Columbia MA Math Camp

Real Analysis

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Motivation

- In real analysis (and also in micro) we often need to use concepts related to limits and convergence
- By saying that a sequence of objects converges to a limiting object, we mean, roughly speaking, that the sequence will get "as close as we want" to the limit.
- To be able to talk about how close 2 objects are, we need the concept of distance.
- Metric spaces are the general framework that capture the concept of distance, but we will focus on Euclidean metric spaces
    - Pretty much the only thing economists work with
    - Easier to visualize

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Definition 1.1
Let X be a set, and $d : X \times X \to \mathbb{R}$ a function. We call d a **metric** *on X if:*

- **Positive Definiteness :** $d(x, y) \geq 0$ *for all* $x, y \in X$, *and* $d(x, y) = 0$ *iff* $x = y$
- **Symmetry :** $d(x, y) = d(y, x)$
- **Triangle Inequality :** $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$

A **metric space** (X, d) *is a set X with a metric d defined on X*

Example : A trivial example is the discrete metric :

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Does this satisfy the properties of a metric? We will however focus on Euclidean metric spaces.

Discrete Metric is actually a metric

Discrete metric : $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

Let us check that the discrete metric satisfies the 3 properties of a metric :

(a) By definition $d(x, y) = 0$ iff $x = y$

(b) Symmetric is also trivial

(c) Take $x, y, z \in X$. If $x = y$, then $d(x, y) = 0 \leq d(x, z) + d(z, y)$. If $x \neq y$, then we must have $d(x, z) = 1$ OR $d(z, y) = 1$. In either case we have :

$$d(x, y) = 1 \leq d(x, z) + d(z, y)$$

Definition 1.2
In $\mathbb{R}^n$*, the* **Euclidean distance** *is the function* $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$*:*

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

Euclidean distance satisfies the 3 properties that we mentioned earlier:

- $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

Let us prove the first two properties. Triangle Inequality is hard to show - requires the Cauchy Schwarz inequality.

Euclidean metric proof

Fact 1
Prove that the Euclidean metric satisfies the first two properties of a metric

Proof.
(a) WTS that $d(x, y) = 0$ iff $x = y$ for any $x, y \in \mathbb{R}^n$.

$(\implies)$: If $d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0$, this implies $\sum_{i=1}^{n} (x_i - y_i)^2 = 0$. Since $(x_i - y_i)^2 \geq 0$ for each i which means that $(x_i - y_i)^2 = 0 \ \forall i$ and thus $x_i = y_i \ \forall i$. Thus $x = y$

$(\impliedby)$: If $x = y$, we have $x_i = y_i$ for each i. Therefore

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0$$

(b) WTS that $d(x, y) = d(y, x)$. This follows from the fact that :

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$$

Definition 1.3
A subset S of $\mathbb{R}^n$ is **bounded** *if there exists $M \in \mathbb{R}$ such that for all $x \in S$, $d(0, x) \leq M$.*

Note: We could have chosen any $a \in \mathbb{R}^n$ in the place of 0. Suppose S is bounded with respect to 0. Then for any $x \in S$ the triangle inequality tells us

$$d(a, x) \leq d(a, 0) + d(0, x) \leq d(a, 0) + M$$

Thus S is bounded with respect to a as well.

Definition 1.4
Let $S \subseteq \mathbb{R}$. A number $M \in \mathbb{R}$ is an **upper bound** *of S if $s \leq M$ for every $s \in S$.*

If no $M' < M$ is an upper bound of S, then M is called the **least upper bound** *or* **supremum** *of S.*

We make one important assumption about the real numbers: every bounded set of real numbers has a least upper bound. This is called the **least upper bound property**.

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Sequences: Definition

Formally, a **sequence** in a set X is a function from $\mathbb{N}$ to X . We denote x_n the image of n, and (x_n) the sequence.

Less formally, a sequence is an **ordered collection** of elements $(x_0, x_1, x_2, ...)$. Many problems in math boil down to understanding the long-term behavior of some sequence.

We typically write sequences as a formula or by enumerating the first few terms.

- $(x_n) = (n)_{n=0}^{\infty}$: $(0, 1, 2, 3, \ldots)$
- $(x_n) = (1)_{n=0}^{\infty}$: $(1, 1, 1, 1, \ldots)$
- $(x_n) = \left(\frac{1+(-1)^n}{2}\right)_{n=0}^{\infty}$: $(1, 0, 1, 0, \ldots)$
- $(x_n) = \left(\frac{1}{n+1}\right)_{n=0}^{\infty}$: $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$

You'll also see people write things like $x_n = n$.

Bounded Sequences

In $\mathbb{R}^n$ we can define properties of sequences that rely on the notion of distance.

Definition 2.1
A sequence (x_n) of $S \subseteq \mathbb{R}^n$ is **bounded** *iff $\{x_0, ..., x_n, ...\}$ is a bounded subset of $\mathbb{R}^n$.*

Are the sequences $x_n = n$ and $x_n = 1/n$ bounded or unbounded?

- The sequence $x_n = n$ is not bounded. For any $M \in \mathbb{R}$, $d(x_n, 0) > M$ for $n > M$.
- The sequence $x_n = \frac{1}{n}$ is bounded: $d(x_n, 0) \leq 1$ for all n

Definition 2.2
A sequence (x_n) of $S \subseteq \mathbb{R}^n$ **converges to a limit** *$\ell \in S$ iff for all $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that for all $n \geq N(\epsilon)$*

$$d(x_n, \ell) < \epsilon$$

We write $x_n \to \ell$ or $\lim_{n\to\infty} x_n = \ell$. If (x_n) does not converge, we say it **diverges**.

This definition is important, so let's unpack it a little:

- The sequence must eventually **get and remain** arbitrarily close to ℓ
- N can be different for each ϵ.
- We require $\ell \in S$. Consider $(x_n) = \left(1, \frac{1}{2}, \frac{1}{3}, ...\right)$. What happens if $S = \mathbb{R}$? If $S = (0, 2)$?

Proposition 2.1
If a sequence (x_n) converges, then (x_n) is bounded.

Proof.
Assume $x_n \to \ell$.

- Take $\epsilon = 1$
- Since $x_n \to \ell$, by definition there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, \ell) < 1$.

This proves that the sequence starting at N is bounded, but we need to deal with the first $N - 1$ terms.

- Define $M \equiv \max\{d(x_1, \ell), ..., d(x_{N-1}, \ell), 1\}$
- For all n, $d(x_n, l) \leq M$

$\square$

Limit (if exists) must be unique

Proposition 2.2
The limit of a sequence (x_n) *is unique provided it exists i.e. if* $x_n \to x$ *and* $x_n \to x'$*, then* $x = x'$

Proof.
We prove this by contradiction. Suppose $x \neq x'$. Thus it must be that $d(x, x') > 0$.

Consider $\varepsilon = \frac{d\left(x, x'\right)}{2}$.

Because $x_n \to x$, there exists N such that $d(x_n, x) < \varepsilon$ for any $n > N$. Similarly, because $x_n \to x'$, there exists N' such that $d(x_n, x') < \varepsilon$ for any $n > N'$. Take $\hat{n} = \max\{N, N'\} + 1$ so that $\hat{n} > N$ and $\hat{n} > N'$. There $d(x_{\hat{n}}, x) < \varepsilon$ and $d(x_{\hat{n}}, x') < \varepsilon$. Thus it must be that :

$$d(x, x_{\hat{n}}) + d\left(x_{\hat{n}}, x'\right) < 2\varepsilon = d\left(x, x'\right)$$

which contradicts the triangle inequality of d. Thus we have reached a logical contradiction and therefore $x = x'$. $\square$

Proposition 2.3
In the Euclidean metric space $(\mathbb{R}, d)$*, suppose there are 2 convergent sequences* $x_n \to x$ *and* $y_n \to y$*. If* $x_n \leq y_n \; \forall n$*, then prove that* $x \leq y$

Proof.
We do this by contradiction. Suppose $x > y$ and set $\varepsilon = \frac{x-y}{2}$. Since $x_n \to x$ and $y_n \to y$, then by definition, $\exists N_x$ and N_y such that $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ $\forall n > N_x$ and $n > N_y$. Take $\hat{n} > \max\{N_x, N_y\}$. Then we must have $|x_{\hat{n}} - x| < \varepsilon$ and $|y_{\hat{n}} - y| < \varepsilon$. Since $\varepsilon = \frac{x-y}{2}$, this implies,$x - \varepsilon = y + \varepsilon$

$$x_{\hat{n}} > x - \varepsilon = y + \varepsilon > y_{\hat{n}}$$

which contradicts $x_n \leq y_n \; \forall n$ $\square$

Convergent sequences: results (cont.)

Here is a result you'll show on your problem set:

Proposition 2.4

A sequence $(x^k) = (x_1^k, ..., x_n^k)$ of $\mathbb{R}^n$ converges to a limit x iff each component converges to the corresponding component of x in $\mathbb{R}$.

The result boils down to the fact that for all $j \in \{1, ..., n\}$:

$$|x_j - x_j^k| \leq \left(\sum_{i=1}^{n} (x_j - x_j^i)^2 \right)^{\frac{1}{2}} \leq n \max_i |x_i - x_i^k|$$

In general, working with the definition of convergence is cumbersome. There are some important results we'll use frequently

Proposition 2.5
Let (x_n) and (y_n) be sequences of $\mathbb{R}$. If $(x_n) \to x$ and $(y_n) \to y$:

(a) $x_n + y_n \to x + y$

(b) $x_n y_n \to xy$

(c) $1/x_n \to 1/x$ *if* $x \neq 0$

Proof.
We'll show the second one and the rest will probably be on your problem set. $\square$

Proposition 2.6
Let (x_n) and (y_n) be sequences of $\mathbb{R}$. If $(x_n) \to x$ and $(y_n) \to y$ then $x_n y_n \to xy$

Proof.
Take any $\varepsilon > 0$. I want to find N such that $|x_n y_n - xy| < \varepsilon$ for any $n > N$.

Because (y_n) is convergent, it is bounded i.e. there exists M such that $|y_n| < M \; \forall n$. Because $x_n \to x$, there exists N_x s.t $|x_n - x| < \frac{\varepsilon}{2M}$. Again since $y_n \to y$, there exists N_y such that $|y_n - y| < \frac{\varepsilon}{2(|x|+1)}$.

Let $N = \max\{N_x, N_y\}$ and I claim this is the N we need to find. This is because for any $n > N$,

$$\begin{aligned}
|x_n y_n - xy| &= |(x_n - x)y_n + (y_n - y)\,x| \\
&\leq |x_n - x|\,|y_n| + |y_n - y|\,|x| \\
&< \frac{\varepsilon}{2M}.M + \frac{\varepsilon}{2\,(|x|+1)}.|x| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

$\square$

Subsequences

Definition 2.3
Let (x_n) be a sequence. A **subsequence** *of (x_n) is a sequence (x_{n_k}) where $n_1 < n_2 < \ldots$ is an increasing sequence of indices.*

- $(x_2, x_3, x_5, \ldots)$ is a subsequence of (x_n)
- $(x_4, x_3, x_2, \ldots)$ is not (the terms are out of order).

Proposition 2.7
A sequence (x_n) converges to a limit ℓ iff all its subsequences converge to the same limit ℓ.

Proof.
$(\Rightarrow)$ Assume $(x_n) \to \ell$ and consider a subsequence (x_{n_k}) of (x_n). Fix $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, \ell) < \epsilon$. Take K such that $n_K \geq N$. Then for all $k \geq K$, $n_k \geq n_K \geq N$, so $d(x_{n_k}, \ell) < \epsilon$.

$(\Leftarrow)$ Since (x_n) is a subsequence of itself, this implication is immediate. $\square$

Proposition 2.8

Bolzano-Weierstrass Theorem *Every bounded sequence of real numbers has a convergent subsequence.*

Proof.
To do this, we will first prove :

- **Monotone Convergence Theorem :** Every increasing (decreasing) and bounded from above (below) real sequence is convergent in $(\mathbb{R},)$
- Every sequence in $\mathbb{R}$ has a monotone subsequence.

Combining these 2 results, we get that the Bolzano-Weierstrass Theorem holds. $\square$

Proposition 2.9
Monotone Convergence Theorem *Every increasing (decreasing) and bounded from above (below) real sequence is convergent in* $(\mathbb{R}, d)$

Proof.
Take any increasing and bounded real sequence (x_n). By the least upper bound property of $\mathbb{R}$, it has a l.u.b.

Let $x := \sup x_1, x_2, \ldots$. We want to show $x_n \to x$. Take any $\varepsilon > 0$. Want to find N s.t. $d(x_n, x) < \varepsilon \; \forall n > N$. Since x is the lub $\implies$ $x - \varepsilon$ is not an upper bound and therefore $\exists$ N s.t. $x_N > x - \varepsilon$. Therefore $\forall n > N$, it must be that :

$$x \geq x_n \geq x_N > x - \varepsilon$$

and therefore $|x_n - x| < \varepsilon$

The proof for a decreasing and bounded from below sequence (x_n) is completed by applying the proof above to $(-x_n)$. $\square$

Proposition 2.10
Every sequence in $\mathbb{R}$ has a monotone subsequence

Proof.
Take any sequence (x_n) in $\mathbb{R}$. Call the term x_n a dominant term if $x_n > x_m$ for all $m \geq n$.

- **Case 1 :** (x_n) has infinitely many dominant terms. Then these dominant terms constitute a decreasing sequence
- **Case 2 :** (x_n) has finitely many dominant terms. Let x_N be the last dominant term. Let $n_1 = N + 1$. So x_{n_1} is not a dominant term. This means there exists $n_2 > n_1$ s.t. $x_{n_2} > x_{n_1}$. Again x_{n_2} is not dominant which implies there exists n_3 st $x_{n_3} > x_{n_2}$ and so on. We thus obtain a strictly increasing sequence
- **Case 3 :** (x_n) has no dominant terms. We can let $n_1 = 1$ and then we're back in the previous case.

$\square$

- When a sequence converges, its terms become closer and closer to the limit as n increases. As a by-product, the terms themselves become closer and closer together.
- The notion of a Cauchy sequence weakens the requirement of convergence by only requiring that the terms get closer together
    - Without necessarily converging to a limit.

Cauchy Sequences (contd)

- How do we actually know when a sequence converges? Do we have to calculate the limit?

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

- Turns out the right notion is that of **Cauchy sequences**
- In many situations, we will show that Cauchy sequences are the same as convergent sequences which will help us show that a sequence is convergent without specifying the limit

Cauchy Sequences (cont.)

Definition 2.4
A sequence (x_n) *is* **Cauchy** *iff:*

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N} \textit{ such that } \forall n, m \geq N(\epsilon), d(x_n, x_m) < \epsilon$$

- Not enough for subsequent terms to be close: **all** terms after a certain point must be close.
- Let's show (a_n) on the previous slide is Cauchy. Fix $\epsilon > 0$, and take $N > 1/\epsilon$:

$$\begin{aligned} d(x_n, x_m) = \sum_{k=n+1}^{m} \frac{1}{k^2} \quad &< \quad \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\ &= \quad \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k} \\ &= \quad \frac{1}{n} - \frac{1}{m} < \epsilon \end{aligned}$$

Cauchy sequences are bounded

Proposition 2.11
If (x_n) is Cauchy, it is bounded

Proof.
As in a proof before, let us take $\varepsilon = 1$. Since (x_n) is Cauchy, there exists N such that $d(x_n, x_m) < 1$ for all $n, m \geq N$. In particular, $d(x_n, x_N) < 1$, so $\{x_n | n \geq N\}$ is bounded.

Since $\{x_n | n < N\}$ is also bounded, the whole sequence is bounded. □

Question: Is the reverse implication true? Counter-example?

Convergence of a subsequence implies convergence

Proposition 2.12
If a subsequence of a Cauchy sequence converges to ℓ, the sequence itself converges to ℓ.

Proof.
Let (x_n) be Cauchy, and suppose $x_{n_k} \to \ell$. Fix $\epsilon > 0$. Since (x_n) is Cauchy, there exists an N such that

$$d(x_n, x_m) < \epsilon/2, \forall n, m \geq N$$

Since $(x_{n_k}) \to \ell$, there exists K such that

$$d(x_{n_k}, \ell) < \epsilon/2, \forall k \geq K$$

Take k such that $N_k \geq \max(N, n_K)$. Then for all $n \geq N_k$:

$$d(x_n, \ell) < d(x_n, x_{n_k}) + d(x_{n_k}, \ell) < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus $x_n \to \ell$.

$\square$

Do Cauchy sequences converge?

Theorem 2.1
In $\mathbb{R}^n$ a sequence converges iff it is Cauchy.

Proof.
($\Rightarrow$) Suppose $x_n \to \ell$. Let $\epsilon > 0$. There exists N such that for all $n \geq N$, $d(x_n, l) < \epsilon/2$. Let $n, m \geq N$. By the triangle inequality,

$$d(x_n, x_m) \leq d(x_n, l) + d(l, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$$

($\Leftarrow$) Since Cauchy sequences are bounded, by Prop. **2.8**, (x_n) has a convergent subsequence. From the result above, (x_n) converges. $\square$

Definition 2.5
Spaces in which Cauchy sequences converge are called **complete metric spaces**

Can you give an example of a Cauchy sequence which does not converge? (**Hint:** Impossible in $\mathbb{R}$.)

Theorem 2.1
In $\mathbb{R}^n$ a sequence converges iff it is Cauchy.

Proof.
($\Rightarrow$) Suppose $x_n \to \ell$. Let $\epsilon > 0$. There exists N such that for all $n \geq N$, $d(x_n, l) < \epsilon/2$. Let $n, m \geq N$. By the triangle inequality,

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Definition 2.5
Spaces in which Cauchy sequences converge are called **complete metric spaces**

Can you give an example of a Cauchy sequence which does not converge? (**Hint:** Impossible in $\mathbb{R}$.)

Consider the rational numbers $\mathbb{Q}$ and the sequence given by :

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

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Open and Closed Sets

- Most of economics deals with sets: budget sets, indifference curves, production sets, the set of unbiased estimators, etc. It's useful to have well-defined language to describe the structure of sets.

- In addition, many common concepts and results (limits, continuity, existence of maxima, etc.) rely on the notion of open and closed sets.

- As a silly example consider the sets $S_1 = [0, 1]$ and $S_2 = (0, 1)$. The points 0 and 1 seem special: they are on the "edge" of both sets. The difference between S_1 and S_2 is whether they contain these points on the edge (turns out to be an important distinction!)

Definition 3.1

In $\mathbb{R}^n$, an **open ball** *with center a and radius $r > 0$ is the subset of all points at a distance less than r from a:*

$$B(a, r) = \{x \in \mathbb{R}^n | d(x, a) < r\}$$

Note: Even though I'm writing $d(x, a)$, I mean the Euclidean metric.

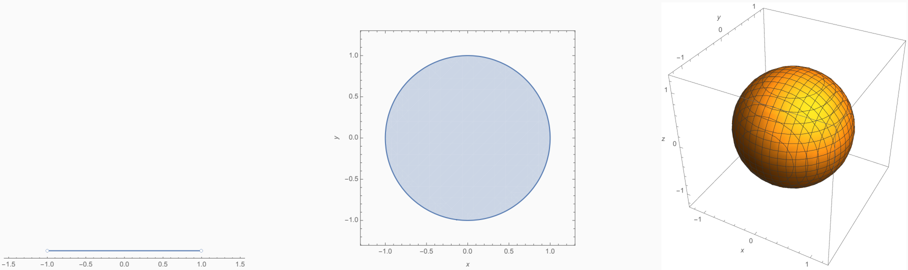

Figure 1: Open balls in $\mathbb{R}, \mathbb{R}^2$, and $\mathbb{R}^3$

Definition 3.2
Let S be a subset of $\mathbb{R}^n$

- *A point $a \in \mathbb{R}^n$ is an* **interior point** *of S if $\exists r > 0$ such that $B(a, r) \subseteq S$.*
- *The* **interior** *of S, noted $int(S)$ is the set of all its interior point.*
- *A subset S is an* **open set** *if all its points are interior points of S, i.e. if $int(S) = S$.*

- Because an interior point of S always belongs to S, $int(S) \subseteq S$.
- To show a set is open, take an arbitrary point $s \in S$ and find a radius such that $B(s, r) \subseteq S$.

Proposition 3.1
Any open ball is an open set.

Proof.
Terminology is not a proof!

Consider an open ball $B(a, r)$. We will show every point is interior.

- Let $x \in B(a, r)$. We need to find a radius $R > 0$ such that $B(x, R) \subseteq B(a, r)$
- Define $R = r - d(a, x) > 0$
- Using the triangle inequality, for any $y \in B(x, R)$:

$$d(y, a) \leq d(y, x) + d(x, a) < r.$$

$\square$

Proposition 3.2

- *Any union (possibly infinite) of open sets is an open set.*
- *Any finite intersection of open sets is an open set.*

Proof.
We will show the first result.

- Let $\{S_i\}$ be a family of open sets and $S \equiv \cup_i S_i$. Need to show $S = int(S)$.
- For any $x \in S$, $x \in S_i$ for some i.
- Since S_i is open, there exists $r > 0$ such that $B(x, r) \subseteq S_i$.
- But $S_i \subseteq S$, so $B(x, r) \subseteq S$, so $x \in int(S)$.

□

Definition 3.3
Let S be a subset of $\mathbb{R}^n$.

- *A point $a \in \mathbb{R}^n$ is a* **closure point** *of S iff for any radius $r > 0$, the open ball $B(a, r)$ around a contains some point of S. In other words, for all $r > 0$, $B(a, r) \cap S \neq \emptyset$*
- *The* **closure** *of a subset S, $cl(S)$, is the set of all its closure points.*
- *A subset S is a* **closed set** *iff $cl(S) = S$.*

A point that belongs to S is always a closure point of S, so $S \subseteq cl(S)$.

Proposition 3.3

- *The complement of* $int(S)$ *is the closure of* S^c : $(int(S))^c = cl(S^c)$.
- *The complement of* $cl(S)$ *is the interior of* S^c : $(cl(S))^c = int(S^c)$.

Proof.
We'll show the first result. The definition of $int(S)$ is:

$$a \in int(S) \Leftrightarrow \exists r > 0 : B(a, r) \subseteq S$$

Take the negation of each side of the equivalence:

$$a \in (int(S))^c \Leftrightarrow \forall r > 0 : B(a, r) \cap S^c \neq \emptyset$$

But the right-hand side is precisely the definition of belonging to $cl(S^c)$!

$\square$

Links between open and closed sets (cont.)

An important theorem follows directly from the previous result

Theorem 3.1
A set S is open (closed) iff its complement is closed (open).

Proof.
Applying the previous result, we see

$$\begin{aligned} S \text{ is open} \quad &\Leftrightarrow \quad int(S) = S \text{ (defn)} \\ &\Leftrightarrow \quad cl(S^c) = S^c \text{ (previous result)} \\ &\Leftrightarrow \quad S^c \text{ is closed (defn)} \end{aligned}$$

A similar proof shows S is closed iff S^c is open. $\square$

Proposition 3.4

- *Any intersection (possibly infinite) of closed sets is a closed set.*
- *Any finite union of closed sets is a closed set.*

Proof.
We'll show the first result:

- For a collection of closed sets $\{S_i\}$, $\cap_i S_i = (\cup_i S^c)^c$ by Morgan's laws.
- Since S_i is closed, S_i^c is open.
- By Proposition 3.2, $\cup_i S_i^c$ is open.
- Therefore $(\cup_i S_i^c)^c$ is closed, finishing the proof.

$\square$

Proposition 3.5
Let $S \subseteq \mathbb{R}^n$. A point x is in $cl(S)$ iff there exists a sequence of elements of S that converges (in $\mathbb{R}^n$) to x.

Proof.
($\Rightarrow$) Assume $x \in cl(S)$. We will construct a sequence of S that converges to x.

- For each $n \in N$ take $x_n \in S \cap B(x, 1/n) \neq \emptyset$ (we can do this because x is a closure point of S)
- To check convergence, fix $\epsilon > 0$. For any $n \geq 1/\epsilon$, $d(x_n, x) < 1/n \leq \epsilon$.

($\Leftarrow$) Assume there exists a sequence x_n of S converging to x.

- Let $r > 0$. We need to show $B(x, r) \cap S \neq \emptyset$.
- Since $x_n \to x$, there exists an N such that for all $n \geq N$, $x_n \in B(x, r)$. Since $x_n \in S$, we see $B(x, r) \cap S \neq \emptyset$

$\square$

The previous result is quite useful since it gives us another approach for determining whether a set is closed.

Corollary 3.1
A set S is closed iff the limits of all convergent sequences of S belong to S.

Example: Show the set $S = [a, \infty)$ is closed.

- Approach: Take a convergent sequence of S, $x_n \to x$. If $x \in S$, then S is closed.
- Since $x_n \geq a$ for all n, from our previous results we know $x \geq a$, so $x \in S$.

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Definition 4.1
A set $S \subseteq \mathbb{R}^n$ is **compact** *iff every sequence of S has a convergent subsequence (in S).*

Examples:

- $\mathbb{R}$?
- $[0, 1]$?
- $(0, 1]$?

Proposition 4.1
If S is a compact subset of $\mathbb{R}^n$, S is bounded.

Proof.
We show the contrapositive. Assume S is not bounded. For each n, let $x_n \in B(0, n)^c$. This sequence has no bounded subsequence, so it has no convergent subsequence. $\square$

Proposition 4.2
If S is a compact subset of $\mathbb{R}^n$, S is closed

Proof.
Let $x \in cl(S)$; we want to show $x \in S$.

- Since $x \in cl(S)$, it is the limit of a sequence $(x_n) \in S^{\mathbb{N}}$.
- Since S is compact, (x_n) has a subsequence (x_{n_k}) that converges to a limit $l \in S$.
- But (x_{n_k}) is a subsequence of the converging sequence (x_n), so l is necessarily equal to x

Thus $x = l \in S$, so S is closed. $\square$

The two previous results have an important implication:

Corollary 4.1
Let S be a subset of $\mathbb{R}$. If S is compact, it has a maximal element.

Proof.

- Since S is compact, it is bounded, meaning it has a least upper bound s.
- By way of contradiction, suppose $s \notin S$. Then $s \in S^c$.
- Since S is closed, S^c is open, so there exists some $\epsilon > 0$ such that $B(s, \epsilon) \subseteq S^c$.
- This implies $s - \epsilon$ is an upper bound for S, contradicting the fact that s is the *least* upper bound.

Therefore we must have $s \in S$. $\square$

How can we tell whether a set is compact?

We've seen that closed and bounded are necessary conditions. Are they sufficient as well?

Theorem 4.1
(Heine-Borel) Let $S \subseteq \mathbb{R}^n$. Then S is compact iff it is closed and bounded.

Proof.
For simplicity we'll show this in $\mathbb{R}$; the proof extends easily.

Assume S is closed and bounded, and let (x_n) be a sequence of S. By Proposition 2.8, (x_n) has a convergent subsequent, $x_{n_k} \to \ell$. Since S is closed, $\ell \in S$. Therefore S is compact. $\square$

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- A sequence (x_n) converges to ℓ if as n gets closer and closer to ∞, x_n gets closer and closer to ℓ.
- Analogously, we say a function f converges to a limit ℓ at a point x_0 if as x gets closer and closer to x_0, $f(x)$ gets closer and closer to ℓ.

Definition 5.1
Let $D \subseteq \mathbb{R}^n$ and let $f : D \to \mathbb{R}^m$ with $x_0 \in cl(D)$. We say $\lim_{x \to x_0} f(x) = \ell$ ("the limit as x approaches x_0 of $f(x)$ is ℓ") if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < d(x, x_0) < \delta \Rightarrow d(f(x), \ell) < \epsilon$$

Example: $f : \mathbb{R} \to \mathbb{R}, f(x) = 3x$. What is $\lim_{x \to 0} f(x)$?

Limit: Discussion

- Another way to put the ϵ, δ criteria is:

$$f(B(x_0, \delta) \setminus \{x_0\}) \subseteq B(\ell, \epsilon)$$

  (Be careful: there are two distances in this equation!)

- What happens when $x = x_0$ is not relevant. Consider $f : [0,1] \to \mathbb{R}$:

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

  What is $\lim_{x \to 1} f(x)$?

- We require $x_0 \in cl(D)$. For a function defined on $(0,1)$ it makes sense to talk about its limit as x approaches 1. What about its limit as x approaches 2?

Continuous functions are functions that preserve limits. Remember this!

Definition 5.2
Let $x_0 \in D$. *Then* f *is* **continuous at** x_0 *iff* $\lim_{x \to x_0} f(x) = f(x_0)$. *That is, for all* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$x \in B(x_0, \delta) \Rightarrow f(x) \in B(f(x_0), \epsilon)$$

We say f *is* **continuous** *if it is continuous at all points of its domain.*

Most functions that we use in economics are continuous:

- Polynomials (x, x^2, etc.)
- Exponential functions ($2^x, e^x$, etc.)
- The log function is continuous for $x > 0$
- Addition, multiplication, and composition of continuous functions
- Division of continuous functions, so long as the denominator $\neq 0$

Sequential characterization of continuity

Proposition 5.1
A function f is continuous at x_0 iff for all $(x_n) \to x_0$, $(f(x_n)) \to f(x_0)$.

Proof.
$(\Rightarrow)$. Suppose f is continuous at x_0, and let (x_n) be a sequence of D that converges to x_0. We want to show $f(x_n) \to f(x_0)$.

- Fix $\epsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that $d(f(x_0), f(x_n)) < \epsilon$ whenever $x \in B(x_0, \delta)$.
- Since $x_n \to x_0$, there exists N such that $d(x_n, x_0) < \delta$ for all $n \geq N$.
- This implies $d(f(x_n), f(x_0)) < \epsilon$ for all $n \geq N$, so $f(x_n) \to f(x_0)$

$(\Leftarrow)$ Suppose f is not continuous at x_0. There exists ϵ such that for every $\delta > 0$, we can find $x \in B(x_0, \delta)$ with $f(x) \notin B(f(x_0), \epsilon)$. Construct (x_n) by taking $\delta = 1/n$ for $n = 1, 2, \ldots.$ We have $x_n \to x$, but $f(x_n) \not\to f(x_0)$. $\square$

Open/closed set characterization of continuity

Theorem 5.1

- *f is continuous iff the inverse image by f of any open set is open*
- *f is continuous iff the inverse image by f of any closed set is closed*

Proof.
($\Rightarrow$) Let $f : D \to \mathbb{R}^m$ be continuous, $V \subseteq \mathbb{R}^m$ open. WTS $f^{-1}(V)$ open

- Let $x \in f^{-1}(V)$; by definition, $f(x_0) \in V$
- Since V is open, there exists $\epsilon > 0$ such that $B(f(x_0), \epsilon) \subseteq V$.
- Since f is continuous, there exists a $\delta > 0$ such that $f(B(x_0, \delta)) \subseteq B(f(x_0), \epsilon) \subseteq V$.
- Therefore $B(x_0, \delta) \subseteq f^{-1}(V)$, so $f^{-1}(V)$ is open

($\Leftarrow$): Try it yourself. $\square$

Result for closed sets follows your problem set: $\left(f^{-1}(S)\right)^c = f^{-1}(S^c)$

Continuous functions preserve compact sets

- The inverse image of a closed set by a continuous function is a closed set.
- Is the image of a closed set by a continuous function a closed set?
- Is the image of a bounded set by a continuous function bounded?

Theorem 5.2
Let K be a compact subset of D and $f : D \to \mathbb{R}^m$ a continuous function. Then $f(K)$ is compact.

Proof.
Let (y_n) be a sequence of $f(K)$.

- By definition there exists $x_n \in K$ such that $y_n = f(x_n)$.
- Since K is compact, (x_n) has a convergent subsequence $x_{n_k} \to \ell \in K$.
- Since f is continuous, $y_{n_k} \equiv f(x_{n_k}) \to f(\ell) \in f(K)$.

Thus $f(K)$ has a convergent subsequence, so $f(K)$ is compact. $\square$

Weierstrass Theorem

A central application of the previous theorem is the existence of a maximizer for a continuous, real-valued function over a compact domain.

Theorem 5.3
(Weierstrass) Let $f : D \to \mathbb{R}$. If f is continuous and D is compact, then f attains a maximum and a minimum.

Proof.
By the previous theorem, $f(D)$ is a compact subset of $\mathbb{R}$. By Theorem 4.1 (Heine-Borel), $f(D)$ has a maximum element. $\square$

Proposition 5.2
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $u \in (f(a), f(b))$, then there exists a $c \in (a, b)$ such that $f(c) = u$.

- If $f(a) = f(b)$ the theorem says nothing. Assume $f(a) < f(b)$
- Let $f(a) < u < f(b)$ and define $S = \{x \in [a, b] | f(x) \leq u\}$.
- S is non-empty and bounded above, so $c \equiv \sup S$ exists
- By continuity, $f(c) \leq u$. Suppose $f(c) < u$.
- Let $T = f^{-1}((f(a), u))$. Since f is continuous, T is open. By assumption, $c \in T$.
- However, c is not an interior point of T, a contradiction. Therefore $f(c) = u$.

Fixed Points

Let $f : \mathbb{R}^n \to \mathbb{R}^n$. A point x is a **fixed point** of f if $f(x) = x$

- Common equation in econ (game theory, dynamic programming)
- General setup: can express roots in terms of fixed points

$$g(x) = 0 \Leftrightarrow \underbrace{g(x) + x}_{\equiv f(x)} = x$$

Question: how do we find fixed points? For a certain class of functions, we can compute them easily

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a **contraction** if there exists a $k \in [0, 1)$ such that for all $x, y \in \mathbb{R}^n$:

$$d(f(x), f(y)) \leq kd(x, y)$$

Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \alpha x + \beta$. Then:

$$\begin{aligned} d(f(x), f(y)) &= |\alpha x + \beta) - (\alpha y + \beta)| \\ &= |\alpha||x - y| \\ &= |\alpha| d(x, y) \end{aligned}$$

Thus f is contraction iff $|\alpha| < 1$.

Contractions are continuous

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction with modulus k and fix $x \in \mathbb{R}^n$; we will show f is continuous at x.
- Fix $\epsilon > 0$. By definition, for any $y \in B_{\epsilon/k}(x)$ we have

$$f(y) \in B_{\epsilon}(f(x))$$

- Thus f is continuous at x (we took $\delta = \epsilon/k$)
- (We didn't need the fact that $k < 1$. Any function satisfying $d(f(x), f(y)) < kd(x, y)$ for some $k \in \mathbb{R}_+$ is continuous. These are called **Lipschitz functions**.)

Theorem 5.4
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction. Then f has a unique fixed point, x^*. Moreover, given any x_0, the sequence $x_{n+1} = f(x_n)$ converges to x^*.

- Show x_n is Cauchy, and therefore converges to some x^*
- By continuity, $f(x_n) \to f(x^*)$. However, $f(x_n) = x_{n+1} \to x^*$. Thus x^* is a fixed point of f
- Show uniqueness, suppose x_1 and x_2 are fixed points. Then

  $$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq kd(x_1, x_2)$$

  Therefore $d(x_1, x_2) = 0$

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A brief recap of metric spaces

Let X be a set, and $d : X \times X \to \mathbb{R}$ a function. We call d a **metric** on X if:

- $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

These are properties which are more general than the Euclidean distance that we have seen

A brief recap of metric spaces (cont.)

- The definitions of results we've developed for $\mathbb{R}^n$ largely carry over to general metric spaces
- One special property of $\mathbb{R}$ we used was the LUB property. This was important for the proofs of:
  - A bounded sequence of $\mathbb{R}^n$ has a convergent subsequence
  - A sequence of $\mathbb{R}^n$ converges iff it is Cauchy, which we used to prove the Contraction Mapping Theorem
  - A compact subset of $\mathbb{R}$ has a maximal element
  - A set $S \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded
  - The intermediate value theorem
- These results don't hold in every metric space, but in some metric spaces similar results hold
  - The contraction mapping theorem holds in any complete metric space

Examples of other metric spaces?

- Let X denote the set of bounded functions from $[0,1]$ to $\mathbb{R}$.
- Define the function $d : X \times X \to \mathbb{R}$ as follows:

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

- Is d a distance? To show triangle inequality, let $f, g, h \in X$:

$$\begin{aligned}
d(f,g) &= \sup_{x \in [0,1]} |f(x) - g(x)| \\
&= \sup_{x \in [0,1]} |f(x) - h(x) + h(x) - g(x)| \\
&\leq \sup_{x \in [0,1]} (|f(x) - h(x)| + |h(x) - g(x)|) \\
&\leq \sup_{x \in [0,1]} |f(x) - h(x)| + \sup_{x \in [0,1]} |h(x) - g(x)| \\
&= d(f,h) + d(h,g)
\end{aligned}$$

Function spaces

- The metric on the previous slide is called the **sup metric**. There are other intuitive notions of distance (e.g. the "average distance" between f and g), but those are a little tricky to worth with (the $d(x, y) = 0 \Leftrightarrow x = y$ property breaks down)
- It turns out that spaces of bounded functions with the sup metric are complete (Cauchy sequences converge)
- Therefore the contraction mapping theorem holds for spaces of bounded functions! This is a powerful result for dynamic programming, which we'll return to later in the semester