Problem 4

4.1

For consumer's problem, substitute the budget constraint into the objective to eliminate C, and the optimal conditions are:

$$\frac{1}{C_t} = \beta(1 + r_{t+1} - \delta)\frac{1}{C_{t+1}}, \tag{1}$$
$$C_t + K_{t+1} = (1 + r_t - \delta)K_t + w_t. \tag{2}$$

For firm's problem, the optimal conditions are (when $L_t = 1$):

$$\alpha K_t^{\alpha-1} = r_t, \tag{3}$$
$$(1-\alpha)K_t^{\alpha} = w_t. \tag{4}$$

Note: as household's and firm's objective functions are concave, the first-order conditions are sufficient for solving the problems.

4.2

Given that $C_t = C_{t+1}$, the steady state values are:

$$r^* = \frac{1}{\beta} - 1 + \delta \;\rightarrow\; K^* = \left(\frac{r^*}{\alpha}\right)^{\frac{1}{\alpha-1}} \;\rightarrow\; w^* = (1-\alpha)(K^*)^{\alpha} \;\rightarrow\; C^* = w^* + (r^* - \delta)K^*.$$

When $\beta = 0.96$, $\alpha = 0.36$, $\delta = 0.06$, these values become

$$r^* = 0.102, \quad K^* = 7.211, \quad w^* = 1.303, \quad C^* = 1.604.$$

4.3

In the following, first construct a vector z of Chebyshev nodes on $[-1, 1]$, and a vector K of associated nodes on $[0.5K^*, 1.5K^*]$, and store the values Chebyshev polynomials in matrix $T(K)$. Suppose $\tilde{C}(K) = \sum_{i=0}^{n} \theta_i T_i(K) = T(K)\theta$ is the approximated consumption policy, so that the Euler equation implies:

$$\begin{aligned}
&\tilde{C}(K') - \beta(1 + \alpha K'^{\alpha-1} - \delta)\tilde{C}(K) \\
=&\tilde{C}(K^{\alpha} + (1-\delta)K - \tilde{C}(K)) - \beta(1 + \alpha(K^{\alpha} + (1-\delta)K - \tilde{C}(K))^{\alpha-1} - \delta)\tilde{C}(K)
\end{aligned}$$

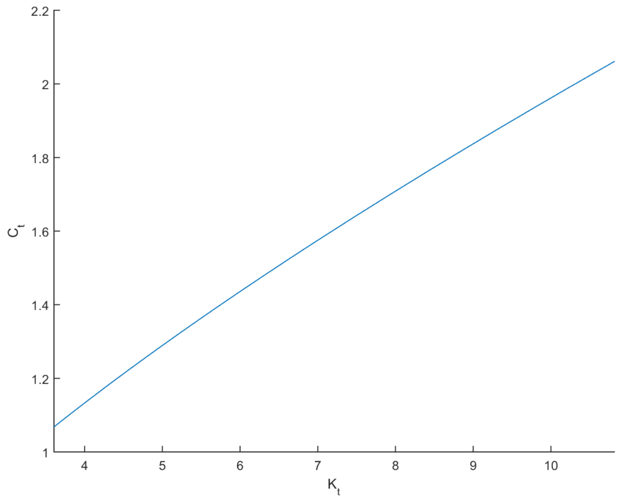
With a initial guess $K = K'$, $\tilde{C}(K) = (r - \delta)K + w = K^{\alpha} - \delta K$, and so $\theta_0 = T(K)\backslash(K^{\alpha} - \delta K)$. Under more general situation, we solve θ by a nonlinear system of equations:

$$T(K_j^{\alpha} + (1-\delta)K_j - T(K_j)\theta)\theta - \beta[1 + \alpha(K_j^{\alpha} + (1-\delta)K_j - T(K_j)\theta)^{\alpha-1} - \delta]T(K_j)\theta = 0, \quad j = 1, \ldots, m.$$

The vector of roots θ^* is obtained by using Matlab function $fsolve$ with initial guess θ_0, and the policy functions are $\tilde{C}(K) = T(K)\theta^*$, and $\tilde{K}'(K) = K^{\alpha} + (1-\delta)K - T(K)\theta^*$.

4.4

The plot of consumption policy on $[0.5K^*, 1.5K^*]$ is shown below.

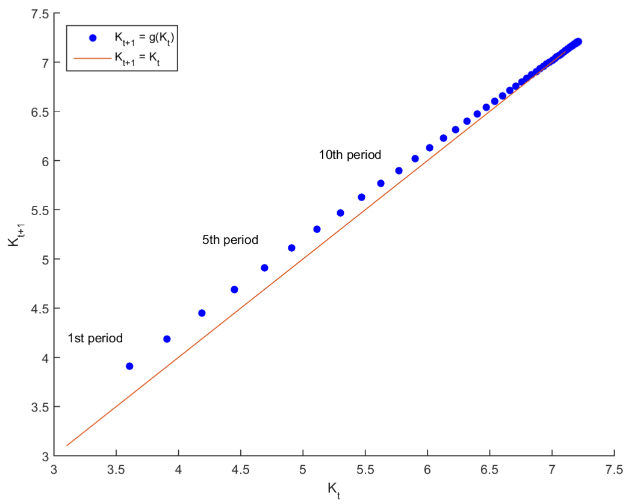


4.5

The path of capital for 100 periods that starts with $K_1 = 0.5K^*$ is shown below. Clearly, the sequence will converge to K^* when the number of period gets larger.

4.4

The plot of consumption policy on $[0.5K^*, 1.5K^*]$ is shown below.



4.6

Given approximated policies, the Euler equation error is

$$E(K;\theta^*) = 1 - \frac{\tilde{C}(\tilde{K}'(K)) - \beta(1 + \alpha \tilde{K}'(K)^{\alpha-1} - \delta)\tilde{C}(K)}{\tilde{C}(K)}.$$

For 1000 random draws, the maximum and average absolute errors are $1.9 * 10^{-7}$ and $5 * 10^{-8}$.

Matlab codes

Problem 4

```
clc; clear all; close all;

%% Parameters
bet = 0.96; alp = 0.36; del = 0.06;

%% The steady state
rs = 1/bet - 1 + del;
Ks = (rs/alp)^(1/(alp-1));
ws = (1-alp)*Ks^alp;
Cs = ws + (rs - del)*Ks;
fprintf('Steady state: r = %4.3f, K = %4.3f, w = %4.3f, C = %4.3f.
    \n', rs, Ks, ws, Cs);

%% Chebyshev nodes and matrix
m = input('m = '); % # nodes (indexed by j)
n = input('n = '); % # Chebyshev functions - 1 (indexed by i)
a = 0.5*Ks; b = 1.5*Ks;

z = -cos( (2*[1:m]-1)/(2*m) * pi )';
K = (z*(b - a) + a + b)/2;

for j = 1:m
    for i = 1:n+1
        T(j,i) = cos((i - 1)*acos(z(j)));
    end
end

%% Chebyshev approximation (when m = n+1!)
tht0 = T\(K.^alp - del.*K); % initial guess: K = Kp
tht = sym('tht', [n+1, 1]);

Kp = K.^alp + (1-del).*K - T*tht;
for j = 1:m
    for i = 1:n+1
        Tp(j,i) = cos((i - 1)*acos( 2*(Kp(j) - a)/(b - a) - 1 ));
    end
end
f = Tp*tht - bet*(1 + alp*Kp.^(alp - 1) - del).*(T*tht);

merr = matlabFunction(f,'Vars',{[tht]});
thts = fsolve(merr,tht0);

syms Ksym; Cpoli = 0;
for i = 1:n+1
    Cpoli = Cpoli+thts(i)*cos((i-1)*acos( 2*(Ksym-a)/(b-a) - 1));
end
Kppoli = Ksym^alp + (1-del)*Ksym - Cpoli;
```

```
mCpoli = matlabFunction(Cpoli); mKppoli = matlabFunction(Kppoli);

%% Plot C against K
figure
hold on
set(gcf,'color','w');
xlabel('K_t'); ylabel('C_t');
fplot(mCpoli, [a,b]);
print('CK','-dpng');

%% Plot Kp against K
t = 100; Ksim(1) = 0.5*Ks;
for i = 1:t
    Ksim(i+1) = mKppoli(Ksim(i));
end

figure
hold on
set(gcf,'color','w');
xlabel('K_t'); ylabel('K_{t+1}');
scatter(Ksim([1:end-1]),Ksim([2:end]),'b','fill'); plot(Ksim(1)
   -0.5:Ksim(end)+0.5,Ksim(1)-0.5:Ksim(end)+0.5);
legend('K_{t+1} = g(K_t)', 'K_{t+1} = K_t', 'Location', 'northwest
   ');
text(Ksim(1)-0.5,Ksim(2)+0.3,'1st period')
text(Ksim(5)-0.5,Ksim(6)+0.3,'5th period')
text(Ksim(10)-0.5,Ksim(11)+0.3,'10th period')
print('KpK','-dpng');

%% Euler equation error
ndraw = 1000;
Krand = 0.5*Ks + Ks.*rand(1,ndraw);

err = zeros(1,ndraw);
for i = 1:ndraw
    err(i) = ( mCpoli( mKppoli(Krand(i)) ) - bet*(1+alp*mKppoli(
       Krand(i))^(alp-1)-del)*mCpoli(Krand(i)) )/mCpoli(Krand(i));
end

fprintf('Maximal and average errors %9.8f, %9.8f. \n', max(abs(err
   )), mean(abs(err)));
```