Problem 4

4.1

The problem is convex, as it maximizes of a concave function (the CRRA utility function) over a convex set ($[\underline{\alpha}, \bar{\alpha}] \in \mathbb{R}$). Thus, a Kuhn-Tucker method can be applied to solve the problem.

4.2

Without the constraint, the first-order condition implies

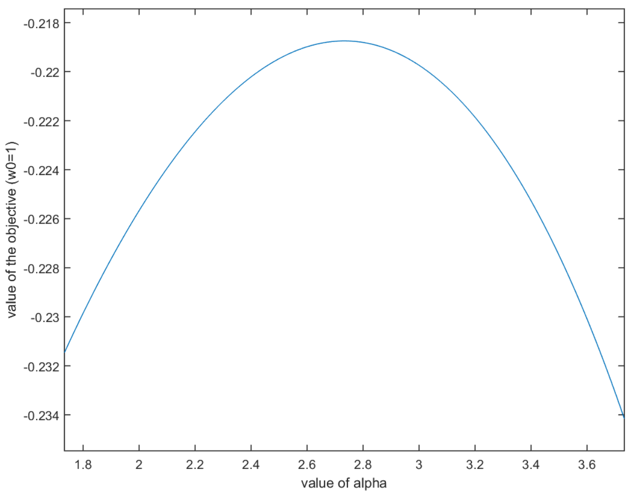
$$\mathbb{E}[w_0^{\phi}(1 + r^f + \alpha(r - r^f))^{\phi-1}(r - r^f)] = 0.$$

But since w_0 is a constant, it can be taken out of the expectation operator, and so the optimal share α is independent of the initial wealth w_0, which is a property of the CRRA utility.

Now the condition can be rewritten as

$$p[1 + r^f + \alpha(r_{low} - r^f)]^{\phi-1}(r_{low} - r^f) + (1 - p)[1 + r^f + \alpha(r_{high} - r^f)]^{\phi-1}(r_{high} - r^f) = 0.$$

Use Matlab function *fzero*, the optimal value is reached at $a^* = 2.7331$, and the plot of α against the value of the objective (wlog $w_0 = 1$) looks like:



4.3

The lower bound $\underline{a} = 0$ implies that investor cannot be on the short position, and the upper bound $\bar{a} = 1$ means that the investor cannot borrow money to buy the risky asset.

In Matlab, *fmincon* finds the minimum of constrained nonlinear multivariable function, while *fminbnd* finds the minimum of single-variable function on fixed interval. Clearly, *fmincon* is far more advanced than what is required for this problem. Starting from the initial value $\alpha = 0.5$, the two methods gave the same result $\alpha^{**} = 1$.

The intuition is straightforward: since the unconstrained optimum $a^* > 1$, and since the function is increasing in $[0, 1]$, the best strategy under the constraint is to hit the right bound, i.e. invest all in the risky asset.

Matlab codes

Problem 4

```
syms a;
rf = 0.02; rlr = -0.08-rf; rhr = 0.12-rf;
p = 0.1; phi = -3;

f = -(p*(1 + rf + a*rlr)^phi + (1 - p)*(1 + rf + a*rhr)^phi)/phi;
df = diff(f,a);
mf = matlabFunction(f);
mdf = matlabFunction(df);
astar1 = fzero(mdf,1);
display(['The unconstrained solution is ', num2str(astar1)]);

figure;
set(gcf,'color','w');
ezplot(-f,[astar1-1,astar1+1]);
xlabel('value of alpha');
ylabel('value of the objective (w0=1)');
title('');
print('plot42','-dpng')

astar2 = fminbnd(mf,0,1);
astar3 = fmincon(mf,0.5,[],[],[],[],0,1);
display(['The function fminbnd and fmincon give the constrained
   solution ', num2str(astar2), ' and ', num2str(astar3)]);
```