Log Linearization Practice Problems: Solutions

Log-linearize the following equations (greek letters are fixed parameters):

1. $f(x)+g(y)=\alpha$ around (X^*,Y^*) satisfying the equation.

   Solution:

$$\begin{aligned}
\widehat{f(x)+g(y)} &= 0 \\
\frac{d(f(x)+g(y))}{f(x^*)+g(y^*)} &= 0 \\
\frac{f'(x^*)dx+g'(y^*)dy}{f(x^*)+g(y^*)} &= 0 \\
\frac{f'(x^*)x^*\hat{x}+g'(y^*)y^*\hat{y}}{f(x^*)+g(y^*)} &= 0
\end{aligned}$$

   We can simplify this by putting things in terms of elasticities:

$$\frac{f'(x^*)x^*}{f(x^*)}\frac{f(x^*)}{f(x^*)+g(y^*)}\hat{x}+\frac{g'(y^*)y^*}{g(y^*)}\frac{g(y^*)}{f(x^*)+g(y^*)}\hat{y} = 0$$

   Defining $s_x=\frac{f(x^*)}{f(x^*)+g(y^*)}$ we get the expression:

$$\epsilon_f(x^*)s_x\hat{x}+\epsilon_g(y^*)(1-s_x)\hat{y} = 0$$

2. $W=\theta C/(1-N)$ around (W^*,C^*,N^*) satisfying the equation.

   Solution:

$$\begin{aligned}
\widehat{W} &= \widehat{\left(\frac{\theta C}{1-N}\right)} \\
\widehat{W} &= \widehat{\theta}+\widehat{C}-\widehat{1-N} \\
\widehat{W} &= \widehat{C}-\frac{d(1-N)}{1-N^*} \\
\widehat{W} &= \widehat{C}+\frac{dN}{1-N^*} \\
\widehat{W} &= \widehat{C}+\frac{N^*}{1-N^*}\widehat{N}
\end{aligned}$$

3. $W=(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}$ around (W^*,K^*,N^*) satisfying the equation.

Solution:

$$\begin{aligned}
\widehat{W} &= \widehat{(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}} \\
\widehat{W} &= \widehat{1-\alpha} + \widehat{\frac{K}{N}^{\alpha}} \\
\widehat{W} &= \alpha\widehat{\left(\frac{K}{N}\right)} \\
\widehat{W} &= \alpha(\widehat{K} - \widehat{N})
\end{aligned}$$

4. $\left(\frac{A}{B}\right)^{-\alpha} = \beta C$ around (A^*, B^*, C^*) satisfying the equation.

Solution:

$$\begin{aligned}
\widehat{\left(\left(\frac{A}{B}\right)^{-\alpha}\right)} &= \widehat{(\beta C)} \\
-\alpha\widehat{\left(\frac{A}{B}\right)} &= \widehat{\beta} + \widehat{C} \\
-\alpha\left(\widehat{A} - \widehat{B}\right) &= \widehat{C}
\end{aligned}$$

5. $S = (1-\delta)K + K^{\alpha}N^{1-\alpha} - C$ around (S^*, K^*, N^*, C^*) satisfying the equation.

Solution: The right hand side is complicated here, so I will use the definition $\hat{x} = dx/x^*$ to simplify.

$$\begin{aligned}
\widehat{S} &= \widehat{\left((1-\delta)K + K^{\alpha}N^{1-\alpha} - C\right)} \\
\widehat{S} &= \frac{d\left((1-\delta)K + K^{\alpha}N^{1-\alpha} - C\right)}{(1-\delta)K^* + K^{*\alpha}N^{*1-\alpha} - C^*} \\
\widehat{S} &= \frac{(1-\delta)dK + d\left(K^{\alpha}N^{1-\alpha}\right) - dC}{S^*}
\end{aligned}$$

To save on notation, I have noted that $S^* = (1-\delta)K^* + K^{*\alpha}N^{*1-\alpha} - C^*$. The only challenge remaining is to calculate $d(K^{\alpha}N^{1-\alpha})$, for which we use the product rule $d(f(X)g(Y)) = f'(X^*)g(Y^*)dX + f(X^*)g'(Y^*)dY$:

$$\begin{aligned}
\widehat{S} &= \frac{(1-\delta)dK + \alpha K^{*\alpha-1}N^{*1-\alpha}dK + (1-\alpha)K^{*\alpha}N^{*-\alpha}dN - dC}{S^*} \\
\widehat{S} &= \frac{(1-\delta)K^*\widehat{K} + \alpha K^{*\alpha}N^{*1-\alpha}\widehat{K} + (1-\alpha)K^{*\alpha}N^{*1-\alpha}\widehat{N} - C^*\widehat{C}}{S^*} \\
\widehat{S} &= s_1\widehat{K} + \alpha s_2\widehat{K} + (1-\alpha)s_2\widehat{N} + (1-s_1-s_2)\widehat{C}
\end{aligned}$$

In the last line I have simplified by setting $s_1 = (1-\delta)K^*/S^*$ and $s_2 = K^{*\alpha}N^{*1-\alpha}/S^*$.

6. $Y = \sum_{i=1}^{N} \alpha_i X_i$ around $(Y^*, X_1^*, ..., X_N^*)$ satisfying the equation

Solution: Again I will use the identity $\widehat{x} = dx/x^*$

$$
\begin{aligned}
dY &= d\left(\sum_{i=1}^{N} \alpha_i X_i\right) \\
dY &= \sum_{i=1}^{N} \alpha_i dX_i \\
Y^*\widehat{Y} &= \sum_{i=1}^{N} \alpha_i X_i^* \widehat{X_i} \\
\widehat{Y} &= \sum_{i=1}^{N} \frac{\alpha_i X_i^*}{Y^*} \widehat{X_i} \\
\widehat{Y} &= \sum_{i=1}^{N} s_i \widehat{X}_i
\end{aligned}
$$

In the last line I have defined $s_i = \frac{\alpha_i X_i^*}{Y^*}$. Note $\sum_{i=1}^{N} s_i = 1$ (write out the definition of Y^* if this isn't clear), as is common in these kinds of linearizations.

7. $Y = \sum_{i=1}^{N} X_i^{\beta_i}$ around $(Y^*, X_1^*, ..., X_N^*)$ satisfying the equation.

Solution: Again, I will start by linearizing and then using the fact that $\widehat{x} = dx/d^*$.

$$
\begin{aligned}
dY &= d\left(\sum_{i=1}^{N} X_i^{\beta_i}\right) \\
Y^*\widehat{Y} &= \sum_{i=1}^{N} d\left(X_i^{\beta_i}\right) \\
Y^*\widehat{Y} &= \sum_{i=1}^{N} \beta_i X_i^{*\beta_i - 1} dX_i \\
Y^*\widehat{Y} &= \sum_{i=1}^{N} \beta_i X_i^{*\beta_i} \widehat{X_i} \\
\widehat{Y} &= \sum_{i=1}^{N} \beta_i \frac{X_i^{*\beta_i}}{Y_i^*} \widehat{X_i} \\
\widehat{Y} &= \sum_{i=1}^{N} \beta_i s_i \widehat{X_i}
\end{aligned}
$$

In the last line I have defined $s_i = X_i^{*\beta_i}/Y^*$. Note $\sum_{i=1}^{N} s_i = 1$.

8. $A = (1 + \alpha B^\epsilon)\left(\frac{C}{D}\right)^{-\beta}$ around (A^*, B^*, C^*, D^*) satisfying the equation.

Solution:

$$\begin{aligned}
\widehat{A} &= \widehat{\left((1+\alpha B^{\epsilon})\left(\tfrac{C}{D}\right)^{-\beta}\right)} \\
\widehat{A} &= \widehat{(1+\alpha B^{\epsilon})} + \widehat{\left(\left(\tfrac{C}{D}\right)^{-\beta}\right)} \\
\widehat{A} &= \frac{d\,(1+\alpha B^{\epsilon})}{1+\alpha B^{*\epsilon}} - \beta\widehat{\left(\tfrac{C}{D}\right)} \\
\widehat{A} &= \frac{\alpha\epsilon B^{*\epsilon-1}dB}{1+\alpha B^{*\epsilon}} - \beta\left(\widehat{C}-\widehat{D}\right) \\
\widehat{A} &= \frac{\alpha\epsilon B^{*\epsilon}\widehat{B}}{1+\alpha B^{*\epsilon}} - \beta\left(\widehat{C}-\widehat{D}\right) \\
\widehat{A} &= \epsilon s_B\widehat{B} - \beta\left(\widehat{C}-\widehat{D}\right)
\end{aligned}$$

In the last step I have defined $s_B = (\alpha B^{*\epsilon})/(1+\alpha B^{*\epsilon})$.

9. The system of equations:

$$\begin{aligned}
C_t^{-\sigma} &= \beta\left(1+\alpha K_{t+1}^{\alpha-1}-\delta\right)C_{t+1}^{-\sigma} \\
K_t^{\alpha} &= K_{t+1}-(1-\delta)K_t + C_t
\end{aligned}$$

around the steady state $K_t = K_{t+1} = K^*$, $C_t = C_{t+1} = C^*$.

Solution: Linearizing the first equation gives:

$$\begin{aligned}
\widehat{C_t^{-\sigma}} &= \widehat{\left(\beta\left(1+\alpha K_{t+1}^{\alpha-1}-\delta\right)C_{t+1}^{-\sigma}\right)} \\
-\sigma\widehat{C}_t &= \widehat{\beta} + \widehat{\left(1+\alpha K_{t+1}^{\alpha-1}-\delta\right)} - \sigma\widehat{C}_{t+1} \\
\sigma\left(\widehat{C}_{t+1}-\widehat{C}_t\right) &= \frac{d\left(1+\alpha K_{t+1}^{\alpha-1}-\delta\right)}{1+\alpha K^{*\alpha-1}-\delta} \\
\sigma\left(\widehat{C}_{t+1}-\widehat{C}_t\right) &= \frac{\alpha d\left(K_{t+1}^{\alpha-1}\right)}{1+\alpha K^{*\alpha-1}-\delta} \\
\sigma\left(\widehat{C}_{t+1}-\widehat{C}_t\right) &= \frac{\alpha(\alpha-1)K^{*\alpha-2}dK_{t+1}}{1+\alpha K^{*\alpha-1}-\delta} \\
\sigma\left(\widehat{C}_{t+1}-\widehat{C}_t\right) &= \frac{\alpha(\alpha-1)K^{*\alpha-1}\widehat{K}_{t+1}}{1+\alpha K^{*\alpha-1}-\delta} \\
\sigma\widehat{C}_{t+1}-\gamma\widehat{K}_{t+1} &= \sigma\widehat{C}_t
\end{aligned}$$

Where γ is the constant in front of $\widehat{K}_{t+1}$ in the second to last equation. Linearizing the second equation gives:

$$\begin{aligned}
\alpha\widehat{K}_t &= \frac{dK_{t+1}-(1-\delta)dK_t+dC_t}{K^*-(1-\delta)K^*+C^*} \\
\alpha\widehat{K}_t &= \frac{K^*\widehat{K}_{t+1}-(1-\delta)K^*\widehat{K}_t+C^*\widehat{C}_t}{\delta K^*+C^*}
\end{aligned}$$

Define $s = K^*/(\delta K^*+C^*)$. Then, rearranging gives

$$(\alpha+(1-\delta)s)\widehat{K}_t-(1-\delta s)\widehat{C}_t = s\widehat{K}_{t+1}$$

Thus we can combine these two equations into one matrix equation that would let us study percentage deviations in period $t+1$ as a function of percentage changes in period t:

$$\begin{pmatrix} \sigma & -\gamma \\ 0 & s \end{pmatrix} \begin{pmatrix} \widehat{C}_{t+1} \\ \widehat{K}_{t+1} \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ -(1-\delta s) & \alpha + (1-\delta)s \end{pmatrix} \begin{pmatrix} \widehat{C}_t \\ \widehat{K}_t \end{pmatrix}$$