1} - \frac{\left(e^{x^2}-x\right)\frac{d}{dx}(2x+1)}{(2x+1)^2} \\ &= \frac{2xe^{x^2}-1}{2x+1} - \frac{2\left(e^{x^2}-x\right)}{(2x+1)^2}\end{aligned}$$

2. Find the second-order Taylor series expansion for $f(x_1, x_2) = \ln(1+x_1+x_2)$ about $(x_1, x_2) = (1, 1)$.

   Solution: We see $f(1,1) = \ln 3$; $f_{x_1}(1,1) = f_{x_2}(1,1) = \frac{1}{3}$ and all three second derivatives are $\frac{-1}{9}$. Thus

   $$f(x_1, x_2) \approx \ln 3 + \frac{1}{3}(x_1-1) + \frac{1}{3}(x_2-1) - \frac{1}{18}(x_1-1)^2 - \frac{1}{9}(x_1-1)(x_2-1) - \frac{1}{18}(x_2-1)^2$$

3. Evaluate the following integrals

   - $\int(2x+4x^3+7x^4)dx$

     Solution: From the linearity of integrals and the power rule:

     $$\int(2x+4x^3+7x^4)dx = x^2 + x^4 + \frac{7}{5}x^5 + C$$

- $\int_1^e \frac{1+\ln x}{x} dx$ (hint: use substitution)

  Solution: Let $u = 1 + \ln(x)$. Then $du = \frac{dx}{x}$. When $x = 1$, $u = 1$, and when $x = e$, $u = 2$. Therefore
  $$\int_1^e \frac{1+\ln(x)}{x} dx = \int_1^2 u du = \left.\frac{u^2}{2}\right|_1^2 = \frac{3}{2}$$

- $\int x^2 e^x dx$ (hint: integrate by parts)

  Solution: Using integration by parts twice we have
  $$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - \int 2xe^x dx \\ &= x^2 e^x - \left(2xe^x - \int 2e^x dx\right) \\ &= x^2 e^x - 2xe^x + 2e^x\end{aligned}$$

4. Calculate the following integrals over the described sets

   - $\int_D 1 dA$, where $D = \{(x, y) \in [0, 1]^2 | y \geq x\}$.

     Solution: This can be represented by the double integral
     $$\begin{aligned}\int_D 1 dA &= \int_{x=0}^{x=1} \left(\int_{y=x}^{y=1} 1 dy\right) dx \\ &= \int_{x=0}^{x=1} (1-x) dx \\ &= x - \left.\frac{x^2}{2}\right|_0^1 \\ &= \frac{1}{2}\end{aligned}$$

   - $\int_D x_1 x_2 dA$, where $D = \{(x_1, x_2) \in R^2; 0 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2\}$

     Solution: This can be represented by the double integral:
     $$\begin{aligned}\int_D x_1 x_2 dA &= \int_{x_1=0}^{x_1=1} \left(\int_{x_2=0}^{x_2=x_1^2} x_1 x_2 dx_2\right) dx_1 \\ &= \int_{x_1=0}^{x_1=1} \left.\frac{1}{2} x_1 x_2^2\right|_0^{x_1^2} dx_1 \\ &= \int_{x_1=0}^{x_1=1} \frac{1}{2} x_1^5 dx_1 \\ &= \left.\frac{1}{12} x_1^6\right|_0^1 \\ &= \frac{1}{12}\end{aligned}$$

5. Suppose $f(x)$ is quasiconcave over the interval $[a,b]$, and define M as the set of maximum points of f; $M = \{m \in [a,b] | f(x) \leq f(m) \forall x \in [a,b]\}$. Show that M is convex.

   Solution: First note that $f(m)$ is constant for all $m \in M$. If not, $f(m_1) > f(m_2)$ for some m_1, m_2, in which m_2 is not a maximum point. Write $f(m) = c$.

   Let $x_1, x_2 \in M$. We need to show $\lambda x_1 + (1-\lambda)x_2 \in M$ for any $\lambda \in [0,1]$. By the quasiconcavity of f we have

   $$f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\} = c$$

   This inequality must bind. If $f(\lambda x_1 + (1-\lambda)x_2) > c$, that contradicts the fact that c is the maximal value of f. Thus $f(\lambda x_1 + (1-\lambda)x_2) = c$. This implies that $\lambda x_1 + (1-\lambda)x_2$ is a maximum point, so M is convex.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $f''(x) > 0$ for all x. We will show that f is strictly convex:

   - Let $x_1 < x_2$, and let $x = \lambda x_1 + (1-\lambda)x_2$ for some $\lambda \in (0,1)$
   - Find a Taylor series expansion for $f(x_1)$, expanding around x. Use the fact that $f'' > 0$ to derive an inequality relating $f(x_1), f(x)$, and $f'(x)$
   - Repeat the above step for $f(x_2)$
   - Combine the two inequalities to show that f is strictly convex

   Solution: Take $\lambda \in [0,1]$ and define $x = \lambda x_1 + (1-\lambda)x_2$. Using the Taylor Theorem we discussed in class:

   $$f(x_1) = f(x) + f'(x)(x_1 - x) + \frac{1}{2}f''(z)(x_1 - x)^2$$

   for some z between x and x_1. Since $f'' > 0$ everywhere, we have

   $$f(x_1) > f(x) + f'(x)(x_1 - x) = f(x) + f'(x)(1-\lambda)(x_2 - x_1)$$

   Similarly, we find

   $$f(x_2) > f(x) - f'(x)\lambda(x_2 - x_1)$$

   Multiply the first equation by λ and the second equation by $1-\lambda$ and add them together (noting that the first derivative terms cancel out):

   $$\lambda f(x_1) + (1-\lambda)f(x_2) > f(\lambda x_1 + (1-\lambda)x_2),$$

   which, by definition, implies f is strictly convex.

7. Calculate the derivative and the Hessian of the following functions. For critical points, use the Hessian to determine whether they are local maxima, minima, or undetermined.

   - $f(x) = x_1^2 + ax_1x_2 + x_2^2$, $|a| < 2$
   - $f(x) = x_1^2 + ax_1x_2 + x_2^2$, $|a| > 2$
   - $f(x) = x^2 - y^2 - xy - x^3$

- $f(x) = x_1^2 + ax_1x_2 + x_2^2$, $|a| < 2$

  Solution: The derivative is:

  $$f'(x) = \begin{pmatrix} 2x1 + ax_2 & 2x_2 + ax_1 \end{pmatrix}$$

  The Hessian is

  $$H(x) = \begin{pmatrix} 2 & a \\ a & 2 \end{pmatrix}$$

  The Hessian is constant. Note its characteristic equation is $(2-\lambda)^2 - a^2 = 0$, or $\lambda = 2 \pm a$. Since $|a| < 2$, we see $\lambda > 0$, so $H(x)$ is positive definite.
  Critical points are where $f'(x) = 0$. This happens when

  $$\begin{pmatrix} 2 & a \\ a & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

  For $|a| < 2$ the matrix above is invertible, so the unique critical point is $x_1 = x_2 = 0$. Since the Hessian is positive definite, this critical point is a local minimum.

- $f(x) = x_1^2 + ax_1x_2 + x_2^2$, $|a| > 2$

  Solution: This question is the same as above; the only difference is that one of the eigenvalues is now negative $|a| > 2$. Thus the Hessian is indeterminate, so the second derivative test is inconclusive (the critical point turns out to be a saddle point in this case).

- $f(x) = x^2 - y^2 - xy - x^3$

  Solution:
  The derivative is:

  $$f'(x) = \begin{pmatrix} 2x - y - 3x^2 & -2y - x \end{pmatrix}$$

  The Hessian is

  $$\begin{pmatrix} 2 - 6x & -1 \\ -1 & -2 \end{pmatrix}$$

  Critical values satisfy

  $$\begin{aligned} 2x - y - 3x^2 &= 0 \\ -2y - x &= 0 \end{aligned}$$

  There are two solutions to this equation: (0,0) and $\left(\frac{5}{6}, -\frac{5}{12}\right)$. At the first critical point the Hessian is

  $$\begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}$$

  The eigenvalues of this matrix are $\pm\sqrt{3}$. This matrix is indefinite, so the second derivative test is inconclusive (this turns out to be a saddle point).

At the second critical point the Hessian is

$$\begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$$

This matrix has eigenvalues $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$, which are both negative, so the matrix is negative definite. Therefore the critical point $\left(\frac{5}{6}, -\frac{5}{12}\right)$ is a local maximum.

8. Find all the candidate maxima/minima of the following functions subject to the given constraints (if using the Lagrange method, after findings all the candidate points, we can determine maxima/minima simply by comparing their values (if maxima/minima exist)).

- $f(x, y) = xy$ subject to $x^2 + y^2 = 2a^2$ $(a > 0)$
  Solution: The Lagrangian is

  $$L(x, y, \lambda) = xy - \lambda(x^2 + y^2 - 2a^2)$$

  The constraint function $g(x, y)$ has derivative

  $$g'(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

  This is rank one everywhere except at the origin. However, at the origin $x^2 + y^2 = 0 \neq 2a^2$, so this point does not satisfy the constraint, meaning we need not worry about the constraint qualification.

  The constraint is continuous and the set of feasible points is bounded (they are a circle of radius $\sqrt{2}a$), so by Weierstrass a maxima and minima exist and will be at a critical point of the Lagrangian.

  The FOC of the Lagrangian with respect to x and y are

  $$\begin{aligned} y - \lambda 2x &= 0 \\ x - \lambda 2y &= 0 \end{aligned}$$

  Combining the second with the first we see $y = 4\lambda^2 y$. This implies $y = 0$ or $4\lambda^2 = 1$, meaning $\lambda = \pm\frac{1}{2}$. We take each case in turn:

  - If $y = 0$, the second equation tells us $x = 0$, which again yields the origin (0,0).
  - If $\lambda = \frac{1}{2}$, then $y = x$, and the constraint tells us $x^2 = a^2$, so we get two candidates: (a, a) and $(-a, -a)$. For both of these points $f(x, y) = a^2 > 0$.
  - If $\lambda = -\frac{1}{2}$, then $y = -x$, and the constraint again tells us $x = \pm a$, so we get two candidates $(a, -a)$ and $(-a, a)$. For both of these points $f(x, y) = -a^2 < 0$.

  We have found all the candidate maximizers/minimizers. Therefore f is maximized at (a, a) and $(-a, -a)$ with a value of a^2, and f is minimized at $f(a, -a)$ and $f(-a, a)$ with a value of $-a^2$.

- $f(x, y) = 1/x + 1/y$ subject to $(1/x)^2 + (1/y)^2 = (1/a)^2$
  Solution: The Lagrangian is

  $$L(x, y, \lambda) = \frac{1}{x} + \frac{1}{y} - \lambda\left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{a^2}\right)$$

The constraint function $g(x, y)$ has derivative

$$g'(x, y) = \begin{pmatrix} -\frac{2}{x^3} & -\frac{2}{y^3} \end{pmatrix}$$

This is rank one everywhere with $x, y \neq 0$. However, since the objective is not defined at points with $x = 0$ or $y = 0$, we need not worry about such points.

The FOC of the Lagrangian with respect to x and y are:

$$-\frac{1}{x^2} + \lambda\frac{2}{x^3} = 0$$
$$-\frac{1}{y^2} + \lambda\frac{2}{y^3} = 0$$

Solving gives $x = y = 2\lambda$. From the budget constraint we see $x = \pm\sqrt{2}a$. Thus our candidate extreme points are $f(\sqrt{2}a, \sqrt{2}a) = \frac{\sqrt{2}}{a}$ and $f(-\sqrt{2}a, -\sqrt{2}a) = -\frac{\sqrt{2}}{a}$. Assuming $a > 0$, the first is our maximum and the second our minimum.

(Note: it isn't immediately obvious that a maximum and minimum is guaranteed to exist. However, making the change of variables $X = 1/x$ and $Y = 1/y$, we quickly see that we are maximizing $X + Y$ over a circle of radius $1/a$, which is indeed bounded).

- $f(x, y) = x + y$ subject to $xy = 16$

  Solution: We can quickly see that this system is unbounded. Substituting the constraint we see the function is $f(x) = x + 16/x$, which is unbounded both above and below. Thus there are no global maxima and minima, although there are two local minima, at $x = \pm 4$. If you run through the Lagrange multiplier argument, it will identify these local minima.

9. Let x be a vector of size n and A a real symmetric matrix of size n. Solve the program

$$\max_x x'Ax \text{ subject to } x'x = 1$$

Solution: We can write this as a Lagrangian:

$$L(x, \lambda) = x'Ax - \lambda(x'x - 1)$$

Differentiating with respect to (the vector) x we find the FOC:

$$2x'A - 2\lambda x' = 0$$

Transposing and solving gives

$$Ax = \lambda x$$

Thus the solution must be an eigenvector of A. For any eigenvector x we see $x'Ax = x'\lambda x = \lambda$ since $x'x = 1$. Thus to maximize $x'Ax$ we need to take x to be the eigenvector associated with the greatest eigenvalue of A.

(Note the constraint qualification always holds since the second derivative of the constraint is the identity matrix.)

10. Let y be a vector of $\mathbb{R}^n$ and X an $n \times k$ matrix with rank k (so $n \geq k$). Solve the program

$$\min_{b \in \mathbb{R}^k} \|y - Xb\|$$

This is nothing but the Ordinary Least Squares estimator you will do in econometrics.

Solution: Minimizing the norm is equivalent to minimizing the square of the norm. That is:

$$\min_b (y - Xb)'(y - Xb) = \min_b y'y - 2y'Xb + b'X'Xb$$

Define $f(b) = y'y - 2y'Xb + b'X'Xb$. We first note:

$$\begin{aligned} f'(b) &= -2y'X + 2b'X'X \\ f''(b) &= 2X'X \end{aligned}$$

(Remember the second derivative is the derivative of the gradient of f). Since $X'X$ is positive definite, this function is convex, so the minimum occurs at a critical point of f. That is, when:

$$-2y'X + 2b'X'X = 0$$

Rearranging, we see

$$b = (X'X)^{-1}X'y,$$

which is also known as the Ordinary Least Squares estimator.

11. Consider a consumer with utility function $u(x_1, ..., x_n) = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$ where $\sum_{i=1}^n \alpha_i = 1$. Each good x_i costs p_i per unit, and the consumer has a budget of M dollars. Find the consumer's optimal mix of consumption. That is, solve the program:

$$\max_{x_1, \ldots, x_n} u(x) \text{ subject to } p \cdot x = M$$

where $p \cdot x = \sum p_i x_i$

Solution: While it's not immediately obvious, this utility function is strictly concave so long as $\alpha_i > 0$ and subject to a linear budget constraint, so a critical point of the Lagrangian will be a unique maximal point.

Let $y = x_1^{\alpha_1} ... x_n^{\alpha_n}$. The first-order conditions of the Lagrangian with respect to x_i are:

$$\alpha_i y / x_i = \lambda p_i$$

Taking the ratio for any two goods $i \neq j$ gives

$$x_i = \frac{\alpha_i}{\alpha_j} \frac{p_j}{p_i} x_j$$

Fix $j = 1$. From the budget constraint we have

$$\sum_{i=1}^n p_i \frac{\alpha_i}{\alpha_1} \frac{p_1}{p_i} x_1 = M$$

That is:

$$x_1 = \frac{\alpha_1}{p_1} M$$

Thus in general,

$$x_i = \frac{\alpha_i}{p_i} M$$

The total utility is

$$\begin{aligned} x_1^{\alpha_1} ... x_n^{\alpha_n} &= \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i} M \right)^{\alpha_i} \\ &= M \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i} \end{aligned}$$