# Columbia MA Math Camp

## Convexity

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**Definition 1.1**
*Let $S \subseteq \mathbb{R}^n$. We say $S$ is* **convex** *if for all $x, y \in S$ and $\lambda \in [0,1]$:*

$$\lambda x + (1-\lambda) y \in S$$

Is the set $S = [0,1]$ convex? What about $S = [0,1)$? What about $S = [0,1) \cup [2,3]$? $S = \{1, 2, 3, \dots\}$ ?

**Notes :**

- In other words, the convex combination of 2 vectors in a set belongs to the same set.
- The intersection of convex sets is convex
- The union of convex sets need not be convex

For finitely many vectors $x_1, x_2, \ldots, x_n$, a **convex combination** is a vector $\sum_{i=1}^{n} \lambda_i x_i$ for scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_+$ such that $\sum_{i=1}^{n} \lambda_i = 1$

**Proposition 1.1**
*Suppose $S \subseteq \mathbb{R}^n$. The set $S$ is convex iff any convex combination of $x_1, x_2, \ldots, x_n \in S$ is also in $S$.*

**Proof:**
$( \Longleftarrow )$ is trivial based on the definition of convex sets.

$( \Longrightarrow )$ If $n = 1$, the statement is trivial.

If $n = 2$, the statement is true by the definition of convexity.

Suppose it is true for $n = k$. This implies that for any set of $k$ vectors $x_1, x_2, \ldots, x_k$, $\sum_{i=1}^{k} \lambda_i x_i \in S$ for all $\lambda_i \geq 0$ such that $\sum \lambda_i = 1$.

# Proof continued...

Now consider $n = k + 1$ . We need to show that $\sum_{i=1}^{k+1} \lambda_i x_i \in S$.

We can rewrite this as :

$$\begin{aligned}
\sum_{i=1}^{k+1} \lambda_i x_i &= \sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1} \\
&= \left(\sum_{i=1}^{k} \lambda_i\right) \left(\sum_{i=1}^{k} \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i} x_i\right) + \lambda_{k+1} x_{k+1} \\
&= \left(\sum_{i=1}^{k} \lambda_i\right) \bar{x} + \lambda_{k+1} x_{k+1} \quad \text{(since it is true for } n = k \text{ i.e. } \sum_{i=1}^{k} \frac{\lambda_i}{\sum_{i=1}^{k} \lambda_i} x_i \in \text{S)} \\
&\in S \quad \text{(Since it is true for } n = 2)
\end{aligned}$$

# Convex and Concave Functions

**Definition 1.2**
*A function $f : \mathbb{R}^n \to \mathbb{R}$ is* **convex** *if for any $x_1, x_2 \in \mathbb{R}^n$ and any $\lambda \in (0, 1)$:*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

- If the inequality is strict, $f$ is **strictly convex**
- If the inequality is reversed, $f$ is **concave**

**Another characterization:** A function $f$ is **convex** if and only if :

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R} | y \geq f(x)\}$$

is **convex**

# Convex Functions: Properties

Convex functions have a whole host of nice properties - people write books on convex analysis. Some include:

- If $f$ and $g$ are **convex (concave)**, $f + g$ is **convex (concave)**
- If $f$ is **convex (concave)** and $g$ is **convex (concave) and increasing**, then $f \circ g$ is **convex (concave)**

Some properties are a little surprising at first glance :

- Convex functions are **continuous**
- Convex functions are **differentiable** almost everywhere

**Definition 1.3**
*Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then*

- $f$ *is* **convex** *iff for all $x_1, x_2 \in \mathbb{R}^n$:*

$$f(x_2) \geq f(x_1) + f'(x_1)(x_2 - x_1)$$

- $f$ *is* **strictly convex** *iff for all $x_1 \neq x_2$:*

$$f(x_2) > f(x_1) + f'(x_1)(x_2 - x_1)$$

Convex functions sit above their tangent lines. The analogous result holds for concave functions (just flip the inequality)

## Characterization for Twice Differentiable Functions

**Definition 1.4**
*Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function. Then*

- *$f$ is convex (concave) iff its Hessian is positive (negative) semi-definite for all $x$*
- *If the Hessian is positive (negative) definite for all $x$, then $f$ is strictly convex (concave)*

**(Proof intuition) :** Use a second-order Taylor series expansion

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}(x-a)^T H(a)(x-a)$$

If $H(a)$ is positive definite, $f$ will sit above its tangent approximation.

# Quasiconcavity

- In micro, we think of preferences that are represented by a utility function: $x$ **is preferred to** $y$ if $u(x) \geq u(y)$
- This is an *ordinal notion*: if $f(\cdot)$ is an **increasing** function, then $f(u(x)) > f(u(y))$, so $f \circ g$ represents the same preferences
- However, convexity is not an ordinal notion. Let $u(x) = x^2$ and $f(x) = \log x$. Then $u$ is *convex* and $f$ an *increasing transformation*, but $f(u(x)) = 2\log x$ is **concave, not convex**
- We will develop a notion of **quasiconcavity (quasiconvexity)** that will be preserved by increasing transformations

**Definition 1.5**
*Let $f : \mathbb{R}^n \to \mathbb{R}$. We say $f$ is* **quasiconvex** *if the lower level set*

$$S_\alpha \equiv \{x | f(x) \leq \alpha\}$$

*is convex for every value $\alpha$. If the upper level sets*

$$U_\alpha \equiv \{x | f(x) \geq \alpha\}$$

*is convex for every $\alpha$, then $f$ is* **quasiconcave**

# Quasiconcave functions - Alternate Characterization

**Definition 1.6**
*A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is* **quasiconvex** *iff for all* $x, y \in \mathbb{R}^n$ *and* $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$$

*If the inequality is strict for* $x \neq y$ *and* $\lambda \in (0, 1)$, $f$ *is* **strictly quasiconvex**

*For* **quasiconcavity**, *we have* $f(\lambda x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}$

- **Convexity $\implies$ Quasiconvexity** : Suppose $f$ is convex. Then for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leq \lambda f(x) + (1-\lambda)f(y) \\ &\leq \max\{f(x), f(y)\} \end{aligned}$$

so $f$ is **quasiconvex**. (Similar argument for Quasiconcavity)

- **Increasing transformation of quasiconvex function is quasiconvex :**

  Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex and $g : \mathbb{R} \to \mathbb{R}$ is an increasing function. Then for all $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$:

$$\begin{aligned} g(f(\lambda x + (1-\lambda)y)) &\leq g(\max\{f(x), f(y)\}) \\ &= \max\{g(f(x)), g(f(y))\} \end{aligned}$$

So $g \circ f$ is **quasiconvex**. Similarly, an *increasing transformation of a quasiconcave function is quasiconcave.*

**Proposition 1.2**
*Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ function. Then $f$ is quasiconcave iff* $f(y) \geq f(x) \Rightarrow f'(x)(y-x) \geq 0.$

**Proof.**
($\Rightarrow$) Suppose $f$ is quasiconcave. Let $x, y \in \mathbb{R}^n$ such that $f(y) \geq f(x)$, and $\lambda \in (0,1)$.

$$f((1-\lambda)x + \lambda y) \geq f(x)$$

Rearranging gives

$$\frac{f(x + \lambda(y-x)) - f(x)}{\lambda} \geq 0$$

Taking $\lambda \to 0$ gives $f'_{y-x}(x) \geq 0$. So $f'(x)(y-x) \geq 0$. $\square$

## Quasiconcave functions : Uniqueness of Maximizer

**Proposition 1.3**
*A* **strictly quasiconcave function** *can have at most one global maximum.*

**Proof :** Suppose there are 2 maximizers $x$ and $y$. If $x \neq y$ are both maximizers, then $f(x) = f(y)$.

However, $f(\lambda x + (1 - \lambda)y) > f(x) = f(y)$ by the definition of strict quasiconcavity which contradicts that $x$ and $y$ are maximizers.