Columbia MA Math Camp

Set Theory

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[a] Material adapted from notes by David Thompson and Xingye Wu

Motivation of Study Set Theory

- Set theory is one of the fundamental building blocks of mathematics
- Many important concepts later such as relations, functions and sequences are defined using the language of sets
- You will encounter a lot of this during your micro course (and math methods of course)

Table of Contents

Small Digression on Proofs

- We went through some basics of logic where we talked about implications like $p \to q$.
- In set theory, you will frequently encounter questions which give you some information and then ask you to show that $p \to q$.
- The way to show it is to assume that p is true and then use the information to show that q must be true
- Sometimes it is easier to prove the contrapositive i.e. try to prove that $\neg q \to \neg p$.
    - Assume that $\neg q$ is true and then proceed to show that $\neg p$ must be true.

Table of Contents

Common Notation

- $\in$: **"in"**; e.g. $x \in \mathbb{N}$ means x is a **natural number**.
- $\forall$: **"for all"**; e.g. $\forall x \in \mathbb{N}$ means for all natural numbers x.
- $\exists$: **"there exists"**; e.g. $\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}$ such that $x + y = 0$
- !: **"unique"**; typically used in conjunction with $\exists$
- $\Rightarrow$: **"implies"**; e.g. $A \Rightarrow B$ means A implies B.
    - We have been using $\rightarrow$. We will switch to $\implies$ from now as this is more commonly used.

Basic Concepts

- A set is a collection of objects and each individual object is called an **element**
- Lowercase letters are used for elements and Capital letters for sets.
- The notation $x \in X$ means that the object x **is an element** of the set X.
- A set is typically written in curly brackets $\{1, 2, 3\}$
    - The order of the elements listed does not matter
- For more complicated sets we use **"set-builder"** notation, e.g.

$$\{x \in \mathbb{N} | x^2 < 100\}$$

- The item before the vertical line defines the domain of our search.
- In the example above, we are searching for natural numbers which satisfy the requirement to the right of the vertical line

Common Sets

- **Common sets:**
  - $\mathbb{N}$: natural numbers $\{0, 1, 2, ...\}$
  - $\mathbb{Z}$: integers $\{..., -2, -1, 0, 1, 2, ...\}$
  - $\mathbb{Q}$: rational numbers; all numbers of form $\frac{p}{q}$ with $p, q \in \mathbb{Z}, q \neq 0$
  - $\mathbb{R}$: real numbers; most of econ happens here
- We do allow a set to contain **no element** at all, and we call it the **empty set** denoted by $\emptyset$
  - The empty set $\emptyset$ is a subset of every set. (Why?)

Comparing sets - Inclusion

- A is a **subset** of B if every element of A is an element of B; write $A \subseteq B$ or $B \supseteq A$
  - In other words, $x \in A \implies x \in B\ \forall x$
- Two sets are **equal** if they contain exactly the same elements. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
- A set A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$. This is sometimes written $A \subset B$ or $A \subsetneq B$
  - Not all sets are comparable. Give me an example?
- Set inclusion is **transitive**: $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$.
- For finite sets, the **cardinality** of a set $|A|$ is the number of elements of A

How to prove Set Inclusion is Transitive?

Lemma 1.1

Set inclusion is transitive i.e. $A \subseteq B$ *and* $B \subseteq C$ *implies* $A \subseteq C$

Proof.

We want to show $A \subseteq C$. By definition, this means that we need to show for any $x \in A$ it must be that $x \in C$. Take any $x \in A$. By definition of $A \subseteq B$ and because $x \in A$, we have that $x \in B$. Again by the definition of $B \subseteq C$, we have $x \in C$. $\square$

Constructing new sets

- The **union** of A and B, $A \cup B$ is the collection of elements in A or B (or both)

$$A \cup B := \{x : x \in A \text{ or } x \in B\}$$

- The **intersection** of A and B, $A \cap B$ is the collection of elements that belong to both A and B

$$A \cap B := \{x : x \in A \text{ and } x \in B\}$$

  If $A \cap B = \emptyset$, then we say that A and B are disjoint.

- The **difference** of A and B, $A \setminus B$ or $A - B$, is the collection of elements in A and not in B.

Can take unions and intersections of big collections of sets, typically indexed by an **index set**. Most common index set is $\mathbb{N}$, e.g.

$$\bigcup_{i \in \{0,1,2,\ldots\}} A_i$$

Some properties - I

Lemma 1.2

$A \cup B = B$ iff $A \subseteq B$ (Note that this is a equivalence because of the iff)

Proof.

We show subset containment both ways i.e. both the $\implies$ and the $\impliedby$.

Let's first prove the $\impliedby$ side.

- Suppose $A \subseteq B$. WTS that $A \cup B = B$. This means that we need to show both $A \cup B \subseteq B$ and $B \subseteq A \cup B$. Let us first show $A \cup B \subseteq B$.
- Take any $x \in A \cup B$. WTS $x \in B$. Because $x \in A \cup B$, then, by definition $\cup$, either $x \in A$ or $x \in B$
- If $x \in A$, then by definition of $x \subseteq B$ we have $x \in B$. So either case, we have $x \in B$. Thus $A \cup B \subseteq B$ is proved. Proving $B \subseteq A \cup B$ is left as an exercise.

Now let us prove the other direction i.e. $\implies$

- Given $A \subseteq B = B$, WTS $A \subseteq B$.
- Take any $x \in A$, WTS that $x \in B$. By definition of $A \cup B$ and $x \in A$, we have $x \in A \cup B$. Because $A \subseteq B = B$, we have $x \in B$. Thus $\implies$ is proved.

Some properties - II

Lemma 1.3

Intersection is distributive with respect to the union:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof.

We show subset containment both ways.

- Let $x \in A \cap (B \cup C)$.
- By definition, $x \in A$ and $x \in B \cup C$, so $x \in B$ or $x \in C$.
- Thus $x \in A \cap B$ or $A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.

Now the other direction

- Let $x \in (A \cap B) \cup (A \cap C)$.
- By definition, $x \in A \cap B$ or $x \in A \cap C$, so $x \in A$ and $x \in B$ or $x \in C$.
- Thus $x \in A \cap (B \cup C)$

$\square$

Complements and DeMorgan's Laws

We normally think of sets living in some larger space Ω. The **complement** of a set A, A^c, is the collection of elements not in A.

$$(A^c)^c = A$$

Complements play nicely with unions and intersections:

$$\begin{aligned} (A \cap B)^c &= A^c \cup B^c \\ (A \cup B)^c &= A^c \cap B^c \end{aligned}$$

Cartesian Products

The **Cartesian product** is our last common method for generating new sets: take all pairs from A and B:

$$A \times B \equiv \{(a, b) | a \in A, b \in B\}$$

Typically work in $\mathbb{R}^n$:

$$\mathbb{R}^n \equiv \{(a_1, ..., a_n) | a_i \in \mathbb{R}\, \forall i = 1, ..., n\}$$

Note order matters: $(2, 1) \neq (1, 2)$