Problem Set 3

Due date: January 25, 2016, before lecture

Instructions: Please hand in a single PDF-file containing all your answers and results. Show the names of the group members on top. Make use of figures and tables and always provide a short interpretation of your results. Include the source code in the appendix. The code should be well documented and readable. From now on, use Github to control the versions when preparing your solutions.

Problem 1: Simple Functions

- Implement the Newton algorithm for any function $f : \mathbf{R} \to \mathbf{R}$
- Calculate the extrema of the two functions

$$\begin{aligned} f_1(x) &= 2x^3 - x^2 - 3x + 2 \\ f_2(x) &= -x\exp(-x). \end{aligned}$$

Problem 2: Consumption Savings Problem

Consider the following life-cycle problem

$$\begin{aligned} &\max_{\{c_t\}_{t=0}^T} \sum_{t=0}^{T} \beta^t u(c_t) \\ \text{s.t. } & u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta} \\ & c_t + a_{t+1} = a_t(1+r) + w_t \\ & a_0 \text{ given and } \ a_{T+1} \geq 0. \end{aligned}$$

- Characterize the analytical solution for any β, r, w_t.
- Compute the numerical solution setting $w_0 = 10, a_0 = 0,\ \ w_t = 0$, for $t \geq 1$, $\beta = 0.99$, $r = 0.05$ and any γ.
- How do you deal with the case $\theta = 1$, numerically?
- How do you deal with the constraint $c_t > 0$ for all t?

Problem 3: The Consumption-Savings Problem with Human Capital

Consider the 2-period consumption-savings problem with human capital:

- c_t, k_t, h_t: cons., stock of physical and human capital, $t = 1, 2$
- i_k, i_h: investment in physical and human capital
- $\delta_k, \delta_h \in [0, 1]$: depreciation rates for physical and human capital
- $k_2 = (1 - \delta_k)k_1 + i_k$ and $h_2 = ((1 - \delta_h)h_1 + i_h)^\eta$
- $k_2 \geq 0$: borrowing constraint on physical capital
- $i_h \geq 0$: irreversibility constraint on human capital

- Optimization problem (substituting in the physical and human capital accumulation functions):

$$\begin{aligned} &\max_{c_1,c_2,i_k,i_h} \left\{ \frac{c_1^{1-\gamma}-1}{1-\gamma} + \beta\, \frac{c_2^{1-\gamma}-1}{1-\gamma} \right\} \\ \text{s.t.} \quad & (1-\delta_k)\; k_1 + i_k \geq 0 \\ & i_h \geq 0 \\ & c_1 + i_k + i_h - \Big(r_k\; k_1 + r_h\; h_1\Big) = 0 \\ & c_2 - \Big((1+r_k)\; ((1-\delta_k)\; k_1 + i_k) + r_h\; ((1-\delta_h)\; h_1 + i_h)^{\eta}\Big) = 0 \end{aligned}$$

- CRRA utility with risk aversion γ, discount factor β
- r_k, r_h: return on physical and human capital
- $\eta \in (0,1)$: decreasing marginal returns to human capital investment
- Parametrization:

Table 1: Parameters

γ	β	r_k	r_h	η	δ_k	δ_h
2.00	0.96	0.10	1.40	0.80	0.05	0.05

1. Derive the Kuhn-Tucker conditions of this optimization problem.
2. Apply the Garcia-Zangwill trick by defining

$$\begin{aligned} \alpha_i^+ &= (\max\{0, \alpha_i\})^k \\ \alpha_i^- &= (\max\{0, -\alpha_i\})^k \end{aligned}$$

   for $k \in \mathbb{N}^+$ and $i \in \{k, h\}$. Verify that then the Kuhn-Tucker conditions write as

$$\begin{aligned} & c_1^{-\gamma} + \mu_1 = 0 \\ & \beta\; c_2^{-\gamma} + \mu_2 = 0 \\ & \alpha_k^+ + \mu_1 - \mu_2\; (1+r_k) = 0 \\ & \alpha_h^+ + \mu_1 - \mu_2\; r_h\; \eta\; ((1-\delta_h)\; h_1 + i_h)^{\eta-1} = 0 \\ & \alpha_k^- - ((1-\delta_k)\; k_1 + i_k) = 0 \\ & \alpha_h^- - i_h = 0 \\ & c_1 - \Big(r_k\; k_1 + r_h\; h_1 - i_k - i_h\Big) = 0 \\ & c_2 - \Big((1+r_k)\; ((1-\delta_k)\; k_1 + i_k) + r_h\; ((1-\delta_h)\; h_1 + i_h)^{\eta}\Big) = 0 \end{aligned}$$

3. How many unknowns? Which?
4. Program the Kuhn-Tucker equations from above.
5. Set $k = 2$. Use `fsolve` to solve the Kuhn-Tucker equations for the following initial endowments
   i) $k_1 = 1, h_1 = 5$
   ii) $k_1 = 1, h_1 = 1$
   iii) $k_1 = 1, h_1 = 0.2$
6. Interpret your results. Which constraints are binding?

Problem 4: A Simple Portfolio Choice Problem

This exercise helps you to get a better understanding of constrained optimization problems. We will deal with a simple portfolio choice problem. The household has an initial endowment of wealth w_0 and $w_1 = w_0(1 + r^p)$ is the terminal wealth for some portfolio return r^p . This portfolio return depends on the investment in one risky and one risk-free asset and may be written as

$$r^p = \alpha^f r^f + \alpha r,$$

where r is the return on the risky asset. We assume that the share invested in the risky asset, α, is constrained:

$$\underline{\alpha} \leq \alpha \leq \overline{\alpha}.$$

Furthermore, we assume that there are only two possible realizations of the return on the risky asset, r_{low} and r_{high}, which are realized with probabilities p and $1-p$. We assume throughout that $r^f = 0.02$, $r_{low} = -0.08$, and $r_{high} = 0.12$. The objective function is given by $\mathbb{E}u(w_1)$. We will again assume a constant relative risk aversion (CRRA) utility function given by

$$u(w_1) = \frac{1}{1-\gamma} w_1^{1-\gamma}$$

- The portfolio shares α^f and α must satisfy

  $$\alpha^f + \alpha = 1.$$

  It is now straightforward to write the portfolio return as

  $$r^p = r^f + \alpha(r - r^f).$$

- We substitute out $w_1 = w_0(1 + r^p)$.

The maximization problem now writes as

$$\max_{\alpha} \mathbb{E}\left[\frac{1}{\phi}\left(w_0(1 + r^f + \alpha(r - r^f))\right)^{\phi}\right]$$
$$\text{s. t.} \quad \underline{\alpha} \leq \alpha \leq \overline{\alpha}$$

for $\phi = 1 - \gamma$.

1. Is the problem convex? If so, what does it imply for the solution method?
2. First, let's assume an unconstrained problem, i.e., $\underline{\alpha} = -\infty$ and $\overline{\alpha} = \infty$.
   a) Show that the optimal portfolio share is independent of initial wealth. This is an important result in portfolio theory due to Merton (1969) and Samuelson (1969).
   b) Solve the problem again for $(\phi = -3, p = 0.1)$ and plot α against the value of the objective function on the interval $[\alpha^\star - 1, \alpha^\star + 1]$. Provide an economic interpretation.
3. Now consider the constrained optimization problem with $\underline{\alpha} = 0$ and $\overline{\alpha} = 1$.
   a) Interpret the constraint economically.
   b) Calculate the solution for $(\phi = -3, p = 0.1)$ using the Matlab solvers `fminbnd` and `fmincon`. Why do the two solvers yield different results?
   c) Compare your results to part 2 and discuss.

References

MERTON, R. C. (1969): "Litetime Portfoliio Selection under Uncertainty: The Continuous-Time Case," *The Review of Economics and Statistics*, 51(3), 247–257.

SAMUELSON, P. A. (1969): "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, 51(3), 239–246.