

# Applied Econometrics

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# Chapter 1

## Before starting

This course covers various econometric topics, including linear regression models, discrete-choice models, and time series analysis. It provides examples or simulations based on R codes.

The R codes use various packages that can be obtained from CRAN. Several pieces of code also involve procedures and data from a companion package (**AEC**). Some of these procedures—those pertaining to VAR models—as well as the associated presentation (in Section 9.3) have been prepared jointly with Kenza Benhima. This **AEC** package is available on GitHub. To install it, one need to employ the **devtools** library:

```
library(devtools)
install_github("jrenne/AEC")
library(AEC)
```

### Useful (R) links:

- Download R:
  - R software: <https://cran.r-project.org> (the basic R software)
  - RStudio: <https://www.rstudio.com> (a convenient R editor)
- Tutorials:
  - Rstudio: <https://dss.princeton.edu/training/RStudio101.pdf> (by Oscar Torres-Reyna)
  - R: [https://cran.r-project.org/doc/contrib/Paradis-rdebuts\\_en.pdf](https://cran.r-project.org/doc/contrib/Paradis-rdebuts_en.pdf) (by Emmanuel Paradis)
  - My own tutorial: [https://jrenne.shinyapps.io/Rtuto\\_publiShiny/](https://jrenne.shinyapps.io/Rtuto_publiShiny/)



# Chapter 2

## Basic statistical results

### 2.1 Cumulative and probability density functions (c.d.f. and p.d.f.)

**Definition 2.1** (Cumulative distribution function (c.d.f.)). The random variable (r.v.)  $X$  admits the cumulative distribution function  $F$  if, for all  $a$ :

$$F(a) = \mathbb{P}(X \leq a).$$

**Definition 2.2** (Probability distribution function (p.d.f.)). A continuous random variable  $X$  admits the probability density function  $f$  if, for all  $a$  and  $b$  such that  $a < b$ :

$$\mathbb{P}(a < X \leq b) = \int_a^b f(x)dx,$$

where  $f(x) \geq 0$  for all  $x$ .

In particular, we have:

$$f(x) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}(x < X \leq x + \varepsilon)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{F(x + \varepsilon) - F(x)}{\varepsilon}. \quad (2.1)$$

and

$$F(a) = \int_{-\infty}^a f(x)dx.$$

This web interface illustrates the link between p.d.f. and c.d.f. Nonparametric estimates of p.d.f. are obtained by kernel-based methods (see this extra material).

**Definition 2.3** (Joint cumulative distribution function (c.d.f.)). The random variables  $X$  and  $Y$  admit the joint cumulative distribution function  $F_{XY}$  if, for all  $a$  and  $b$ :

$$F_{XY}(a, b) = \mathbb{P}(X \leq a, Y \leq b).$$

**Definition 2.4** (Joint probability density function (p.d.f.)). The continuous random variables  $X$  and  $Y$  admit the joint p.d.f.  $f_{XY}$ , where  $f_{XY}(x, y) \geq 0$  for all  $x$  and  $y$ , if:

$$\mathbb{P}(a < X \leq b, c < Y \leq d) = \int_a^b \int_c^d f_{XY}(x, y)dxdy, \quad \forall a \leq b, c \leq d.$$

In particular, we have:

$$\begin{aligned} & f_{XY}(x, y) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}(x < X \leq x + \varepsilon, y < Y \leq y + \varepsilon)}{\varepsilon^2} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{F_{XY}(x + \varepsilon, y + \varepsilon) - F_{XY}(x, y + \varepsilon) - F_{XY}(x + \varepsilon, y) + F_{XY}(x, y)}{\varepsilon^2}. \end{aligned}$$

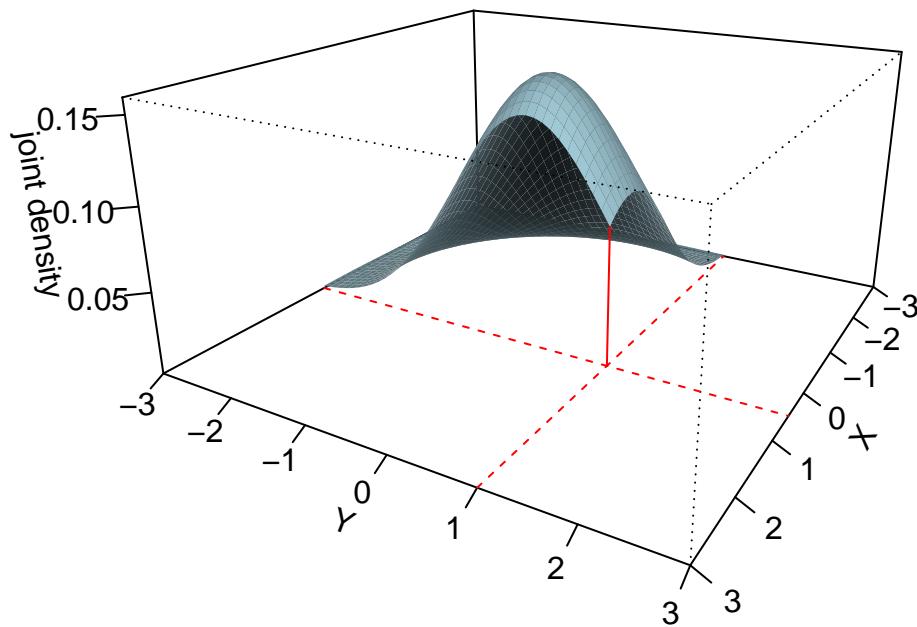


Figure 2.1: The volume between the horizontal plane ( $z = 0$ ) and the surface is equal to  $F_{XY}(0.5, 1) = \mathbb{P}(X < 0.5, Y < 1)$ .

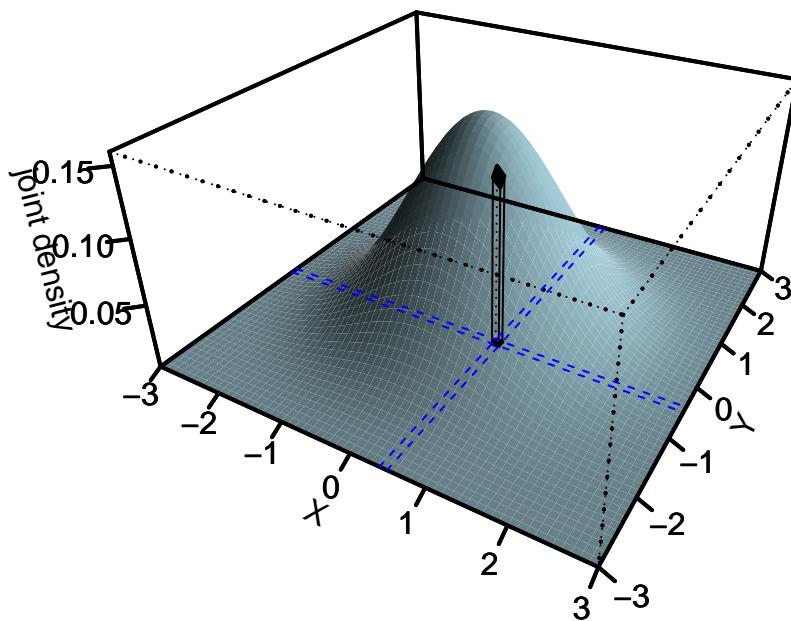


Figure 2.2: Assume that the basis of the black column is defined by those points whose  $x$ -coordinates are between  $x$  and  $x + \varepsilon$  and  $y$ -coordinates are between  $y$  and  $y + \varepsilon$ . Then the volume of the black column is equal to  $\mathbb{P}(x < X \leq x + \varepsilon, y < Y \leq y + \varepsilon)$ , which is approximately equal to  $f_{XY}(x, y)\varepsilon^2$  if  $\varepsilon$  is small.

**Definition 2.5** (Conditional probability distribution function). If  $X$  and  $Y$  are continuous r.v., then the distribution of  $X$  conditional on  $Y = y$ , which we denote by  $f_{X|Y}(x, y)$ , satisfies:

$$f_{X|Y}(x, y) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}(x < X \leq x + \varepsilon | Y = y)}{\varepsilon}.$$

**Proposition 2.1** (Conditional probability distribution function). *We have*

$$f_{X|Y}(x, y) = \frac{f_{XY}(x, y)}{f_Y(y)}.$$

*Proof.* We have:

$$\begin{aligned} f_{X|Y}(x, y) &= \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}(x < X \leq x + \varepsilon | Y = y)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{P}(x < X \leq x + \varepsilon | y < Y \leq y + \varepsilon) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \frac{\mathbb{P}(x < X \leq x + \varepsilon, y < Y \leq y + \varepsilon)}{\mathbb{P}(y < Y \leq y + \varepsilon)} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \frac{\varepsilon^2 f_{XY}(x, y)}{\varepsilon f_Y(y)}. \end{aligned}$$

□

**Definition 2.6** (Independent random variables). Consider two r.v.,  $X$  and  $Y$ , with respective c.d.f.  $F_X$  and  $F_Y$ , and respective p.d.f.  $f_X$  and  $f_Y$ .

These random variables are independent if and only if (iff) the joint c.d.f. of  $X$  and  $Y$  (see Def. 2.3) is given by:

$$F_{XY}(x, y) = F_X(x) \times F_Y(y),$$

or, equivalently, iff the joint p.d.f. of  $(X, Y)$  (see Def. 2.4) is given by:

$$f_{XY}(x, y) = f_X(x) \times f_Y(y).$$

We have the following:

1. If  $X$  and  $Y$  are independent,  $f_{X|Y}(x, y) = f_X(x)$ . This implies, in particular, that  $\mathbb{E}(g(X)|Y) = \mathbb{E}(g(X))$ , where  $g$  is any function.
2. If  $X$  and  $Y$  are independent, then  $\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X))\mathbb{E}(h(Y))$  and  $\text{Cov}(g(X), h(Y)) = 0$ , where  $g$  and  $h$  are any functions.

It is important to note that the absence of correlation between two variables is not a sufficient condition to have independence. Consider for instance the case where  $X = Y^2$ , with  $Y \sim \mathcal{N}(0, 1)$ . In this case, we have  $\text{Cov}(X, Y) = 0$ , but  $X$  and  $Y$  are not independent. Indeed, we have for instance  $\mathbb{E}(Y^2 \times X) = 3$ , which is not equal to  $\mathbb{E}(Y^2) \times \mathbb{E}(X) = 1$ . (If  $X$  and  $Y$  were independent, we should have  $\mathbb{E}(Y^2 \times X) = \mathbb{E}(Y^2) \times \mathbb{E}(X)$  according to point 2 above.)

## 2.2 Law of iterated expectations

**Proposition 2.2** (Law of iterated expectations). *If  $X$  and  $Y$  are two random variables and if  $\mathbb{E}(|X|) < \infty$ , we have:*

$$\boxed{\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y)).}$$

*Proof.* (in the case where the p.d.f. of  $(X, Y)$  exists) Let us denote by  $f_X$ ,  $f_Y$  and  $f_{XY}$  the probability distribution functions (p.d.f.) of  $X$ ,  $Y$  and  $(X, Y)$ , respectively. We have:

$$f_X(x) = \int f_{XY}(x, y) dy.$$

Besides, we have (Bayes equality, Prop. 2.1):

$$f_{XY}(x, y) = f_{X|Y}(x, y)f_Y(y).$$

Therefore:

$$\begin{aligned} \mathbb{E}(X) &= \int xf_X(x)dx = \int x \underbrace{\int f_{XY}(x, y)dy}_{=f_X(x)} dx = \int \int xf_{X|Y}(x, y)f_Y(y)dydx \\ &= \int \underbrace{\left( \int xf_{X|Y}(x, y)dx \right)}_{\mathbb{E}[X|Y=y]} f_Y(y)dy = \mathbb{E}(\mathbb{E}[X|Y]). \end{aligned}$$

□

**Example 2.1** (Mixture of Gaussian distributions). By definition,  $X$  is drawn from a mixture of Gaussian distributions if:

$$X = B \times Y_1 + (1 - B) \times Y_2,$$

where  $B$ ,  $Y_1$  and  $Y_2$  are three independent variables drawn as follows:

$$B \sim \text{Bernoulli}(p), \quad Y_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), \quad \text{and} \quad Y_2 \sim \mathcal{N}(\mu_2, \sigma_2^2).$$

Figure 2.3 displays the pdfs associated with three different mixtures of Gaussian distributions. (This web-interface allows to produce the pdf associated for any other parameterization.)

$$p=0.5, \mu_1=-1, \sigma_1=1, \mu_2=2, \sigma_2=1, \quad p=0.1, \mu_1=-2, \sigma_1=0.5, \mu_2=2, \sigma_2=1, \quad p=0.3, \mu_1=-2, \sigma_1=2, \mu_2=1, \sigma_2=1,$$

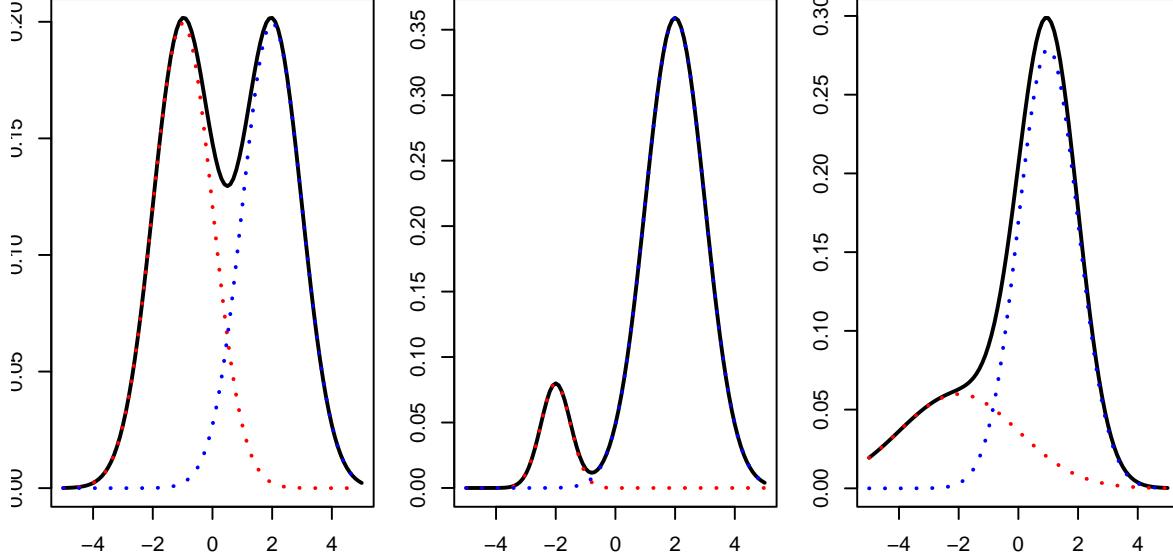


Figure 2.3: Example of pdfs of mixtures of Gaussian distributions.

The law of iterated expectations gives:

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|B)) = \mathbb{E}(B\mu_1 + (1 - B)\mu_2) = p\mu_1 + (1 - p)\mu_2.$$

**Example 2.2** (Buffon (1733)'s needles). Suppose we have a floor made of parallel strips of wood, each the same width [ $w = 1$ ]. We drop a needle, of length  $1/2$ , onto the floor. What is the probability that the needle crosses the grooves of the floor?

Let's define the random variable  $X$  by

$$X = \begin{cases} 1 & \text{if the needle crosses a line} \\ 0 & \text{otherwise.} \end{cases}$$

Conditionally on  $\theta$ , it can be seen that we have  $\mathbb{E}(X|\theta) = \cos(\theta)/2$  (see Figure 2.4).

It is reasonable to assume that  $\theta$  is uniformly distributed on  $[-\pi/2, \pi/2]$ , therefore:

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\theta)) = \mathbb{E}(\cos(\theta)/2) = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \cos(\theta) \left( \frac{d\theta}{\pi} \right) = \frac{1}{\pi}.$$

[This web-interface allows to simulate the present experiment (select Worksheet “Buffon’s needles”).]

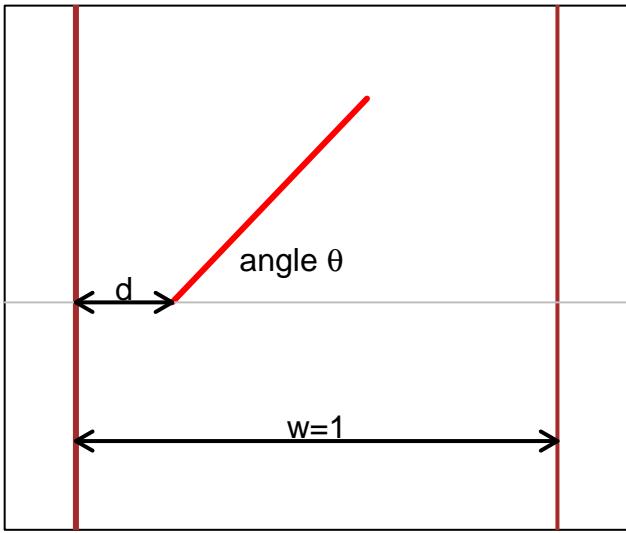


Figure 2.4: Schematic representation of the problem.

## 2.3 Law of total variance

**Proposition 2.3** (Law of total variance). *If  $X$  and  $Y$  are two random variables and if the variance of  $X$  is finite, we have:*

$$\mathbb{V}ar(X) = \mathbb{E}(\mathbb{V}ar(X|Y)) + \mathbb{V}ar(\mathbb{E}(X|Y)).$$

*Proof.* We have:

$$\begin{aligned} \mathbb{V}ar(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= \mathbb{E}(\mathbb{E}(X^2|Y)) - \mathbb{E}(X)^2 \\ &= \mathbb{E}(\mathbb{E}(X^2|Y) - \mathbb{E}(X|Y)^2) + \mathbb{E}(\mathbb{E}(X|Y)^2) - \mathbb{E}(X)^2 \\ &= \underbrace{\mathbb{E}(\mathbb{E}(X^2|Y) - \mathbb{E}(X|Y)^2)}_{\mathbb{V}ar(X|Y)} + \underbrace{\mathbb{E}(\mathbb{E}(X|Y)^2) - \mathbb{E}(\mathbb{E}(X|Y))^2}_{\mathbb{V}ar(\mathbb{E}(X|Y))}. \end{aligned}$$

□

**Example 2.3** (Mixture of Gaussian distributions (cont'd)). Consider the case of a mixture of Gaussian distributions (Example 2.1). We have:

$$\begin{aligned} \mathbb{V}ar(X) &= \mathbb{E}(\mathbb{V}ar(X|B)) + \mathbb{V}ar(\mathbb{E}(X|B)) \\ &= p\sigma_1^2 + (1-p)\sigma_2^2 + p(1-p)(\mu_1 - \mu_2)^2. \end{aligned}$$

## 2.4 About consistent estimators

The objective of econometrics is to estimate parameters out of data observations (samples). Examples of parameters of interest include, among many others: causal effect of a variable on another, elasticities, parameters defining some distribution of interest, preference parameters (risk aversion)...

Except in degenerate cases, the estimates are different from the “true” (or *population*) value. A good estimator is expected to converge to the true value when the sample size increases. That is, we are interested in the *consistency* of the estimator.

Denote by  $\hat{\theta}_n$  the estimate of  $\theta$  based on a sample of length  $n$ . We say that  $\hat{\theta}$  is a consistent estimator of  $\theta$  if, for any  $\varepsilon > 0$  (even if very small), the probability that  $\hat{\theta}_n$  is in  $[\theta - \varepsilon, \theta + \varepsilon]$  goes to 1 when  $n$  goes to  $\infty$ . Formally:

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\hat{\theta}_n \in [\theta - \varepsilon, \theta + \varepsilon]) = 1.$$

That is,  $\hat{\theta}$  is a consistent estimator if  $\hat{\theta}_n$  converges in probability (Def. 10.16) to  $\theta$ . Note that there exist different types of stochastic convergence.<sup>1</sup>

<sup>1</sup> Appendix 10.3.3 notably provides the definitions of the convergence in distribution (Def. 10.19) and the mean-square convergence (Def. 10.17).

**Example 2.4** (Example of non-convergent estimator). Assume that  $X_i \sim i.i.d.\text{Cauchy}$  with a location parameter of 1 and a scale parameter of 1 (Def. 10.14). The sample mean  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$  does not converge in probability. This is because a Cauchy distribution has no mean; hence the law of large numbers (Theorem 3.1) does not apply.

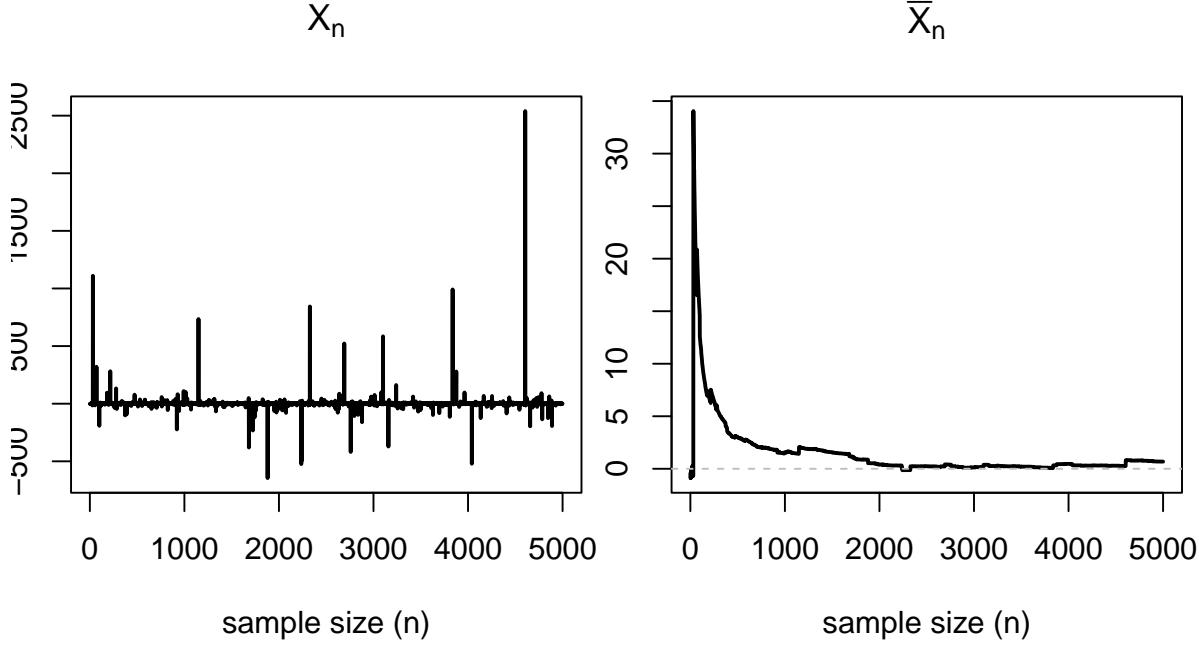


Figure 2.5: Simulation of  $\bar{X}_n$  when  $X_i \sim i.i.d.\text{Cauchy}$ .

# Chapter 3

## Central Limit Theorem

The law of large numbers (LLN) and the central limit theorem (CLT) are two fundamental theorems of probability. The former states that the sample mean converges in probability to the population mean (when it exists). The latter states that the distribution of the sum (or average) of a large number of independent, identically distributed (i.i.d.) variables with finite variance is approximately normal, regardless of the underlying distribution (as long as it features a finite variance).

### 3.1 Law of large numbers

**Definition 3.1** (Sample and Population). In statistics, a *sample* refers to a finite set of observations drawn from a *population*.

Most of the time, it is necessary to use samples for research, because it is impractical to study the whole population. Typically, it is rare to observe the population mean, and we often use instead sample means (or averages). The sample average of  $\{x_1, \dots, x_n\}$  is given by:

$$\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i.$$

This sample mean is an estimator of the true (population) mean  $\mathbb{E}(x_i)$  (assuming the latter exists). According to the law of large numbers (Theorem 3.1), this estimator is *consistent*. (see Def. 10.16 for the definition of convergence in probability.)

**Theorem 3.1** (Law of large numbers). *The sample mean is a consistent estimator of the population mean. In other words, the sample mean converges in probability to the population mean.*

*Proof.* See Appendix 10.3.4. □

**Example 3.1.** Suppose one is interested in the average number of doctor consultations over 12 months, for Swiss people. One can get an estimate by computing the sample average of a large dataset, for instance using the Swiss Household Panel (SHP):

Figure 3.1 shows the empirical distribution of the number of doctor visits.

```
library(AEC) # to load the shp data-frame.  
Nb.doct.visits <- shp$p19c15  
Nb.doct.visits <- Nb.doct.visits[Nb.doct.visits>=0] # remove irrelevant observations  
par(plt=c(.15,.95,.2,.95))  
barplot(table(Nb.doct.visits),xlim=c(0,25))  
  
cbind(mean(Nb.doct.visits),var(Nb.doct.visits))  
  
## [,1] [,2]  
## [1,] 5.253816 79.50575
```

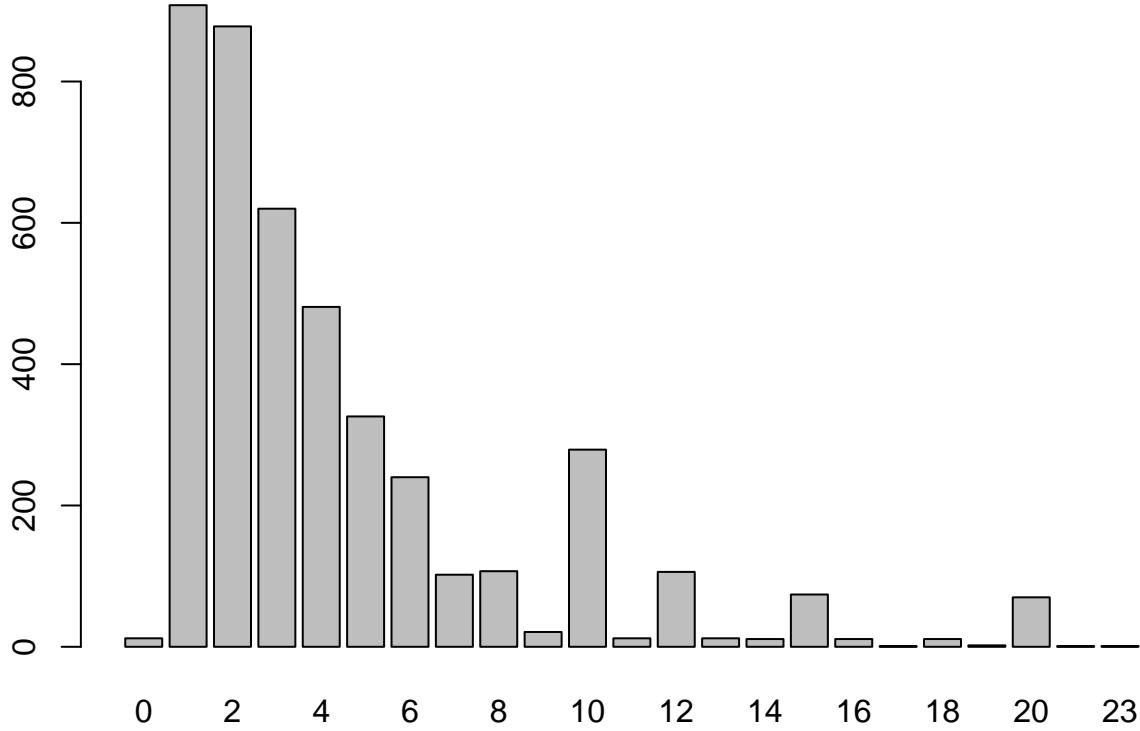


Figure 3.1: Distribution of the number of doctor visits. Source: Swiss Household Panel.

Let us compute the sample mean and variance:

```
cbind(mean(Nb.doct.visits), var(Nb.doct.visits))
```

```
##          [,1]      [,2]
## [1,] 5.253816 79.50575
```

How to know whether the sample mean is a good estimate of the true (population) average? In other words, how to measure the accuracy of the estimator?

A first answer is provided by the Chebychev inequality. Assume the  $x_i$ 's are independently drawn from a distribution of mean  $\mu$  and variance  $\sigma^2$ . We have:

$$\mathbb{E}(\bar{x}_n) = \mu \quad \text{and} \quad \mathbb{V}ar(\mu - \bar{x}_n) = \frac{1}{n}\sigma^2 \xrightarrow{n \rightarrow \infty} 0.$$

The Chebychev inequality (Proposition 10.11) states that, for any  $\varepsilon > 0$  (even very small), we have:

$$\mathbb{P}(|\bar{x}_n - \mu| > \varepsilon) \leq \frac{\mathbb{V}ar(\mu - \bar{x}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \xrightarrow{n \rightarrow \infty} 0.$$

Consider for instance the estimation of the average number of doctor visits (Example 3.1). In this example, the sample length is  $n = 4389$ , we have  $\bar{x}_n = 5.25$ , and the variance  $\sigma^2$  was approximately equal to 79.51. Therefore, taking  $\varepsilon = 0.5$ , we have that  $\mathbb{P}(\bar{x}_n - 0.5 < \mu < \bar{x}_n + 0.5)$  is lower than  $\frac{\sigma^2}{n\varepsilon^2} \approx 0.072$ .

However, this only gives bounds for such probabilities. The central limit theorem provides richer information, as it gives the approximate distribution of the estimation error.

## 3.2 Central Limit Theorem (CLT)

**Theorem 3.2** (Lindberg-Levy Central limit theorem, CLT). *If  $x_n$  is an i.i.d. sequence of random variables with mean  $\mu$  and (finite) variance, then:*

$$\sqrt{n}(\bar{x}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \quad \text{where} \quad \bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i.$$

[“ $\xrightarrow{d}$ ” denotes the convergence in distribution (see Def. 10.19)]

*Proof.* See Appendix 10.3.4. □

According to the CLT, when  $n$  is large and whatever the distribution of the  $x_i$ 's (as long as it features an average  $\mu$  and a variance  $\sigma^2$ ), the error between  $\bar{x}_n$  and  $\mu$  can be seen as a random variable that is normally distributed; it is of mean zero and its standard deviation is  $\sigma/\sqrt{n}$ . (We then say that the convergence rate is *in*  $\sqrt{n}$ .) The knowledge of the quantiles of the normal distribution can then be used to compute approximate confidence intervals for  $\mu$  (based on the sample  $\{x_1, \dots, x_n\}$ ).

This web interface illustrates the CLT with simulations (select the “CLT” block).

The CL theorem was first postulated by the mathematician Abraham de Moivre. In an article published in 1733, he used the normal distribution to approximate the distribution of the number of heads resulting from many tosses of a fair coin. This finding was nearly forgotten until Pierre-Simon Laplace recalled it in his *Théorie analytique des probabilités*, published in 1812. Laplace expanded De Moivre's finding by approximating the binomial distribution with the normal distribution.

**Exercise 3.1.** Consider Buffon's experiment (Example 2.2). We have seen that  $\pi$  can be estimated by counting the fraction of times the needles cross the grooves of the floor.<sup>1</sup> Using the CLT, determine the (approximate) minimal number  $n$  of needles that one has to throw to get an estimate  $\hat{\pi} (= 1/\bar{X}_n)$  of  $\pi$  that is such that there is a 95% chance that  $\pi \in [\hat{\pi} - 0.01, \hat{\pi} + 0.01]$ ?

**Exercise 3.2.** Assume you have a lot of time and a coin. How would you approximate the 100 percentiles of the  $\mathcal{N}(0, 1)$  distribution?

The CLT can be extended to the case where each  $x_i$  is a vector. (In that case, and if the dimension of  $x_i$  is  $m$ , then  $\text{Var}(x_i)$  is a  $m \times m$  matrix.) Here is the multivariate CLT:

**Theorem 3.3** (Multivariate Central limit theorem, CLT). *If  $\mathbf{x}_n$  is an i.i.d. sequence of  $m$ -dimensional random variables with mean  $\mu$  and variance  $\Sigma$ , where  $\Sigma$  is a positive definite matrix, then:*

$$\sqrt{n}(\bar{\mathbf{x}}_n - \mu) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma), \quad \text{where} \quad \bar{\mathbf{x}}_n = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i.$$

### 3.3 Comparison of sample means

As said at the beginning of this chapter, the CLT is a key statistical result that has numerous applications. An important one is the statistical comparison of sample means:

Consider two samples  $\{m_1, \dots, m_{n_m}\}$  and  $\{w_1, \dots, w_{n_w}\}$ . Assume that the  $m_i$ 's (respectively the  $w_i$ 's) are i.i.d., with mean  $\mu_m$  and variance  $\sigma_m^2$  and of  $w_i$ 's (resp. with mean  $\mu_w$  and variance  $\sigma_w^2$ ).

If  $n$  is large, according to the CLT, we approximately have:

$$\sqrt{n_m}(\bar{m} - \mu_m) \sim \mathcal{N}(0, \sigma_m^2) \quad \text{and} \quad \sqrt{n_w}(\bar{w} - \mu_w) \sim \mathcal{N}(0, \sigma_w^2),$$

or

$$\bar{m} \sim \mathcal{N}\left(\mu_m, \frac{\sigma_m^2}{n_m}\right) \quad \text{and} \quad \bar{w} \sim \mathcal{N}\left(\mu_w, \frac{\sigma_w^2}{n_w}\right).$$

if the  $m_i$ 's and the  $w_i$ 's are independent, then  $\bar{m}$  and  $\bar{w}$  also are, and we get:

$$\bar{m} - \bar{w} \sim \mathcal{N}(\mu_m - \mu_w, \sigma^2) \quad \text{with} \quad \sigma^2 = \frac{\sigma_m^2}{n_m} + \frac{\sigma_w^2}{n_w}.$$

This can be used to test the null hypothesis  $H_0 : \mu_m - \mu_w = 0$  (see Section 4). Indeed under this assumption, the probability that  $\bar{m} - \bar{w}$  is in  $[-1.96\sigma, 1.96\sigma]$  (respectively in  $[-2.58\sigma, 2.58\sigma]$ ) is approximately 0.95 (respectively 0.99) [see Table 10.1].

For instance, if we find, using our sample, that  $\bar{m} - \bar{w}$  is not in  $[-2.58\sigma, 2.58\sigma]$ , then we reject the null hypothesis at the 1% significance level.

---

<sup>1</sup>More precisely, the expectation of this fraction is equal to  $1/\pi$ .

**Example 3.2** (Comparison of sample means). Let us use the Swiss Household Panel (SHP) dataset to test the equality between men and women incomes, for Swiss people under the age of 35.

The next lines of codes construct two samples of incomes; one for men (`income.m`), and one for women (`income.w`).

```
library(AEC);library(logKDE)
dataset.m <- subset(shp,(sex19==1)&(age19<=35))
dataset.w <- subset(shp,(sex19==2)&(age19<=35))
income.m <- dataset.m$i19ptotg
income.w <- dataset.w$i19ptotg
income.m <- income.m[income.m>=0] # remove irrelevant data
income.w <- income.w[income.w>=0] # remove irrelevant data
```

Figure 3.2 displays the empirical densities of incomes for men and women. Vertical dashed lines indicate sample means:

```
par(plt=c(.1,.95,.2,.95)) # adjust margins
plot(logdensity(income.w),lwd=2,xlim=c(0,200000),main="",
     xlab="Yearly gross income (in CHF)",ylab="")
lines(logdensity(income.m),col="red",lwd=2)
abline(v=mean(income.m),col="red",lty=3,lwd=2)
abline(v=mean(income.w),col="black",lty=3,lwd=2)
legend("topright",c("men yearly income","women"),lty=1,lwd=2,col=c("red","black"))
```

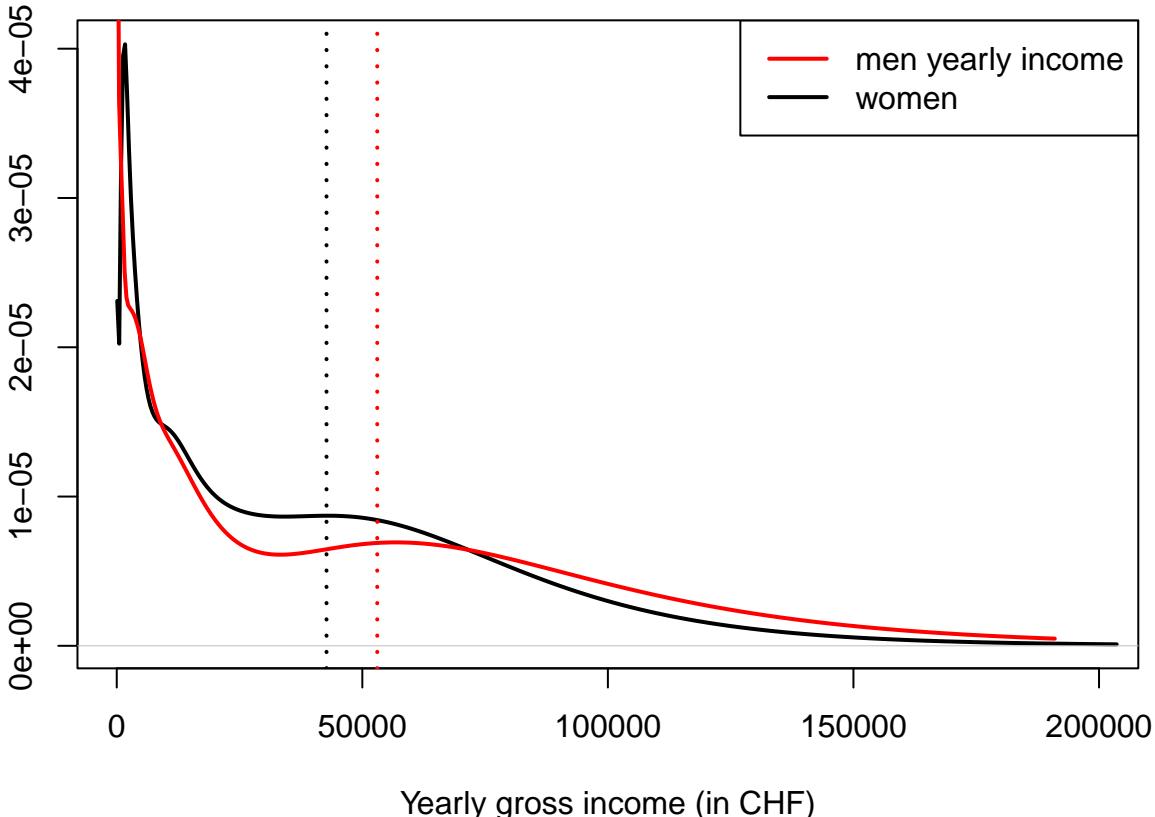


Figure 3.2: Distribution of yearly gross incomes for Swiss residents under the age of 35. Source: SHP. Vertical dashed lines indicates the sample mean.

Let us compute  $\bar{m}$ ,  $\bar{w}$ , and  $\sigma$ :

```
m.bar <- mean(income.m)
w.bar <- mean(income.w)
sigma <- sqrt(var(income.m)/length(income.m) + var(income.w)/length(income.w))
print(c(m.bar,w.bar,sigma))

## [1] 53035.023 42708.227 2427.425
```

Under the “equality assumption”, there would be a probability of only 5% (respectively 1%) for  $\bar{m} - \bar{w}$  not to be in [-4758,4758] (respectively in [-6263,6263]). Since  $\bar{m} - \bar{w}$  is equal to 10327, we reject the null hypothesis at the 5%, and even 1%, significance levels.



# Chapter 4

## Statistical tests

We run a statistical test when we want to know whether some hypothesis about a vector of parameters  $\theta$  —that is imperfectly observed— is consistent with data that are seen as random, and whose randomness depend on  $\theta$ .

Typically, assume you observe a sample  $\mathbf{x} = \{x_1, \dots, x_n\}$  where the  $x_i$ 's are i.i.d., of mean  $\mu$  and variance  $\sigma^2$  (these two parameters being unobserved). One may want to know whether  $\mu = 0$  or, maybe, whether  $\sigma = 1$ .

The hypothesis the researcher wants to test is called the *null hypothesis*. It is often denoted by  $H_0$ . It is a conjecture about a given property of a population. Without loss of generality, it can be stated as:

$$H_0 : \{\theta \in \Theta\}.$$

It can also be defined through a function  $h$  (say):<sup>1</sup>

$$H_0 : h(\theta) = 0.$$

The *alternative hypothesis*, often denoted  $H_1$  is then defined by  $H_1 : \{\theta \in \Theta^c\}$ .<sup>2</sup>

The ingredients of a statistical test are:

- a vector of (unknown) parameters ( $\theta$ ),
- a **test statistic**, that is a function of the sample components ( $S(\mathbf{x})$ , say), and
- a **critical region** ( $\Omega$ , say), defined as a set of implausible values of  $S$  under  $H_0$ .

To implement a test statistic, we need to know the distribution of  $S$  under the null hypothesis  $H_0$ . Equipped, with such a distribution, we compute  $S(\mathbf{x})$  and we look at its location; more specifically, we look whether it lies within the critical region  $\Omega$ . The latter corresponds to “implausible” regions of this distribution (typically the tails). If  $S \in \Omega$ , then we reject the null hypothesis. Loosely speaking; it amounts to saying: *if  $H_0$  were true, it would have been unlikely to get such a large (or small) draw for  $S$ .*

Hence, there are two possible outcomes for a statistical test:

- $H_0$  is rejected if  $S \in \Omega$ ;
- $H_0$  is not rejected if  $S \notin \Omega$ .

Except in extreme cases, there is always a non-zero probability to reject  $H_0$  while it is true (**Type I error**, or **false positive**), or to fail to reject it while it is false (**Type II error**, or **false negative**).

This vocabulary is widely used. For instance, the notions of false positive and false negative are used in the context of Early Warning Signals (see Example ??).

---

<sup>1</sup>We can relate the previous and the next equation through  $\Theta = \{\theta, s.t. h(\theta) = 0\}$ .

<sup>2</sup>Note that researchers may want to find support for the null hypothesis  $H_0$  as well as for the alternative hypothesis  $H_1$ .

**Example 4.1** (Early Warning Signals). These are approaches that aim to detect the occurrence of crises in advance. See, e.g., ECB (2014), for applications to financial crises.

To implement such approaches, researchers look for signals/indices forecasting crises. Suppose one index ( $W$ , say) tends to be large before financial crises; one may define an EWS by predicting a crisis when  $W > a$ , where  $a$  is a given threshold. It is easily seen that the lower (respectively higher)  $a$ , the larger the fraction of FP (resp. of FN).

## 4.1 Size and power of a test

**Definition 4.1** (Size and Power of a test). For a given test,

- the probability of type-I errors, denoted by  $\alpha$ , is called the **size**, or **significance level**, of the test,
- the **power** of a test is equal to  $1 - \beta$ , where  $\beta$  is the probability of type-II errors.

Formally, the previous definitions can be written as follows:

$$\alpha = \mathbb{P}(S \in \Omega | H_0) \quad (\text{Proba. of a false positive}) \quad (4.1)$$

$$\beta = \mathbb{P}(S \notin \Omega | H_1) \quad (\text{Proba. of a false negative}). \quad (4.2)$$

The power is the probability that the test will lead to the rejection of the null hypothesis if the latter is false. Therefore, for a given size, we prefer tests with high power.

In most cases, there is a trade-off between size and power, which is easily understood in the EWS context (Example 4.1): increasing the threshold  $a$  reduces the fraction of FP (thereby reducing the size of the test), but it increases the fraction of FN (thereby reducing the power of the test).

## 4.2 The different types of statistical tests

How to determine the critical region? Loosely speaking, we want the critical region to be a set of “implausible” values of the test statistic  $S$  under the null hypothesis  $H_0$ . The lower the size of the test ( $\alpha$ ), the more implausible these values. Recall that, by definition of the size of the test,  $\alpha = \mathbb{P}(S \in \Omega | H_0)$ . That is, if  $\alpha$  is small, there is only a small probability that  $S$  lies in  $\Omega$  under  $H_0$ .

Consider the case where, under  $H_0$ , the distribution of the test statistic is symmetrical (e.g., normal distribution or Student-t distribution). In this case, the critical region is usually defined by the union of the two tails of the distribution. The test is then said to be a **two-tailed test** or a **two-sided test**. This situation is illustrated by Figures 4.1 and 4.2. (Use this web interface to explore alternative situations.)

Figure 4.2 also illustrates the notion of *p-value* (in the case of a two-sided test). The p-value can be defined as the value of the size of the test  $\alpha$  that is such that the computed test statistic,  $S$ , is at the “frontier” of the critical region. Given this definition, if the p-value is smaller (respectively larger) than the size of the test, we reject (resp. cannot reject) the null hypothesis at the  $\alpha$  significance level.

Figures 4.3 and 4.4 illustrate the **one-tailed**, or **one-sided** situation. These tests are typically employed when the distribution of the test statistic under the null hypothesis has a support on  $\mathbb{R}^+$  (e.g., the chi-square distribution  $\chi^2$ , see Def. 10.13). Figure 4.4 also illustrates the notion of p-value associated with a one-sided statistical test.

**Example 4.2** (A practical illustration of size and power). Consider a factory that produces metal cylinders whose diameter has to be equal to 1,cm. The tolerance is  $a = 0.01$  cm. More precisely, more than 90% of the pieces have to satisfy the tolerance for the whole production (say 1.000.000 pieces) to be bought by the client.

The production technology is such that a proportion  $\theta$  (imperfectly known) of the pieces does not satisfy the tolerance. The (population) parameter  $\theta$  could be computed by measuring

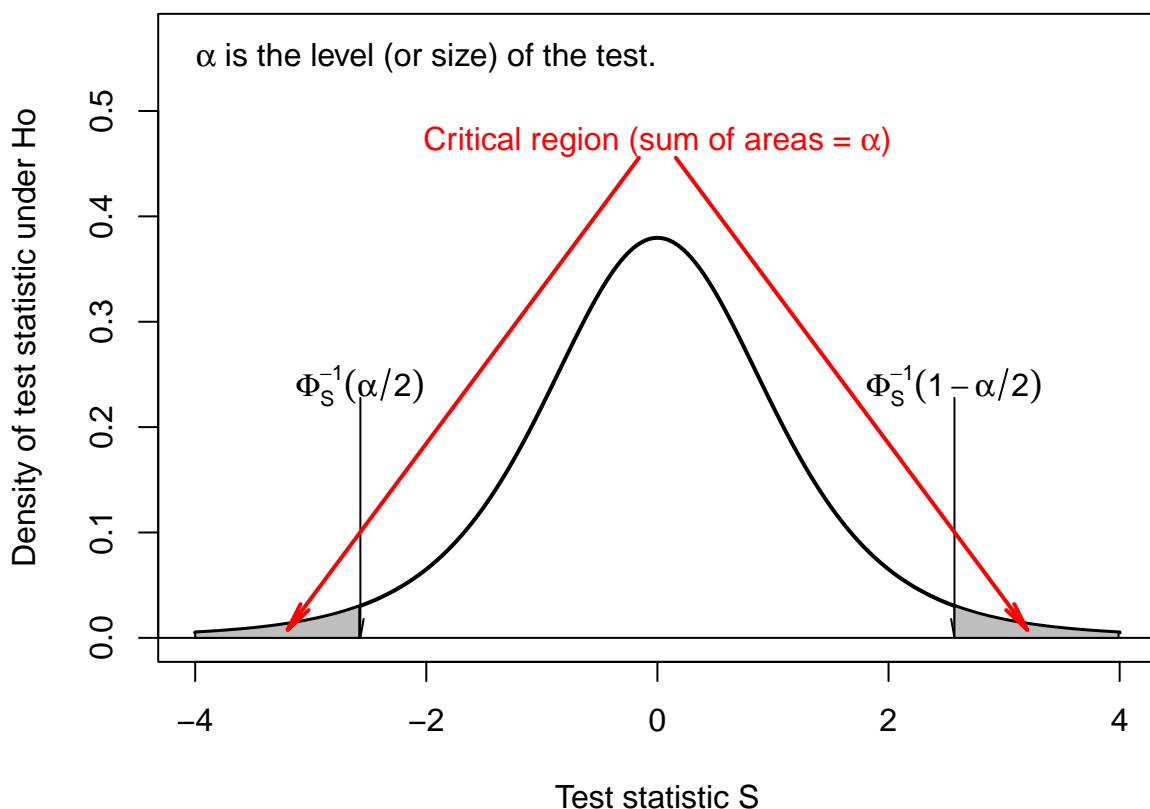


Figure 4.1: Two-sided test. Under  $H_0$ ,  $S \sim t(5)$ .  $\alpha$  is the size of the test.

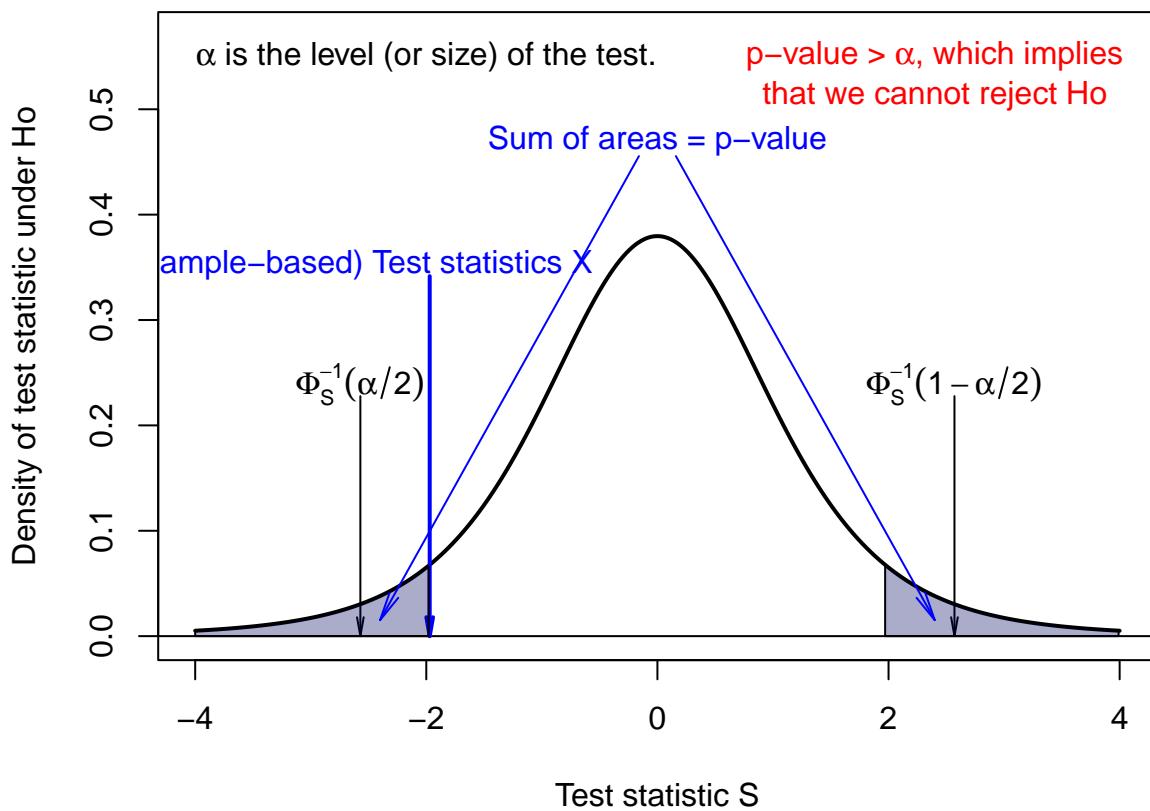


Figure 4.2: Two-sided test. Under  $H_0$ ,  $S \sim t(5)$ .  $\alpha$  is the size of the test.

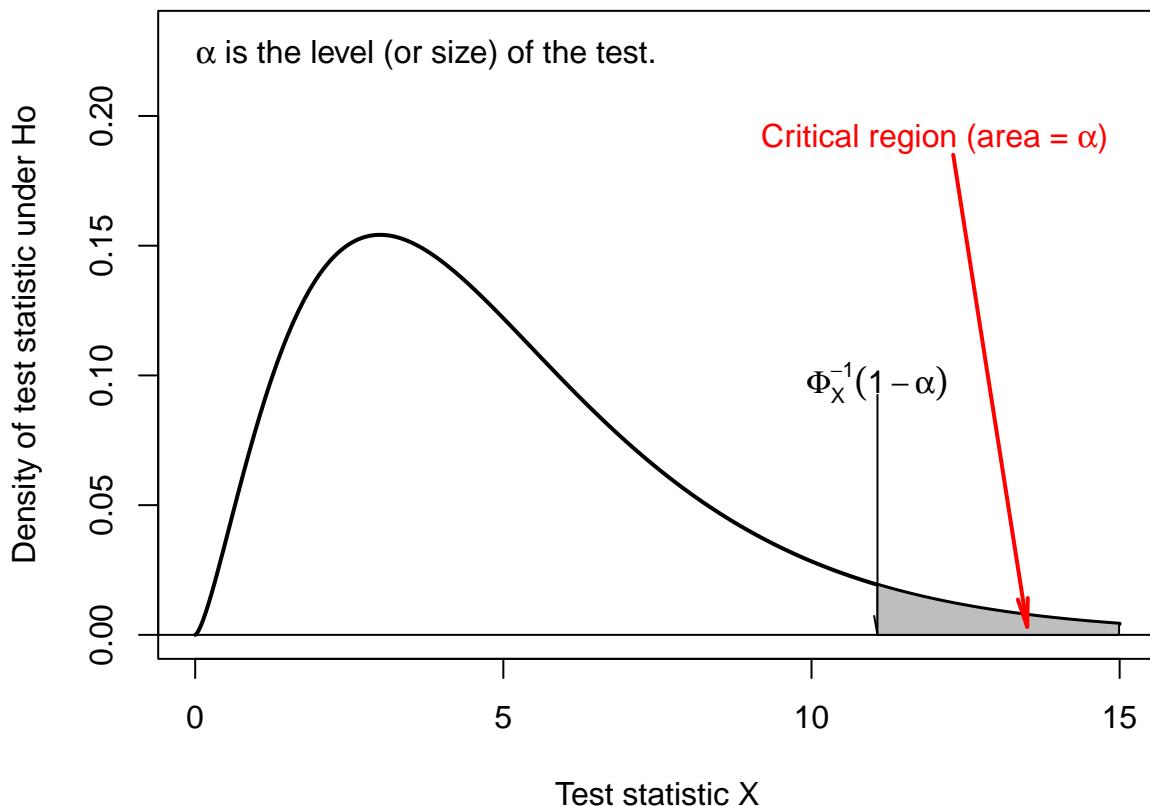


Figure 4.3: One-sided test. Under  $H_0$ ,  $S \sim \chi^2(5)$ .  $\alpha$  is the size of the test.

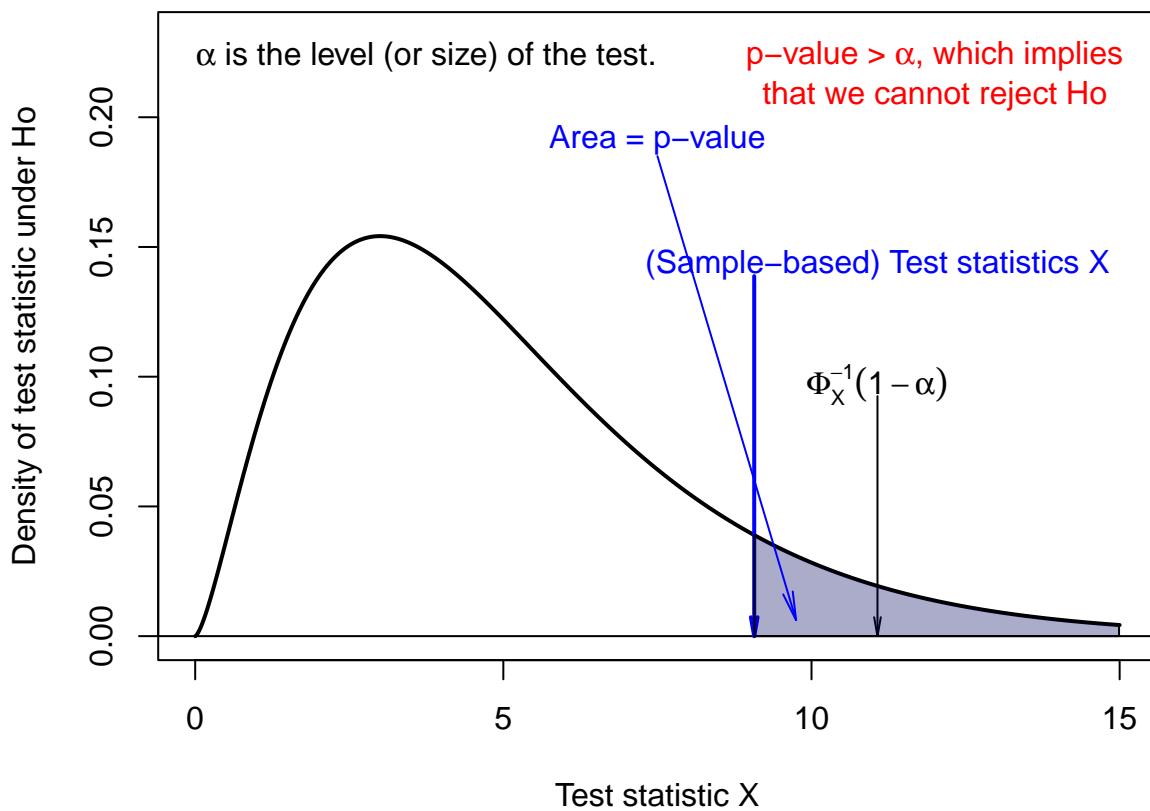


Figure 4.4: One-sided test. Under  $H_0$ ,  $S \sim \chi^2(5)$ .  $\alpha$  is the size of the test.

all the pieces but this would be costly. Instead, it is decided that  $n \ll 1.000.000$  pieces will be measured. In this context, the null hypothesis  $H_0$  is  $\theta < 10\%$ . The producing firm would like  $H_0$  to be true.

Let us denote by  $d_i$  the binary indicator defined as:

$$d_i = \begin{cases} 0 & \text{if the size of the } i^{\text{th}} \text{ cylinder is in } [1 - a, 1 + a]; \\ 1 & \text{otherwise.} \end{cases}$$

We set  $x_n = \sum_{i=1}^n d_i$ . That is,  $x_n$  is the number of measured pieces that do not satisfy the tolerance (out of  $n$ ).

The decision rule is: accept  $H_0$  if  $\frac{x_n}{n} \leq b$ , reject otherwise. A natural choice for  $b$  is  $b = 0.1$ . However, this would not be a conservative choice, since it remains likely that  $x_n < 0.1$  even if  $\mathbb{E}(x_n) = \theta > 0.1$  (especially if  $n$  is small). Hence, if one chooses  $b = 0.1$ , the probability of false negative may be high.

In this simple example, the size and the power of the test can be computed analytically. The test statistic is  $S_n = \frac{x_n}{n}$  and the critical region is  $\Omega = [b, 1]$ . The probability to reject  $H_0$  is:

$$\mathbb{P}_\theta(S_n \in \Omega) = \sum_{i=b \times n+1}^n C_n^i \theta^i (1-\theta)^{n-i}.$$

When  $\theta < 0.1$ , the previous expression gives the size of the test, while it gives the power of the test when  $\theta > 0.1$ . Figure ?? shows how this probability depends on  $\theta$  for two sample sizes:  $n = 100$  (upper plot) and  $n = 500$  (lower plot).

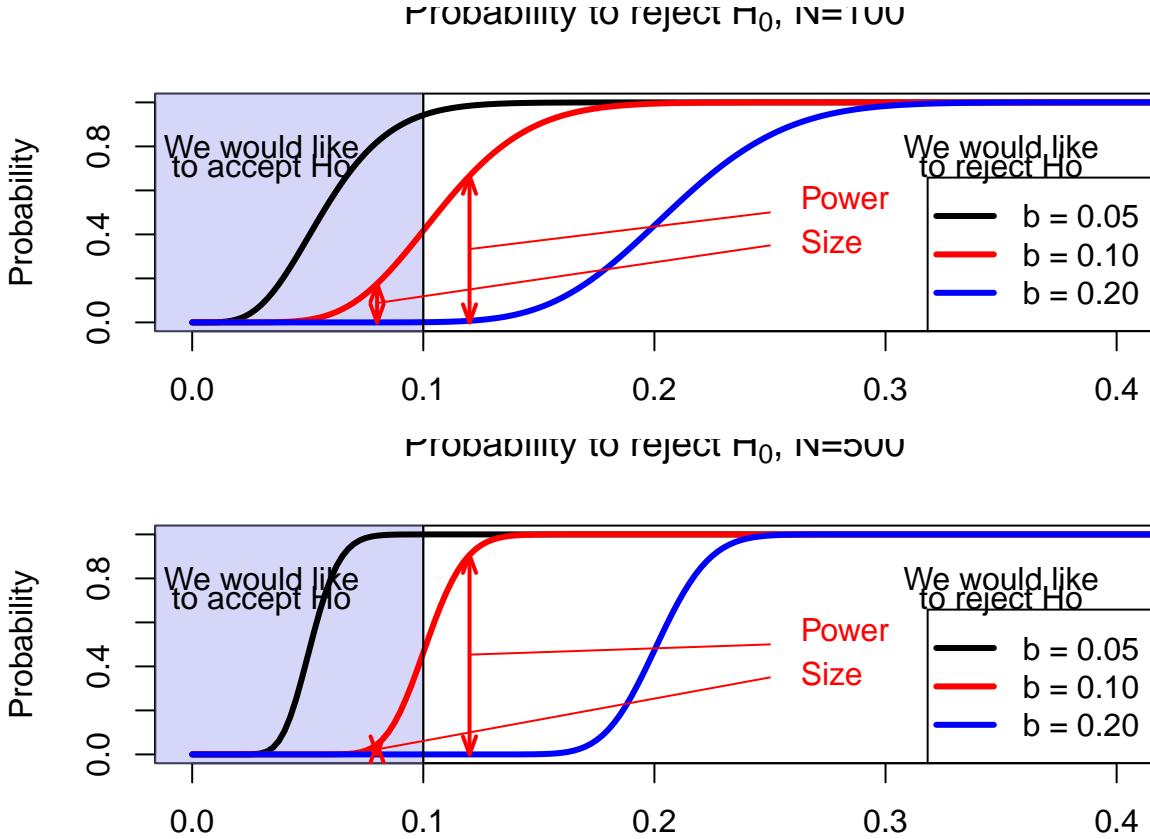


Figure 4.5: Factory example.

(Alternative situations can be explored by using this web interface.)

### 4.3 Asymptotic properties of statistical tests

In some cases, the distribution of the test statistic is known only asymptotically. For instance, it may be the case that the distribution of a given test statistic becomes normal only in large samples (e.g., related to the CLT, see Section 3). The level of the test is then not known for small sample sizes, but only asymptotically, i.e., when  $n$  becomes infinitely large. That defines the asymptotic level of the test:

**Definition 4.2** (Asymptotic level). An asymptotic test with critical region  $\Omega_n$  has an asymptotic level equal to  $\alpha$  if:

$$\sup_{\theta \in \Theta} \lim_{n \rightarrow \infty} \mathbb{P}_\theta(S_n \in \Omega_n) = \alpha.$$

**Example 4.3** (The factory example). Let us come back to the factory example (Example 4.2). Because  $S_n = \bar{d}_n$ , and since  $\mathbb{E}(d_i) = \theta$  and  $\text{Var}(d_i) = \theta(1 - \theta)$ , the CLT (Theorem 10.5) leads to:

$$S_n \sim \mathcal{N}\left(\theta, \frac{1}{n}\theta(1 - \theta)\right) \quad \text{or} \quad \frac{\sqrt{n}(S_n - \theta)}{\sqrt{\theta(1 - \theta)}} \sim \mathcal{N}(0, 1).$$

Hence,  $\mathbb{P}_\theta(S_n \in \Omega_n) = \mathbb{P}_\theta(S_n > b) \approx 1 - \Phi\left(\frac{\sqrt{n}(b - \theta)}{\sqrt{\theta(1 - \theta)}}\right)$ . Since function  $\theta \rightarrow 1 - \Phi\left(\frac{\sqrt{n}(b - \theta)}{\sqrt{\theta(1 - \theta)}}\right)$  increases w.r.t.  $\theta$ , we have:

$$\begin{aligned} \sup_{\theta \in \Theta = [0, 0.1]} \mathbb{P}_\theta(S_n > b_n) &= \mathbb{P}_{\theta=0.1}(S_n \in \Omega_n) \approx \\ &1 - \Phi\left(\frac{\sqrt{n}(b_n - 0.1)}{0.3}\right). \end{aligned}$$

Hence, if we set  $b_n = 0.1 + 0.3\Phi^{-1}(1 - \alpha)/\sqrt{n}$ , we have  $\sup_{\theta \in \Theta = [0, 0.1]} \mathbb{P}_\theta(S_n > b_n) \approx \alpha$  for large values of  $n$ .

Let us now turn to the power of the test. Although it is often difficult to compute the power of the test, it is sometimes feasible to demonstrate that the power of the test converges to one when  $n$  goes to  $+\infty$ . In that case, if  $H_0$  is false, the probability to reject it tends to be close to one in large samples.

**Definition 4.3** (Asymptotically consistent test). An asymptotic test with critical region  $\Omega_n$  is consistent if:

$$\forall \theta \in \Theta^c, \quad \mathbb{P}_\theta(S_n \in \Omega_n) \rightarrow 1.$$

**Example 4.4** (The factory example). Let us come back to the factory example (Example 4.2). We proceed under the assumption that  $\theta > 0.1$  and we consider  $b_n = b = 0.1$ . We still have:

$$\mathbb{P}_\theta(S_n \in \Omega_n) = \mathbb{P}_\theta(S_n > b) \approx 1 - \Phi\left(\frac{\sqrt{n}(b - \theta)}{\sqrt{\theta(1 - \theta)}}\right).$$

Because  $\frac{\sqrt{n}(b - \theta)}{\sqrt{\theta(1 - \theta)}} \xrightarrow[n \rightarrow \infty]{} -\infty$ , we have

$$\mathbb{P}_\theta(S_n > b) \approx 1 - \underbrace{\Phi\left(\frac{\sqrt{n}(b - \theta)}{\sqrt{\theta(1 - \theta)}}\right)}_{\xrightarrow[n \rightarrow \infty]{} 0} \rightarrow 1.$$

Therefore, with  $b_n = b = 0.1$ , the test is consistent.

## 4.4 Example: Normality tests

Because many statistical results are valid only if the underlying data is normally distributed, researchers often have to conduct normality tests. Under the null hypothesis, the data at hand ( $\mathbf{y} = \{y_1, \dots, y_n\}$ , say) are drawn from a Gaussian distribution. A popular normality test is the Jarque-Bera test. It consists in verifying that the sample skewness and kurtosis of the  $y_i$ 's are consistent with those of the normal distribution. Let us first define the skewness and kurtosis of a random variable.

Let  $f$  be the p.d.f. of  $Y$ . The  $k^{th}$  **standardized moment** of  $Y$  is defined as:

$$\psi_k = \frac{\mu_k}{\left(\sqrt{\text{Var}(Y)}\right)^k},$$

where  $\mathbb{E}(Y) = \mu$  and

$$\mu_k = \mathbb{E}[(Y - \mu)^k] = \int_{-\infty}^{\infty} (y - \mu)^k f(y) dy$$

is the  $k^{th}$  **central moment** of  $Y$ . In particular,  $\mu_2 = \text{Var}(Y)$ . Therefore:

$$\psi_k = \frac{\mu_k}{(\mu_2^{1/2})^k},$$

The **skewness** of  $Y$  corresponds to  $\psi_3$  and the **kurtosis** to  $\psi_4$  (Def. 10.6).

**Proposition 4.1** (Skewness and kurtosis of the normal distribution). *For a Gaussian var., the skewness ( $\psi_3$ ) is 0 and the kurtosis ( $\psi_4$ ) is 3.*

*Proof.* For a centered Gaussian distribution,  $(-y)^3 f(-y) = -y^3 f(y)$ . This implies that

$$\begin{aligned} \int_{-\infty}^{\infty} y^3 f(y) dy &= \int_{-\infty}^0 y^3 f(y) dy + \int_0^{\infty} y^3 f(y) dy \\ &= - \int_0^{\infty} y^3 f(y) dy + \int_0^{\infty} y^3 f(y) dy = 0, \end{aligned}$$

which leads to the skewness result.

Moreover, for a Gaussian distribution,  $df(y)/dy = -yf(y)$  and therefore  $\frac{d}{dy}(y^3 f(y)) = 3y^2 f(y) - y^4 f(y)$ . Partial integration leads to the kurtosis result.  $\square$

Let us now introduce the sample analog of standardized moments. The  $k^{th}$  **central sample moment** of  $Y$  is given by:

$$m_k = \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^k,$$

and the  $k^{th}$  **standardized sample moment** of  $Y$  is given by:

$$g_k = \frac{m_k}{m_2^{k/2}}.$$

**Proposition 4.2** (Consistency of central sample moments). *If the  $k^{th}$  central moment of  $Y$ , exists and if the  $y_i$ 's are i.i.d., then the sample central moment  $m_k$  is a consistent estimate of the central moment  $\mu_k$ .*

**Proposition 4.3** (Asymptotic distribution of 3rd-order sample central moment of a normal distribution). *If  $y_i \sim$  i.i.d.  $\mathcal{N}(\mu, \sigma^2)$ , then  $\sqrt{n}g_3 \xrightarrow{d} \mathcal{N}(0, 6)$ .*

*Proof.* See, e.g. Lehmann (1999).  $\square$

**Proposition 4.4** (Asymptotic distribution of 4th-order sample central moment of a normal distribution). *If  $y_i \sim$  i.i.d.  $\mathcal{N}(\mu, \sigma^2)$ , then  $\sqrt{n}(g_4 - 3) \xrightarrow{d} \mathcal{N}(0, 24)$ .*

**Proposition 4.5** (Joint asymptotic distribution of 3rd and 4th-order sample central moments of a normal distribution). *If  $y_i \sim$  i.i.d.  $\mathcal{N}(\mu, \sigma^2)$ , then the vector  $(\sqrt{n}g_3, \sqrt{n}(g_4 - 3))$  is asymptotically bivariate Gaussian. Further its elements are uncorrelated and therefore independent.*

The Jarque-Bera statistic is defined by:

$$JB = n \left( \frac{g_3^2}{6} + \frac{(g_4 - 3)^2}{24} \right) = \frac{n}{6} \left( g_3^2 + \frac{(g_4 - 3)^2}{4} \right).$$

**Proposition 4.6** (Jarque-Bera asympt. distri.). *If  $y_i \sim$  i.i.d.  $\mathcal{N}(\mu, \sigma^2)$ ,  $JB \xrightarrow{d} \chi^2(2)$ .*

*Proof.* This directly derives from Proposition 4.5.  $\square$

**Example 4.5** (Consistency of the Jarque-Bera normality test). This example illustrates the consistency of the JB test (see Def. 10.8).

For each row of matrix  $\mathbf{x}$ , the function `JB` (defined below) computes the Jarque-Bera test statistic. (That is, each row is considered as a given sample.)

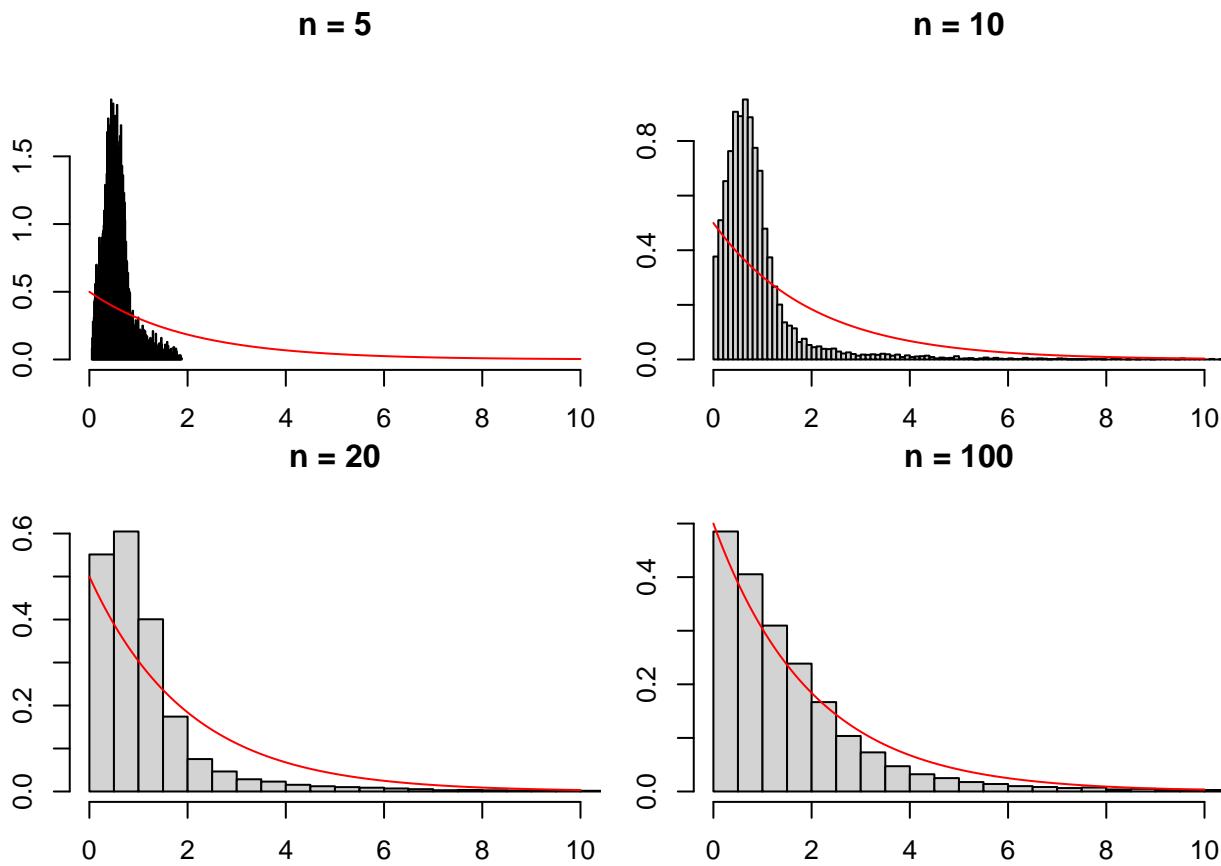
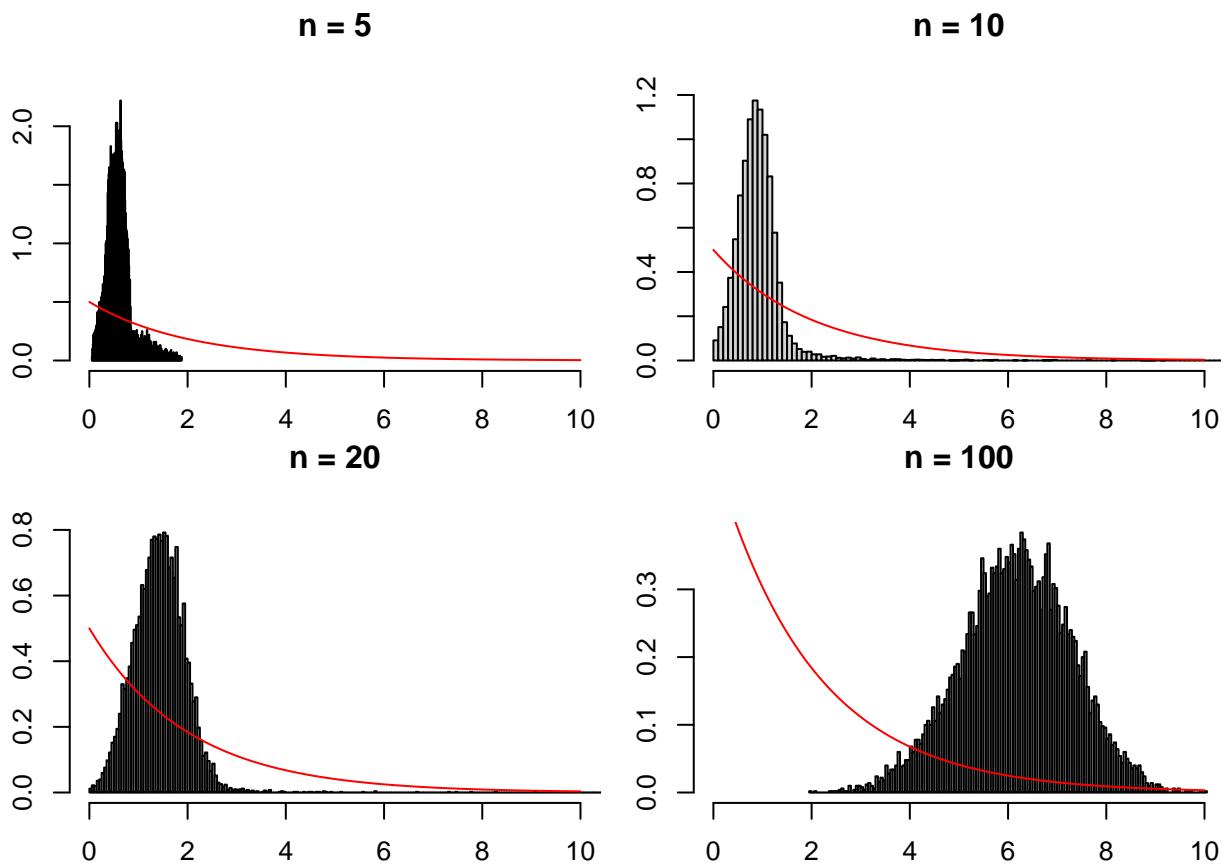
```
JB <- function(x){
  N <- dim(x)[1] # number of samples
  n <- dim(x)[2] # sample size
  x.bar <- apply(x,1,mean)
  x.x.bar <- x - matrix(x.bar,N,n)
  m.2 <- apply(x.x.bar,1,function(x){mean(x^2)})
  m.3 <- apply(x.x.bar,1,function(x){mean(x^3)})
  m.4 <- apply(x.x.bar,1,function(x){mean(x^4)})
  g.3 <- m.3/m.2^(3/2)
  g.4 <- m.4/m.2^(4/2)
  return(n*(g.3^2/6 + (g.4-3)^2/24))
}
```

Let us first consider the case where  $H_0$  is satisfied. Figure 4.6 displays, for different sample sizes  $n$ , the distribution of the JB statistics when the  $y_i$ 's are normal, consistently with  $H_0$ . It appears that when  $n$  grows, the distribution indeed converges to the  $\chi^2(2)$  distribution (as stated by Proposition 4.6).

```
all.n <- c(5,10,20,100)
nb.sim <- 10000
y <- matrix(rnorm(nb.sim*max(all.n)),nb.sim,max(all.n))

par(mfrow=c(2,2));par(plt=c(.1,.95,.15,.8))
for(i in 1:length(all.n)){
  n <- all.n[i]
  hist(JB(y[,1:n]),nclass = 200,freq = FALSE,
        main=paste("n = ",toString(n),sep=""),xlim=c(0,10))
  xx <- seq(0,10,by=.01)
  lines(xx,dchisq(xx,df = 2),col="red")
}
```

Now, replace `rnorm` with `runif`. The  $y_i$ 's are then drawn from a uniform distribution.  $H_0$  is not satisfied. Figure 4.7 then shows that, when  $n$  grows, the distributions of the JB statistic shift to the right. This results in the consistency of the JB test (see Def. 10.8).

Figure 4.6: Distribution of the JB test statistic under  $H_0$  (normality).Figure 4.7: Distribution of the JB test statistic when the  $y_i$ 's are drawn from a uniform distribution (hence  $H_0$  is not satisfied).



# Chapter 5

## Linear Regressions

**Definition 5.1.** A linear regression model is of the form:

$$y_i = \beta' \mathbf{x}_i + \varepsilon_i, \quad (5.1)$$

where  $\mathbf{x}_i = [x_{i,1}, \dots, x_{i,K}]'$  is a vector of dimension  $K \times 1$ .

For entity  $i$ , the  $x_{i,k}$ 's, for  $k \in \{1, \dots, K\}$ , are explanatory variables, regressors, or covariates. The variable of interest,  $y_i$ , is often called dependent variable, or regressand. The last term of the specification, namely  $\varepsilon_i$ , is called error, or disturbance.

The researcher is usually interested in the components of vector  $\beta$ , denoted by  $\beta_k$ ,  $k \in \{1, \dots, K\}$ . She usually aims at estimating these coefficients based on observations of  $\{y_i, \mathbf{x}_i\}$ ,  $i \in \{1, \dots, n\}$ , which constitutes a *sample*. In the following, we will denote the sample length by  $n$ .

To have an intercept in the specification (5.1), one has to set  $x_{i,1} = 1$  for all  $i$ ;  $\beta_1$  then corresponds to the intercept.

### 5.1 Hypotheses

In this section, we introduce different assumptions regarding the covariates and/or the errors. The properties of the estimators used by the researcher depend on which of these assumptions are satisfied.

**Hypothesis 5.1** (Full rank). There is no exact linear relationship among the independent variables (the  $x_{i,k}$ 's, for a given  $i \in \{1, \dots, n\}$ ).

Intuitively, when Hypothesis 5.1 is satisfied, then the estimation of the model parameters is unfeasible since, for any value of  $\beta$ , some changes in the explanatory variables will be exactly compensated by other changes in another set of explanatory variables, preventing the identification of these effects.

Let us denote by  $\mathbf{X}$  the matrix containing all explanatory variables, of dimension  $n \times K$ . (That is, row  $i$  of  $\mathbf{X}$  is  $\mathbf{x}'_i$ .) The following hypothesis concerns the relationship between the errors (gathered in  $\varepsilon$ , a  $n$ -dimensional vector) and the explanatory variables  $\mathbf{X}$ :

**Hypothesis 5.2** (Conditional mean-zero assumption).

$$\mathbb{E}(\varepsilon|\mathbf{X}) = 0. \quad (5.2)$$

Hypothesis 5.2 has important implications:

**Proposition 5.1.** Under Hypothesis 5.2:

- i.  $\mathbb{E}(\varepsilon_i) = 0$ ;
- ii. The  $x_{ij}$ 's and the  $\varepsilon_i$ 's are uncorrelated, i.e.  $\forall i, j \quad \text{Corr}(x_{ij}, \varepsilon_i) = 0$ .

*Proof.* Let us prove (i) and (ii):

i. By the law of iterated expectations:

$$\mathbb{E}(\varepsilon) = \mathbb{E}(\mathbb{E}(\varepsilon|\mathbf{X})) = \mathbb{E}(0) = 0.$$

ii.  $\mathbb{E}(x_{ij}\varepsilon_i) = \mathbb{E}(\mathbb{E}(x_{ij}\varepsilon_i|\mathbf{X})) = \mathbb{E}(x_{ij} \underbrace{\mathbb{E}(\varepsilon_i|\mathbf{X})}_{=0}) = 0.$

□

The next two hypotheses (5.3 and 5.4) concern the stochastic properties of the errors  $\varepsilon_i$ :

**Hypothesis 5.3** (Homoskedasticity).

$$\forall i, \quad \text{Var}(\varepsilon_i|\mathbf{X}) = \sigma^2.$$

The following lines of code generate a figure comparing two situations: Panel (a) of Figure 5.1 corresponds to a situation of homoskedasticity, and Panel (b) corresponds to a situation of heteroskedasticity. Let us be more specific. In the two plots, we have  $X_i \sim \mathcal{N}(0, 1)$  and  $\varepsilon_i^* \sim \mathcal{N}(0, 1)$ . In Panel (a) (homoskedasticity):

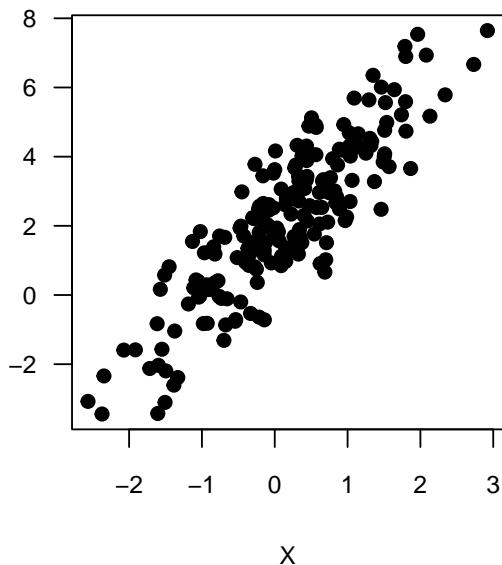
$$Y_i = 2 + 2X_i + \varepsilon_i^*.$$

In Panel (b) (heteroskedasticity):

$$Y_i = 2 + 2X_i + (2\mathbb{I}_{\{X_i < 0\}} + 0.2\mathbb{I}_{\{X_i \geq 0\}})\varepsilon_i^*$$

```
N <- 200
X <- rnorm(N); eps <- rnorm(N)
par(mfrow=c(1,2), plt=c(.2,.95,.2,.8))
Y <- 2 + 2*X + eps
plot(X,Y,pch=19,main="(a) Homoskedasticity",
     las=1,cex.lab=.8,cex.axis=.8,cex.main=.8,)
Y <- 2 + 2*X + eps*( (X<0)*2 + (X>=0)*.2 )
plot(X,Y,pch=19,main="(b) Heteroskedasticity",
     las=1,cex.lab=.8,cex.axis=.8,cex.main=.8,)
```

(a) Homoskedasticity



(b) Heteroskedasticity

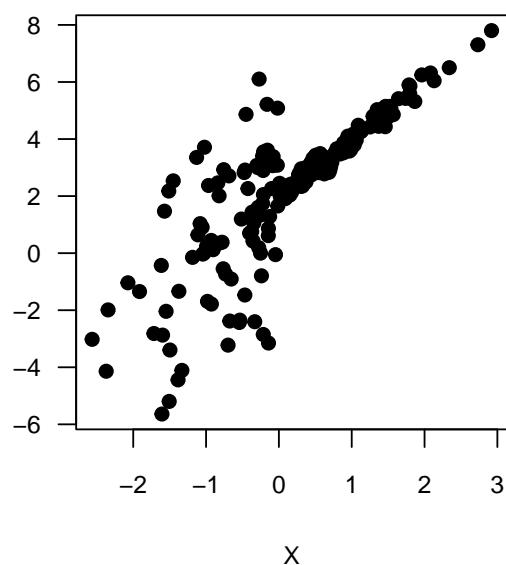


Figure 5.1: Homoskedasticity vs heteroskedasticity. See text for the exact specifications.

Figure 5.2 shows a real-data situation of heteroskedasticity, based on data taken from the Swiss Household Panel. The sample is restricted to persons (i) that are younger than 35 year in 2019, and (ii) that have completed at least 19 years of study. The figure shows that the dispersion of yearly income increases with age.

```

library(AEC)
table(shp$edyear19)

## 
##      8     9    10    12    13    14    16    19    21
##     70   325   350  1985   454   117   990  1263   168

shp_higherEd <- subset(shp, (edyear19>18)&age19<35)
plot(i19wyg/1000~age19,data=shp_higherEd,pch=19,las=1,
     xlab="Age",ylab="Yearly work income")
abline(lm(i19wyg/1000~age19,data=shp_higherEd),col="red",lwd=2)

```

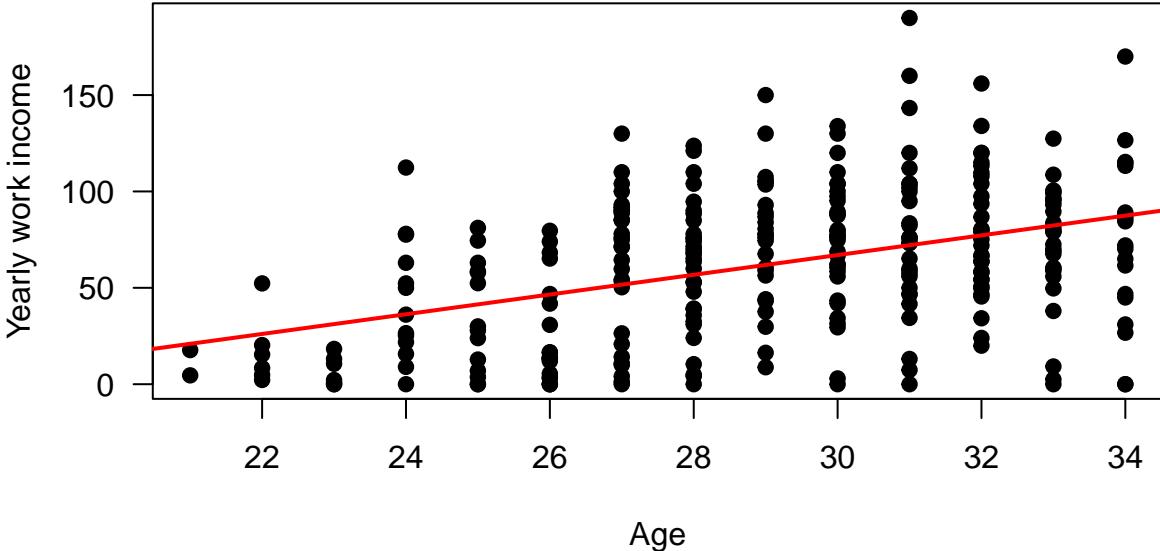


Figure 5.2: Income versus age. Data are from the Swiss Household Panel. The sample is restricted to persons that have completed at least 19 years of study. The figure shows that the dispersion of yearly income increases with age.

The next assumption concerns the correlation of the errors across entities.

**Hypothesis 5.4** (Uncorrelated errors).

$$\forall i \neq j, \quad \text{Cov}(\varepsilon_i, \varepsilon_j | \mathbf{X}) = 0.$$

We will often need to work with the covariance matrix of the errors. Proposition 5.2 give the specific form of the covariance matrix of the errors —conditional on  $\mathbf{X}$ — when both Hypotheses 5.3 and 5.4 are satisfied:

**Proposition 5.2.** *If Hypotheses 5.3 and 5.4 hold, then:*

$$\text{Var}(\varepsilon | \mathbf{X}) = \sigma^2 \text{Id},$$

where  $\text{Id}$  is the  $n \times n$  identity matrix.

We will sometimes assume that errors are Gaussian—or normal. We will then invoke Hypothesis 5.5:

**Hypothesis 5.5** (Normal distribution).

$$\forall i, \quad \varepsilon_i \sim \mathcal{N}(0, \sigma^2).$$

## 5.2 Least square estimation

### 5.2.1 Derivation of the OLS formula

In this section, we will present and study the properties of the most popular estimation approach, namely the **Ordinary Least Squares (OLS)** approach. As suggested by its

name, the OLS estimator of  $\beta$  is defined as the vector  $\mathbf{b}$  that minimizes the sum of squared residuals. (The *residuals* are the estimates of the *errors*  $\varepsilon_i$ .)

For a given vector of coefficients  $\mathbf{b} = [b_1, \dots, b_K]',$  the sum of squared residuals is:

$$f(\mathbf{b}) = \sum_{i=1}^n \left( y_i - \sum_{j=1}^K x_{i,j} b_j \right)^2 = \sum_{i=1}^n (y_i - \mathbf{x}'_i \mathbf{b})^2.$$

Minimizing this sum amounts to minimizing:

$$f(\mathbf{b}) = (\mathbf{y} - \mathbf{X}\mathbf{b})'(\mathbf{y} - \mathbf{X}\mathbf{b}).$$

Since (see Def. ?? and Proposition 10.7):

$$\frac{\partial f}{\partial \mathbf{b}}(\mathbf{b}) = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\mathbf{b},$$

it comes that a necessary first-order condition (FOC) is:

$$\mathbf{X}'\mathbf{X}\mathbf{b} = \mathbf{X}'\mathbf{y}. \quad (5.3)$$

Under Assumption 5.1,  $\mathbf{X}'\mathbf{X}$  is invertible. Hence:

$$\boxed{\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}.}$$

Vector  $\mathbf{b}$  minimizes the sum of squared residuals. ( $f$  is a non-negative quadratic function, it therefore admits a minimum.)

We have:

$$\mathbf{y} = \underbrace{\mathbf{X}\mathbf{b}}_{\text{fitted values } (\hat{\mathbf{y}})} + \underbrace{\mathbf{e}}_{\text{residuals}}$$

The estimated residuals are:

$$\mathbf{e} = \mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{M}\mathbf{y}, \quad (5.4)$$

where  $\mathbf{M} := \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$  is called the **residual maker** matrix.

Moreover, the fitted values  $\hat{\mathbf{y}}$  are given by:

$$\hat{\mathbf{y}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{P}\mathbf{y}, \quad (5.5)$$

where  $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$  is a **projection matrix**.

These matrices  $\mathbf{M}$  and  $\mathbf{P}$  are such that:

- $\mathbf{MX} = \mathbf{0}$ : if one regresses one of the explanatory variables on  $\mathbf{X}$ , the residuals are null.
- $\mathbf{My} = \mathbf{M}\varepsilon$  (because  $\mathbf{y} = \mathbf{X}\beta + \varepsilon$  and  $\mathbf{MX} = \mathbf{0}$ ).

Here are some additional properties of  $\mathbf{M}$  and  $\mathbf{P}$ :

- $\mathbf{M}$  is symmetric ( $\mathbf{M} = \mathbf{M}'$ ) and idempotent ( $\mathbf{M} = \mathbf{M}^2 = \mathbf{M}^k$  for  $k > 0$ ).
- $\mathbf{P}$  is symmetric and idempotent.
- $\mathbf{PX} = \mathbf{X}$ .
- $\mathbf{PM} = \mathbf{MP} = \mathbf{0}$ .
- $\mathbf{y} = \mathbf{Py} + \mathbf{My}$  (decomposition of  $\mathbf{y}$  into two orthogonal parts).

It is easily checked that  $\mathbf{X}'\mathbf{e} = 0$ . Each column of  $\mathbf{X}$  is therefore orthogonal to  $\mathbf{e}$ . In particular, if an intercept is included in the regression ( $x_{i,1} \equiv 1$  for all  $i$ 's, i.e., the first column of  $\mathbf{X}$  is filled with ones), the average of the residuals is null.

**Example 5.1** (Bivariate case). Consider a bivariate situation, where we regress  $y_i$  on a constant and an explanatory variable  $w_i$ . We have  $K = 2$ , and  $\mathbf{X}$  is a  $n \times 2$  matrix whose  $i^{th}$  row is  $[x_{i,1}, x_{i,2}]$ , with  $x_{i,1} = 1$  (to account for the intercept) and with  $w_i = x_{i,2}$  (say).

We have:

$$\begin{aligned}\mathbf{X}'\mathbf{X} &= \begin{bmatrix} n & \sum_i w_i \\ \sum_i w_i & \sum_i w_i^2 \end{bmatrix}, \\ (\mathbf{X}'\mathbf{X})^{-1} &= \frac{1}{n \sum_i w_i^2 - (\sum_i w_i)^2} \begin{bmatrix} \sum_i w_i^2 & -\sum_i w_i \\ -\sum_i w_i & n \end{bmatrix}, \\ (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} &= \frac{1}{n \sum_i w_i^2 - (\sum_i w_i)^2} \begin{bmatrix} \sum_i w_i^2 \sum_i y_i - \sum_i w_i \sum_i w_i y_i \\ -\sum_i w_i \sum_i y_i + n \sum_i w_i y_i \end{bmatrix} \\ &= \frac{1}{\frac{1}{n} \sum_i (w_i - \bar{w})^2} \begin{bmatrix} \frac{\bar{y}}{n} \sum_i w_i^2 - \frac{\bar{w}}{n} \sum_i w_i y_i \\ \frac{1}{n} \sum_i (w_i - \bar{w})(y_i - \bar{y}) \end{bmatrix}.\end{aligned}$$

It can be seen that the second element of  $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$  is:

$$b_2 = \frac{\overline{\text{Cov}(W, Y)}}{\overline{\text{Var}(W)}},$$

where  $\overline{\text{Cov}(W, Y)}$  and  $\overline{\text{Var}(W)}$  are sample estimates.

Since there is a constant in the regression, we have  $b_1 = \bar{y} - b_2 \bar{w}$ .

### 5.2.2 Properties of the OLS estimate (small sample)

The OLS properties stated in Proposition 5.3 are valid for any sample size  $n$ :

**Proposition 5.3** (Properties of the OLS estimator). *We have:*

- i. Under Assumptions 5.1 and 5.2, the OLS estimator is linear and unbiased.
- ii. Under Hypotheses 5.1 to 5.4, the conditional covariance matrix of  $\mathbf{b}$  is:  $\text{Var}(\mathbf{b}|\mathbf{X}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$ .

*Proof.* Under Hypothesis 5.1,  $\mathbf{X}'\mathbf{X}$  can be inverted. We have:

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon.$$

- i. Let us consider the expectation of the last term, i.e.  $\mathbb{E}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon)$ . Using the law of iterated expectations, we obtain:

$$\mathbb{E}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon) = \mathbb{E}(\mathbb{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'|\mathbf{X}]) = \mathbb{E}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbb{E}[\varepsilon|\mathbf{X}]).$$

By Hypothesis 5.2, we have  $\mathbb{E}[\varepsilon|\mathbf{X}] = 0$ . Hence  $\mathbb{E}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon) = 0$  and result (i) follows.

- ii.  $\text{Var}(\mathbf{b}|\mathbf{X}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbb{E}(\varepsilon\varepsilon'|\mathbf{X})\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$ . By Prop. 5.2, if 5.3 and 5.4 hold, then we have  $\mathbb{E}(\varepsilon\varepsilon'|\mathbf{X}) = \text{Var}(\varepsilon|\mathbf{X}) = \sigma^2 Id$ .

□

Together, Hypotheses 5.1 to 5.4 form the so-called Gauss-Markov set of assumptions. Under these assumptions, the OLS estimator feature the lowest possible variance within the family of linear unbiased estimates of  $\beta$ :

**Theorem 5.1** (Gauss-Markov Theorem). *Under Assumptions 5.1 to 5.4, for any vector  $w$ , the minimum-variance linear unbiased estimator of  $w'\beta$  is  $w'\mathbf{b}$ , where  $\mathbf{b}$  is the least squares estimator. (BLUE: Best Linear Unbiased Estimator.)*

*Proof.* Consider  $\mathbf{b}^* = Cy$ , another linear unbiased estimator of  $\beta$ . Since it is unbiased, we must have  $\mathbb{E}(Cy|\mathbf{X}) = \mathbb{E}(C\mathbf{X}\beta + C\varepsilon|\mathbf{X}) = \beta$ . We have  $\mathbb{E}(C\varepsilon|\mathbf{X}) = C\mathbb{E}(\varepsilon|\mathbf{X}) = 0$  (by 5.2). Therefore  $\mathbf{b}^*$  is unbiased if  $\mathbb{E}(C\mathbf{X})\beta = \beta$ . This has to be the case for any  $\beta$ , which implies that we must have  $C\mathbf{X} = \mathbf{I}$ . Let us compute  $\text{Var}(\mathbf{b}^*|\mathbf{X})$ . For this, we introduce  $D = C - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ , which is such that  $Dy = \mathbf{b}^* - \mathbf{b}$ . The fact that  $C\mathbf{X} = \mathbf{I}$  implies that  $D\mathbf{X} = \mathbf{0}$ . We have  $\text{Var}(\mathbf{b}^*|\mathbf{X}) = \text{Var}(Cy|\mathbf{X}) = \text{Var}(C\varepsilon|\mathbf{X}) = \sigma^2 CC'$  (by Assumptions 5.3 and 5.4, see Prop. 5.2). Using  $C = D + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$  and exploiting the fact that  $D\mathbf{X} = \mathbf{0}$  leads to:

$$\text{Var}(\mathbf{b}^*|\mathbf{X}) = \sigma^2 [(D + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')(D + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')'] = \text{Var}(\mathbf{b}|\mathbf{X}) + \sigma^2 \mathbf{D}\mathbf{D}'.$$

Therefore, we have

$$\begin{aligned} \text{Var}(w'\mathbf{b}^*|\mathbf{X}) &= w'\text{Var}(\mathbf{b}|\mathbf{X})w + \sigma^2 w'\mathbf{D}\mathbf{D}'w \\ &\geq w'\text{Var}(\mathbf{b}|\mathbf{X})w = \text{Var}(w'\mathbf{b}|\mathbf{X}). \end{aligned}$$

□

The Frish-Waugh theorem (Theorem 5.2) reveals the relationship between the OLS estimator and the notion of partial correlation coefficient. Consider the linear least square regression of  $\mathbf{y}$  on  $\mathbf{X}$ . We introduce the notations:

- $\mathbf{b}^{\mathbf{y}/\mathbf{X}}$ : OLS estimates of  $\beta$ ,
- $\mathbf{M}^{\mathbf{X}}$ : residual-maker matrix of any regression on  $\mathbf{X}$  (given by  $\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ ),
- $\mathbf{P}^{\mathbf{X}}$ : projection matrix of any regression on  $\mathbf{X}$  (given by  $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ ).

Let us split the set of explanatory variables into two:  $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2]$ . With obvious notations:  $\mathbf{b}^{\mathbf{y}/\mathbf{X}} = [\mathbf{b}'_1, \mathbf{b}'_2]'$ .

**Theorem 5.2** (Frisch-Waugh Theorem). *We have:*

$$\mathbf{b}_2 = \mathbf{b}^{\mathbf{M}^{\mathbf{X}_1}\mathbf{y}/\mathbf{M}^{\mathbf{X}_1}\mathbf{X}_2}.$$

*Proof.* The minimization of the least squares leads to (these are first-order conditions, see Eq. (5.3)):

$$\begin{bmatrix} \mathbf{X}'_1\mathbf{X}_1 & \mathbf{X}'_1\mathbf{X}_2 \\ \mathbf{X}'_2\mathbf{X}_1 & \mathbf{X}'_2\mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}'_1\mathbf{y} \\ \mathbf{X}'_2\mathbf{y} \end{bmatrix}.$$

Use the first-row block of equations to solve for  $\mathbf{b}_1$  first; it comes as a function of  $\mathbf{b}_2$ . Then use the second set of equations to solve for  $\mathbf{b}_2$ , which leads to:

$$\begin{aligned} \mathbf{b}_2 &= [\mathbf{X}'_2\mathbf{X}_2 - \mathbf{X}'_2\mathbf{X}_1(\mathbf{X}'_1\mathbf{X}_1)\mathbf{X}'_1\mathbf{X}_2]^{-1}\mathbf{X}'_2(Id - \mathbf{X}_1(\mathbf{X}'_1\mathbf{X}_1)\mathbf{X}'_1)\mathbf{y} \\ &= [\mathbf{X}'_2\mathbf{M}^{\mathbf{X}_1}\mathbf{X}_2]^{-1}\mathbf{X}'_2\mathbf{M}^{\mathbf{X}_1}\mathbf{y}. \end{aligned}$$

Using the fact that  $\mathbf{M}^{\mathbf{X}_1}$  is idempotent and symmetric leads to the result. □

This suggests a second way of estimating  $\mathbf{b}_2$ :

1. Regress  $Y$  on  $X_1$ , regress  $X_2$  on  $X_1$ .
2. Regress the residuals associated with the former regression on those associated with the latter regressions.

This is illustrated by the following code, where we run different regressions involving the number of Google searches for “parapluie” (*umbrella* in French). In the broad specification, we regress it on French precipitations and month dummies. Next, we deseasonalize both the dependent variable and the precipitations by regressing them on the month dummies. As stated by Theorem 5.2, regressing deseasonalized Google searches on deseasonalized precipitations give the same coefficient as in the baseline regression.

```

library(AEC)
dummies <- as.matrix(parapluie[, 4:14])
eq_all <- lm(parapluie~dummies+precip, data=parapluie)
deseas_parapluie <- lm(parapluie~dummies, data=parapluie)$residuals
deseas_precip <- lm(precip~dummies, data=parapluie)$residuals
eq_frac <- lm(deseas_parapluie~deseas_precip-1)
stargazer::stargazer(eq_all, eq_frac, omit=c(1:11, "Constant"), type="text",
                      omit.stat = c("f", "ser"), digits=5,
                      add.lines=list(c('Monthly dummy', 'Yes', 'No')))

## 
## =====
##             Dependent variable:
## -----
##          parapluie deseas_parapluie
##          (1)           (2)
## -----
##  precip      0.13001***  

##                (0.03594)
## 
##  deseas_precip      0.13001***  

##                (0.03277)
## 
## -----
##  Monthly dummy   Yes        No
##  Observations    72         72
##  R2              0.51793   0.18148
##  Adjusted R2     0.41988   0.16995
## =====
## Note: *p<0.1; **p<0.05; ***p<0.01

```

When  $b_2$  is scalar (and then  $\mathbf{X}_2$  is of dimension  $n \times 1$ ), Theorem 5.2 gives the expression of the **partial regression coefficient**  $b_2$ :

$$b_2 = \frac{\mathbf{X}'_2 M^{\mathbf{X}_1} \mathbf{y}}{\mathbf{X}'_2 M^{\mathbf{X}_1} \mathbf{X}_2}.$$

### 5.2.3 Goodness of fit

Define the total variation in  $y$  as the sum of squared deviations (from the sample mean):

$$TSS = \sum_{i=1}^n (y_i - \bar{y})^2.$$

We have:

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e} = \hat{\mathbf{y}} + \mathbf{e}$$

In the following, we assume that the regression includes a constant (i.e. for all  $i$ ,  $x_{i,1} = 1$ ). Denote by  $\mathbf{M}^0$  the matrix that transforms observations into deviations from sample means. Using that  $\mathbf{M}^0\mathbf{e} = \mathbf{e}$  and that  $\mathbf{X}'\mathbf{e} = 0$ , we have:

$$\begin{aligned}
 \underbrace{\mathbf{y}'\mathbf{M}^0\mathbf{y}}_{\text{Total sum of sq.}} &= (\mathbf{X}\mathbf{b} + \mathbf{e})'\mathbf{M}^0(\mathbf{X}\mathbf{b} + \mathbf{e}) \\
 &= \underbrace{\mathbf{b}'\mathbf{X}'\mathbf{M}^0\mathbf{X}\mathbf{b}}_{\text{"Explained" sum of sq.}} + \underbrace{\mathbf{e}'\mathbf{e}}_{\text{Sum of sq. residuals}} \\
 TSS &= Expl.SS + SSR.
 \end{aligned}$$

We can now define the coefficient of determination:

$$\text{Coefficient of determination} = \frac{Expl.SS}{TSS} = 1 - \frac{SSR}{TSS} = 1 - \frac{\mathbf{e}'\mathbf{e}}{\mathbf{y}'\mathbf{M}^0\mathbf{y}}.$$

(5.6)

It can be shown (Greene (2003), Section 3.5) that:

$$\text{Coefficient of determination} = \frac{[\sum_{i=1}^n (y_i - \bar{y})(\hat{y}_i - \bar{y})]^2}{\sum_{i=1}^n (y_i - \bar{y})^2 \sum_{i=1}^n (\hat{y}_i - \bar{y})^2}.$$

That is, the  $R^2$  is the sample squared correlation between  $y$  and the (regression-implied)  $y$ 's predictions.

The higher the  $R^2$ , the higher the goodness of fit of a model. One however has to be cautious with  $R^2$ . Indeed, it is easy to increase it: it suffices to add explanatory variables. As stated by Proposition 5.5, adding an explanatory variable (even if it does not truly relate to the dependent variable) mechanically results in an increase in the  $R^2$ . In the limit, taking any set of  $n$  non-linearly-dependent explanatory variables (i.e., variables satisfying Hypothesis 5.1) results in a  $R^2$  equal to one.

**Proposition 5.4** (Change in SSR when a variable is added). *We have:*

$$\mathbf{u}'\mathbf{u} = \mathbf{e}'\mathbf{e} - c^2(\mathbf{z}^{*\prime}\mathbf{z}^*) \quad (\leq \mathbf{e}'\mathbf{e}) \quad (5.7)$$

where (i)  $\mathbf{u}$  and  $\mathbf{e}$  are the residuals in the regressions of  $\mathbf{y}$  on  $[\mathbf{X}, \mathbf{z}]$  and of  $\mathbf{y}$  on  $\mathbf{X}$ , respectively, (ii)  $c$  is the regression coefficient on  $\mathbf{z}$  in the former regression and where  $\mathbf{z}^*$  are the residuals in the regression of  $\mathbf{z}$  on  $\mathbf{X}$ .

*Proof.* The OLS estimates  $[\mathbf{d}', \mathbf{c}]'$  in the regression of  $\mathbf{y}$  on  $[\mathbf{X}, \mathbf{z}]$  satisfies (first-order cond., Eq. (5.3)):

$$\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{z} \\ \mathbf{z}'\mathbf{X} & \mathbf{z}'\mathbf{z} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{y} \\ \mathbf{z}'\mathbf{y} \end{bmatrix}.$$

Hence, in particular  $\mathbf{d} = \mathbf{b} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{z}c$ , where  $\mathbf{b}$  is the OLS of  $\mathbf{y}$  on  $\mathbf{X}$ . Substituting in  $\mathbf{u} = \mathbf{y} - \mathbf{X}\mathbf{d} - \mathbf{z}c$ , we get  $\mathbf{u} = \mathbf{e} - \mathbf{z}^*c$ . We therefore have:

$$\mathbf{u}'\mathbf{u} = (\mathbf{e} - \mathbf{z}^*c)(\mathbf{e} - \mathbf{z}^*c) = \mathbf{e}'\mathbf{e} + c^2(\mathbf{z}^{*\prime}\mathbf{z}^*) - 2c\mathbf{z}^*\mathbf{e}. \quad (5.8)$$

Now  $\mathbf{z}^{*\prime}\mathbf{e} = \mathbf{z}^{*\prime}(\mathbf{y} - \mathbf{X}\mathbf{b}) = \mathbf{z}^{*\prime}\mathbf{y}$  because  $\mathbf{z}^*$  are the residuals in an OLS regression on  $\mathbf{X}$ . Since  $c = (\mathbf{z}^{*\prime}\mathbf{z}^*)^{-1}\mathbf{z}^{*\prime}\mathbf{y}^*$  (by an application of Theorem 5.2), we have  $(\mathbf{z}^{*\prime}\mathbf{z}^*)c = \mathbf{z}^{*\prime}\mathbf{y}^*$  and, therefore,  $\mathbf{z}^{*\prime}\mathbf{e} = (\mathbf{z}^{*\prime}\mathbf{z}^*)c$ . Inserting this in Eq. (5.8) leads to the results.  $\square$

**Proposition 5.5** (Change in the coefficient of determination when a variable is added). *Denoting by  $R_W^2$  the coefficient of determination in the regression of  $\mathbf{y}$  on some variable  $\mathbf{W}$ , we have:*

$$R_{\mathbf{X}, \mathbf{z}}^2 = R_{\mathbf{X}}^2 + (1 - R_{\mathbf{X}}^2)(r_{yz}^{\mathbf{X}})^2,$$

where  $r_{yz}^{\mathbf{X}}$  is the coefficient of partial correlation (see Definition 10.5).

*Proof.* Let's use the same notations as in Prop. 5.4. Theorem 5.2 implies that  $c = (\mathbf{z}^{*\prime}\mathbf{z}^*)^{-1}\mathbf{z}^{*\prime}\mathbf{y}^*$ . Using this in Eq. (5.7) gives  $\mathbf{u}'\mathbf{u} = \mathbf{e}'\mathbf{e} - (\mathbf{z}^{*\prime}\mathbf{y}^*)^2/(\mathbf{z}^{*\prime}\mathbf{z}^*)$ . Using the definition of the partial correlation (Eq. (10.2)), we get  $\mathbf{u}'\mathbf{u} = \mathbf{e}'\mathbf{e}(1 - (r_{yz}^{\mathbf{X}})^2)$ . The results is obtained by dividing both sides of the previous equation by  $\mathbf{y}'\mathbf{M}_0\mathbf{y}$ .  $\square$

Figure 5.3, below, illustrates the fact that one can obtain an  $R^2$  of one by regressing a sample of length  $n$  on any set of  $n$  linearly-independent variables.

```
n <- 30; Y <- rnorm(n); X <- matrix(rnorm(n^2), n, n)
all_R2 <- NULL; all_adjR2 <- NULL
for(j in 0:(n-1)){
  if(j==0){eq <- lm(Y~1)}else{eq <- lm(Y~X[,1:j])}
  all_R2 <- c(all_R2, summary(eq)$r.squared)
  all_adjR2 <- c(all_adjR2, summary(eq)$adj.r.squared)
}
par(plt=c(.15,.95,.25,.95))
plot(all_R2, pch=19, ylim=c(min(all_adjR2, na.rm = TRUE), 1),
     xlab="number of regressors", ylab="R2")
points(all_adjR2, pch=3); abline(h=0, col="light grey", lwd=2)
legend("topleft", c("R2", "Adjusted R2"),
      lty=NaN, col=c("black"), pch=c(19, 3), lwd=2)
```

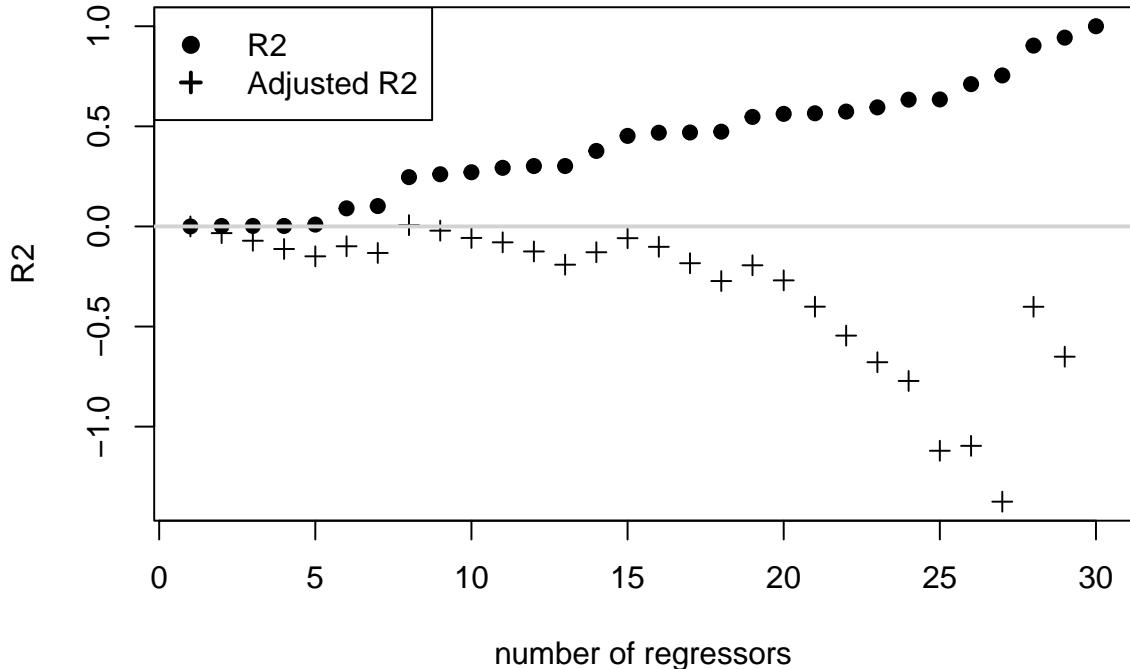


Figure 5.3: This figure illustrates the monotonous increase in the  $R^2$  as a function of the number of explanatory variables. In the true model, there is no explanatory variables, i.e.,  $y_i = \varepsilon_i$ . We then take (independent) regressors and regress  $y$  on the latter, progressively increasing the set of regressors.

In order to address the risk of adding irrelevant explanatory variables, measures of **adjusted  $R^2$**  have been proposed. Compared to the standard  $R^2$ , these measures add penalties that depend on the number of covariates employed in the regression. A common adjusted  $R^2$  measure, denoted by  $\bar{R}^2$ , is the following:

$$\bar{R}^2 = 1 - \frac{\mathbf{e}'\mathbf{e}/(n-K)}{\mathbf{y}'\mathbf{M}^0\mathbf{y}/(n-1)} = 1 - \frac{n-1}{n-K}(1-R^2).$$

#### 5.2.4 Inference and confidence intervals (in small sample)

Under the normality assumption (Assumption 5.5), we know the distribution of  $\mathbf{b}$  (conditional on  $\mathbf{X}$ ). Indeed,  $\mathbf{b} = \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon$ . Therefore, conditional on  $\mathbf{X}$ , vector  $\mathbf{b}$  is an affine combination of Gaussian variables—the components of  $\varepsilon$ . As a result, it is also Gaussian. Its distribution is therefore completely characterized by its mean and variance, and we have:

$$\mathbf{b}|\mathbf{X} \sim \mathcal{N}(\beta, \sigma^2(\mathbf{X}'\mathbf{X})^{-1}). \quad (5.9)$$

Eq. (5.9) can be used to conduct inference and tests. However, in practice, we do not know  $\sigma^2$  (which is a population parameter). The following proposition gives an unbiased estimate of  $\sigma^2$ .

**Proposition 5.6.** *Under 5.1 to 5.4, an unbiased estimate of  $\sigma^2$  is given by:*

$$s^2 = \frac{\mathbf{e}'\mathbf{e}}{n-K}. \quad (5.10)$$

(It is sometimes denoted by  $\sigma_{OLS}^2$ )

*Proof.* We have:

$$\begin{aligned} \mathbb{E}(\mathbf{e}'\mathbf{e}|\mathbf{X}) &= \mathbb{E}(\varepsilon'\mathbf{M}\varepsilon|\mathbf{X}) = \mathbb{E}(\text{Tr}(\varepsilon'\mathbf{M}\varepsilon)|\mathbf{X}) \\ &= \text{Tr}(\mathbf{M}\mathbb{E}(\varepsilon\varepsilon'|\mathbf{X})) = \sigma^2\text{Tr}(\mathbf{M}). \end{aligned}$$

(Note that we have  $\mathbb{E}(\varepsilon\varepsilon'|\mathbf{X}) = \sigma^2 Id$  by Assumptions 5.3 and 5.4, see Prop. 5.2.) Moreover:

$$\begin{aligned} \text{Tr}(\mathbf{M}) &= n - \text{Tr}(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') \\ &= n - \text{Tr}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}) = n - \text{Tr}(Id_{K \times K}), \end{aligned}$$

which leads to the result.  $\square$

Two results will prove important to produce inference:

- i. We know the conditional distribution of  $s^2$  (Prop. 5.7).
- ii.  $s^2$  and  $\mathbf{b}$  are independent random variables (Prop. 5.8).

**Proposition 5.7.** *Under 5.1 to 5.5, we have:  $\frac{s^2}{\sigma^2}|\mathbf{X} \sim \frac{1}{n-K}\chi^2(n-K)$ .*

*Proof.* We have  $\mathbf{e}'\mathbf{e} = \varepsilon'\mathbf{M}\varepsilon$ .  $\mathbf{M}$  is an idempotent symmetric matrix. Therefore it can be decomposed as  $PDP'$  where  $D$  is a diagonal matrix and  $P$  is an orthogonal matrix. As a result  $\mathbf{e}'\mathbf{e} = (P'\varepsilon)'D(P'\varepsilon)$ , i.e.  $\mathbf{e}'\mathbf{e}$  is a weighted sum of independent squared Gaussian variables (the entries of  $P'\varepsilon$  are independent because they are Gaussian —under 5.5— and uncorrelated). The variance of each of these i.i.d. Gaussian variable is  $\sigma^2$ . Because  $\mathbf{M}$  is an idempotent symmetric matrix, its eigenvalues are either 0 or 1, and its rank equals its trace (see Propositions 10.3 and 10.4). Further, its trace is equal to  $n - K$  (see proof of Eq. (5.10)). Therefore  $D$  has  $n - K$  entries equal to 1 and  $K$  equal to 0. Hence,  $\mathbf{e}'\mathbf{e} = (P'\varepsilon)'D(P'\varepsilon)$  is a sum of  $n - K$  squared independent Gaussian variables of variance  $\sigma^2$ . Therefore  $\frac{\mathbf{e}'\mathbf{e}}{\sigma^2} = (n - K)\frac{s^2}{\sigma^2}$  is a sum of  $n - K$  squared i.i.d. standard normal variables. The result follows by the definition of the chi-square distribution (see Def. 10.13).  $\square$

**Proposition 5.8.** *Under Hypotheses 5.1 to 5.5,  $\mathbf{b}$  and  $s^2$  are independent.*

*Proof.* We have  $\mathbf{b} = \beta + [\mathbf{X}'\mathbf{X}]^{-1}\mathbf{X}\varepsilon$  and  $s^2 = \varepsilon'\mathbf{M}\varepsilon/(n - K)$ . Hence  $\mathbf{b}$  is an affine combination of  $\varepsilon$  and  $s^2$  is a quadratic combination of the same Gaussian shocks. One can write  $s^2$  as  $s^2 = (\mathbf{M}\varepsilon)'(\mathbf{M}\varepsilon)/(n - K)$  and  $\mathbf{b}$  as  $\beta + \mathbf{T}\varepsilon$ . Since  $\mathbf{T}\mathbf{M} = 0$ ,  $\mathbf{T}\varepsilon$  and  $\mathbf{M}\varepsilon$  are independent (because two uncorrelated Gaussian variables are independent), therefore  $\mathbf{b}$  and  $s^2$ , which are functions of two sets of independent variables, are independent.  $\square$

Consistently with Eq. (5.9), under Hypotheses 5.1 to 5.5, the  $k^{th}$  entry of  $\mathbf{b}$  satisfies:

$$b_k|\mathbf{X} \sim \mathcal{N}(\beta_k, \sigma^2 v_k),$$

where  $v_k$  is the  $k^{th}$  component of the diagonal of  $(\mathbf{X}'\mathbf{X})^{-1}$ .

Moreover, we have (Prop. 5.7):

$$\frac{(n - K)s^2}{\sigma^2}|\mathbf{X} \sim \chi^2(n - K).$$

As a result (using Propositions 5.7 and 5.8), we have:

$$t_k = \frac{\frac{b_k - \beta_k}{\sqrt{\sigma^2 v_k}}}{\sqrt{\frac{(n - K)s^2}{\sigma^2(n - K)}}} = \frac{b_k - \beta_k}{\sqrt{s^2 v_k}} \sim t(n - K), \quad (5.11)$$

where  $t(n - K)$  denotes a Student  $t$  distribution with  $n - K$  degrees of freedom (see Def. 10.12).<sup>1</sup>

Note that  $s^2 v_k$  is not exactly the conditional variance of  $b_k$ : The variance of  $b_k$  conditional on  $\mathbf{X}$  is  $\sigma^2 v_k$ . However  $s^2 v_k$  is an unbiased estimate of  $\sigma^2 v_k$  (by Prop. 5.6).

The previous result (Eq. (5.11)) can be extended to any linear combinations of elements of  $\mathbf{b}$ . (Eq. (5.11) is for its  $k^{th}$  component only.) Let us consider  $\alpha'\mathbf{b}$ , the OLS estimate of  $\alpha'\beta$ . From Eq. (5.9), we have:

$$\alpha'\mathbf{b}|\mathbf{X} \sim \mathcal{N}(\alpha'\beta, \sigma^2 \alpha'(\mathbf{X}'\mathbf{X})^{-1}\alpha).$$

Therefore:

$$\frac{\alpha'\mathbf{b} - \alpha'\beta}{\sqrt{\sigma^2 \alpha'(\mathbf{X}'\mathbf{X})^{-1}\alpha}}|\mathbf{X} \sim \mathcal{N}(0, 1).$$

Using the same approach as the one used to derive Eq. (5.11), one can show that Props. 5.7 and 5.8 also imply that:

$$\frac{\alpha'\mathbf{b} - \alpha'\beta}{\sqrt{s^2 \alpha'(\mathbf{X}'\mathbf{X})^{-1}\alpha}} \sim t(n - K). \quad (5.12)$$

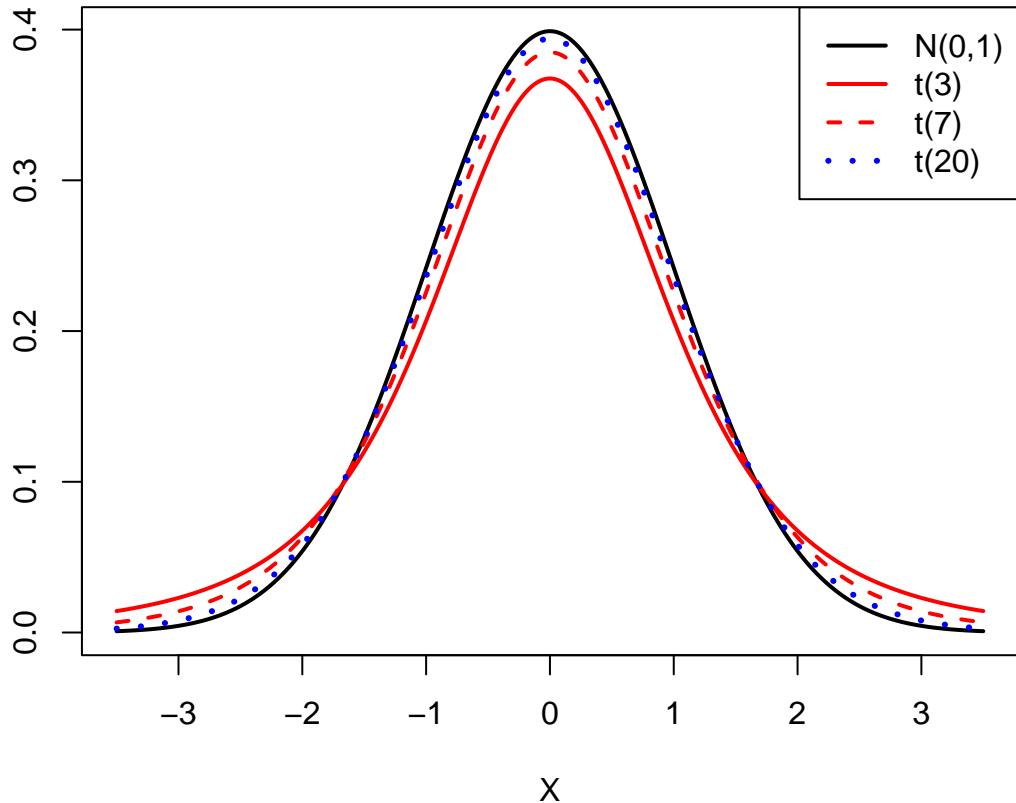


Figure 5.4: The higher the degree of freedom, the closer the distribution of  $t(\nu)$  gets to the normal distribution. (Convergence in distribution.)

What precedes is widely exploited for statistical inference in the context of linear regressions. Indeed, Eq. (5.11) gives a sense of the distances between  $b_k$  and  $\beta_k$  that can be deemed as “likely” (or, conversely, “unlikely”). For instance, it implies that, if  $\sqrt{v_k s^2}$  is equal to 1 (say), then the probability to obtain  $b_k$  smaller than  $\beta_k - 4.587 \times \sqrt{v_k s^2}$  or larger than  $\beta_k + 4.587 \times \sqrt{v_k s^2}$  is equal to 0.1% when  $n - K = 10$ .

That means for instance that, under the assumption that  $\beta_k = 0$ , it would be extremely unlikely to have obtained  $b_k / \sqrt{v_k s^2}$  smaller than -4.587 or larger than 4.587. More generally, this shows that the **t-statistic**, i.e., the ratio  $b_k / \sqrt{v_k s^2}$ , is the test statistic associated with the null hypothesis:

$$H_0 : \beta_k = 0.$$

Under the null hypothesis, the test statistic follows a Student-t distribution with  $n - K$  degrees of freedom. The **t-statistic** is therefore of particular importance, and, as a result, it is routinely reported in regression outputs (see Example 5.2).

**Example 5.2** (Education and income). Consider regression that aims at determining covariates of households’ income. This example makes use of data from the Swiss Household Panel (SHP); `edyear19` is the number of years of education and `age19` is the age of the respondent, as of 2019.

```
library(AEC)
library(sandwich)
shp$income <- shp$i19ptotn/1000
shp$female <- 1*(shp$sex19==2)
eq <- lm(income ~ edyear19 + age19 + I(age19^2) + female, data=shp)
lmtest::coeftest(eq)

##
## t test of coefficients:
##
##             Estimate Std. Error t value Pr(>|t|)    
## (Intercept) -71.9738073   5.7082456 -12.609 < 2.2e-16 ***
##
```

<sup>1</sup>We have  $\frac{b_k - \beta_k}{\sqrt{\sigma^2 v_k}} | \mathbf{X} \sim \mathcal{N}(0, 1)$  and  $\frac{(n-K)s^2}{\sigma^2} | \mathbf{X} \sim \chi^2(n - K)$ . These two distributions do not depend on  $\mathbf{X} \Rightarrow$  the *marginal distribution* of  $t_k$  is also  $t$ .

```

## edyear19      4.8442661  0.2172320 22.300 < 2.2e-16 ***
## age19        3.2386215  0.2183812 14.830 < 2.2e-16 ***
## I(age19^2)   -0.0289498  0.0020915 -13.842 < 2.2e-16 ***
## female       -31.8089006  1.4578004 -21.820 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

The last two columns of the previous table give the t-statistic and the p-values associated with t-tests, whose size- $\alpha$  critical region is:

$$\left] -\infty, -\Phi_{t(n-K)}^{-1} \left( 1 - \frac{\alpha}{2} \right) \right] \cup \left[ \Phi_{t(n-K)}^{-1} \left( 1 - \frac{\alpha}{2} \right), +\infty \right[.$$

We recall that the **p-value** is defined as the probability that  $|Z| > |t|$ , where  $t$  is the (computed) t-statistics and where  $Z \sim t(n - K)$ . That is, in the context of the t-test, the p-value is given by  $2(1 - \Phi_{t(n-K)}(|t_k|))$ . See this webpage for details regarding the link between critical regions, p-value, and test outcomes.

Now, suppose we want to compute a (symmetrical) *confidence interval*  $[I_{d,1-\alpha}, I_{u,1-\alpha}]$  that is such that  $\mathbb{P}(\beta_k \in [I_{d,1-\alpha}, I_{u,1-\alpha}]) = 1 - \alpha$ . That is, we want to have:  $\mathbb{P}(\beta_k < I_{d,1-\alpha}) = \frac{\alpha}{2}$  and  $\mathbb{P}(\beta_k > I_{u,1-\alpha}) = \frac{\alpha}{2}$ . Let us focus on  $I_{d,1-\alpha}$  to start with. Using Eq. (5.11), i.e.,  $t_k = \frac{b_k - \beta_k}{\sqrt{s^2 v_k}} \sim t(n - K)$ , we have:

$$\begin{aligned} \mathbb{P}(\beta_k < I_{d,1-\alpha}) = \frac{\alpha}{2} &\Leftrightarrow \\ \mathbb{P}\left(\frac{b_k - \beta_k}{\sqrt{s^2 v_k}} > \frac{b_k - I_{d,1-\alpha}}{\sqrt{s^2 v_k}}\right) = \frac{\alpha}{2} &\Leftrightarrow \mathbb{P}\left(t_k > \frac{b_k - I_{d,1-\alpha}}{\sqrt{s^2 v_k}}\right) = \frac{\alpha}{2} \Leftrightarrow \\ 1 - \mathbb{P}\left(t_k \leq \frac{b_k - I_{d,1-\alpha}}{\sqrt{s^2 v_k}}\right) = \frac{\alpha}{2} &\Leftrightarrow \frac{b_k - I_{d,1-\alpha}}{\sqrt{s^2 v_k}} = \Phi_{t(n-K)}^{-1} \left( 1 - \frac{\alpha}{2} \right), \end{aligned}$$

where  $\Phi_{t(n-K)}(\alpha)$  is the c.d.f. of the  $t(n - K)$  distribution (Table 10.2).

Doing the same for  $I_{u,1-\alpha}$ , we obtain:

$$\begin{aligned} [I_{d,1-\alpha}, I_{u,1-\alpha}] = \\ \left[ b_k - \Phi_{t(n-K)}^{-1} \left( 1 - \frac{\alpha}{2} \right) \sqrt{s^2 v_k}, b_k + \Phi_{t(n-K)}^{-1} \left( 1 - \frac{\alpha}{2} \right) \sqrt{s^2 v_k} \right]. \end{aligned}$$

Using the results presented in Example 5.2, we can compute lower and upper bounds of 95% confidence intervals for the estimated parameters as follows:

```

n <- length(eq$residuals); K <- length(eq$coefficients)
lower.b <- eq$coefficients + qt(.025, df=n-K)*sqrt(diag(vcov(eq)))
upper.b <- eq$coefficients + qt(.975, df=n-K)*sqrt(diag(vcov(eq)))
cbind(lower.b,upper.b)

```

```

##           lower.b      upper.b
## (Intercept) -83.16413225 -60.78348237
## edyear19     4.41840914  5.27012310
## age19        2.81051152  3.66673148
## I(age19^2)   -0.03304986 -0.02484977
## female       -34.66674188 -28.95105932

```

### 5.2.5 Testing a set of linear restrictions

We sometimes want to test if a set of restrictions are *jointly* consistent with the data at hand. Let us formalize such a set of ( $J$ ) linear restrictions:

$$\begin{aligned} r_{1,1}\beta_1 + \cdots + r_{1,K}\beta_K &= q_1 \\ &\vdots \\ r_{J,1}\beta_1 + \cdots + r_{J,K}\beta_K &= q_J. \end{aligned} \tag{5.13}$$

In matrix form, we get:

$$\mathbf{R}\beta = \mathbf{q}. \quad (5.14)$$

Define the *discrepancy vector*  $\mathbf{m} = \mathbf{R}\mathbf{b} - \mathbf{q}$ . Under the null hypothesis:

$$\begin{aligned} \mathbb{E}(\mathbf{m}|\mathbf{X}) &= \mathbf{R}\beta - \mathbf{q} = 0 \quad \text{and} \\ \mathbb{V}ar(\mathbf{m}|\mathbf{X}) &= \mathbf{R}\mathbb{V}ar(\mathbf{b}|\mathbf{X})\mathbf{R}'. \end{aligned}$$

With these notations, the assumption to test is:

$$H_0 : \mathbf{R}\beta - \mathbf{q} = 0 \text{ against } H_1 : \mathbf{R}\beta - \mathbf{q} \neq 0. \quad (5.15)$$

Under Hypotheses 5.1 to 5.4, we have  $\mathbb{V}ar(\mathbf{m}|\mathbf{X}) = \sigma^2 \mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'$  (see Prop. 5.3). If we add the normality assumption (Hypothesis 5.5), we have:

$$W = \mathbf{m}'\mathbb{V}ar(\mathbf{m}|\mathbf{X})^{-1}\mathbf{m} \sim \chi^2(J). \quad (5.16)$$

If  $\sigma^2$  was known, we could then conduct a *Wald test* (directly exploiting Eq. (5.16)). But this is not the case in practice and we cannot compute  $W$ . We can, however, approximate it by replacing  $\sigma^2$  by  $s^2$  (given in Eq. (5.10)). The distribution of this new statistic is not  $\chi^2(J)$  any more; it is an  $\mathcal{F}$  *distribution* (whose quantiles are shown in Table 10.4), and the test is called *F test*.

**Proposition 5.9.** *Under Hypotheses 5.1 to 5.5 and if Eq. (5.15) holds, we have:*

$$F = \frac{W \sigma^2}{J s^2} = \frac{\mathbf{m}'(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')^{-1}\mathbf{m}}{s^2 J} \sim \mathcal{F}(J, n - K), \quad (5.17)$$

where  $\mathcal{F}$  is the distribution of the F-statistic (see Def. 10.11).

*Proof.* According to Eq. (5.16),  $W/J \sim \chi^2(J)/J$ . Moreover, the denominator  $(s^2/\sigma^2)$  is  $\sim \chi^2(n-K)$ . Therefore,  $F$  is the ratio of a r.v. distributed as  $\chi^2(J)/J$  and another distributed as  $\chi^2(n-K)/(n-K)$ . It remains to verify that these r.v. are independent. Under  $H_0$ , we have  $\mathbf{m} = \mathbf{R}(\mathbf{b} - \beta) = \mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon$ . Therefore  $\mathbf{m}'(\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')^{-1}\mathbf{m}$  is of the form  $\varepsilon'\mathbf{T}\varepsilon$  with  $\mathbf{T} = \mathbf{D}'\mathbf{C}\mathbf{D}$  where  $\mathbf{D} = \mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$  and  $\mathbf{C} = (\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')^{-1}$ . Under Hypotheses 5.1 to 5.4, the covariance between  $\mathbf{T}\varepsilon$  and  $\mathbf{M}\varepsilon$  is  $\sigma^2\mathbf{T}\mathbf{M} = \mathbf{0}$ . Therefore, under 5.5, these variables are Gaussian variables with 0 covariance. Hence they are independent.  $\square$

For large  $n - K$ , the  $\mathcal{F}_{J,n-K}$  distribution converges to  $\mathcal{F}_{J,\infty} = \chi^2(J)/J$ . This implies that, in large samples, the F-statistic approximately has a  $\chi^2$  distribution. In other words, one can approximately employ Eq. (5.16) to perform a Wald test (one just has to replace  $\sigma^2$  with  $s^2$  when computing  $\mathbb{V}ar(\mathbf{m}|\mathbf{X})$ ).

The following proposition proposes another equivalent computation of the F-statistic, based on the  $R^2$  of the restricted and unrestricted linear models.

**Proposition 5.10.** *The F-statistic defined by Eq. (5.17) is also equal to:*

$$F = \frac{(R^2 - R_*^2)/J}{(1 - R^2)/(n - K)} = \frac{(SSR_{\text{restr}} - SSR_{\text{unrestr}})/J}{SSR_{\text{unrestr}}/(n - K)}, \quad (5.18)$$

where  $R_*^2$  is the coef. of determination (Eq. (5.6)) of the “restricted regression” (SSR: sum of squared residuals.)

*Proof.* Let's denote by  $\mathbf{e}_* = \mathbf{y} - \mathbf{X}\mathbf{b}_*$  the vector of residuals associated to the *restricted regression* (i.e.  $\mathbf{R}\mathbf{b}_* = \mathbf{q}$ ). We have  $\mathbf{e}_* = \mathbf{e} - \mathbf{X}(\mathbf{b}_* - \mathbf{b})$ . Using  $\mathbf{e}'\mathbf{X} = 0$ , we get  $\mathbf{e}'_*\mathbf{e}_* = \mathbf{e}'\mathbf{e} + (\mathbf{b}_* - \mathbf{b})'\mathbf{X}'\mathbf{X}(\mathbf{b}_* - \mathbf{b}) \geq \mathbf{e}'\mathbf{e}$ .

By Proposition 10.5 (in Appendix 10.2), we have:  $\mathbf{b}_* - \mathbf{b} = -(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\{\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\}^{-1}(\mathbf{R}\mathbf{b} - \mathbf{q})$ . Therefore:

$$\mathbf{e}'_*\mathbf{e}_* - \mathbf{e}'\mathbf{e} = (\mathbf{R}\mathbf{b} - \mathbf{q})'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{R}\mathbf{b} - \mathbf{q}).$$

This implies that the F statistic defined in Prop. 5.9 is also equal to:

$$\frac{(\mathbf{e}'_*\mathbf{e}_* - \mathbf{e}'\mathbf{e})/J}{\mathbf{e}'\mathbf{e}/(n - K)},$$

which leads to the result.  $\square$

The null hypothesis  $H_0$  (Eq. (5.15)) of the F-test is rejected if  $F$ —defined by Eq. (5.17) or (5.18)—is higher than  $\mathcal{F}_{1-\alpha}(J, n - K)$ . (Hence, this test is a one-sided test.)

### 5.2.6 Large Sample Properties

Even if we relax the normality assumption (Hypothesis 5.5), we can approximate the finite-sample behavior of the estimators by using *large-sample* or *asymptotic properties*.

To begin with, we proceed under Hypothesis 5.1 to 5.4. (We will see, later on, how to deal with —partial— relaxations of Hypothesis 5.3 and 5.4.)

Under regularity assumptions, and under Hypotheses 5.1 to 5.4, even if the residuals are not normally-distributed, the least square estimators can be *asymptotically normal* and inference can be performed in the same way as in small samples when Hypotheses 5.1 to 5.5 hold. This derives from Prop. 5.11 (below). The F-test (Prop. 5.10) and the t-test (Eq. (5.11)) can then be performed.

**Proposition 5.11.** *Under Assumptions 5.1 to 5.4, and assuming further that:*

$$Q = \text{plim}_{n \rightarrow \infty} \frac{\mathbf{X}'\mathbf{X}}{n}, \quad (5.19)$$

*and that the  $(\mathbf{x}_i, \varepsilon_i)$ 's are independent (across entities  $i$ ), we have:*

$$\sqrt{n}(\mathbf{b} - \beta) \xrightarrow{d} \mathcal{N}(0, \sigma^2 Q^{-1}). \quad (5.20)$$

*Proof.* Since  $\mathbf{b} = \beta + \left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1} \left(\frac{\mathbf{X}'\varepsilon}{n}\right)$ , we have:  $\sqrt{n}(\mathbf{b} - \beta) = \left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1} \left(\frac{1}{\sqrt{n}}\right) \mathbf{X}'\varepsilon$ . Since  $f : A \rightarrow A^{-1}$  is a continuous function (for  $A \neq \mathbf{0}$ ),  $\text{plim}_{n \rightarrow \infty} \left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1} = \mathbf{Q}^{-1}$  (see Prop. 10.12). Let us denote by  $V_i$  the vector  $\mathbf{x}_i\varepsilon_i$ . Because the  $(\mathbf{x}_i, \varepsilon_i)$ 's are independent, the  $V_i$ 's are independent as well. Their covariance matrix is  $\sigma^2 \mathbb{E}(\mathbf{x}_i \mathbf{x}_i') = \sigma^2 Q$ . Applying the multivariate central limit theorem on vectors  $V_i$  gives  $\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \varepsilon_i \right) = \left( \frac{1}{\sqrt{n}} \right) \mathbf{X}'\varepsilon \xrightarrow{d} \mathcal{N}(0, \sigma^2 Q)$ . An application of Slutsky's theorem (Prop. 10.12) then leads to the results.  $\square$

In practice,  $\sigma^2$  is approximated by  $s^2 = \frac{\mathbf{e}'\mathbf{e}}{n-K}$  (Eq. (5.10)) and  $\mathbf{Q}^{-1}$  by  $\left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1}$ . That is, the covariance matrix of the estimator is approximated by:

$$\boxed{\widehat{\text{Var}}(\mathbf{b}) = s^2 (\mathbf{X}'\mathbf{X})^{-1}.} \quad (5.21)$$

Eqs. (5.19) and (5.20) respectively correspond to convergences in probability and in distribution (see Definitions 10.16 and 10.19, respectively).

## 5.3 Common pitfalls in linear regressions

### 5.3.1 Multicollinearity

Consider the model:  $y_i = \beta_1 x_{i,1} + \beta_2 x_{i,2} + \varepsilon_i$ , where all variables are zero-mean and  $\text{Var}(\varepsilon_i) = \sigma^2$ . We have:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} \sum_i x_{i,1}^2 & \sum_i x_{i,1} x_{i,2} \\ \sum_i x_{i,1} x_{i,2} & \sum_i x_{i,2}^2 \end{bmatrix},$$

therefore:

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{\sum_i x_{i,1}^2 \sum_i x_{i,2}^2 - (\sum_i x_{i,1} x_{i,2})^2} \begin{bmatrix} \sum_i x_{i,2}^2 & -\sum_i x_{i,1} x_{i,2} \\ -\sum_i x_{i,1} x_{i,2} & \sum_i x_{i,1}^2 \end{bmatrix}.$$

The inverse of the upper-left parameter of  $(\mathbf{X}'\mathbf{X})^{-1}$  is:

$$\sum_i x_{i,1}^2 - \frac{(\sum_i x_{i,1} x_{i,2})^2}{\sum_i x_{i,2}^2} = \sum_i x_{i,1}^2 (1 - \text{correl}_{1,2}^2), \quad (5.22)$$

where  $\text{correl}_{1,2}$  is the sample correlation between  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .

Hence, the closer to one  $\text{correl}_{1,2}$ , the higher the variance of  $b_1$  (recall that the variance of  $b_1$  is the upper-left component of  $\sigma^2(\mathbf{X}'\mathbf{X})^{-1}$ ). That is, if some of our regressors are close to a linear combination of the other ones, then the confidence intervals will tend to be wide, which typically reduces the power of the t-test (we tend to fail to reject the null hypothesis that the coefficients are different from zero).

### 5.3.2 Omitted variables

Consider the following model (the “True model”):

$$\mathbf{y} = \underbrace{\mathbf{X}_1}_{n \times K_1} \underbrace{\boldsymbol{\beta}_1}_{K_1 \times 1} + \underbrace{\mathbf{X}_2}_{n \times K_2} \underbrace{\boldsymbol{\beta}_2}_{K_2 \times 1} + \varepsilon$$

If one computes  $\mathbf{b}_1$  by regressing  $\mathbf{y}$  on  $\mathbf{X}_1$  only, one gets:

$$\mathbf{b}_1 = (\mathbf{X}'_1 \mathbf{X}_1)^{-1} \mathbf{X}'_1 \mathbf{y} = \beta_1 + (\mathbf{X}'_1 \mathbf{X}_1)^{-1} \mathbf{X}'_1 \mathbf{X}_2 \beta_2 + (\mathbf{X}'_1 \mathbf{X}_1)^{-1} \mathbf{X}'_1 \varepsilon.$$

This results in the omitted-variable formula:

$$\mathbb{E}(\mathbf{b}_1 | \mathbf{X}) = \beta_1 + \underbrace{(\mathbf{X}'_1 \mathbf{X}_1)^{-1} (\mathbf{X}'_1 \mathbf{X}_2)}_{K_1 \times K_2} \beta_2.$$

(Each column of  $(\mathbf{X}'_1 \mathbf{X}_1)^{-1} (\mathbf{X}'_1 \mathbf{X}_2)$  are the OLS regressors obtained when regressing the columns of  $\mathbf{X}_2$  on  $\mathbf{X}_1$ .) Unless the variables included in  $\mathbf{X}_1$  are orthogonal to those in  $\mathbf{X}_2$ , we obtain a bias. A way to address this potential pitfall is to introduce “controls” in the specification.

**Example 5.3.** Let us use the California Test Score dataset (in the package **AER**). Assume we want to measure the effect of the students-to-teacher ratio (**str**) on student test scores (**testscr**). The following regressions show that the effect is lower when controls are added.

```
library(AER); data("CASchools")
CASchools$str <- CASchools$students/CASchools$teachers
CASchools$testscr <- .5 * (CASchools$math + CASchools$read)
eq1 <- lm(testscr~str,data=CASchools)
eq2 <- lm(testscr~str+lunch,data=CASchools)
eq3 <- lm(testscr~str+lunch+english,data=CASchools)
stargazer::stargazer(eq1,eq2,eq3,type="text",
                      no.space = TRUE,omit.stat=c("f","ser"))

## 
## =====
##             Dependent variable:
## -----
##                  testscr
##      (1)        (2)        (3)
## -----
## str       -2.280***   -1.117***   -0.998***  

##           (0.480)     (0.240)     (0.239)  

## lunch      -0.600***   -0.547***  

##           (0.017)     (0.022)  

## english      -0.122***  

##           (0.032)  

## Constant    698.933*** 702.911*** 700.150***  

##           (9.467)     (4.700)     (4.686)
## -----
## Observations   420        420        420  

## R2          0.051        0.767        0.775  

## Adjusted R2  0.049        0.766        0.773
## =====
## Note:          *p<0.1; **p<0.05; ***p<0.01
```

### 5.3.3 Irrelevant variable

Consider the *true model*:

$$\mathbf{y} = \mathbf{X}_1 \boldsymbol{\beta}_1 + \varepsilon,$$

while the *estimated model* is:

$$\mathbf{y} = \mathbf{X}_1\beta_1 + \mathbf{X}_2\beta_2 + \varepsilon$$

The estimates are unbiased. However, adding irrelevant explanatory variables increases the variance of the estimate of  $\beta_1$  (compared to the case where one uses the correct explanatory variables). This is the case unless the correlation between  $\mathbf{X}_1$  and  $\mathbf{X}_2$  is null, see Eq. (5.22).

In other words, the estimator is *inefficient*, i.e., there exists an alternative consistent estimator whose variance is lower. The inefficiency problem can have serious consequences when testing hypotheses such as  $H_0 : \beta_1 = 0$ . Due to the loss of power, we might wrongly infer that the  $\mathbf{X}_1$  variables are not “relevant” (*Type-II error, False Negative*).

## 5.4 Instrumental Variables

The conditional mean zero assumption (Hypothesis 5.2), according to which  $\mathbb{E}(\varepsilon|\mathbf{X}) = 0$  —which implies in particular that  $\mathbf{x}_i$  and  $\varepsilon_i$  are uncorrelated— is sometimes not consistent with the considered economic framework. When it is the case, the parameters of interest may still be estimated consistently by resorting to instrumental variable techniques.

Consider the following model:

$$y_i = \mathbf{x}_i'\beta + \varepsilon_i, \quad \text{where } \mathbb{E}(\varepsilon_i) = 0 \text{ and } \mathbf{x}_i \perp \varepsilon_i. \quad (5.23)$$

Let us illustrate how this situation may result in biased OLS estimate. Consider for instance the situation where:

$$\mathbb{E}(\varepsilon_i) = 0 \quad \text{and} \quad \mathbb{E}(\varepsilon_i \mathbf{x}_i) = \gamma, \quad (5.24)$$

in which case we have  $\mathbf{x}_i \perp \varepsilon_i$  (consistently with Eq. (5.23)).

By the law of large numbers,  $\text{plim}_{n \rightarrow \infty} \mathbf{X}'\varepsilon/n = \gamma$ . If  $\mathbf{Q}_{xx} := \text{plim} \mathbf{X}'\mathbf{X}/n$ , the OLS estimator is not consistent because

$$\mathbf{b} = \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon \xrightarrow{p} \beta + \mathbf{Q}_{xx}^{-1}\gamma \neq \beta.$$

Let us now introduce the notion of instruments.

**Definition 5.2** (Instrumental variables). The  $L$ -dimensional random variable  $\mathbf{z}_i$  is a valid set of instruments if:

- a.  $\mathbf{z}_i$  is correlated to  $\mathbf{x}_i$ ;
- b. we have  $\mathbb{E}(\varepsilon|\mathbf{Z}) = 0$  and
- c. the orthogonal projections of the  $\mathbf{x}_i$ 's on the  $\mathbf{z}_i$ 's are not multicollinear.

Point c implies in particular that the dimension of  $\mathbf{z}_i$  has to be at least as large as that of  $\mathbf{x}_i$ . If  $\mathbf{z}_i$  is a valid set of instruments, we have:

$$\text{plim} \left( \frac{\mathbf{Z}'\mathbf{y}}{n} \right) = \text{plim} \left( \frac{\mathbf{Z}'(\mathbf{X}\beta + \varepsilon)}{n} \right) = \text{plim} \left( \frac{\mathbf{Z}'\mathbf{X}}{n} \right) \beta.$$

Indeed, by the law of large numbers,  $\frac{\mathbf{Z}'\varepsilon}{n} \xrightarrow{p} \mathbb{E}(\mathbf{z}_i\varepsilon_i) = 0$ .

If  $L = K$ , the matrix  $\frac{\mathbf{Z}'\mathbf{X}}{n}$  is of dimension  $K \times K$  and we have:

$$\left[ \text{plim} \left( \frac{\mathbf{Z}'\mathbf{X}}{n} \right) \right]^{-1} \text{plim} \left( \frac{\mathbf{Z}'\mathbf{y}}{n} \right) = \beta.$$

By continuity of the inverse function (everywhere but at 0):  $\left[ \text{plim} \left( \frac{\mathbf{Z}'\mathbf{X}}{n} \right) \right]^{-1} = \text{plim} \left( \frac{\mathbf{Z}'\mathbf{X}}{n} \right)^{-1}$ . The Slutsky Theorem (Prop. 10.12) further implies that:

$$\text{plim} \left( \frac{\mathbf{Z}'\mathbf{X}}{n} \right)^{-1} \text{plim} \left( \frac{\mathbf{Z}'\mathbf{y}}{n} \right) = \text{plim} \left( \left( \frac{\mathbf{Z}'\mathbf{X}}{n} \right)^{-1} \frac{\mathbf{Z}'\mathbf{y}}{n} \right).$$

Hence  $\mathbf{b}_{iv}$  is consistent if it is defined by:

$$\boxed{\mathbf{b}_{iv} = (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\mathbf{y}.}$$

**Proposition 5.12** (Asymptotic distribution of the IV estimator). *If  $\mathbf{z}_i$  is a  $L$ -dimensional random variable that constitutes a valid set of instruments (see Def. 5.2) and if  $L = K$ , then the asymptotic distribution of  $\mathbf{b}_{iv}$  is:*

$$\mathbf{b}_{iv} \xrightarrow{d} \mathcal{N} \left( \beta, \frac{\sigma^2}{n} [Q_{xz} Q_{zz}^{-1} Q_{zx}]^{-1} \right)$$

where  $\text{plim } \mathbf{Z}'\mathbf{Z}/n =: \mathbf{Q}_{zz}$ ,  $\text{plim } \mathbf{Z}'\mathbf{X}/n =: \mathbf{Q}_{zx}$ ,  $\text{plim } \mathbf{X}'\mathbf{Z}/n =: \mathbf{Q}_{xz}$ .

*Proof.* The proof is very similar to that of Prop. 5.11, the starting point being that  $\mathbf{b}_{iv} = \beta + (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\varepsilon$ .  $\square$

When  $L = K$ , we have:

$$[Q_{xz} Q_{zz}^{-1} Q_{zx}]^{-1} = Q_{zx}^{-1} Q_{zz} Q_{xz}^{-1}.$$

In practice, to estimate  $\text{Var}(\mathbf{b}_{iv}) = \frac{\sigma^2}{n} Q_{zx}^{-1} Q_{zz} Q_{xz}^{-1}$ , we replace  $\sigma^2$  by:

$$s_{iv}^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \mathbf{x}'_i \mathbf{b}_{iv})^2.$$

What about when  $L > K$ ? In this case, we proceed as follows:

1. Regress  $\mathbf{X}$  on the space spanned by  $\mathbf{Z}$  and
2. Regress  $\mathbf{y}$  on the fitted values  $\hat{\mathbf{X}} := \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}$ .

These two-step approach is called **Two-Stage Least Squares (2SLS)**. It results in:

$$\boxed{\mathbf{b}_{iv} = [\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}.} \quad (5.25)$$

In this case, Prop. 5.12 still holds, with  $\mathbf{b}_{iv}$  given by Eq. (5.25).

If the instruments do not properly satisfy Condition (a) in Def. 5.2 (i.e. if  $\mathbf{x}_i$  and  $\mathbf{z}_i$  are only loosely related), the instruments are said to be **weak** (see, e.g., Stock and Yogo (2005), available here or Andrews et al. (2019)). A simple standard way to test for weak instruments consist in looking at the F-statistic associated with the first stage of the estimation. The easier it is to reject the null hypothesis (large test statistic), the less weak—or the stronger—the instruments.

The Durbin-Wu-Hausman test (Durbin (1954), Wu (1973), Hausman (1978)) can be used to test if IV necessary. (IV techniques are required if  $\text{plim}_{n \rightarrow \infty} \mathbf{X}'\varepsilon/n \neq 0$ .) Hausman (1978) proposes a test of the efficiency of estimators. Under the null hypothesis two estimators,  $\mathbf{b}_0$  and  $\mathbf{b}_1$ , are consistent but  $\mathbf{b}_0$  is (asymptotically) efficient relative to  $\mathbf{b}_1$ . Under the alternative hypothesis,  $\mathbf{b}_1$  (IV in the present case) remains consistent but not  $\mathbf{b}_0$  (OLS in the present case). That is, when we reject the null hypothesis, it means that the OLS estimator is not consistent, potentially due to endogeneity issue.

The test statistic is:

$$H = (\mathbf{b}_1 - \mathbf{b}_0)'MPI(\text{Var}(\mathbf{b}_1) - \text{Var}(\mathbf{b}_0))(\mathbf{b}_1 - \mathbf{b}_0),$$

where  $MPI$  is the Moore-Penrose pseudo-inverse. Under the null hypothesis,  $H \sim \chi^2(q)$ , where  $q$  is the rank of  $\text{Var}(\mathbf{b}_1) - \text{Var}(\mathbf{b}_0)$ .

**Example 5.4** (Estimation of price elasticity). See e.g. WHO and estimation of tobacco price elasticity of demand.

We want to estimate what is the effect on demand of an *exogenous increase* in prices of cigarettes (say).

The model is:

$$\begin{aligned} \log(\text{demand}) &= \alpha_0 + \alpha_1 \underbrace{\times p_t}_{\text{log(price)}} + \alpha_2 \underbrace{\times w_t}_{\text{income}} + \varepsilon_t^d \\ \log(\text{supply}) &= \gamma_0 + \gamma_1 \times p_t + \gamma_2 \underbrace{\times y_t}_{\text{cost factors}} + \varepsilon_t^s, \end{aligned}$$

where  $\mathbf{y}_t, w_t, \varepsilon_t^s \sim \mathcal{N}(0, \sigma_s^2)$  and  $\varepsilon_t^d \sim \mathcal{N}(0, \sigma_d^2)$  are independent.

Equilibrium:  $q_t^d = q_t^s$ . This implies that prices are **endogenous**:

$$p_t = \frac{\alpha_0 + \alpha_2 w_t + \varepsilon_t^d - \gamma_0 - \gamma_2 \mathbf{y}_t - \varepsilon_t^s}{\gamma_1 - \alpha_1}.$$

In particular we have  $\mathbb{E}(p_t \varepsilon_t^d) = \frac{\sigma_d^2}{\gamma_1 - \alpha_1} \neq 0 \Rightarrow$  Regressing by OLS  $q_t^d$  on  $p_t$  gives biased estimates (see Eq. (5.24)).

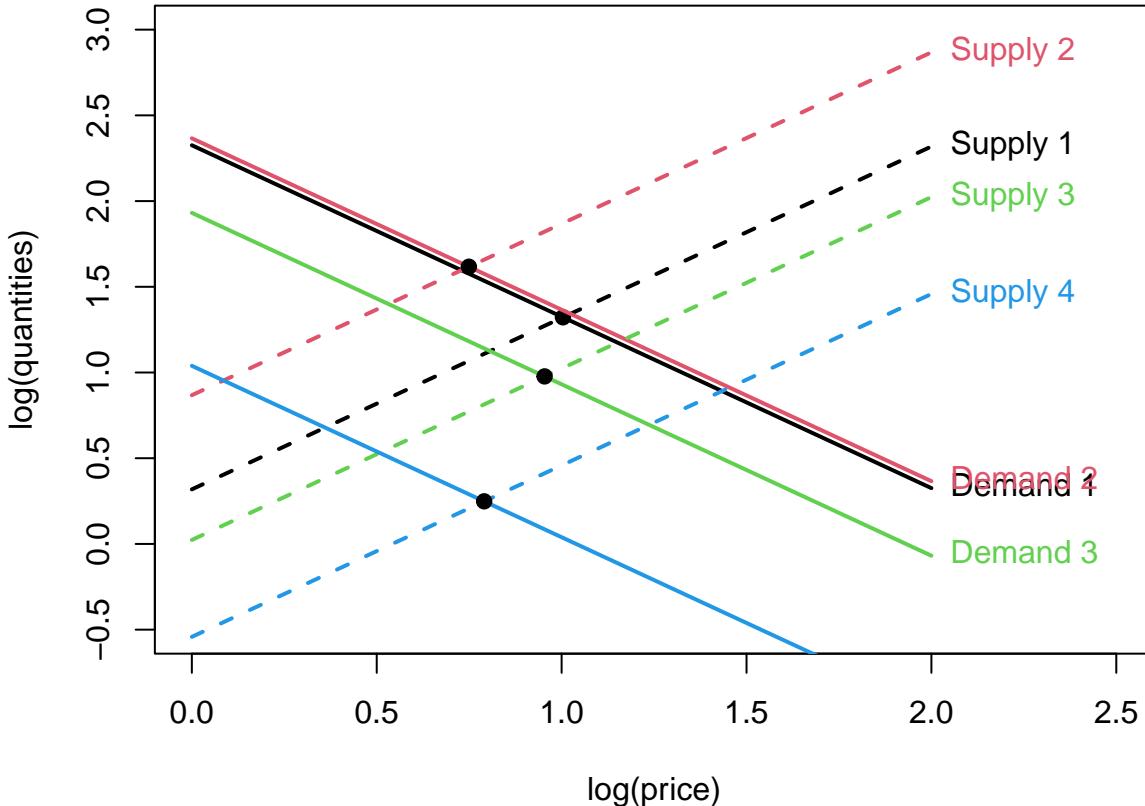


Figure 5.5: This figure illustrates the situation prevailing when estimating a price-elasticity (and the price is endogenous).

Let us use IV regressions to estimate the price elasticity of cigarette demand. For that purpose, we use the `CigarettesSW` dataset of package `AER` (these data are used by Stock and Watson (2003)). This panel dataset documents cigarette consumption for the 48 continental US States from 1985–1995. The instrument is the real tax on cigarettes arising from the state’s general sales tax. The rationale is that larger general sales tax drives cigarette prices up, but the general tax is not determined by other forces affecting  $\varepsilon_t^d$ .

```
data("CigarettesSW", package = "AER")
CigarettesSW$rprice <- with(CigarettesSW, price/cpi)
CigarettesSW$rincome <- with(CigarettesSW, income/population/cpi)
CigarettesSW$tdiff <- with(CigarettesSW, (taxs - tax)/cpi)

## model
eq.IV1 <- ivreg(log(packs) ~ log(rprice) + log(rincome) +
  log(rincome) + tdiff + I(tax/cpi),
  data = CigarettesSW, subset = year == "1995")
eq.IV2 <- ivreg(log(packs) ~ log(rprice) | tdiff,
  data = CigarettesSW, subset = year == "1995")
eq.no.IV <- lm(log(packs) ~ log(rprice) + log(rincome),
  data = CigarettesSW, subset = year == "1995")
stargazer::stargazer(eq.no.IV, eq.IV1, eq.IV2, type="text", no.space = TRUE,
  omit.stat=c("f", "ser"))

## =====
```

```

##                               Dependent variable:
## -----
##                               log(packs)
##      OLS      instrumental
##                  variable
##      (1)      (2)      (3)
## -----
## log(rprice) -1.407*** -1.277*** -1.084*** 
##             (0.251)   (0.263)   (0.317)
## log(rincome)  0.344     0.280
##             (0.235)   (0.239)
## Constant    10.342*** 9.895*** 9.720*** 
##             (1.023)   (1.059)   (1.514)
## -----
## Observations   48       48       48
## R2            0.433    0.429    0.401
## Adjusted R2   0.408    0.404    0.388
## -----
## Note:          *p<0.1; **p<0.05; ***p<0.01

```

```
summary(eq.IV1, diagnostics = TRUE)$diagnostics
```

```

##           df1 df2 statistic      p-value
## Weak instruments 2  44  244.7337536 1.444054e-24
## Wu-Hausman      1  44   3.0678163 8.682505e-02
## Sargan          1  NA   0.3326221 5.641191e-01

```

The last three tests are interpreted as follows:

- Since the p-value of the first test is small, we reject the null hypothesis according to which the instrument is weak.
- The small p-value of the Wu-Hausman test implies that we reject the null hypothesis according to which the OLS estimates are consistent (at the 10% level only, though).
- No over-identification (misspecification) is detected by the Sargan test (large p-value).

**Example 5.5** (Education and wage). In this example, we make use of another dataset proposed by Stock and Watson (2003), namely the `CollegeDistance` dataset.<sup>2</sup> the objective is to estimate the effect of education on wages. Education choice is suspected to be an endogenous variable, which calls for an IV strategy. The instrumental variable is the distance to college (see, e.g., Dee (2004)).

```

library(sem)
data("CollegeDistance", package = "AER")
eq.1st.stage <- lm(education ~ urban + gender + ethnicity + unemp + distance,
                    data = CollegeDistance)
CollegeDistance$ed.pred<- predict(eq.1st.stage)
eq.2nd.stage <- lm(wage ~ urban + gender + ethnicity + unemp + ed.pred,
                     data = CollegeDistance)
eqOLS <- lm(wage ~ urban + gender + ethnicity + unemp + education,
              data=CollegeDistance)
eq2SLS <- ivreg(wage ~ urban + gender + ethnicity + unemp + education |
                  urban + gender + ethnicity + unemp + distance,
                  data=CollegeDistance)
stargazer::stargazer(eq.1st.stage, eq.2nd.stage, eq2SLS, eqOLS,
                      type="text", no.space = TRUE, omit.stat = c("f", "ser"))

```

---

<sup>2</sup>Cross-section data from the High School and Beyond survey conducted by the Department of Education in the 80s. The survey includes students from approximately 1,100 high schools.

```

## 
## =====
##          Dependent variable:
## -----
##          education      wage
##          OLS        OLS   instrumental    OLS
##                      variable
##          (1)       (2)       (3)       (4)
## -----
## urbanyes      -0.092     0.046     0.046     0.070
##                  (0.065)    (0.045)    (0.060)    (0.045)
## genderfemale   -0.025    -0.071*   -0.071   -0.085**
##                  (0.052)    (0.037)    (0.050)    (0.037)
## ethnicityafam  -0.524*** -0.227*** -0.227** -0.556***
##                  (0.072)    (0.073)    (0.099)    (0.052)
## ethnicityhispanic -0.275*** -0.351*** -0.351*** -0.544***
##                  (0.068)    (0.057)    (0.077)    (0.049)
## unemp         0.010     0.139*** 0.139*** 0.133*** 
##                  (0.010)    (0.007)    (0.009)    (0.007)
## distance      -0.087*** 
##                  (0.012)
## ed.pred        0.647*** 
##                  (0.101)
## education      0.647*** 0.005
##                  (0.136)   (0.010)
## Constant      14.061*** -0.359   -0.359   8.641*** 
##                  (0.083)   (1.412)   (1.908)   (0.157)
## -----
## Observations  4,739     4,739     4,739     4,739
## R2            0.023     0.117    -0.612     0.110
## Adjusted R2   0.022     0.116    -0.614     0.109
## =====
## Note: *p<0.1; **p<0.05; ***p<0.01

```

## 5.5 General Regression Model (GRM) and robust covariance matrices

The statistical inference presented above relies on strong assumptions regarding the stochastic properties of the errors. Namely, they are assumed to be mutually uncorrelated (Hypothesis 5.4) and homoskedastic (Hypothesis 5.3).

The objective of this section is to present approaches aimed at adjusting the estimate of the covariance matrix of the OLS estimator ( $(\mathbf{X}'\mathbf{X})^{-1}s^2$ , see Eq. (5.21)), when the previous hypotheses do not hold.

### 5.5.1 Presentation of the General Regression Model (GRM)

It will prove useful to introduce the following notation:

$$\text{Var}(\varepsilon|\mathbf{X}) = \mathbb{E}(\varepsilon\varepsilon'|\mathbf{X}) = \Sigma. \quad (5.26)$$

Note that Eq. (5.26) is more general than Hypothesis 5.3 and 5.4 because the diagonal entries of  $\Sigma$  may be different (as opposed to under Hypothesis 5.3), and the non-diagonal entries of  $\Sigma$  can be non-null (as opposed to under Hypothesis 5.4).

**Definition 5.3** (General Regression Model (GRM)). Hypothesis 5.1 and 5.2, together with Eq. (5.26), form the General Regression Model (GRM) framework.

Naturally, a regression model where Hypotheses 5.1 to 5.4 hold is a specific case of the GRM framework.

The GRM context notably encompasses situations of heteroskedasticity and autocorrelation:

- Heteroskedasticity:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \sigma_n^2 \end{bmatrix}. \quad (5.27)$$

- Autocorrelation:

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho_{2,1} & \dots & \rho_{n,1} \\ \rho_{2,1} & 1 & & \vdots \\ \vdots & & \ddots & \rho_{n,n-1} \\ \rho_{n,1} & \rho_{n,2} & \dots & 1 \end{bmatrix}. \quad (5.28)$$

**Example 5.6** (Auto-regressive processes). Autocorrelation is common in time-series contexts (see Section 9). In a time-series context, subscript  $i$  refers to a date.

Assume for instance that:

$$y_i = \mathbf{x}'_i \beta + \varepsilon_i \quad (5.29)$$

with

$$\varepsilon_i = \rho \varepsilon_{i-1} + v_i, \quad v_i \sim \mathcal{N}(0, \sigma_v^2). \quad (5.30)$$

In this case, we are in the GRM context, with:

$$\Sigma = \frac{\sigma_v^2}{1 - \rho^2} \begin{bmatrix} 1 & \rho & \dots & \rho^{n-1} \\ \rho & 1 & & \vdots \\ \vdots & & \ddots & \rho \\ \rho^{n-1} & \rho^{n-2} & \dots & 1 \end{bmatrix}. \quad (5.31)$$

In some cases —in particular when one assumes a parametric formulation for  $\Sigma$ — one can determine a better (more accurate) estimator than the OLS one. This approach is called Generalized Least Squares (GLS), which we present below.

### 5.5.2 Generalized Least Squares

Assume  $\Sigma$  is known (“feasible GLS”). Because  $\Sigma$  is symmetric positive, it admits a spectral decomposition of the form  $\Sigma = \mathbf{C}\Lambda\mathbf{C}'$ , where  $\mathbf{C}$  is an orthogonal matrix (i.e.  $\mathbf{C}\mathbf{C}' = Id$ ) and  $\Lambda$  is a diagonal matrix (the diagonal entries are the eigenvalues of  $\Sigma$ ).

We have  $\Sigma = (\mathbf{P}\mathbf{P}')^{-1}$  with  $\mathbf{P} = \mathbf{C}\Lambda^{-1/2}$ . Consider the transformed model:

$$\mathbf{P}'\mathbf{y} = \mathbf{P}'\mathbf{X}\beta + \mathbf{P}'\varepsilon \quad \text{or} \quad \mathbf{y}^* = \mathbf{X}^*\beta + \varepsilon^*.$$

The variance of  $\varepsilon^*$  is the identity matrix  $Id$ . In the transformed model, OLS is BLUE (Gauss-Markow Theorem 5.1).

The **Generalized least squares** estimator of  $\beta$  is:

$$\mathbf{b}_{GLS} = (\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{y}. \quad (5.32)$$

We have:

$$\mathbb{V}ar(\mathbf{b}_{GLS} | \mathbf{X}) = (\mathbf{X}'\Sigma^{-1}\mathbf{X})^{-1}.$$

However, in general,  $\Sigma$  is unknown. The GLS estimator is then said to be *infeasible*. Some structure is required. Assume  $\Sigma$  admits a parametric form  $\Sigma(\theta)$ . The estimation becomes *feasible* (FGLS) if one replaces  $\Sigma(\theta)$  by  $\hat{\Sigma}(\hat{\theta})$ , where  $\hat{\theta}$  is a consistent estimator of  $\theta$ . In that case, the FGLS is asymptotically efficient (see Example 5.7).

When  $\Sigma$  has no obvious structure: the OLS (or IV) is the only estimator available. Under regularity assumptions, it remains unbiased, consistent, and asymptotically normally distributed, but not efficient. Standard inference procedures are no longer appropriate.

**Example 5.7** (GLS in the auto-correlation case). Consider the case presented in Example 5.6. Because the OLS estimate  $\mathbf{b}$  of  $\beta$  is consistent, the estimates  $e_i$  of the  $\varepsilon_i$ 's also are. Consistent estimators of  $\rho$  and  $\sigma_v$  are then obtained by regressing the  $e_i$ 's on the  $e_{i-1}$ 's. Using these estimates in Eq. (5.31) provides a consistent estimate of  $\Sigma$ . Applying these steps recursively gives an efficient estimator of  $\beta$  (Cochrane and Orcutt (1949)).

### 5.5.3 Asymptotic properties of the OLS estimator in the GRM framework

Since  $\mathbf{b} = \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon$  and  $\text{Var}(\varepsilon|\mathbf{X}) = \Sigma$ , we have:

$$\text{Var}(\mathbf{b}|\mathbf{X}) = \frac{1}{n} \left( \frac{1}{n} \mathbf{X}'\mathbf{X} \right)^{-1} \left( \frac{1}{n} \mathbf{X}'\Sigma\mathbf{X} \right) \left( \frac{1}{n} \mathbf{X}'\mathbf{X} \right)^{-1}. \quad (5.33)$$

Therefore, the conditional covariance matrix of the OLS estimator is not  $\sigma^2(\mathbf{X}'\mathbf{X})^{-1}$  any longer, and using  $s^2(\mathbf{X}'\mathbf{X})^{-1}$  for inference may be misleading. Below, we will see how to construct appropriate estimates of the covariance matrix of  $\mathbf{b}$ . Before that, let us prove that the OLS estimator remains consistent in the GRM framework.

**Proposition 5.13** (Consistency of the OLS estimator in the GRM framework). *If  $\text{plim}(\mathbf{X}'\mathbf{X}/n)$  and  $\text{plim}(\mathbf{X}'\Sigma\mathbf{X}/n)$  are finite positive definite matrices, then  $\text{plim}(\mathbf{b}) = \beta$ .*

*Proof.* We have  $\text{Var}(\mathbf{b}) = \mathbb{E}[\text{Var}(\mathbf{b}|\mathbf{X})] + \text{Var}[\mathbb{E}(\mathbf{b}|\mathbf{X})]$ . Since  $\mathbb{E}(\mathbf{b}|\mathbf{X}) = \beta$ ,  $\text{Var}[\mathbb{E}(\mathbf{b}|\mathbf{X})] = 0$ . Eq. (5.33) implies that  $\text{Var}(\mathbf{b}|\mathbf{X}) \rightarrow 0$ . Hence  $\mathbf{b}$  converges in mean square, and therefore in probability (see Prop. 10.13).  $\square$

Prop. 5.14 gives the asymptotic distribution of the OLS estimator in the GRM framework.

**Proposition 5.14** (Asymptotic distribution of the OLS estimator in the GRM framework). *If  $Q_{xx} = \text{plim}(\mathbf{X}'\mathbf{X}/n)$  and  $Q_{x\Sigma x} = \text{plim}(\mathbf{X}'\Sigma\mathbf{X}/n)$  are finite positive definite matrices, then:*

$$\sqrt{n}(\mathbf{b} - \beta) \xrightarrow{d} \mathcal{N}(0, Q_{xx}^{-1}Q_{x\Sigma x}Q_{xx}^{-1}).$$

The IV estimator also features a normal asymptotic distribution:

**Proposition 5.15** (Asymptotic distribution of the IV estimator in the GRM framework). *If regressors and IV variables are “well-behaved”, then:*

$$\mathbf{b}_{iv} \xrightarrow{a} \mathcal{N}(\beta, \mathbf{V}_{iv}),$$

where

$$\mathbf{V}_{iv} = \frac{1}{n}(\mathbf{Q}^*) \text{plim} \left( \frac{1}{n} \mathbf{Z}'\Sigma\mathbf{Z} \right) (\mathbf{Q}^*)',$$

with

$$\mathbf{Q}^* = [\mathbf{Q}_{xz}\mathbf{Q}_{zz}^{-1}\mathbf{Q}_{zx}]^{-1}\mathbf{Q}_{xz}\mathbf{Q}_{zz}^{-1}.$$

For practical purposes, one needs to have estimates of  $\Sigma$  in Props. 5.14 or 5.15. The complication comes from the fact that  $\Sigma$  is of dimension  $n \times n$ , and its estimation—based on a sample of length  $n$ —is therefore infeasible in the general case. Notwithstanding, looking at Eq. (5.33), it appears that one can focus on the estimation of  $Q_{x\Sigma x} = \text{plim}(\mathbf{X}'\Sigma\mathbf{X}/n)$  (or  $\text{plim}(\frac{1}{n}\mathbf{Z}'\Sigma\mathbf{Z})$  in the IV case). This matrix being of dimension  $K \times K$ , its estimation is easier.

We have:

$$\frac{1}{n} \mathbf{X}'\Sigma\mathbf{X} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \sigma_{i,j} \mathbf{x}_i \mathbf{x}_j'. \quad (5.34)$$

The so-called **robust covariance matrices** are estimates of the previous matrix. Their computation is based on the fact that if  $\mathbf{b}$  is consistent, then the  $e_i$ 's are consistent estimators of the  $\varepsilon_i$ 's.

In the following sections (5.5.4 and 5.5.5), we present two types of robust covariance matrices.

### 5.5.4 HAC-robust covariance matrices

When only heteroskedasticity prevails, i.e., when matrix  $\Sigma$  is as in Eq. (5.27), then one can use the formula proposed by White (1980) to estimate  $\frac{1}{n}\mathbf{X}'\Sigma\mathbf{X}$  (see Example 5.8). When the residuals feature both heteroskedasticity and auto-correlation, then one can use the Newey and West (1987) approach (see Example 5.9).

**Example 5.8** (Heteroskedasticity). This is the case of Eq. (5.27). We have  $\sigma_{i,j} = 0$  for  $i \neq j$ . Hence, in this case, we then need to estimate  $\frac{1}{n} \sum_{i=1}^n \sigma_i^2 \mathbf{x}_i \mathbf{x}_i'$ . White (1980) has shown that, under general conditions:

$$\text{plim} \left( \frac{1}{n} \sum_{i=1}^n \sigma_i^2 \mathbf{x}_i \mathbf{x}_i' \right) = \text{plim} \left( \frac{1}{n} \sum_{i=1}^n e_i^2 \mathbf{x}_i \mathbf{x}_i' \right). \quad (5.35)$$

The estimator of  $\frac{1}{n}\mathbf{X}'\Sigma\mathbf{X}$  therefore is:

$$M_{HC0} = \frac{1}{n} \mathbf{X}' \begin{bmatrix} e_1^2 & 0 & \dots & 0 \\ 0 & e_2^2 & & \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & e_n^2 \end{bmatrix} \mathbf{X}. \quad (5.36)$$

where the  $e_i$  are the OLS residuals of the regression. The previous estimator is often called **HC0**. The **HC1** estimator, due to MacKinnon and White (1985), is obtained by applying an adjustment factor  $n/(n - K)$  for the number of degrees of freedom (as in Prop. 5.6). That is:

$$M_{HC1} = \frac{n}{n - K} M_{HC0}. \quad (5.37)$$

We can illustrate the influence of heteroskedasticity using simulations. Consider the following model:

$$y_i = x_i + \varepsilon_i, \quad \varepsilon_i \sim \mathcal{N}(0, x_i^2),$$

where the  $x_i$ 's are i.i.d.  $t(6)$ .

Here is a simulated sample ( $n = 200$ ) of this model:

```
n <- 200
x <- rt(n, df=6)
y <- x + x*rnorm(n)
par(plt=c(.1,.95,.1,.95))
plot(x,y,pch=19)
```

We simulate 1000 samples of the same model with  $n = 200$ . For each sample, we compute the OLS estimate of  $\beta$  ( $= 1$ ). For each of the 1000 OLS estimations, we employ (a) the standard OLS variance formula ( $s^2(\mathbf{X}'\mathbf{X})^{-1}$ ) and (b) the White formula to estimate the variance of  $b$ . For each formula, we compute the average of the 1000 resulting standard deviations and compare these with the standard deviation of the 1000 OLS estimate of  $\beta$ .

```
n <- 200 # sample size
N <- 1000 # number of simulated samples
XX <- matrix(rt(n*N, df=6), n, N)
YY <- matrix(XX + XX*rnorm(n), n, N)
all_b <- NULL; all_V_OLS <- NULL; all_V_White <- NULL
for(j in 1:N){
  Y <- matrix(YY[,j], ncol=1)
  X <- matrix(XX[,j], ncol=1)
  b <- solve(t(X) %*% X) %*% t(X) %*% Y
  e <- Y - X %*% b
  S <- 1/n * t(X) %*% diag(c(e^2)) %*% X
  V_OLS <- solve(t(X) %*% X) * var(e)
  V_White <- 1/n * (solve(1/n*t(X) %*% X)) %*% S %*% (solve(1/n*t(X) %*% X))
  all_b <- c(all_b, b)}
```

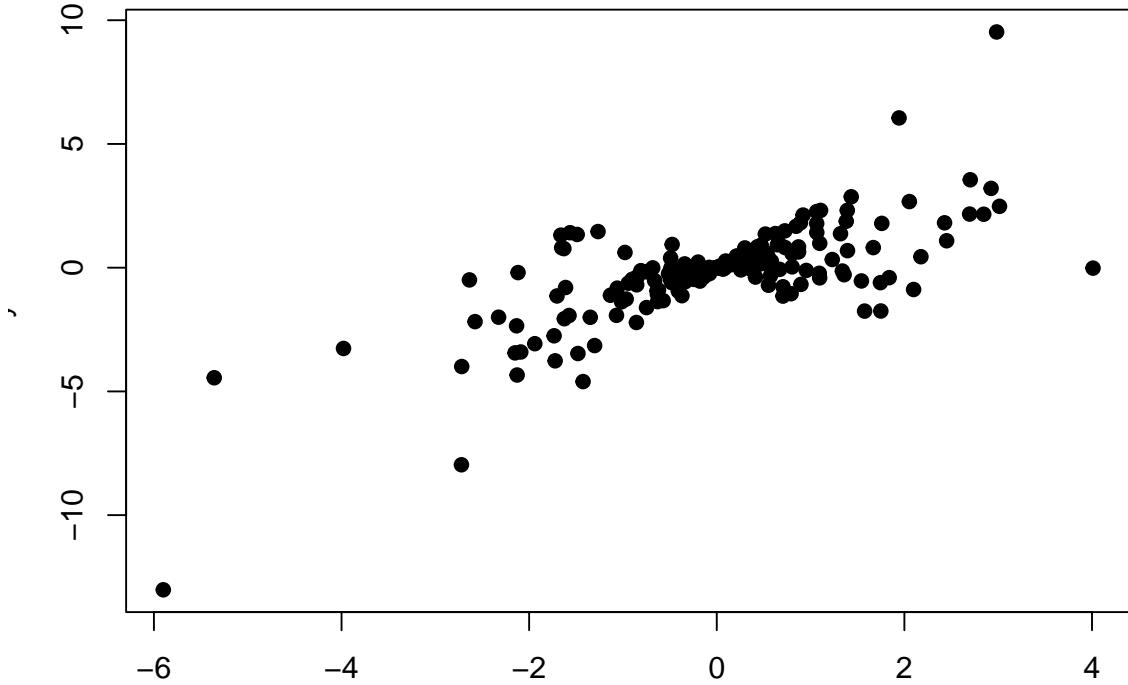


Figure 5.6: Situation of heteroskedasticity. The model is  $y_i = x_i + \varepsilon_i$ ,  $\varepsilon_i \sim \mathcal{N}(0, x_i^2)$ , where the  $x_i$ 's are i.i.d.  $t(6)$ .

```

all_V_OLS <- c(all_V_OLS,V_OLS)
all_V_White <- c(all_V_White,V_White)
}
c(sd(all_b),mean(sqrt(all_V_OLS)),mean(sqrt(all_V_White)))

```

```
## [1] 0.14024423 0.06748804 0.13973431
```

The results show that the White formula yields, on average, an estimated standard deviation that is much closer to the “true” value than the standard OLS formula. The latter underestimate the standard deviation of  $b$ .

In the following example, we regress GDP growth rates from the Jordà et al. (2017) database on a systemic financial crisis dummy. We compute the HC0- and HC1-based standard deviations of the parameter estimate, and compare it to the one based on the standard OLS formula. The adjusted standard deviations are close to the one provided by the non-adjusted OLS formula.

```

library(lmtest)
library(sandwich)
nT <- dim(JST)[1]
JST$growth <- NaN
JST$growth[2:nT] <- log(JST$rgdpbarro[2:nT]/JST$rgdpbarro[1:(nT-1)])
JST.red <- subset(JST,year>1950)
JST.red$iso <- as.factor(JST.red$iso)
JST.red$year <- as.factor(JST.red$year)
eq <- lm(growth~crisisJST+iso,data=JST.red)
rbind(coeftest(eq)[2,],
      coeftest(eq, vcov = vcovHC(eq, type = "HC0"))[2,],
      coeftest(eq, vcov = vcovHC(eq, type = "HC1"))[2,])

```

```

##           Estimate Std. Error   t value    Pr(>|t|) 
## [1,] -0.02481424 0.005490411 -4.519560 6.789284e-06
## [2,] -0.02481424 0.005605452 -4.426804 1.040602e-05
## [3,] -0.02481424 0.005648268 -4.393247 1.212105e-05

```

**Example 5.9** (Heteroskedasticity and Autocorrelation (HAC)). Newey and West (1987) have proposed a formula to address both heteroskedasticity and auto-correlation of the residuals (Eqs. (5.27) and (5.28)). They show that, if the correlation between terms  $i$  and  $j$  gets sufficiently small when  $|i - j|$  increases:

$$\begin{aligned} \text{plim} \left( \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \sigma_{i,j} \mathbf{x}_i \mathbf{x}'_j \right) &\approx \\ \text{plim} \left( \frac{1}{n} \sum_{t=1}^n e_t^2 \mathbf{x}_t \mathbf{x}'_t + \frac{1}{n} \sum_{\ell=1}^L \sum_{t=\ell+1}^n w_\ell e_t e_{t-\ell} (\mathbf{x}_t \mathbf{x}'_{t-\ell} + \mathbf{x}_{t-\ell} \mathbf{x}'_t) \right), \end{aligned} \quad (5.38)$$

where  $w_\ell = 1 - \ell/(L + 1)$  (with  $L$  large).

Let us illustrate the influence of autocorrelation using simulations. We consider the following model:

$$y_i = x_i + \varepsilon_i, \quad \varepsilon_i \sim \mathcal{N}(0, x_i^2), \quad (5.39)$$

where the  $x_i$ 's and the  $\varepsilon_i$ 's are such that:

$$x_i = 0.8x_{i-1} + u_i \quad \text{and} \quad \varepsilon_i = 0.8\varepsilon_{i-1} + v_i, \quad (5.40)$$

where the  $u_i$ 's and the  $v_i$ 's are i.i.d.  $\mathcal{N}(0, 1)$ .

We simulate 500 samples of the same model with  $n = 200$ . For each sample, we compute the OLS estimate of  $\beta (=1)$ . For each of the 1000 OLS estimations, we employ (a) the standard OLS variance formula ( $s^2(\mathbf{X}'\mathbf{X})^{-1}$ ), (b) the White formula, and (c) the Newey-West formula to estimate the variance of  $b$ . For each formula, we compute the average of the 500 resulting standard deviations and compare these with the standard deviation of the 500 OLS estimate of  $\beta$ .

```
library(AEC)
n <- 100 # sample length
nb.sim <- 500 # number of simulated samples
all.b <- NULL; all.OLS.stdv.b <- NULL
all.Whi.stdv.b <- NULL; all.NW.stdv.b <- NULL
for(i in 1:nb.sim){
  eps <- rnorm(n); x <- rnorm(n)
  for(i in 2:n){
    eps[i] <- eps[i] + .8*eps[i-1]
    x[i] <- x[i] + .8*x[i-1]
  }
  y <- x + eps
  eq <- lm(y~x)
  all.b <- c(all.b, eq$coefficients[2])
  all.OLS.stdv.b <- c(all.OLS.stdv.b, summary(eq)$coefficients[2,2])
  X <- cbind(rep(1,n),x)
  XX <- t(X) %*% X; XX_1 <- solve(XX)
  E2 <- diag(eq$residuals^2)
  White.V <- XX_1 %*% (t(X) %*% E2 %*% X) %*% XX_1
  all.Whi.stdv.b <- c(all.Whi.stdv.b, sqrt(White.V[2,2]))
  # HAC:
  U <- X * cbind(eq$residuals,eq$residuals)
  XSigmaX <- NW.LongRunVariance(U,5)
  NW.V <- 1/n * (n*XX_1) %*% XSigmaX %*% (n*XX_1)
  all.NW.stdv.b <- c(all.NW.stdv.b, sqrt(NW.V[2,2]))
}
cbind(sd(all.b),mean(all.OLS.stdv.b),
      mean(all.Whi.stdv.b),mean(all.NW.stdv.b))

##          [,1]      [,2]      [,3]      [,4]
## [1,] 0.201172 0.1013048 0.0962689 0.146974
```

The results show that the Newey-West formula yields, on average, an estimated standard deviation that is closer to the “true” value than the standard OLS formula. The latter underestimate the standard deviation of  $b$ .

What precedes suggest that, when the residuals feature autocorrelation, it is important to use appropriately adjusted covariance matrices to make statistical inference. How to detect autocorrelation in residuals? A popular test has been proposed by Durbin and Watson (1950) and Durbin and Watson (1951). The Durbin-Watson test statistic is:

$$DW = \frac{\sum_{i=2}^n (e_i - e_{i-1})^2}{\sum_{i=1}^n e_i^2} = 2(1 - r) - \underbrace{\frac{e_1^2 + e_n^2}{\sum_{i=1}^n e_i^2}}_{\xrightarrow{p} 0},$$

where  $r$  is the slope in the regression of the  $e_i$ 's on the  $e_{i-1}$ 's, i.e.:

$$r = \frac{\sum_{i=2}^n e_i e_{i-1}}{\sum_{i=1}^{n-1} e_i^2}.$$

( $r$  is a consistent estimator of  $\text{Cor}(\varepsilon_i, \varepsilon_{i-1})$ , i.e.  $\rho$  in Eq. (5.30).)

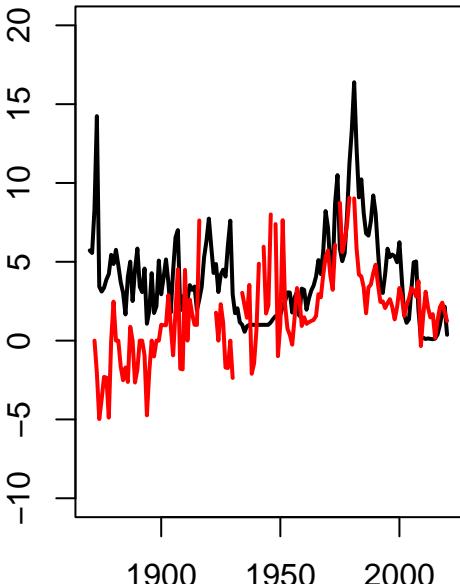
The one-sided test for  $H_0: \rho = 0$  against  $H_1: \rho > 0$  is carried out by comparing  $DW$  to values  $d_L(T, K)$  and  $d_U(T, K)$ :

$$\begin{cases} \text{If } DW < d_L, & \text{the null hypothesis is rejected;} \\ \text{if } DW > d_U, & \text{the hypothesis is not rejected;} \\ \text{If } d_L \leq DW \leq d_U, & \text{no conclusion is drawn.} \end{cases}$$

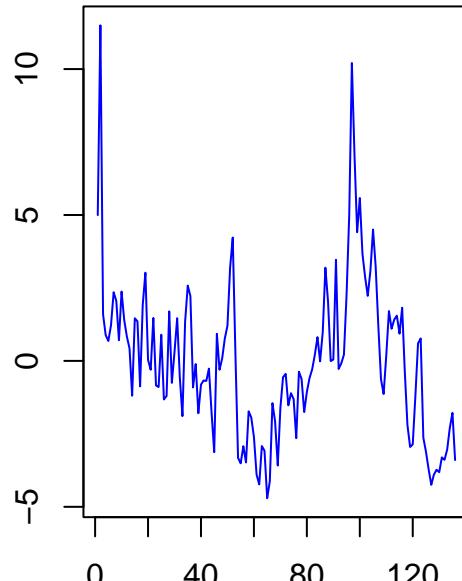
**Example 5.10** (Durbin-Watson test). We regress the short-term nominal US interest rate on inflation. We then employ the Durbin-Watson test to see whether the residuals are auto-correlated (which is quite obviously the case).

```
library(car)
data <- subset(JST, iso=="USA"); T <- dim(data)[1]
data$infl <- c(NaN, 100*log(data$cpi[2:T]/data$cpi[1:(T-1)]))
data$infl[(data$infl < -5) | (data$infl > 10)] <- NaN
par(mfrow=c(1,2))
plot(data$year, data$stir, ylim=c(-10, 20), type="l", lwd=2, xlab="", 
     ylab="", main="Nominal rate and inflation")
lines(data$year, data$infl, col="red", lwd=2)
eq <- lm(stir~infl, data=data)
plot(eq$residuals, type="l", col="blue", main="Residuals", xlab="", ylab="")
```

Nominal rate and inflation



Residuals



```
durbinWatsonTest(eq)
```

```
##   lag Autocorrelation D-W Statistic p-value
##   1      0.7321902    0.4984178     0
## Alternative hypothesis: rho != 0
```

### 5.5.5 Cluster-robust covariance matrices

The present section is based on MacKinnon et al. (2022); another useful reference is Cameron and Miller (2014). We will see how one can approximate  $Q_{x\Sigma x} = \text{plim}(\mathbf{X}'\Sigma\mathbf{X}/n)$  (see Prop. 5.14) when the dataset can be decomposed into *clusters*. These clusters constitute a partition of the sample, and they are such that the error terms may be correlated within a given cluster, but not across different clusters. A cluster may, e.g., gathers entities from a given geographical area, from the same industry, or same age cohort.

The OLS estimator satisfies:

$$\mathbf{b} = \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\varepsilon. \quad (5.41)$$

Consider a set  $\{n_1, n_2, \dots, n_G\}$  s.t.  $n = \sum_g n_g$ , on which is based the following decomposition of  $\mathbf{X}$ :

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_G \end{bmatrix}.$$

With these notations, Eq. (5.41) rewrites:

$$\mathbf{b} - \beta = \left( \sum_{g=1}^G \mathbf{X}'_g \mathbf{X}_g \right)^{-1} \sum_{g=1}^G \mathbf{s}_g, \quad (5.42)$$

where

$$\mathbf{s}_g = \mathbf{X}'_g \varepsilon_g \quad (5.43)$$

denotes the score vector (of dimension  $K \times 1$ ) associated with the  $g^{th}$  cluster.

If the model is correctly specified then  $\mathbb{E}(\mathbf{s}_g) = 0$  for all clusters  $g$ . Note that Eq. (5.42) is valid for any partition of  $\{1, \dots, n\}$ . Dividing the sample into clusters becomes meaningful if we further assume that the following hypothesis holds:

**Hypothesis 5.6** (Clusters). We have:

$$(i) \mathbb{E}(\mathbf{s}_g \mathbf{s}'_g) = \Sigma_g, \quad (ii) \mathbb{E}(\mathbf{s}_g \mathbf{s}'_q) = 0, \quad g \neq q,$$

where  $s_g$  is defined in Eq. (5.43).

The real assumption here is (ii); the first one simply gives a notation for the covariance matrix of the score associated with the  $g^{th}$  cluster. Remark that these covariance matrices can differ across clusters. That is, *cluster-based inference is robust against both heteroskedasticity and intra-cluster dependence without imposing any restrictions on the (unknown) form of either of them*.

Naturally, matrix  $\Sigma_g$  depends on the covariance structure of the  $\varepsilon$ 's. In particular, if  $\Omega_g = \mathbb{E}(\varepsilon_g \varepsilon'_g | \mathbf{X}_g)$ , then we have  $\Sigma_g = \mathbb{E}(\mathbf{X}'_g \Omega_g \mathbf{X}_g)$ .

Under Hypothesis 5.6, it comes that the conditional covariance matrix of  $\mathbf{b}$  is:

$$(\mathbf{X}'\mathbf{X})^{-1} \left( \sum_{g=1}^G \Sigma_g \right) (\mathbf{X}'\mathbf{X})^{-1} \quad (5.44)$$

Let us denote by  $\varepsilon_{g,i}$  the error associated with the  $i^{th}$  component of vector  $\varepsilon_g$ . Consider the special case where  $\mathbb{E}(\varepsilon_{g,i} \varepsilon_{g,j} | \mathbf{X}_g) = \sigma^2 \mathbb{I}_{\{i=j\}}$ , then Eq. (5.44) gives the standard expression  $\sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$  (see Eq. (5.9)).

If we have  $\mathbb{E}(\varepsilon_{gi}\varepsilon_{gj}|\mathbf{X}_g) = \sigma_{gi}^2\mathbb{I}_{\{i=j\}}$ , then we fall in the case addressed by the White formula (see Example 5.8). That is, in this case, the conditional covariance matrix of  $\mathbf{b}$  is:

$$(\mathbf{X}'\mathbf{X})^{-1} \left( \mathbf{X}' \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \sigma_n^2 \end{bmatrix} \mathbf{X} \right) (\mathbf{X}'\mathbf{X})^{-1}.$$

As in White (1980), the natural way to approach the conditional covariance given in Eq. (5.44) consists in replacing the  $\Sigma_g$  matrices by their sample equivalent, i.e.  $\widehat{\Sigma}_g = \mathbf{X}'_g \mathbf{e}_g \mathbf{e}'_g \mathbf{X}_g$ . Adding corrections for the number of degrees of freedom, this leads to the following estimate of the covariance matrix of  $\mathbf{b}$ :

$$\frac{G(n-1)}{(G-1)(n-K)} (\mathbf{X}'\mathbf{X})^{-1} \left( \sum_{g=1}^G \widehat{\Sigma}_g \right) (\mathbf{X}'\mathbf{X})^{-1}. \quad (5.45)$$

The previous estimate is CRCV1 in MacKinnon et al. (2022). Note that we indeed find the White-MacKinnon estimator (Eq. (5.37)) when  $G = n$ .

Remark that if there was only one cluster ( $G = 1$ ), and neglecting the degree-of-freedom correction, we would have:

$$(\mathbf{X}'\mathbf{X})^{-1} (\mathbf{X}'\mathbf{e}\mathbf{e}'\mathbf{X}) (\mathbf{X}'\mathbf{X})^{-1} = 0$$

because  $\mathbf{X}'\mathbf{e} = 0$ . Hence, having large clusters does not necessarily increase variance.

Often, when working with panel data (see Chapter 6), we want to cluster in different dimensions. A typical case is when the data are indexed by both individuals ( $i$ ) and time ( $t$ ). In that case, we may indeed suspect that: (a) the residuals are correlated across clusters of dates (e.g., with monthly data, a cluster may be one year) and (b) the residuals are correlated across clusters of individuals (e.g., with data at the county level a cluster may be a state). In this case, one can employ **two-way clustering**.

Formally, consider two distinct partitions of the data: one through index  $g$ , with  $g \in \{1, \dots, G\}$ , and the other through index  $h$ , with  $h \in \{1, \dots, H\}$ . Accordingly, we denote by  $\mathbf{X}_{g,h}$  the submatrix of  $\mathbf{X}$  that contains the explanatory variables corresponding to clusters  $g$  and  $h$  (e.g., the firms of a given country  $g$  at a given date  $h$ ). We also denote by  $\mathbf{X}_{g,\bullet}$  (respectively  $\mathbf{X}_{\bullet,h}$ ) the submatrix of  $\mathbf{X}$  containing all explanatory variables pertaining to cluster  $g$ , for all possible values of  $h$  (resp. to cluster  $h$ , for all possible values of  $g$ ).

We will make the following assumption:

**Hypothesis 5.7** (Two-way clusters). We have:

$$\begin{aligned} \mathbb{E}(\mathbf{s}_{g,\bullet}\mathbf{s}'_{g,\bullet}) &= \Sigma_g, & \mathbb{E}(\mathbf{s}_{\bullet,h}\mathbf{s}'_{\bullet,h}) &= \Sigma_h^*, & \mathbb{E}(\mathbf{s}_{g,h}\mathbf{s}'_{g,h}) &= \Sigma_{g,h}, \\ \mathbb{E}(\mathbf{s}_{g,h}\mathbf{s}'_{q,k}) &= 0 \text{ if } g \neq q \text{ and } h \neq k. \end{aligned}$$

**Proposition 5.16** (Covariance of scores in the two-way-cluster setup). *Under this assumption, the matrix of covariance of the scores is given by:*

$$\Sigma = \sum_{g=1}^G \Sigma_g + \sum_{h=1}^H \Sigma_h^* - \sum_{g=1}^G \sum_{h=1}^H \Sigma_{g,h}.$$

(The last term on the right-hand side must be subtracted in order to avoid double counting.)

*Proof.* We have:

$$\begin{aligned} \Sigma &= \sum_{g=1}^G \sum_{q=1}^G \sum_{h=1}^H \sum_{k=1}^H \mathbf{s}_{g,h}\mathbf{s}'_{q,k} \\ &= \sum_{g=1}^G \underbrace{\left( \sum_{h=1}^H \sum_{k=1}^H \mathbf{s}_{g,h}\mathbf{s}'_{g,k} \right)}_{=\Sigma_g} + \sum_{h=1}^H \underbrace{\left( \sum_{g=1}^G \sum_{q=1}^G \mathbf{s}_{g,h}\mathbf{s}'_{q,h} \right)}_{=\Sigma_h^*} - \sum_{g=1}^G \sum_{h=1}^H \mathbf{s}_{g,h}\mathbf{s}'_{g,h}, \end{aligned}$$

which gives the result.  $\square$

The asymptotic theory can be based on two different approaches: (i) large number of clusters (common case), and (ii) fixed number of clusters but large number of observations in each cluster (see Subsections 4.1 and 4.2 in MacKinnon et al. (2022)). The more variable the  $N_g$ 's (clusters' sizes are heterogeneous in terms of size), the less reliable asymptotic inference based on Eq. (5.45), especially when a very few clusters are unusually large, or when the distribution of the data is heavy-tailed. These issues are somehow mitigated when the clusters have an approximate factor structure.

In practice,  $\Sigma$  is estimated by:

$$\widehat{\Sigma} = \sum_{g=1}^G \widehat{\mathbf{s}}_{g,\bullet} \widehat{\mathbf{s}}'_{g,\bullet} + \sum_{h=1}^H \widehat{\mathbf{s}}_{\bullet,h} \widehat{\mathbf{s}}'_{\bullet,h} - \sum_{g=1}^G \sum_{h=1}^H \widehat{\mathbf{s}}_{g,h} \widehat{\mathbf{s}}'_{g,h},$$

and we use:

$$\widehat{\text{Var}}(\mathbf{b}) = (\mathbf{X}'\mathbf{X})^{-1} \widehat{\Sigma} (\mathbf{X}'\mathbf{X})^{-1}.$$

As an alternative to the asymptotic approximation to the distribution of a statistic of interest, one can resort to bootstrap approximation (see Section 5 of MacKinnon et al. (2022)). In R, the package `fwildclusterboot` allows to implement such approaches.<sup>3</sup>

Let us come back to the analysis of the effect of systemic financial crises on GDP growth. Clustering the data at the country level and, further, at both the country and time levels gives the following:

```
eq <- lm(growth~crisisJST+iso, data=JST.red)
rbind(coeftest(eq)[2,],
      coeftest(eq, vcov = vcovHC(eq, type = "HC1"))[2,],
      coeftest(eq, vcov = vcovCL(eq, cluster = JST.red[, c("iso")]))[2,],
      coeftest(eq, vcov = vcovCL(eq, cluster = JST.red[, c("iso", "year")]))[2,])

##           Estimate Std. Error   t value    Pr(>|t|) 
## [1,] -0.02481424 0.005490411 -4.519560 6.789284e-06
## [2,] -0.02481424 0.005648268 -4.393247 1.212105e-05
## [3,] -0.02481424 0.005847708 -4.243413 2.365508e-05
## [4,] -0.02481424 0.006546931 -3.790209 1.577572e-04
```

## 5.6 Shrinkage methods

Choosing the appropriate explanatory variables is often complicated, especially in the presence of many potentially relevant covariates. Keeping a large number of covariates results in large standard deviations for the estimated parameters (see Section 5.3.3). In order to address this issue, shrinkage methods have been designed. The objective of these methods is to help select a limited number of variables by shrinking the regression coefficients of the less useful variables towards zero. The two best-known shrinkage techniques are **ridge regression** and the **lasso** approach.<sup>4</sup> In both cases (ridge and lasso), the OLS minimization problem (see Section 5.2), i.e.,

$$\mathbf{b} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^n (y_i - \mathbf{x}'_i \beta)^2 \quad (5.46)$$

is replaced with the following:

$$\mathbf{b}_\lambda = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^n (y_i - \mathbf{x}'_i \beta)^2 + \lambda f(\beta), \quad (5.47)$$

where  $\lambda f(\beta)$  is a penalty term that positively depends on the “size” of the components of  $\beta$ . This term is called the *shrinkage penalty* term.

<sup>3</sup>See, e.g., this tutorial by Alexander Fischer.

<sup>4</sup>See Tibshirani (2011) for a review of the lasso approach. See also Section 6.2 of James et al. (2013).

Specifically, assuming that the vector  $\mathbf{x}_i$  of potential covariates is of dimension  $K \times 1$ , we have:

$$\begin{aligned} f(\beta) &= \sum_{j=1}^K \beta_j^2 \quad \text{in the ridge case } (\ell_2 \text{ norm}), \\ f(\beta) &= \sum_{j=1}^K |\beta_j| \quad \text{in the lasso case } (\ell_1 \text{ norm}). \end{aligned}$$

In most cases, we do not want to involve the intercept in the set of parameters to shrink, and the preceding equations are respectively replaced with:

$$\begin{aligned} f(\beta) &= \sum_{j=2}^K \beta_j^2 \quad (\text{ridge}), \\ f(\beta) &= \sum_{j=2}^K |\beta_j| \quad (\text{lasso}). \end{aligned}$$

The nature of the penalty —based on either the  $\ell_1$  or the  $\ell_2$  norm— implies a different behaviour of the parameter estimates when  $\lambda$  —the *tuning parameter*— grows. In the ridge regression, coefficient estimates go to zero (shrinkage); in the lasso case, some coefficients reach zero when  $\lambda$  reach some values. In other words, while ridge regression achieve shrinkage, lasso regressions achieve shrinkage *and* variable selection.

Parameter  $\lambda$  has to be determined separately from the minimization problem of Eq. (5.47). One can combine standard criteria (e.g., BIC or Akaike) for this purpose.

In R, one can use the `glmnet` package to run ridge and lasso regressions. In the following example, we employ this package to model interest rates proposed to debtors, using data extracted from the Lending Club platform.

To begin with, let us define the variables we want to consider:

```
library(AEC)
library(glmnet)
credit$owner <- 1*(credit$home_ownership=="OWN")
credit$renter <- 1*(credit$home_ownership=="MORTGAGE")
credit$verification_status <- 1*(credit$verification_status=="Not Verified")
credit$emp_length_10 <- 1*(credit$emp_length_10)
credit$log_annual_inc <- log(credit$annual_inc)
credit$log_funded_amnt <- log(credit$funded_amnt)
credit$annual_inc2 <- (credit$annual_inc)^2
credit$funded_amnt2 <- (credit$funded_amnt)^2
x <- subset(credit,
             select = c(delinq_2yrs,annual_inc,annual_inc2,log_annual_inc,
                        dti,installment,verification_status,funded_amnt,
                        funded_amnt2,log_funded_amnt,pub_rec,emp_length_10,
                        owner,renter,pub_rec_bankruptcies,revol_util,revol_bal))
```

Let us standardize the data:

```
y <- scale(credit$int_rate)
x <- scale(x)
```

Next, we define the set of  $\lambda$  we will use, and run the ridge and lasso regressions:

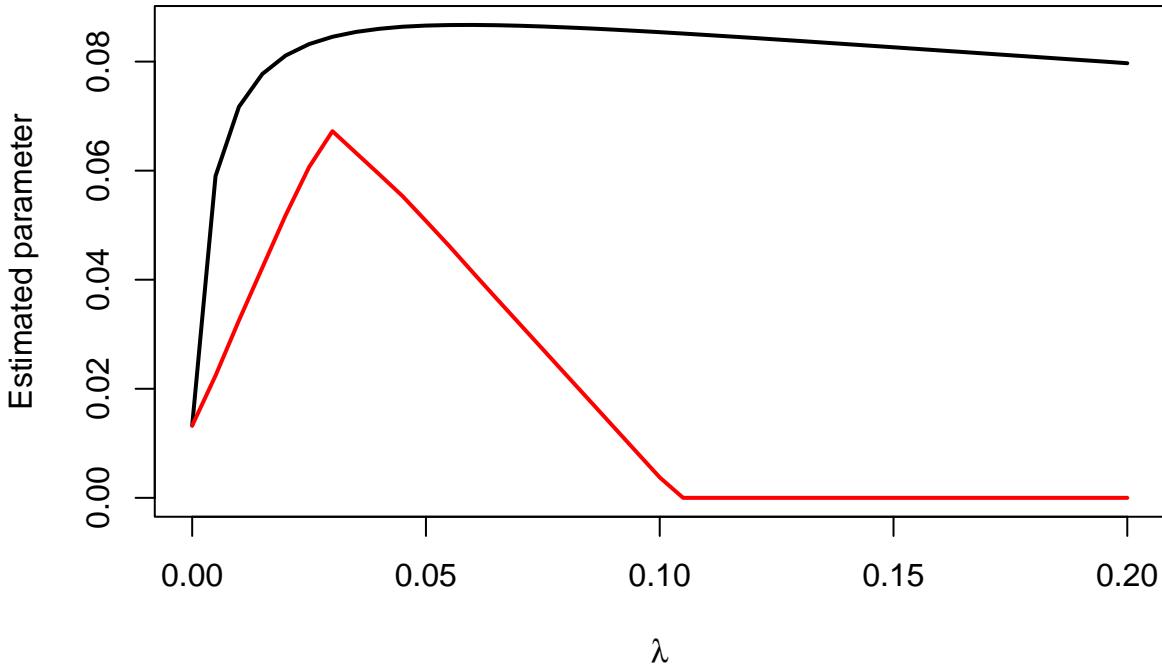
```
grid.lambda <- seq(0,.2,by=.005)
result.ridge <- glmnet(x, y, alpha = 0, lambda = grid.lambda)
result.lasso <- glmnet(x, y, alpha = 1, lambda = grid.lambda)
```

The following figure shows how estimated parameters depend on  $\lambda$ :

```

variab <- 6
plot(result.ridge$lambda,coef(result.ridge)[variab,],type="l",
      ylim=c(min(coef(result.ridge)[variab,],coef(result.lasso)[variab,]),
             max(coef(result.ridge)[variab,],coef(result.lasso)[variab,])),
      xlab=expression(lambda),ylab="Estimated parameter",lwd=2)
lines(result.lasso$lambda,coef(result.lasso)[variab,],col="red",lwd=2)

```



Let us take two values of  $\lambda$  and see the associated estimated parameters in the context of lasso regressions:

```

i <- 20; j <- 40
cbind(result.lasso$lambda[i],result.lasso$lambda[j])

```

```

##      [,1]  [,2]
## [1,] 0.105 0.005

```

```

cbind(coef(result.lasso)[,i],coef(result.lasso)[,j])

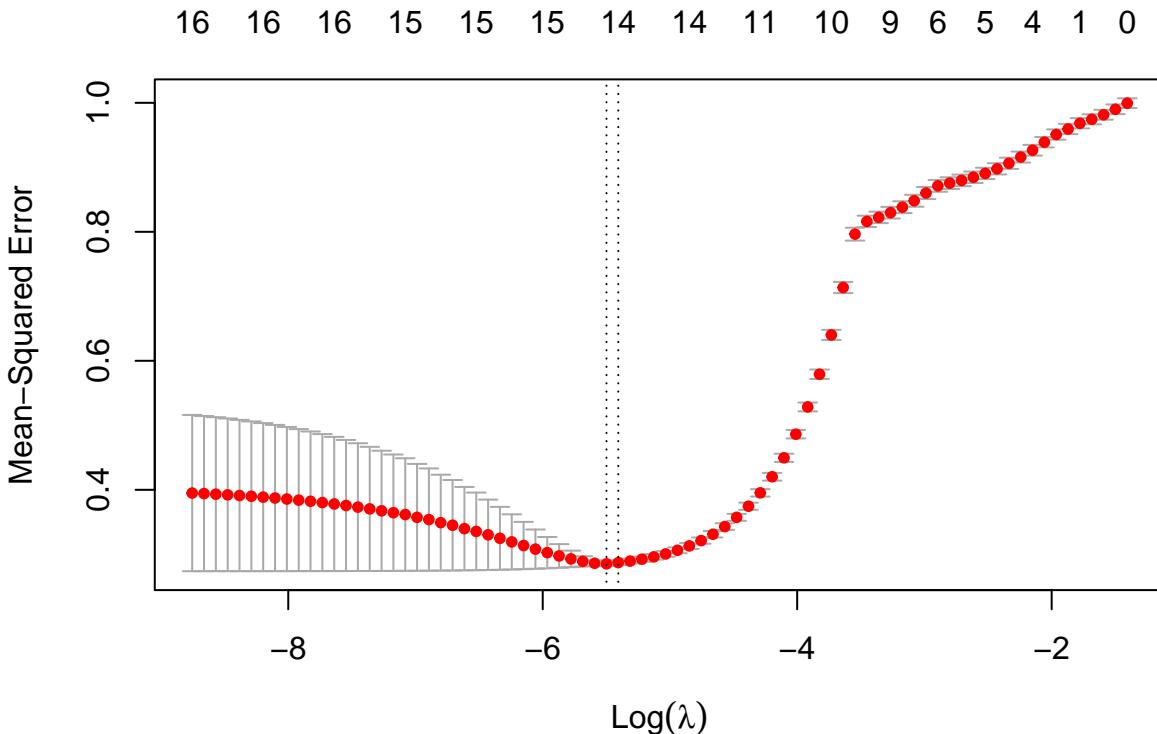
```

	[,1]	[,2]
## (Intercept)	-1.044971e-15	1.088731e-14
## delinq_2yrs	6.308870e-02	6.893527e-02
## annual_inc	0.000000e+00	4.595653e-03
## annual_inc2	0.000000e+00	0.000000e+00
## log_annual_inc	0.000000e+00	-3.612382e-02
## dti	0.000000e+00	2.242246e-02
## installment	1.476796e-01	8.228729e+00
## verification_status	0.000000e+00	-9.750047e-04
## funded_amnt	0.000000e+00	-7.309169e+00
## funded_amnt2	0.000000e+00	-4.711846e-01
## log_funded_amnt	0.000000e+00	-2.460932e-01
## pub_rec	3.390816e-02	5.997252e-02
## emp_length_10	0.000000e+00	-1.924941e-02
## owner	0.000000e+00	-2.444599e-02
## renter	-3.882640e-02	-6.243087e-02
## pub_rec_bankruptcies	0.000000e+00	0.000000e+00
## revol_util	0.000000e+00	0.000000e+00
## revol_bal	0.000000e+00	2.402268e-03

```
# Compute values of y predicted by the model, for all lambdas:
pred1 <- predict(result.lasso,as.matrix(x))
# Compute values of y predicted by the model, for a specific value:
pred2 <- predict(result.lasso,as.matrix(x),s=0.085)
```

The `glmnet` package (see Hastie et al. (2021)) also offers tools to implement cross-validation:

```
# cross validation (cv):
cvglmnet <- cv.glmnet(as.matrix(x),y)
plot(cvglmnet)
```



```
# lambda.min: lambda that gives minimum mean cross-validated error
cvglmnet$lambda.min
```

```
## [1] 0.004091039
```

```
# lambda.1se: largest lambda s.t. cost within the one-std-dev cv-based band
cvglmnet$lambda.1se
```

```
## [1] 0.00448991
```

```
coef(cvglmnet, s = "lambda.min") # associated parameters
```

```
## 18 x 1 sparse Matrix of class "dgCMatrix"
##                               s1
## (Intercept)      1.128990e-14
## delinq_2yrs     6.625860e-02
## annual_inc      6.054923e-03
## annual_inc2     .
## log_annual_inc -3.795151e-02
## dti             2.065272e-02
## installment     8.518961e+00
## verification_status -2.894435e-03
## funded_amnt    -7.560853e+00
## funded_amnt2   -5.041923e-01
## log_funded_amnt -2.537567e-01
```

```
## pub_rec           5.804333e-02
## emp_length_10    -1.894666e-02
## owner            -2.484192e-02
## renter           -6.039831e-02
## pub_rec_bankruptcies .
## revol_util        .
## revol_bal         3.093752e-03

# predicted values of y for specific values of x:
predict(cvglmnet, newx = as.matrix(x)[1:5,], s = "lambda.min")

##          lambda.min
## 21529  0.34496384
## 21547 -0.04553753
## 21579  0.56455499
## 21583 -0.20696954
## 21608 -0.12165356
```



# Chapter 6

## Panel regressions

### 6.1 Specification and notations

A standard panel situation is as follows: the sample covers a lot of “entities”, indexed by  $i \in \{1, \dots, n\}$ , with  $n$  large, and, for each entity, we observe different variables over a small number of periods  $t \in \{1, \dots, T\}$ . This is a *longitudinal dataset*.

The linear panel regression model is:

$$y_{i,t} = \underbrace{\mathbf{x}'_{i,t}}_{K \times 1} \underbrace{\beta}_{\text{Individual effects}} + \underbrace{\mathbf{z}'_i \alpha}_{\text{Individual effects}} + \varepsilon_{i,t}. \quad (6.1)$$

When running panel regressions, the usual objective is to estimate  $\beta$ .

Figure 6.1 illustrates a panel-data situation. The model is  $y_i = \alpha_i + \beta x_{i,t} + \varepsilon_{i,t}$ ,  $t \in \{1, 2\}$ . On Panel (b), blue dots are for  $t = 1$ , red dots are for  $t = 2$ . The lines relate the dots associated with the same entity  $i$ . What is remarkable in the simulated model is that, while the unconditional correlation between  $y$  and  $x$  is negative, the conditional correlation (conditional on  $\alpha_i$ ) is positive. Indeed, the sign of this conditional correlation is the sign of  $\beta$ , which is positive in the simulated example ( $\beta = 5$ ). In other words, if one did not know the panel nature of the data, that would be tempting to say that  $\beta < 0$ , but this is not the case, due to **fixed effects** (the  $\alpha_i$ 's) that are negatively correlated to the  $x_i$ 's.

```
T <- 2; n <- 12 # 2 periods and 12 entities
alpha <- 5*rnorm(n) # draw fixed effects
x.1 <- rnorm(n) - .5*alpha # note: x_i's correlate to alpha_i's
x.2 <- rnorm(n) - .5*alpha
beta <- 5; sigma <- .3
y.1 <- alpha + x.1 + sigma*rnorm(n); y.2 <- alpha + x.2 + sigma*rnorm(n)
x <- c(x.1,x.2) # pooled x
y <- c(y.1,y.2) # pooled y
par(mfrow=c(1,2))
plot(x,y,col="black",pch=19,xlab="x",ylab="y",main="(a)")
plot(x,y,col="black",pch=19,xlab="x",ylab="y",main="(b)")
points(x.1,y.1,col="blue",pch=19);points(x.2,y.2,col="red",pch=19)
for(i in 1:n){lines(c(x.1[i],x.2[i]),c(y.1[i],y.2[i]))}
```

Figure 6.2 presents the same type of plot based on the Cigarette Consumption Panel dataset (**CigarettesSW** dataset, used in Stock and Watson (2003)). This dataset documents the average consumption of cigarettes in 48 continental US states for two dates (1985 and 1995).

We will make use of the following notations:

$$\mathbf{y}_i = \underbrace{\begin{bmatrix} y_{i,1} \\ \vdots \\ y_{i,T} \end{bmatrix}}_{T \times 1}, \quad \varepsilon_i = \underbrace{\begin{bmatrix} \varepsilon_{i,1} \\ \vdots \\ \varepsilon_{i,T} \end{bmatrix}}_{T \times 1}, \quad \mathbf{x}_i = \underbrace{\begin{bmatrix} \mathbf{x}'_{i,1} \\ \vdots \\ \mathbf{x}'_{i,T} \end{bmatrix}}_{T \times K}, \quad \mathbf{X} = \underbrace{\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix}}_{(nT) \times K}.$$

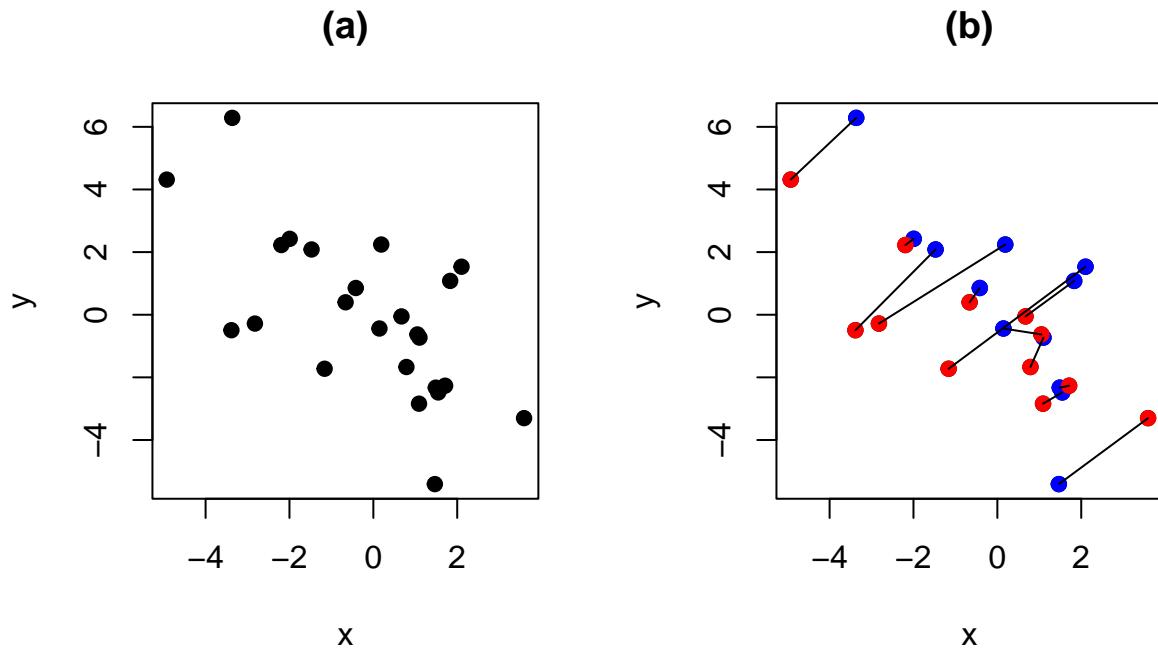


Figure 6.1: The data are the same for both panels. On Panel (b), blue dots are for  $t = 1$ , red dots are for  $t = 2$ . The lines relate the dots associated to the same entity  $i$ .

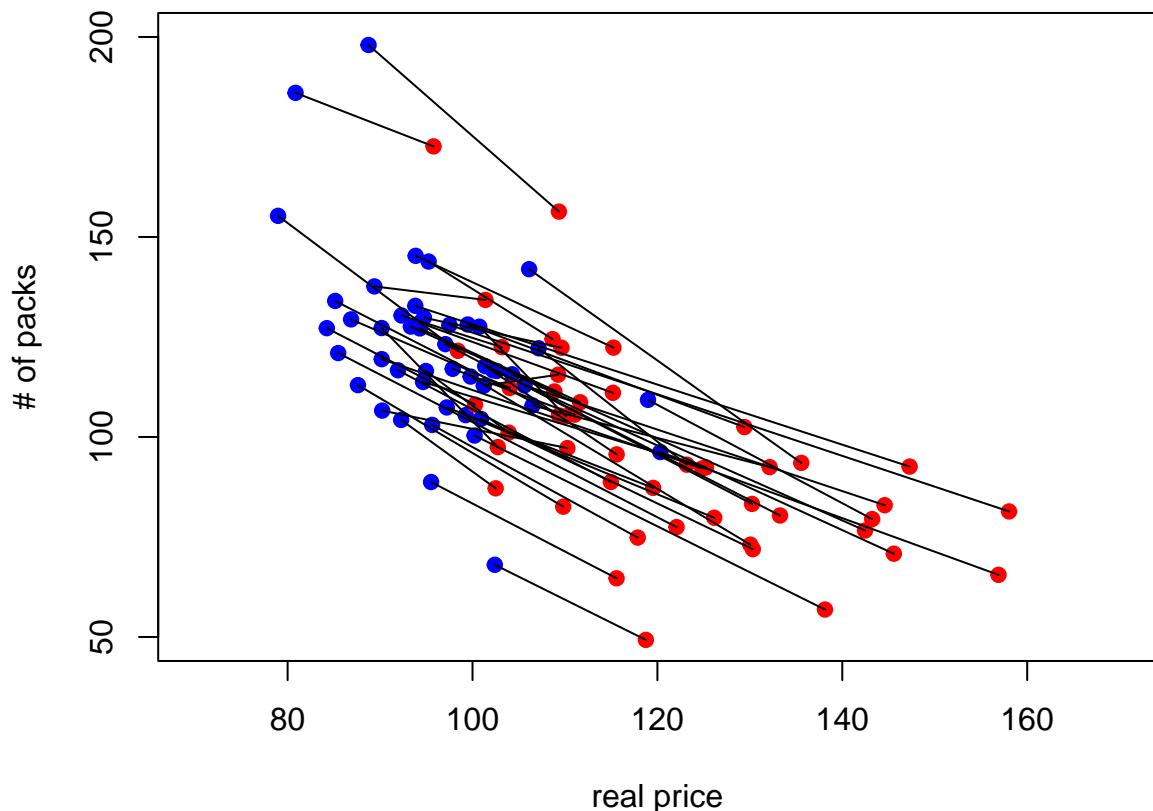


Figure 6.2: Cigarette consumption versus real price in the CigarettesSW panel dataset.

$$\begin{aligned}\tilde{\mathbf{y}}_i &= \begin{bmatrix} y_{i,1} - \bar{y}_i \\ \vdots \\ y_{i,T} - \bar{y}_i \end{bmatrix}, \quad \tilde{\varepsilon}_i = \begin{bmatrix} \varepsilon_{i,1} - \bar{\varepsilon}_i \\ \vdots \\ \varepsilon_{i,T} - \bar{\varepsilon}_i \end{bmatrix}, \\ \tilde{\mathbf{x}}_i &= \begin{bmatrix} \mathbf{x}'_{i,1} - \bar{\mathbf{x}}'_i \\ \vdots \\ \mathbf{x}'_{i,T} - \bar{\mathbf{x}}'_i \end{bmatrix}, \quad \tilde{\mathbf{X}} = \begin{bmatrix} \tilde{\mathbf{x}}_1 \\ \vdots \\ \tilde{\mathbf{x}}_n \end{bmatrix}, \quad \tilde{\mathbf{Y}} = \begin{bmatrix} \tilde{\mathbf{y}}_1 \\ \vdots \\ \tilde{\mathbf{y}}_n \end{bmatrix},\end{aligned}$$

where

$$\bar{y}_i = \frac{1}{T} \sum_{t=1}^T y_{i,t}, \quad \bar{\varepsilon}_i = \frac{1}{T} \sum_{t=1}^T \varepsilon_{i,t} \quad \text{and} \quad \bar{\mathbf{x}}_i = \frac{1}{T} \sum_{t=1}^T \mathbf{x}_{i,t}.$$

## 6.2 Three standard cases

There are three typical situations:

- **Pooled regression:**  $\mathbf{z}_i \equiv 1$ . This case amounts to the case studied in Chapter 5.
- **Fixed Effects** (Section 6.3):  $\mathbf{z}_i$  is unobserved, but correlates with  $\mathbf{x}_i \Rightarrow \mathbf{b}$  is biased and inconsistent in the OLS regression of  $\mathbf{y}$  on  $\mathbf{X}$  (omitted variable, see Section 5.3.2).
- **Random Effects** (Section 6.4):  $\mathbf{z}_i$  is unobserved, but uncorrelated with  $\mathbf{x}_i$ . The model writes:

$$y_{i,t} = \mathbf{x}'_{i,t} \beta + \alpha + \underbrace{u_i + \varepsilon_{i,t}}_{\text{compound error}},$$

where  $\alpha = \mathbb{E}(\mathbf{z}'_i \alpha)$  and  $u_i = \mathbf{z}'_i \alpha - \mathbb{E}(\mathbf{z}'_i \alpha) \perp \mathbf{x}_i$ . In that case, the OLS is consistent, but not efficient. GLS can be used to gain efficiencies over OLS (see Section 5.5.2 for a presentation of the GLS approach).

## 6.3 Estimation of Fixed-Effects Models

**Hypothesis 6.1** (Fixed-effect model). We assume that:

- There is no perfect multicollinearity among the regressors.
- $\mathbb{E}(\varepsilon_{i,t} | \mathbf{X}) = 0$ , for all  $i, t$ .
- We have:

$$\mathbb{E}(\varepsilon_{i,t} \varepsilon_{j,s} | \mathbf{X}) = \begin{cases} \sigma^2 & \text{if } i = j \text{ and } s = t, \\ 0 & \text{otherwise.} \end{cases}$$

These assumptions are analogous to those introduced in the standard linear regression:

(i)  $\leftrightarrow$  Hyp. 5.1, (ii)  $\leftrightarrow$  Hyp. 5.2, (iii)  $\leftrightarrow$  Hyp. 5.3 + 5.4.

In matrix form, for a given  $i$ , the model writes:

$$\mathbf{y}_i = \mathbf{x}_i \beta + \mathbf{1} \alpha_i + \varepsilon_i,$$

where  $\mathbf{1}$  is a  $T$ -dimensional vector of ones.

This is the **Least Square Dummy Variable (LSDV)** model:

$$\mathbf{y} = [\mathbf{X} \quad \mathbf{D}] \begin{bmatrix} \beta \\ \alpha \end{bmatrix} + \varepsilon, \tag{6.2}$$

with:

$$\mathbf{D} = \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1 \end{bmatrix}}_{(nT \times n)}.$$

The linear regression (Eq. (6.2)) —with the dummy variables— satisfies the Gauss-Markov conditions (Theorem 5.1). Hence, in this context, the OLS estimator is the *best linear unbiased estimator* (BLUE).

Denoting by  $\mathbf{Z}$  the matrix  $[\mathbf{X} \quad \mathbf{D}]$ , and by  $\mathbf{b}$  and  $\mathbf{a}$  the respective OLS estimates of  $\beta$  and of  $\alpha$ , we have:

$$\boxed{\begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} = [\mathbf{Z}'\mathbf{Z}]^{-1}\mathbf{Z}'\mathbf{y}.} \quad (6.3)$$

The asymptotical distribution of  $[\mathbf{b}', \mathbf{a}']'$  derives from the standard OLS context: Prop. 5.11 can be used after having replaced  $\mathbf{X}$  by  $\mathbf{Z} = [\mathbf{X} \quad \mathbf{D}]$ .

We have:

$$\boxed{\begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} \xrightarrow{d} \mathcal{N} \left( \begin{bmatrix} \beta \\ \alpha \end{bmatrix}, \sigma^2 \frac{Q^{-1}}{nT} \right),} \quad (6.4)$$

where

$$Q = \text{plim}_{nT \rightarrow \infty} \frac{1}{nT} \mathbf{Z}'\mathbf{Z},$$

assuming the previous limit exists.

In practice, an estimator of the covariance matrix of  $[\mathbf{b}', \mathbf{a}']'$  is:

$$s^2 (\mathbf{Z}'\mathbf{Z})^{-1} \quad \text{with} \quad s^2 = \frac{\mathbf{e}'\mathbf{e}}{nT - K - n},$$

where  $\mathbf{e}$  is the  $(nT) \times 1$  vector of OLS residuals.

There is an alternative way of expressing the LSDV estimators. It involves the residual-maker matrix matrix  $\mathbf{M}_D = \mathbf{I} - \mathbf{D}(\mathbf{D}'\mathbf{D})^{-1}\mathbf{D}'$  (see Eq. (5.4)), which acts as an operator that removes entity-specific means, i.e.:

$$\tilde{\mathbf{Y}} = \mathbf{M}_D \mathbf{Y}, \quad \tilde{\mathbf{X}} = \mathbf{M}_D \mathbf{X} \quad \text{and} \quad \tilde{\varepsilon} = \mathbf{M}_D \varepsilon.$$

With these notations, using the Frisch-Waugh theorem (Theorem 5.2), we get another expression for the estimator  $\mathbf{b}$  appearing in Eq. (6.3):

$$\boxed{\mathbf{b} = [\mathbf{X}'\mathbf{M}_D\mathbf{X}]^{-1}\mathbf{X}'\mathbf{M}_D\mathbf{y}.} \quad (6.5)$$

This amounts to regressing the  $\tilde{y}_{i,t}$ 's ( $= y_{i,t} - \bar{y}_i$ ) on the  $\tilde{\mathbf{x}}_{i,t}$ 's ( $= \mathbf{x}_{i,t} - \bar{\mathbf{x}}_i$ ).

The estimate of  $\alpha$  is given by:

$$\boxed{\mathbf{a} = (\mathbf{D}'\mathbf{D})^{-1}\mathbf{D}'(\mathbf{y} - \mathbf{X}\mathbf{b}),} \quad (6.6)$$

which is obtained by developing the second row of

$$\boxed{\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{D} \\ \mathbf{D}'\mathbf{X} & \mathbf{D}'\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{Y} \\ \mathbf{D}'\mathbf{Y} \end{bmatrix},}$$

which are the first-order conditions resulting from the least squares problem (see Eq. (5.3)).

One can use different types of fixed effects in the same regression. Typically, one can have time and entity fixed effects. In that case, the model writes:

$$y_{i,t} = \mathbf{x}'_i \beta + \alpha_i + \gamma_t + \varepsilon_{i,t}.$$

The LSDV approach (Eq. (6.2)) can still be resorted to. It suffices to extend the  $\mathbf{Z}$  matrix with additional columns (then called *time dummies*):

$$\mathbf{y} = [\mathbf{X} \quad \mathbf{D} \quad \mathbf{C}] \begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix} + \varepsilon, \quad (6.7)$$

with:

$$\mathbf{C} = \begin{bmatrix} \delta_1 & \delta_2 & \dots & \delta_{T-1} \\ \vdots & \vdots & & \vdots \\ \delta_1 & \delta_2 & \dots & \delta_{T-1} \end{bmatrix},$$

where the  $T$ -dimensional vector  $\delta_t$  (the *time dummy*) is

$$[0, \dots, 0, \underbrace{1}_{t^{th} \text{ entry}}, 0, \dots, 0]'$$

Using state and year fixed effects in the `CigarettesSW` panel dataset yields the following results:

```
CigarettesSW$rincome <- with(CigarettesSW, income/population/cpi)
eq.pooled <- lm(log(packs)~log(rprice)+log(rincome), data=CigarettesSW)
eq.LSDV <- lm(log(packs)~log(rprice)+log(rincome)+state,
                 data=CigarettesSW)
CigarettesSW$year <- as.factor(CigarettesSW$year)
eq.LSDV2 <- lm(log(packs)~log(rprice)+log(rincome)+state+year,
                 data=CigarettesSW)
stargazer::stargazer(eq.pooled, eq.LSDV, eq.LSDV2, type="text", no.space = TRUE,
                      omit=c("state", "year"),
                      add.lines=list(c('State FE', 'No', 'Yes', 'Yes'),
                                    c('Year FE', 'No', 'No', 'Yes')),
                      omit.stat=c("f", "ser"))
```

```
##
## =====
##             Dependent variable:
## -----
##                  log(packs)
##          (1)      (2)      (3)
## -----
## log(rprice) -1.334*** -1.210*** -1.056***
##           (0.135)   (0.114)   (0.149)
## log(rincome)  0.318**   0.121     0.497
##           (0.136)   (0.190)   (0.304)
## Constant     10.067*** 9.954*** 8.360*** 
##           (0.516)   (0.264)   (1.049)
## -----
## State FE      No       Yes      Yes
## Year FE      No       No       Yes
## Observations 96       96       96
## R2            0.552    0.966    0.967
## Adjusted R2   0.542    0.929    0.931
## =====
## Note: *p<0.1; **p<0.05; ***p<0.01
```

**Example 6.1** (Housing prices and interest rates). In this example, we want to estimate the effect of short and long-term interest rate on housing prices. The data come from the Jordà et al. (2017) dataset (see this website).

```
library(AEC); library(sandwich)
data(JST); JST <- subset(JST, year>1950); N <- dim(JST)[1]
JST$hpreal <- JST$hpnom/JST$cpi # real house price index
JST$dpreal <- 100*log(JST$hpreal/c(NaN, JST$hpreal[1:(N-1)]))
# Put NA's when change in country:
JST$dpreal[c(0, JST$iso[2:N] != JST$iso[1:(N-1)])] <- NaN
JST$dpreal[abs(JST$dpreal)>30] <- NaN # remove extreme price change
JST$YEAR <- as.factor(JST$year) # to have time fixed effects
```

```

eq1_noFE <- lm(dhpreal ~ stir + ltrate, data=JST)
eq1_FE   <- lm(dhpreal ~ stir + ltrate + iso + YEAR, data=JST)
eq2_noFE <- lm(dhpreal ~ I(ltrate-stir), data=JST)
eq2_FE   <- lm(dhpreal ~ I(ltrate-stir) + iso + YEAR, data=JST)
vcov_cluster1_noFE <- vcovHC(eq1_noFE, cluster = JST[, c("iso", "YEAR")])
vcov_cluster1_FE   <- vcovHC(eq1_FE, cluster = JST[, c("iso", "YEAR")])
vcov_cluster2_noFE <- vcovHC(eq2_noFE, cluster = JST[, c("iso", "YEAR")])
vcov_cluster2_FE   <- vcovHC(eq2_FE, cluster = JST[, c("iso", "YEAR")])
robust_se_FE1_noFE <- sqrt(diag(vcov_cluster1_noFE))
robust_se_FE1_FE   <- sqrt(diag(vcov_cluster1_FE))
robust_se_FE2_noFE <- sqrt(diag(vcov_cluster2_noFE))
robust_se_FE2_FE   <- sqrt(diag(vcov_cluster2_FE))
stargazer::stargazer(eq1_noFE, eq1_FE, eq2_noFE, eq2_FE, type = "text",
                      column.labels = c("no FE", "with FE", "no FE", "with FE"),
                      omit = c("iso", "YEAR", "Constant"), keep.stat = "n",
                      add.lines=list(c('Country FE', 'No', 'Yes', 'No', 'Yes'),
                                    c('Year FE', 'No', 'Yes', 'No', 'Yes')),
                      se = list(robust_se_FE1_noFE, robust_se_FE1_FE,
                                robust_se_FE2_noFE, robust_se_FE2_FE))

```

```

##
## =====
##             Dependent variable:
## -----
##                  dhpreal
##          no FE    with FE    no FE    with FE
##          (1)      (2)      (3)      (4)
## -----
## stir           0.485***  0.532***  

##                 (0.149)  (0.170)
## 
## ltrate         -0.690*** -0.384**  

##                 (0.164)  (0.182)
## 
## I(ltrate - stir)           -0.476*** -0.475***  

##                               (0.145)  (0.159)
## 
## -----
## Country FE      No       Yes      No       Yes
## Year FE        No       Yes      No       Yes
## Observations   1,141    1,141    1,141    1,141
## =====
## Note:          *p<0.1; **p<0.05; ***p<0.01

```

## 6.4 Estimation of random effects models

Here, the individual effects are assumed to be not correlated to other variables (the  $\mathbf{x}_i$ 's). In that context, the OLS estimator is consistent. However, it is not efficient. The GLS approach can be employed to gain efficiency.

**Random-effect models** write:

$$y_{i,t} = \mathbf{x}'_{it}\beta + (\alpha + \underbrace{u_i}_{\substack{\text{Random} \\ \text{heterogeneity}}}) + \varepsilon_{i,t},$$

with

$$\begin{aligned}\mathbb{E}(\varepsilon_{i,t}|\mathbf{X}) &= \mathbb{E}(u_i|\mathbf{X}) = 0, \\ \mathbb{E}(\varepsilon_{i,t}\varepsilon_{j,s}|\mathbf{X}) &= \begin{cases} \sigma_\varepsilon^2 & \text{if } i = j \text{ and } s = t, \\ 0 & \text{otherwise.} \end{cases} \\ \mathbb{E}(u_i u_j|\mathbf{X}) &= \begin{cases} \sigma_u^2 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \\ \mathbb{E}(\varepsilon_{i,t} u_j|\mathbf{X}) &= 0 \quad \text{for all } i, j \text{ and } t.\end{aligned}$$

Introducing the notations  $\eta_{i,t} = u_i + \varepsilon_{i,t}$  and  $\eta_i = [\eta_{i,1}, \dots, \eta_{i,T}]'$ , we have  $\mathbb{E}(\eta_i|\mathbf{X}) = \mathbf{0}$  and  $\mathbb{V}ar(\eta_i|\mathbf{X}) = \Gamma$ , where

$$\Gamma = \begin{bmatrix} \sigma_\varepsilon^2 + \sigma_u^2 & \sigma_u^2 & \sigma_u^2 & \dots & \sigma_u^2 \\ \sigma_u^2 & \sigma_\varepsilon^2 + \sigma_u^2 & \sigma_u^2 & \dots & \sigma_u^2 \\ \vdots & & \ddots & & \vdots \\ \sigma_u^2 & \sigma_u^2 & \sigma_u^2 & \dots & \sigma_\varepsilon^2 + \sigma_u^2 \end{bmatrix} = \sigma_\varepsilon^2 \mathbf{I} + \sigma_u^2 \mathbf{1} \mathbf{1}'.$$

Denoting by  $\Sigma$  the covariance matrix of  $\eta = [\eta'_1, \dots, \eta'_n]'$ , we have:

$$\Sigma = \mathbf{I} \otimes \Gamma.$$

If we knew  $\Sigma$ , we would apply (feasible) GLS (Eq. (5.32), in Section 5.5.2):

$$\beta = (\mathbf{X}' \Sigma^{-1} \mathbf{X})^{-1} \mathbf{X}' \Sigma^{-1} \mathbf{y}.$$

(As explained in Section 5.5.2, this amounts to regressing  $\Sigma^{-1/2}' \mathbf{y}$  on  $\Sigma^{-1/2}' \mathbf{X}$ .)

It can be checked that  $\Sigma^{-1/2} = \mathbf{I} \otimes (\Gamma^{-1/2})$  where

$$\Gamma^{-1/2} = \frac{1}{\sigma_\varepsilon} \left( \mathbf{I} - \frac{\theta}{T} \mathbf{1} \mathbf{1}' \right), \quad \text{with} \quad \theta = 1 - \frac{\sigma_\varepsilon}{\sqrt{\sigma_\varepsilon^2 + T \sigma_u^2}}.$$

Hence, if we knew  $\Sigma$ , we would transform the data as follows:

$$\Gamma^{-1/2} \mathbf{y}_i = \frac{1}{\sigma_\varepsilon} \begin{bmatrix} y_{i,1} - \theta \bar{y}_i \\ y_{i,2} - \theta \bar{y}_i \\ \vdots \\ y_{i,T} - \theta \bar{y}_i \end{bmatrix}.$$

What about when  $\Sigma$  is unknown? One can take deviations from group means to remove heterogeneity:

$$y_{i,t} - \bar{y}_i = [\mathbf{x}_{i,t} - \bar{\mathbf{x}}_i]' \beta + (\varepsilon_{i,t} - \bar{\varepsilon}_i). \quad (6.8)$$

The previous equation can be consistently estimated by OLS. (Although the residuals are correlated across  $t$ 's for the observations pertaining to a given entity, the OLS remain consistent; see Prop. 5.13.)

We have  $\mathbb{E} \left[ \sum_{i=1}^T (\varepsilon_{i,t} - \bar{\varepsilon}_i)^2 \right] = (T-1)\sigma_\varepsilon^2$ .

The  $\varepsilon_{i,t}$ 's are not observed but  $\mathbf{b}$ , the OLS estimator of  $\beta$  in Eq. (6.8), is a consistent estimator of  $\beta$ . Using an adjustment for the degrees of freedom, we can approximate their variance with:

$$\hat{\sigma}_e^2 = \frac{1}{nT - n - K} \sum_{i=1}^n \sum_{t=1}^T (e_{i,t} - \bar{e}_i)^2.$$

What about  $\sigma_u^2$ ? We can exploit the fact that OLS are consistent in the pooled regression:

$$\text{plim } s_{\text{pooled}}^2 = \text{plim} \frac{\mathbf{e}' \mathbf{e}}{nT - K - 1} = \sigma_u^2 + \sigma_\varepsilon^2,$$

and therefore use  $s_{\text{pooled}}^2 - \hat{\sigma}_e^2$  as an approximation to  $\sigma_u^2$ .

Let us come back to Example 6.1 (relationship between changes in housing prices and interest rates). In the following, we use the random effect specification; and compare the results with those obtained with the pooled regression and with the fixed-effect model. For that, we use the function `p1m` of the package of the same name. (Note that `eq.FE` is similar to `eq1` in Example 6.1.)

```

library(plm);library(stargazer)
eq.RE <- plm(dhpreal ~ stir + ltrate,data=JST,index=c("iso","YEAR"),
              model="random",effect="twoways")
eq.FE <- plm(dhpreal ~ stir + ltrate,data=JST,index=c("iso","YEAR"),
              model="within",effect="twoways")
eq0   <- plm(dhpreal ~ stir + ltrate,data=JST,index=c("iso","YEAR"),
              model="pooling")
stargazer(eq0, eq.RE, eq.FE, type = "text",no.space = TRUE,
          column.labels=c("Pooled","Random Effect","Fixed Effects"),
          add.lines=list(c('State FE','No','Yes','Yes'),
                        c('Year FE','No','Yes','Yes')),
          omit.stat=c("f","ser"))

## -----
## =====
##             Dependent variable:
## -----
##                   dhpreal
## Pooled    Random Effect Fixed Effects
## (1)        (2)        (3)
## -----
## stir      0.485***   0.456***   0.532***  

##           (0.114)     (0.019)     (0.134)  

## ltrate    -0.690***  -0.541***  -0.384***  

##           (0.127)     (0.020)     (0.145)  

## Constant  4.103***   3.341***  

##           (0.421)     (0.096)
## -----
## State FE   No        Yes       Yes  

## Year FE   No        Yes       Yes  

## Observations 1,141    1,141    1,141  

## R2         0.027    0.024    0.015  

## Adjusted R2 0.025    0.022    -0.067
## =====
## Note: *p<0.1; **p<0.05; ***p<0.01

```

One can run an Hausman (1978) test in order to check whether or not the fixed-effect model is needed. Indeed, if this is not the case (i.e., if the covariates are not correlated to the disturbances), then it is preferable to use the random-effect estimation as the latter is more efficient.

```
phtest(eq.FE,eq.RE)
```

```

##  

##  Hausman Test  

##  

##  data: dhpreal ~ stir + ltrate  

##  chisq = 3.8386, df = 2, p-value = 0.1467  

##  alternative hypothesis: one model is inconsistent

```

The p-value being high, we do not reject the null hypothesis according to which the covariates and the errors are uncorrelated. We should therefore prefer the random-effect model.

**Example 6.2** (Spatial data). This example makes use of Airbnb prices (Zürich, 22 June 2017), collected from Tom Slee's website. The covariates are the number of bedrooms and the number of people that can be accommodated. We consider the use of district fixed effects. Figure 6.3 shows the price to explain (the size of the circles is proportional to the prices). The white lines delineate the 12 districts of the city.

Let us regress prices on the covariates as well as on district dummies:

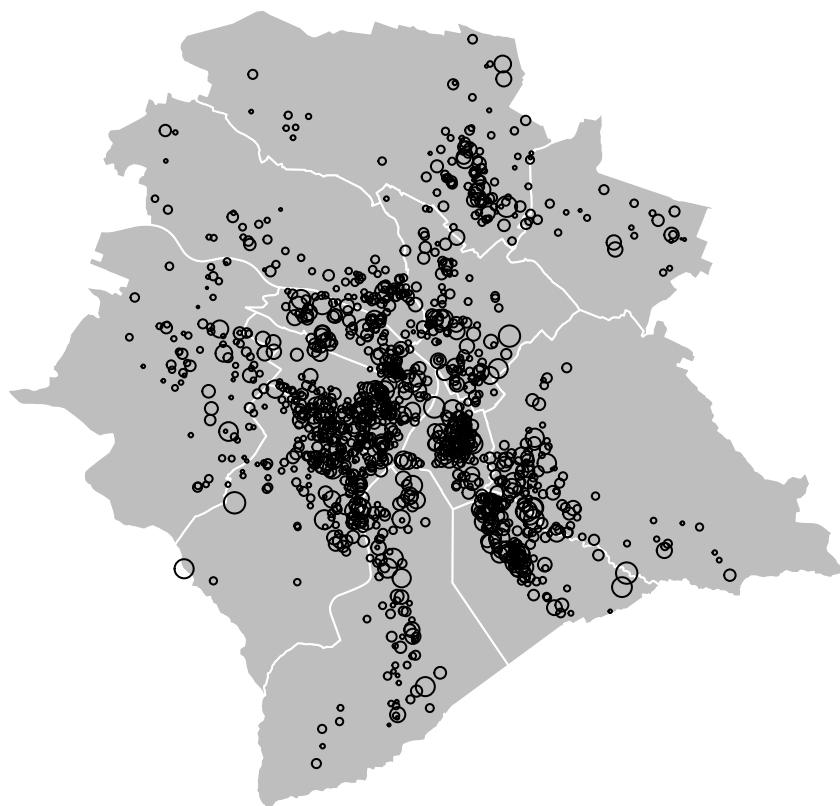


Figure 6.3: Airbnb prices for the Zurich area, 22 June 2017. The size of the circles is proportional to the prices. White lines delineate the 12 districts of the city.

```

eq_noFE <- lm(price~bedrooms+accommodates,data=airbnb)
eq_FE    <- lm(price~bedrooms+accommodates+neighborhood,data=airbnb)
# Adjust standard errors:
cov_FE      <- vcovHC(eq_FE, cluster = airbnb[, c("neighborhood")])
robust_se_FE <- sqrt(diag(cov_FE))
cov_noFE     <- vcovHC(eq_noFE, cluster = airbnb[, c("neighborhood")])
robust_se_noFE <- sqrt(diag(cov_noFE))
stargazer::stargazer(eq_FE, eq_noFE, eq_FE, eq_noFE, type = "text",
                      column.labels = c("FE (no HAC)", "No FE (no HAC)",
                      "FE (with HAC)", "No FE (with HAC)"),
                      omit = c("neighborhood"), no.space = TRUE,
                      omit.labels = c("District FE"), keep.stat = "n",
                      se = list(NULL, NULL, robust_se_FE, robust_se_noFE))

##
## -----
##                               Dependent variable:
## -----
##                               price
## FE (no HAC)  No FE (no HAC)  FE (with HAC)  No FE (with HAC)
##          (1)           (2)           (3)           (4)
## -----
## ## bedrooms   7.229***       5.629**       7.229***       5.629***
##                 (2.135)        (2.194)        (2.052)        (2.073)
## ## accommodates 16.426***     17.449***     16.426***     17.449***
##                   (1.284)        (1.323)        (1.431)        (1.428)
## ## Constant    95.118***     68.417***     95.118***     68.417***
##                   (5.323)        (3.223)        (5.664)        (3.527)
## -----
## ## District FE Yes           No           Yes           No
## -----
## ## Observations 1,321         1,321         1,321         1,321
## -----
## ## Note: *p<0.1; **p<0.05; ***p<0.01

```

Figure 6.4 compares the residuals with and without fixed effects. The sizes of the circles are proportional to the absolute values of the residuals, the color indicates the sign (blue for positive).

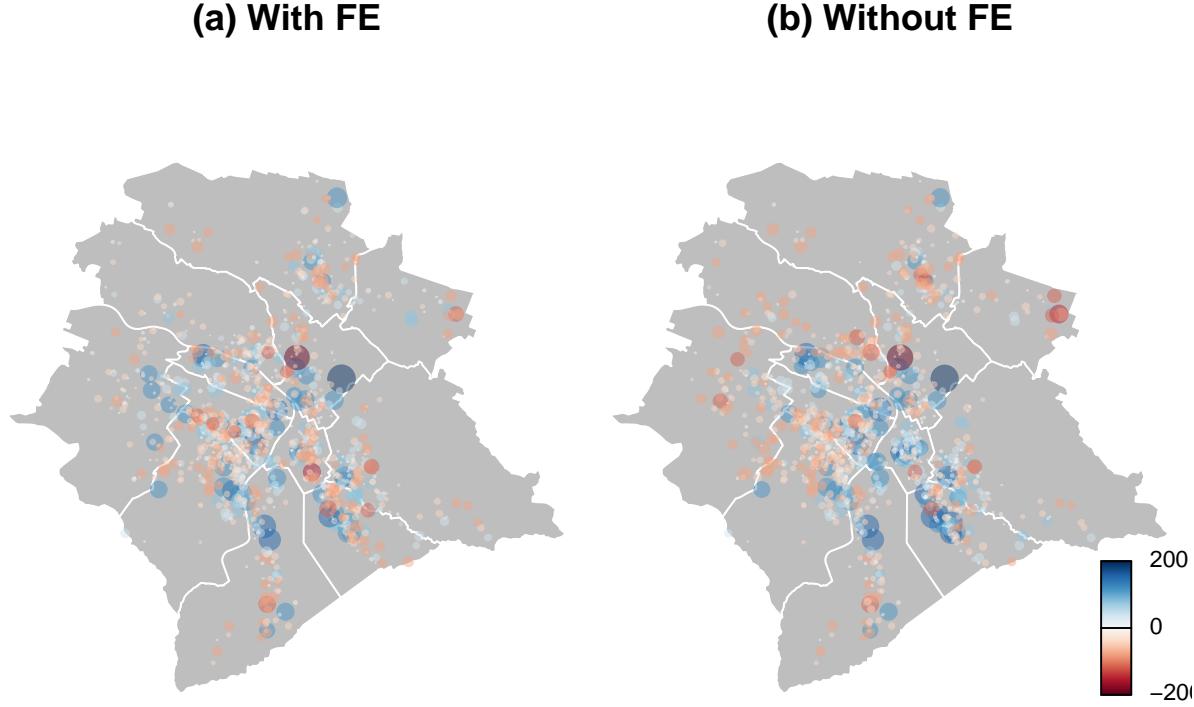


Figure 6.4: Regression residuals. The sizes of the circles are proportional to the absolute values of the residuals, the color indicates the sign (blue for negative).

With fixed effects, the colors are better balanced within each district.

## 6.5 Dynamic Panel Regressions

In what precedes, it has been assumed that there is no correlation between the observations indexed by  $(i, t)$  and those indexed by  $(j, s)$  as long as  $j \neq i$  or  $t \neq s$ . If one suspects that the errors  $\varepsilon_{i,t}$  are correlated (across entities  $i$  for a given date  $t$ , or across dates for a given entity, or both), then one should employ a robust covariance matrix (see Section 5.5.5).

In several cases, auto-correlation in the variable of interest may stem from an auto-regressive specification. That is, Eq. (6.1) is then replaced by:

$$y_{i,t} = \rho y_{i,t-1} + \underset{K \times 1}{\mathbf{x}'_{i,t}} \underset{\text{Individual effects}}{\beta} + \alpha_i + \varepsilon_{i,t}. \quad (6.9)$$

In that case, even if the explanatory variables  $\mathbf{x}_{i,t}$  are uncorrelated to the errors  $\varepsilon_{i,t}$ , we have that the additional *explanatory variable*  $y_{i,t-1}$  correlates to the errors  $\varepsilon_{i,t-1}, \varepsilon_{i,t-2}, \dots, \varepsilon_{i,1}$ . As a result, the LSDV estimate of the model parameters  $\{\rho, \beta\}$  may be biased, even if  $n$  is large. To see this, notice that the LSDV regression amounts to regressing  $\tilde{\mathbf{y}}$  on  $\tilde{\mathbf{X}}$  (see Eq. (6.5)), where the elements of  $\tilde{\mathbf{X}}$  are the explanatory variables to which we subtract their within-sample means. In particular, we have:

$$\tilde{y}_{i,t-1} = y_{i,t-1} - \frac{1}{T} \sum_{s=1}^T y_{i,s-1},$$

which correlates to the corresponding error, that is:

$$\tilde{\varepsilon}_{i,t} = \varepsilon_{i,t} - \frac{1}{T} \sum_{s=1}^T \varepsilon_{i,s}.$$

The previous equation shows that the *within-group* estimator (LSDV) introduces all realizations of the  $\varepsilon_{i,t}$  errors into the transformed error term ( $\tilde{\varepsilon}_{i,t}$ ). As a result, in large- $n$  fixed- $T$  panels, it is consistent only if all the right-hand-side variables of the regression are strictly

exogenous (i.e., do not correlate to past, present, and future errors  $\varepsilon_{i,t}$ ).<sup>1</sup> This is not the case when there are lags of  $y_{i,t}$  on the right-hand side of the regression formula.

The following simulation illustrate this bias. The  $x$ -coordinates of the dots are the fixed effects  $\alpha_i$ 's, and the  $y$ -coordinates are their LSDV estimates. The blue line is the 45-degree line.

```
n <- 400; T <- 10
rho <- 0.8; sigma <- .5
alpha <- rnorm(n)
y <- alpha / (1 - rho) + sigma^2 / (1 - rho^2) * rnorm(n)
all_y <- y
for(t in 2:T){
  y <- rho * y + alpha + sigma * rnorm(n)
  all_y <- rbind(all_y, y)
}
y <- c(all_y[2:T,]); y_1 <- c(all_y[1:(T-1),])
D <- diag(n) %x% rep(1, T-1)
Z <- cbind(c(y_1), D)
b <- solve(t(Z) %*% Z) %*% t(Z) %*% y
a <- b[2:(n+1)]
plot(alpha, a)
lines(c(-10, 10), c(-10, 10), col="blue")
```

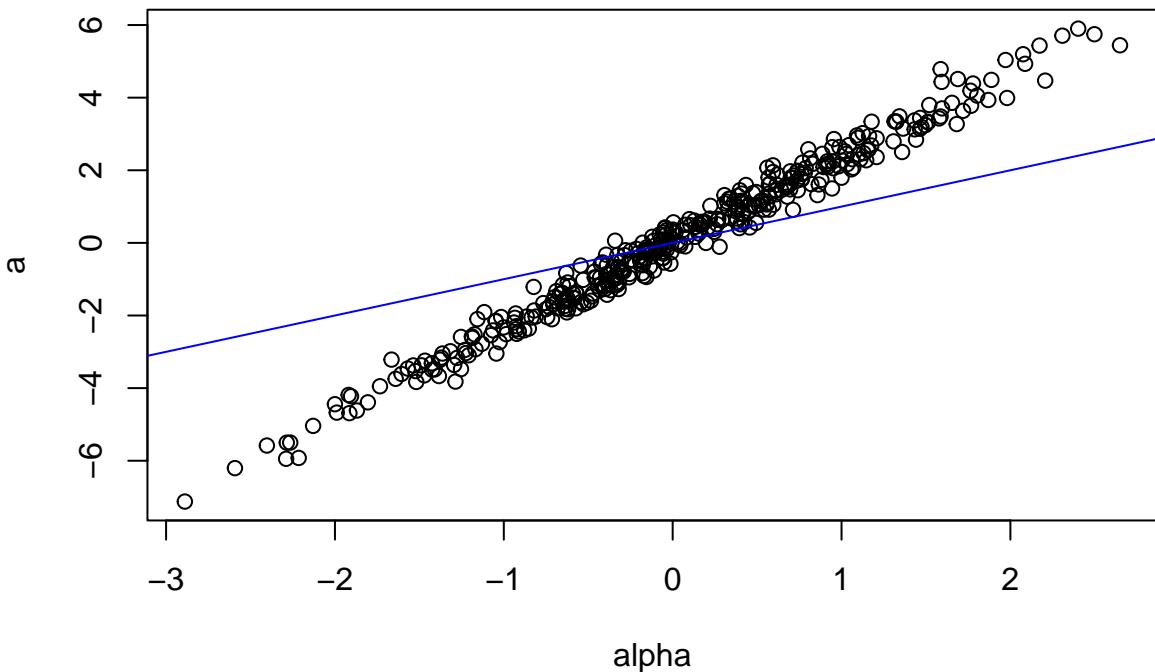


Figure 6.5: illustration of the bias pertaining to the LSDV estimation approach in the presence of auto-correlation of the depend variable.

In the previous example, the estimate of  $\rho$  (whose true value is 0.8) is 0.531.

To address this, one can resort to instrumental-variable regressions. Anderson and Hsiao (1982) have, in particular, proposed a first-differenced Two Stage Least Squares (2SLS) estimator (see Eq. (5.25) in Section 5.4). This estimation is based on the following transformation of the model:

$$\Delta y_{i,t} = \rho \Delta y_{i,t-1} + (\Delta \mathbf{x}_{i,t})' \beta + \Delta \varepsilon_{i,t}. \quad (6.10)$$

The OLS estimates of the parameters are biased because  $\varepsilon_{i,t-1}$  —which is part of the error  $\Delta \varepsilon_{i,t}$ — is correlated to  $y_{i,t-1}$  —which is part of the “explanatory variable”, namely  $\Delta y_{i,t-1}$ . But consistent estimates can be obtained using 2SLS with instrumental variables that are correlated with  $\Delta y_{i,t}$  but orthogonal to  $\Delta \varepsilon_{i,t}$ . One can for instance use  $\{y_{i,t-2}, \mathbf{x}_{i,t-2}\}$  as instruments. Note that this approach can be implemented only if there are more than 3 time observations per entity  $i$ .

---

<sup>1</sup>Although the bias may vanish for large  $T$ 's, it does not if  $n$  only goes to infinity.

If the explanatory variables  $\mathbf{x}_{i,t}$  are assumed to be predetermined (i.e., do not contemporaneous correlate with the errors  $\varepsilon_{i,t}$ ), then  $\mathbf{x}_{i,t-1}$  can be added to the instruments associated with  $\Delta y_{i,t}$ . Further, if these variables (the  $\mathbf{x}_{i,t}$ 's) are exogenous (i.e., do not contemporaneous correlate with any of the errors  $\varepsilon_{i,s}$ ,  $\forall s$ ), then  $\mathbf{x}_{i,t}$  also constitute a valid instrument.

Using the previous (simulated) example, this approach consists in the following steps:

```
Dy <- c(all_y[3:T,]) - c(all_y[2:(T-1),])
Dy_1 <- c(all_y[2:(T-1),]) - c(all_y[1:(T-2),])
y_2 <- c(all_y[1:(T-2),])
Z <- matrix(y_2, ncol=1)
Pz <- Z %*% solve(t(Z) %*% Z) %*% t(Z)
Dy_1hat <- Pz %*% Dy_1
rho_2SLS <- solve(t(Dy_1hat) %*% Dy_1hat) %*% t(Dy_1hat) %*% Dy
```

While the OLS estimate of  $\rho$  (whose true value is 0.8) was 0.531, we obtain here `rho_2SLS` = 0.89.

Let us come back to the general case (with covariates  $\mathbf{x}_{i,k}$ 's). For  $t = 3$ ,  $y_{i,1}$  (and  $\mathbf{x}_{i,1}$ ) is the only possible instrument. However, for  $t = 4$ , one could use  $y_{i,2}$  and  $y_{i,1}$  (as well as  $\mathbf{x}_{i,2}$  and  $\mathbf{x}_{i,1}$ ). More generally, defining matrix  $Z_i$  as follows:

$$Z_i = \begin{bmatrix} \mathbf{z}'_{i,1} & 0 & \dots \\ 0 & \mathbf{z}'_{i,1} & \mathbf{z}'_{i,2} & 0 & \dots \\ 0 & 0 & 0 & \mathbf{z}'_{i,1} & \mathbf{z}'_{i,2} & \mathbf{z}'_{i,3} & 0 & \dots \\ \vdots & & & & & & & \\ 0 & \dots & & & & & 0 & \mathbf{z}'_{i,1} & \dots & \mathbf{z}'_{i,T-2} \end{bmatrix},$$

where  $\mathbf{z}_{i,t} = [y_{i,t}, \mathbf{x}'_{i,t}]'$ , we have the moment conditions:<sup>2</sup>

$$\mathbb{E}(Z'_i \Delta \varepsilon_i) = 0,$$

with  $\Delta \varepsilon_i = [\Delta \varepsilon_{i,3}, \dots, \Delta \varepsilon_{i,T}]'$ .

These restrictions are used in the GMM approach employed by Arellano and Bond (1991). Specifically, a GMM estimator of the model parameters is given by:

$$\operatorname{argmin} \left( \frac{1}{n} \sum_{i=1}^n Z'_i \Delta \varepsilon_i \right)' W_n \left( \frac{1}{n} \sum_{i=1}^n Z'_i \Delta \varepsilon_i \right),$$

using the weighting matrix

$$W_n = \left( \frac{1}{n} \sum_{i=1}^n Z'_i \widehat{\Delta \varepsilon_i} \widehat{\Delta \varepsilon_i}' Z_i \right)^{-1},$$

where the  $\widehat{\Delta \varepsilon_i}$ 's are consistent estimates of the  $\Delta \varepsilon_i$ 's that result from a preliminary estimation. In this sense, this estimator is a two-step GMM one.

If the disturbances are homoskedastic, then it can be shown that an asymptotically equivalent (efficient) GMM estimator can be obtained by using:

$$W_{1,n} = \left( \frac{1}{n} Z'_i H Z_i \right)^{-1},$$

where  $H$  is is  $(T-2) \times (T-2)$  matrix of the form:

$$H = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}.$$

---

<sup>2</sup>If  $\mathbf{x}_{i,t}$  is predetermined (exogenous), we can use  $\mathbf{z}_{i,t} = [y_{i,t}, \mathbf{x}_{i,t+1}, \mathbf{x}'_{i,t}]'$  (respectively  $\mathbf{z}_{i,t} = [y_{i,t}, \mathbf{x}_{i,t+2}, \mathbf{x}_{i,t+1}, \mathbf{x}'_{i,t}]'$ ).

It is straightforward to extend these GMM methods to cases where there is more than one lag of the dependent variable on the right-hand side of the equation or in cases where disturbances feature limited moving-average serial correlation.

The pdynmc package allows to run these GMM approaches (see Fritsch et al. (2019)). The following lines of code allow to replicate the results of Arellano and Bond (1991):

```

library(pdynmc)
data(EmplUK, package = "plm")
dat <- EmplUK
dat[,c(4:7)] <- log(dat[,c(4:7)])
m1 <- pdynmc(dat = dat, # name of the dataset
               varname.i = "firm", # name of the cross-section identifier
               varname.t = "year", # name of the time-series identifiers
               use.mc.diff = TRUE, # use moment conditions from equations in differences?
               use.mc.lev = FALSE, # use moment conditions from equations in levels? (i.e.
               use.mc.nonlin = FALSE, # use nonlinear (quadratic) moment conditions?
               include.y = TRUE, # instruments should be derived from the lags of the dependent variable
               varname.y = "emp", # name of the dependent variable in the dataset
               lagTerms.y = 2, # number of lags of the dependent variable
               fur.con = TRUE, # further control variables (covariates) are included?
               fur.con.diff = TRUE, # include further control variables in equations from differences?
               fur.con.lev = FALSE, # include further control variables in equations from levels?
               varname.reg.fur = c("wage", "capital", "output"), # covariate(s) -in the difference equations
               lagTerms.reg.fur = c(1,2,2), # number of lags of the further controls
               include.dum = TRUE, # A logical variable indicating whether dummy variables are included
               dum.diff = TRUE, # A logical variable indicating whether dummy variables are included in the difference equations
               dum.lev = FALSE, # A logical variable indicating whether dummy variables are included in the level equations
               varname.dum = "year",
               w.mat = "iid.err", # One of the character strings c('"iid.err"', '"identities"')
               std.err = "corrected",
               estimation = "onestep", # One of the character strings c('"onestep"', '"two-step"')
               opt.meth = "none" # numerical optimization procedure. When no nonlinear method is specified
)
summary(m1,digits=3)

## 
## Dynamic linear panel estimation (onestep)
## Estimation steps: 1
##
## Coefficients:
##              Estimate Std.Err.rob z-value.rob Pr(>|z.rob|)
## L1.emp      0.686226  0.144594   4.746    < 2e-16 ***
## L2.emp     -0.085358  0.056016  -1.524    0.12751
## L0.wage    -0.607821  0.178205  -3.411    0.00065 ***
## L1.wage     0.392623  0.167993   2.337    0.01944 *
## L0.capital  0.356846  0.059020   6.046    < 2e-16 ***
## L1.capital -0.058001  0.073180  -0.793    0.42778
## L2.capital -0.019948  0.032713  -0.610    0.54186
## L0.output    0.608506  0.172531   3.527    0.00042 ***
## L1.output   -0.711164  0.231716  -3.069    0.00215 **
## L2.output    0.105798  0.141202   0.749    0.45386
## 1979        0.009554  0.010290   0.929    0.35289
## 1980        0.022015  0.017710   1.243    0.21387
## 1981       -0.011775  0.029508  -0.399    0.68989
## 1982       -0.027059  0.029275  -0.924    0.35549
## 1983       -0.021321  0.030460  -0.700    0.48393
## 1976       -0.007703  0.031411  -0.245    0.80646
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

```
##  
## 41 total instruments are employed to estimate 16 parameters  
## 27 linear (DIF)  
## 8 further controls (DIF)  
## 6 time dummies (DIF)  
##  
## J-Test (overid restrictions): 70.82 with 25 DF, pvalue: <0.001  
## F-Statistic (slope coeff): 528.06 with 10 DF, pvalue: <0.001  
## F-Statistic (time dummies): 14.98 with 6 DF, pvalue: 0.0204
```

We generate novel results (`m2`) by replacing “`onestep`” with “`twostep`” (in the `estimation` field). The resulting estimated coefficients are:

```
##      L1.emp      L2.emp      L0.wage      L1.wage      L0.capital    L1.capital  
##  0.62870890 -0.06518800 -0.52575951  0.31128961  0.27836190  0.01409950  
##  L2.capital    L0.output    L1.output    L2.output      1979        1980  
## -0.04024847  0.59192286 -0.56598515  0.10054264  0.01121551  0.02306871  
##          1981        1982        1983        1976  
## -0.02135806 -0.03111604 -0.01799335 -0.02336762
```

Arellano and Bond (1991) have proposed a specification test. If the model is correctly specified, then the errors of Eq. (6.10)—that is the first-difference equation—should feature non-zero first-order auto-correlations, but zero higher-order autocorrelations.

Function `mtest.fct` of package `pdynmc` implements this test. Here is its result in the present case:

```
mtest.fct(m1, order=3)
```

```
##  
##  Arellano and Bond (1991) serial correlation test of degree 3  
##  
## data: 1step GMM Estimation  
## normal = 0.045945, p-value = 0.9634  
## alternative hypothesis: serial correlation of order 3 in the error terms
```

One can also implement the Hansen J-test of the over-identifying restrictions (see Section 7.1.3):

```
jtest.fct(m1)
```

```
##  
##  J-Test of Hansen  
##  
## data: 1step GMM Estimation  
## chisq = 70.82, df = 25, p-value = 2.905e-06  
## alternative hypothesis: overidentifying restrictions invalid
```

```
jtest.fct(m2)
```

```
##  
##  J-Test of Hansen  
##  
## data: 2step GMM Estimation  
## chisq = 31.381, df = 25, p-value = 0.1767  
## alternative hypothesis: overidentifying restrictions invalid
```

## 6.6 Introduction to program evaluation

This section briefly introduces the econometrics of program evaluation. Program evaluation refer to the analysis of the causal effects of some “treatments” in a broad sense. These treatment can, e.g., correspond to the implementation (or announcement) of policy measures. A comprehensive review is proposed by Abadie and Cattaneo (2018). A seminal book on the subject is that of Angrist and Pischke (2008).

### 6.6.1 Presentation of the problem

To begin with, let us consider a single entity. To simplify notations, we drop the entity index ( $i$ ). Let us denote by  $Y$  the outcome variable (for the variable of interest), by  $W$  is a binary variable indicating whether the considered entity has received treatment ( $W = 1$ ) or not ( $W = 0$ ), and by  $X$  a vector of covariates, assumed to be predetermined relative to the treatment. That is,  $W$  and  $X$  could be correlated, but the values of  $X$  have been determined before that of  $W$  (in such a way that the realization of  $W$  does not affect  $X$ ). Typically,  $X$  contains characteristics of the considered entity.

We are interested in the effect of the treatment, that is:

$$Y_1 - Y_0,$$

where  $Y_1$  correspond to the outcome obtained under treatment, while  $Y_0$  is the outcome obtained without it. Notice that we have:

$$Y = (1 - W)Y_0 + WY_1.$$

The problem is that observing  $(Y, W, X)$  is not sufficient to observe the treatment effect  $Y_1 - Y_0$ . Additional assumptions are needed to estimate it, or, more precisely, its expectations (*average treatment effect*):

$$ATE = \mathbb{E}(Y_1 - Y_0).$$

Importantly,  $ATE$  is different from the following quantity:

$$\alpha = \underbrace{\mathbb{E}(Y|W = 1)}_{=\mathbb{E}(Y_1|W=1)} - \underbrace{\mathbb{E}(Y|W = 0)}_{=\mathbb{E}(Y_0|W=0)},$$

that is easier to estimate. Indeed, a consistent estimate of  $\alpha$  is the difference between the means of the outcome variables in two sub-samples: one containing only the treated entities (this gives an estimate of  $\mathbb{E}(Y_1|W = 1)$ ) and the other containing only the non-treated entities (this gives an estimate of  $\mathbb{E}(Y_0|W = 0)$ ). Coming back to  $ATE$ , the problem is that we won’t have direct information regarding  $\mathbb{E}(Y_0|W = 1)$  and  $\mathbb{E}(Y_1|W = 0)$ . However, these two conditional expectations are part of  $ATE$ . Indeed,  $ATE = \mathbb{E}(Y_1) - \mathbb{E}(Y_0)$ , and:

$$\mathbb{E}(Y_1) = \mathbb{E}(Y_1|W = 0)\mathbb{P}(W = 0) + \mathbb{E}(Y_1|W = 1)\mathbb{P}(W = 1) \quad (6.11)$$

$$\mathbb{E}(Y_0) = \mathbb{E}(Y_0|W = 0)\mathbb{P}(W = 0) + \mathbb{E}(Y_0|W = 1)\mathbb{P}(W = 1). \quad (6.12)$$

### 6.6.2 Randomized controlled trials (RCTs)

In the context of Randomized controlled trials (RCTs), entities are randomly assigned to receive the treatment. As a result, we have  $\mathbb{E}(Y_1) = \mathbb{E}(Y_1|W = 0) = \mathbb{E}(Y_1|W = 1)$  and  $\mathbb{E}(Y_0) = \mathbb{E}(Y_0|W = 0) = \mathbb{E}(Y_0|W = 1)$ . Using this into Eqs. (6.11) and (6.12) yields  $ATE = \alpha$ .

Therefore, in this context, estimating  $\mathbb{E}(Y_1 - Y_0)$  amounts to computing the difference between two sample means, namely (a) the sample mean of the subset of  $Y_i$ ’s corresponding to the entities for which  $W_i = 1$ , and (b) the one for which  $W_i = 0$ .

More accurate estimates can be obtained through regressions. Assume that the model reads:

$$Y_i = W_i\beta_1 + X'_i\beta_z + \varepsilon_i,$$

where  $\mathbb{E}(\varepsilon_i|X_i) = 0$  (and  $W_i$  is independent from  $X_i$  and  $\varepsilon_i$ ). In this case, we obtain a consistent estimate of  $\beta_1$  by regressing  $\mathbf{y}$  on  $\mathbf{Z} = [\mathbf{w}, \mathbf{X}]$ .

### 6.6.3 Difference-in-Difference (DiD) approach

The DiD approach is a popular methodology implemented in cases where  $W$  cannot be considered as an independent variable. It exploits two dimensions: entities ( $i$ ), and time ( $t$ ). To simplify the exposition, we consider only two periods here ( $t = 0$  and  $t = 1$ ).

Consider the following model:

$$Y_{i,t} = W_{i,t}\beta_1 + \mu_i + \delta_t + \varepsilon_{i,t} \quad (6.13)$$

The parameter of interest is  $\beta_1$ , which is the treatment effect (recall that  $W_{i,t} \in \{0, 1\}$ ). Usually, for all entities  $i$ , we have  $W_{i,t=0} = 0$ . But only some of them are treated on date 1, i.e.,  $W_{i,1} \in \{0, 1\}$ .

The disturbance  $\varepsilon_{i,t}$  affects the outcome, but we assume that it does not relate to the selection for treatment; therefore,  $\mathbb{E}(\varepsilon_{i,t}|W_{i,t}) = 0$ . By contrast, we do not exclude some correlation between  $W_{i,t}$  (for  $t = 1$ ) and  $\mu_i$ ; hence,  $\mu_i$  may constitute a *confounder*. Finally, we suppose that the micro-variables  $W_i$  do not affect the time fixed effects  $\delta_t$ , such that  $\mathbb{E}(\delta_t|W_{i,t}) = \mathbb{E}(\delta_t)$ .

We have:

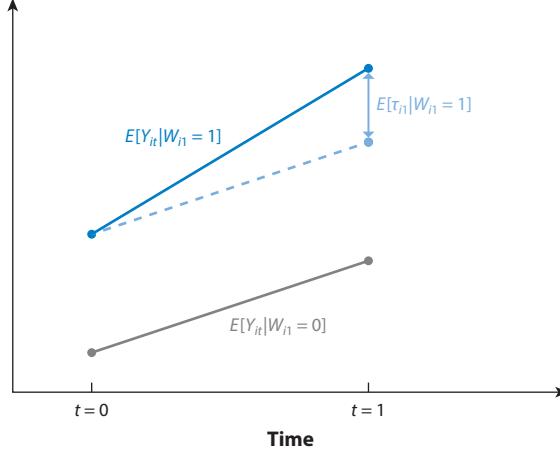
$$\begin{aligned} \mathbb{E}(Y_{i,1}|W_{i,1} = 1) &= \beta_1 + \mathbb{E}(\mu_i|W_{i,1} = 1) + \mathbb{E}(\delta_1|W_{i,1} = 1) + \mathbb{E}(\varepsilon_{i,1}) \\ \mathbb{E}(Y_{i,0}|W_{i,1} = 1) &= \mathbb{E}(\mu_i|W_{i,1} = 1) + \mathbb{E}(\delta_0|W_{i,1} = 1) + \mathbb{E}(\varepsilon_{i,0}) \\ \mathbb{E}(Y_{i,1}|W_{i,1} = 0) &= \mathbb{E}(\mu_i|W_{i,1} = 0) + \mathbb{E}(\delta_1|W_{i,1} = 0) + \mathbb{E}(\varepsilon_{i,1}) \\ \mathbb{E}(Y_{i,0}|W_{i,1} = 0) &= \mathbb{E}(\mu_i|W_{i,1} = 0) + \mathbb{E}(\delta_0|W_{i,1} = 0) + \mathbb{E}(\varepsilon_{i,0}). \end{aligned}$$

and, under our assumptions, it can be checked that:

$$\beta_1 = \mathbb{E}(\Delta Y_{i,1}|W_{i,1} = 1) - \mathbb{E}(\Delta Y_{i,1}|W_{i,1} = 0),$$

where  $\Delta Y_{i,1} = Y_{i,1} - Y_{i,0}$ . Therefore, in this context, the treatment effect appears to be a difference (of two conditional expectations) of difference (of the outcome variable, through time).

This is illustrated by Figure 6.6, which represents the generic DiD framework.



**Figure 5**

Identification in a difference-in-differences model. The dashed line represents the outcome that the treated units would have experienced in the absence of the treatment.

Figure 6.6: Source: Abadie et al., (1998).

In practice, implementing this approach consists in running a linear regression of the type of Eq. (6.13). These regressions also usually involve controls on top of the fixed effects  $\mu_i$ . As illustrated in the next subsection, the parameter of interest ( $\beta_1$ ) is often associated with an interaction term.

### 6.6.4 Application of the DiD approach

This example is based on the data used in Meyer et al. (1995). This dataset is part of the `wooldridge` package. This paper examines the effect of workers' compensation for injury on time out of work. It exploits a **natural experiment** approach of comparing individuals injured before and after increases in the maximum weekly benefit amount. Specifically, in 1980, the cap on weekly earnings covered by worker's compensation was increased in Kentucky and Michigan. Let us check whether this new policy was followed by an increase in the amount of time workers spent unemployed (for example, higher compensation may reduce workers' incentives to avoid injury).

As shown in Figure 6.7, the measure has only affected high-earning workers. The idea exploited by Meyer et al. (1995) was to compare the increase in time out of work before-after 1980 for higher-earnings workers on the one hand (entities who received the treatment) and low-earnings workers on the other hand (control group).

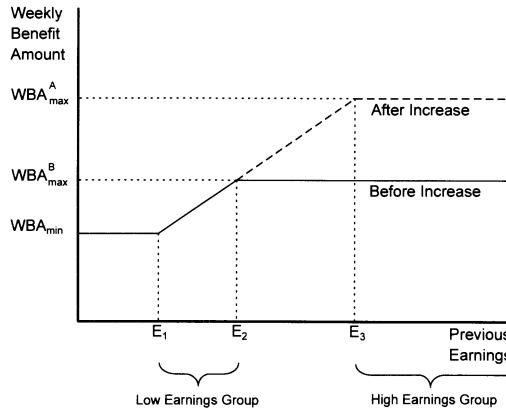


FIGURE 1. TEMPORARY TOTAL BENEFIT SCHEDULE  
BEFORE AND AFTER AN INCREASE IN  
THE MAXIMUM WEEKLY BENEFIT

Figure 6.7: Source: Meyer et al., (1995).

The next lines of codes replicate some of their results. The dependent variable is the logarithm of the duration of benefits. For more information use `?injury`, after having loaded the `wooldridge` library.

In the table of results below, the parameter of interest is the one associated with the interaction term `afchng:highearn`. Columns 2 and 3 correspond to the first two column of Table 6 in Meyer et al. (1995).

```
library(wooldridge)
data(injury)
injury <- subset(injury,ky==1)
injury$indust <- as.factor(injury$indust)
injury$injtype <- as.factor(injury$injtype)
#names(injury)
eq1 <- lm(log(durat) ~ afchng + highearn + afchng*highearn,data=injury)
eq2 <- lm(log(durat) ~ afchng + highearn + afchng*highearn +
    lprewage*highearn + male + married + lage + ltotmed + hosp +
    indust + injtype,data=injury)
eq3 <- lm(log(durat) ~ afchng + highearn + afchng*highearn +
    lprewage*highearn + male + married + lage + indust +
    injtype,data=injury)
stargazer::stargazer(eq1,eq2,eq3,type="text",
    omit=c("indust","injtype","Constant"),no.space = TRUE,
    add.lines = list(c("industry dummy","no","yes","yes"),
        c("injury dummy","no","yes","yes")),
    order = c(1,2,18,3:17,19,20),omit.stat = c("f","ser"))
```

##

```

## =====
##             Dependent variable:
## -----
##                   log(durat)
##             (1)      (2)      (3)
## -----
## afchng          0.008    -0.004    0.016
##                  (0.045)   (0.038)   (0.045)
## highearn       0.256***   -0.595   -1.522
##                  (0.047)   (0.930)   (1.099)
## afchng:highearn 0.191***  0.162***  0.215***  

##                  (0.069)   (0.059)   (0.069)
## lprewage        0.207**   0.258**  

##                  (0.088)   (0.104)
## male            -0.070*   -0.072  

##                  (0.039)   (0.046)
## married         0.055    0.051  

##                  (0.035)   (0.041)
## lage            0.244***  0.252***  

##                  (0.044)   (0.052)
## ltotmed         0.361***  

##                  (0.011)
## hosp            0.252***  

##                  (0.044)
## highearn:lprewage 0.065    0.232  

##                  (0.158)   (0.187)
## -----
## industry dummy   no      yes     yes
## injury dummy     no      yes     yes
## Observations     5,626   5,347   5,347
## R2               0.021   0.319   0.049
## Adjusted R2      0.020   0.316   0.046
## =====
## Note:           *p<0.1; **p<0.05; ***p<0.01

```

# Chapter 7

## Estimation Methods

This chapter presents three approaches to estimate parametric models: the General Method of Moments (GMM), the Maximum Likelihood approach (ML), and the Bayesian approach. The general context is the following: You observe a sample  $\mathbf{y} = \{y_1, \dots, y_n\}$ , you assume that these data have been generated by a model parameterized by  $\theta \in \mathbb{R}^K$ , and you want to estimate this vector  $\theta_0$ .

### 7.1 Generalized Method of Moments (GMM)

#### 7.1.1 Definition of the GMM estimator

We denote by  $y_i$  a  $p \times 1$  vector of variables; by  $\theta$  an  $K \times 1$  vector of parameters, and by  $h(y_i; \theta)$  a continuous  $r \times 1$  vector-valued function.

We denote by  $\theta_0$  the true value of  $\theta$  and we assume that  $\theta_0$  satisfies:

$$\mathbb{E}[h(y_i; \theta_0)] = \mathbf{0}.$$

We denote by  $\underline{y}_i$  the information contained in the current and past observations of  $y_i$ , that is:  $\underline{y}_i = \{y_i, y_{i-1}, \dots, y_1\}$ . We denote by  $g(\underline{y}_n; \theta)$  the sample average of the  $h(y_i; \theta)$  vectors, i.e.:

$$g(\underline{y}_n; \theta) = \frac{1}{n} \sum_{i=1}^n h(y_i; \theta).$$

The intuition behind the GMM estimator is the following: choose  $\theta$  so as to make the sample moment as close as possible to their population values, that is 0.

**Definition 7.1.** A GMM estimator of  $\theta_0$  is given by:

$$\hat{\theta}_n = \operatorname{argmin}_{\theta} g(\underline{y}_n; \theta)' W_n g(\underline{y}_n; \theta),$$

where  $W_n$  is a positive definite matrix (that may depend on  $\underline{y}_n$ ).

In the specific case where  $K = r$  (the dimension of  $\theta$  is the same as that of  $h(y_i; \theta)$  —or of  $g(\underline{y}_n; \theta)$ — then  $\hat{\theta}_n$  satisfies:

$$g(\underline{y}_n; \hat{\theta}_n) = \mathbf{0}.$$

Under regularity and identification conditions, this estimator is consistent, that is  $\hat{\theta}_n$  converges towards  $\theta_0$  in probability, which we denote by:

$$\operatorname{plim}_n \hat{\theta}_n = \theta_0, \quad \text{or} \quad \hat{\theta}_n \xrightarrow{P} \theta_0, \tag{7.1}$$

i.e.  $\forall \varepsilon > 0, \lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\theta}_n - \theta_0| > \varepsilon) = 0$  (this is Definition 10.16).

Definition 7.1 involves a positive definite matrix  $W_n$ . While one can take any positive definite matrix to have consistency (Eq. (7.1)), it can be shown that the GMM estimator achieves the

minimum asymptotic variance when  $W_n$  is the inverse of matrix  $S$ , the latter being defined by:

$$S = \text{Asy.Var}(\sqrt{n}g(\underline{y}_n; \hat{\theta}_n)).$$

In this case,  $W_n$  is said to be the *optimal weighting matrix*.

The intuition behind this result is the same that underlies Generalized Least Squares (see Section 5.5.2), that is: it is beneficial to use a criterion in which the weights are inversely proportional to the variances of the moments.

If  $h(x_i; \theta_0)$  is not correlated to  $h(x_j; \theta_0)$ , for  $i \neq j$ , then we have:

$$S = \text{Var}(h(x_i; \theta_0)),$$

which can be approximated by

$$\hat{\Gamma}_{0,n} = \frac{1}{n} \sum_{i=1}^n h(x_i; \hat{\theta}_n)h(x_i; \hat{\theta}_n)'.$$

In a time series context, we often have correlation between  $x_i$  and  $x_{i+k}$ , especially for small  $k$ 's. In this case, and if the time series  $\{y_i\}$  is covariance stationary (see Def. 9.4), then we have:

$$S := \sum_{\nu=-\infty}^{\infty} \Gamma_{\nu},$$

where  $\Gamma_{\nu} := \mathbb{E}[h(x_i; \theta_0)h(x_{i-\nu}; \theta_0)']$ . Matrix  $S$  is called the **long-run variance** of process  $\{y_i\}$  (see Def. 9.9).

For  $\nu \geq 0$ , let us define  $\hat{\Gamma}_{\nu,n}$  by:

$$\hat{\Gamma}_{\nu,n} = \frac{1}{n} \sum_{i=\nu+1}^n h(x_i; \hat{\theta}_n)h(x_{i-\nu}; \hat{\theta}_n)',$$

then  $S$  can be approximated by the Newey and West (1987) formula (similar to Eq. (9.6)):

$$\hat{\Gamma}_{0,n} + \sum_{\nu=1}^q \left[ 1 - \frac{\nu}{q+1} \right] (\hat{\Gamma}_{\nu,n} + \hat{\Gamma}'_{\nu,n}). \quad (7.2)$$

### 7.1.2 Asymptotic distribution of the GMM estimator

We have:

$$\boxed{\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0, V)}, \quad (7.3)$$

where  $V = (DS^{-1}D')^{-1}$ , with

$$D := \mathbb{E} \left( \frac{\partial h(x_i; \theta)}{\partial \theta'} \right) \Big|_{\theta=\theta_0}.$$

Matrix  $V$  can be approximated by

$$\hat{V}_n = (\hat{D}_n \hat{S}_n^{-1} \hat{D}'_n)^{-1}, \quad (7.4)$$

where  $\hat{S}_n$  is given by Eq. (7.2) and

$$\hat{D}'_n := \frac{\partial g(\underline{y}_n; \theta)}{\partial \theta'} \Big|_{\theta=\hat{\theta}_n}.$$

In practice, the previous matrix is computed numerically.

### 7.1.3 Testing hypotheses in the GMM framework

A first important test is the one concerning the validity of the moment restrictions (Sargan-Hansen test; Sargan (1958) and Hansen (1982)). Assume that the number of restrictions imposed is larger than the number of parameters to estimate ( $r > K$ ). In this case, the restrictions are said to be over-identifying.

Under correct specification, we asymptotically have:

$$\sqrt{n}g(\underline{y}_n; \theta_0) \sim \mathcal{N}(0, S).$$

As a result, it comes that:

$$J_n = (\sqrt{n}g(\underline{y}_n; \theta_0))' S^{-1} (\sqrt{n}g(\underline{y}_n; \theta_0)) \quad (7.5)$$

asymptotically follows a  $\chi^2$  distribution. The number of degrees of freedom is equal to  $r - K$ . (Note that, for  $r = K$ , we have, as expected,  $J = 0$ .) That is, asymptotically:

$$J_n \sim \chi^2(r - K).$$

The GMM framework also allows to easily test linear restrictions on the parameters. First, given Eq. (7.3), Wald tests (see Eq. (5.16) in Section 5.2.5) are readily available. Second, one can also resort to a test equivalent to the *likelihood ratio tests* (see Definition 7.8). More precisely, consider an unconstrained model and a constrained version of this model, the number of restrictions being equal to  $k$ . If the two models are estimated by considering the same moment constraints, and the same weighting matrix —using Eq. (7.4), based on the unrestricted model—, then we have that:

$$n \left[ (g(\underline{y}_n; \hat{\theta}_n^*) - g(\underline{y}_n; \hat{\theta}_n)) \right] \sim \chi^2(k),$$

where  $\hat{\theta}_n^*$  is the constrained estimate of  $\theta_0$ .

### 7.1.4 Example: Estimation of the Stochastic Discount Factor (s.d.f.)

Under the no-arbitrage assumption, there exists a random variable  $\mathcal{M}_{t,t+1}$  (a s.d.f.) such that

$$\mathbb{E}_t(\mathcal{M}_{t,t+1} R_{t+1}) = 1$$

for any (gross) asset return  $R_t$ . In the following,  $R_t$  denotes a  $n_r$ -dimensional vector of gross returns.

We consider the following specification of the s.d.f.:

$$\mathcal{M}_{t,t+1} = 1 - \mathbf{b}'_M(F_{t+1} - \mathbb{E}_t(F_{t+1})), \quad (7.6)$$

where  $F_t$  is a vector of factors. Eq. (7.6) then reads:

$$\mathbb{E}_t([1 - \mathbf{b}'_M(F_{t+1} - \mathbb{E}_t(F_{t+1}))] R_{t+1}) = 1.$$

Assume that the date- $t$  information set is  $\mathcal{I}_t = \{\mathbf{z}_t, \mathcal{I}_{t-1}\}$ , where  $\mathbf{z}_t$  is a vector of variables observed on date  $t$ . (We then have  $\mathbb{E}_t(\bullet) \equiv \mathbb{E}(\bullet | \mathcal{I}_t)$ .)

We can use  $\mathbf{z}_t$  as an instrument. Indeed, we have:

$$\begin{aligned} & \mathbb{E}(z_{i,t} [\mathbf{b}'_M \{F_{t+1} - \mathbb{E}_t(F_{t+1})\} R_{t+1} - R_{t+1} + 1]) \\ &= \mathbb{E}(\mathbb{E}_t \{z_{i,t} [\mathbf{b}'_M \{F_{t+1} - \mathbb{E}_t(F_{t+1})\} R_{t+1} - R_{t+1} + 1]\}) \\ &= \mathbb{E}(z_{i,t} \underbrace{\mathbb{E}_t \{\mathbf{b}'_M \{F_{t+1} - \mathbb{E}_t(F_{t+1})\} R_{t+1} - R_{t+1} + 1\}}_{1 - \mathbb{E}_t(\mathcal{M}_{t,t+1} R_{t+1}) = 0}) = 0. \end{aligned} \quad (7.7)$$

We have then converted a conditional moment condition into a unconditional one (which we need to implement the GMM approach described above). However, at that stage, we

can still not directly use the GMM formulas because of the conditional expectation  $\mathbb{E}_t(F_{t+1})$  that appears in  $\mathbb{E}(z_{i,t}[\mathbf{b}'_M\{F_{t+1} - \mathbb{E}_t(F_{t+1})\}R_{t+1} - R_{t+1} + 1]) = 0$ .

To go further, let us assume that:

$$\mathbb{E}_t(F_{t+1}) = \mathbf{b}_F \mathbf{z}_t.$$

We can then easily estimate matrix  $\mathbf{b}_F$  (of dimension  $n_F \times n_z$ ) by OLS. Note here that these OLS can be seen as a special GMM case. Indeed, as was done in Eq. (7.7), we can show that, for the  $j^{th}$  component of  $F_t$ , we have:

$$\mathbb{E}([F_{j,t+1} - \mathbf{b}_{F,j}\mathbf{z}_t]\mathbf{z}_t) = 0,$$

where  $\mathbf{b}_{F,j}$  denotes the  $j^{th}$  row of  $\mathbf{b}_F$ . This yields the OLS formula.

Equipped with  $\mathbf{b}_F$ , we rely on the following moment restrictions to estimate  $\mathbf{b}_M$ :

$$\mathbb{E}(z_{i,t}[\mathbf{b}'_M\{F_{t+1} - \mathbf{b}_F \mathbf{z}_t\}R_{t+1} - R_{t+1} + 1]) = 0.$$

Specifically, the number of restrictions is  $n_R \times n_z$ . Let us implement this approach in the U.S. context, using data extracted from the FRED database. In factor  $F_t$ , we use the changes in the VIX and in the personal consumption expenditures. The returns ( $R_t$ ) are based on the Wilshire 5000 Price Index (a stock price index) and on the ICE BofA BBB US Corporate Index Total Return Index (a bond return index).

```
library(fredr)
fredr_set_key("df65e14c054697a52b4511e77fcfa1f3")
start_date <- as.Date("1990-01-01"); end_date <- as.Date("2022-01-01")
f <- function(ticker){
  fredr(series_id = ticker,
        observation_start = start_date, observation_end = end_date,
        frequency = "m", aggregation_method = "avg")
}
vix <- f("VIXCLS") # VIX
pce <- f("PCE") # Personal consumption expenditures
sto <- f("WILL5000PRFC") # Wilshire 5000 Full Cap Price Index
bdr <- f("BAMLCCOA4BBBTRIV") # ICE BofA BBB US Corp. Index Tot. Return
T <- dim(vix)[1]
dvix <- c(vix$value[3:T]/vix$value[2:(T-1)]) # change in VIX t+1
dpce <- c(pce$value[3:T]/pce$value[2:(T-1)]) # change in PCE t+1
dsto <- c(sto$value[3:T]/sto$value[2:(T-1)]) # return t+1
dbdr <- c(bdr$value[3:T]/bdr$value[2:(T-1)]) # return t+1
dvix_1 <- c(vix$value[2:(T-1)]/vix$value[1:(T-2)]) # change in VIX t
dpce_1 <- c(pce$value[2:(T-1)]/pce$value[1:(T-2)]) # change in PCE t
dsto_1 <- c(sto$value[2:(T-1)]/sto$value[1:(T-2)]) # return t
dbdr_1 <- c(bdr$value[2:(T-1)]/bdr$value[1:(T-2)]) # return t
```

Define the matrices containing the  $F_{t+1}$ ,  $\mathbf{z}_t$ , and  $R_{t+1}$  vectors:

```
F_tp1 <- cbind(dvix, dpce)
Z      <- cbind(1, dvix_1, dpce_1, dsto_1, dbdr_1)
b_F <- t(solve(t(Z) %*% Z) %*% t(Z) %*% F_tp1)
F_innov <- F_tp1 - Z %*% t(b_F)
R_tp1 <- cbind(dsto, dbdr)
n_F <- dim(F_tp1)[2]; n_R <- dim(R_tp1)[2]; n_z <- dim(Z)[2]
```

Function `f_aux` compute the  $h(x_t; \theta)$  and the  $g(y_T; \theta)$ ; function `f2beMin` is the function to be minimized.

```

f_aux <- function(theta){
  b_M <- matrix(theta[1:n_F], ncol=1)
  R_aux <- matrix(F_innov %*% b_M, T-2, n_R) * R_tp1 - R_tp1 + 1
  H <- (R_aux %x% matrix(1, 1, n_z)) * (matrix(1, 1, n_R) %x% Z)
  g <- matrix(apply(H, 2, mean), ncol=1)
  return(list(g=g, H=H))
}

f2beMin <- function(theta, W){# function to be minimized
  res <- f_aux(theta)
  return(t(res$g) %*% W %*% res$g)
}

```

Now, let's minimize this function, using use the BFGS numerical algorithm (part of the `optim` wrapper). We run 5 iterations (where  $W$  is updated).

```

library(AEC)
theta <- c(rep(0, n_F)) # initial value
for(i in 1:10){# recursion on W
  res <- f_aux(theta)
  W <- solve(NW.LongRunVariance(res$H, q=6))
  res.optim <- optim(theta, f2beMin, W=W,
                      method="BFGS", # could be "Nelder-Mead"
                      control=list(trace=FALSE, maxit=200), hessian=TRUE)
  theta <- res.optim$par
}

```

Finally, let's compute the standard deviation of the parameter estimates, using Eq. (7.4):

```

eps <- .0001
g0 <- f_aux(theta)$g
D <- NULL
for(i in 1:length(theta)){
  theta.i <- theta
  theta.i[i] <- theta.i[i] + eps
  gi <- f_aux(theta.i)$g
  D <- cbind(D, (gi-g0)/eps)
}
V <- 1/T * solve(t(D) %*% W %*% D)
std.dev <- sqrt(diag(V)); t.stud <- theta/std.dev
cbind(theta, std.dev, t.stud)

##           theta      std.dev      t.stud
## [1,] -0.7180716  0.4646617 -1.5453642
## [2,] -11.2042452 17.1039449 -0.6550679

```

The Hansen statistic can be used to test the model (see Eq. (7.5)). If the model is correct, we have:

$$Tg(\underline{y}_T; \theta)' S^{-1} g(\underline{y}_T; \theta) \sim i.i.d. \chi^2(J - K),$$

where  $J$  is the number of moment constraints ( $n_z \times n_r$  here) and  $K$  is the number of estimated parameters ( $= n_F$  here).

```

g <- f_aux(theta)$g
Hansen_stat <- T * t(g) %*% W %*% g
pvalue <- pchisq(q = Hansen_stat, df = n_R*n_z - n_F)
pvalue

##          [,1]
## [1,] 0.8789782

```

## 7.2 Maximum Likelihood Estimation

### 7.2.1 Intuition

Intuitively, the *Maximum Likelihood Estimation (MLE)* consists in looking for the value of  $\theta$  that is such that the probability of having observed  $\mathbf{y}$  (the sample at hand) is the highest possible.

To set an example, assume that the time periods between the arrivals of two customers in a shop, denoted by  $y_i$ , are i.i.d. and follow an exponential distribution, i.e.  $y_i \sim i.i.d. \mathcal{E}(\lambda)$ . You have observed these arrivals for some time, thereby constituting a sample  $\mathbf{y} = \{y_1, \dots, y_n\}$ . You want to estimate  $\lambda$  (i.e. in that case, the vector of parameters is simply  $\theta = \lambda$ ).

The density of  $Y$  (one observation) is  $f(y; \lambda) = \frac{1}{\lambda} \exp(-y/\lambda)$ . Fig. 7.1 represents such density functions for different values of  $\lambda$ .

Your 200 observations are reported at the bottom of Fig. 7.1 (red bars). You build the histogram and display it on the same chart.

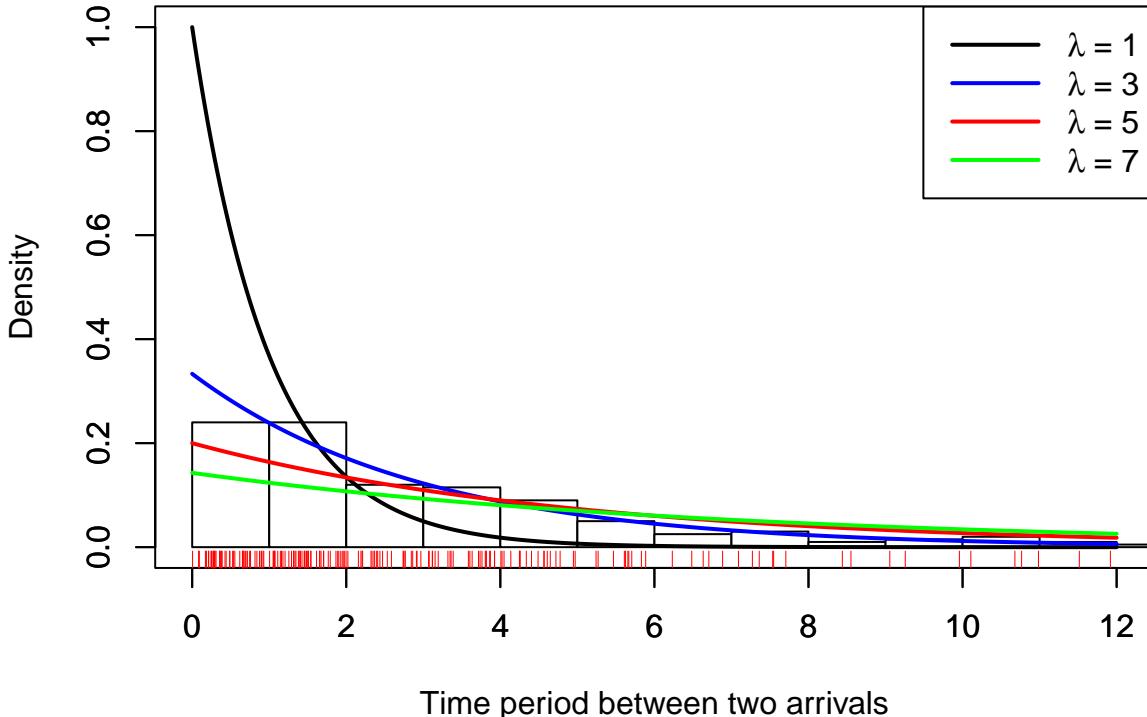


Figure 7.1: The red ticks, at the bottom, indicate observations (there are 200 of them). The histogram is based on these 200 observations

What is your estimate of  $\lambda$ ? Intuitively, one is led to take the  $\lambda$  for which the (theoretical) distribution is the closest to the histogram (that can be seen as an “empirical distribution”). This approach is consistent with the idea of picking the  $\lambda$  for which the probability of observing the values included in  $\mathbf{y}$  is the highest.

Let us be more formal. Assume that you have only four observations:  $y_1 = 1.1$ ,  $y_2 = 2.2$ ,  $y_3 = 0.7$  and  $y_4 = 5.0$ . What was the probability of jointly observing:

- $1.1 - \varepsilon \leq Y_1 < 1.1 + \varepsilon$ ,
- $2.2 - \varepsilon \leq Y_2 < 2.2 + \varepsilon$ ,
- $0.7 - \varepsilon \leq Y_3 < 0.7 + \varepsilon$ , and
- $5.0 - \varepsilon \leq Y_4 < 5.0 + \varepsilon$ ?

Because the  $y_i$ 's are i.i.d., this probability is  $\prod_{i=1}^4 (2\varepsilon f(y_i, \lambda))$ . The next plot shows the probability (divided by  $16\varepsilon^4$ , which does not depend on  $\lambda$ ) as a function of  $\lambda$ .

The value of  $\lambda$  that maximizes the probability is 2.26.

Let us come back to the example with 200 observations:

In that case, the value of  $\lambda$  that maximizes the probability is 3.42.

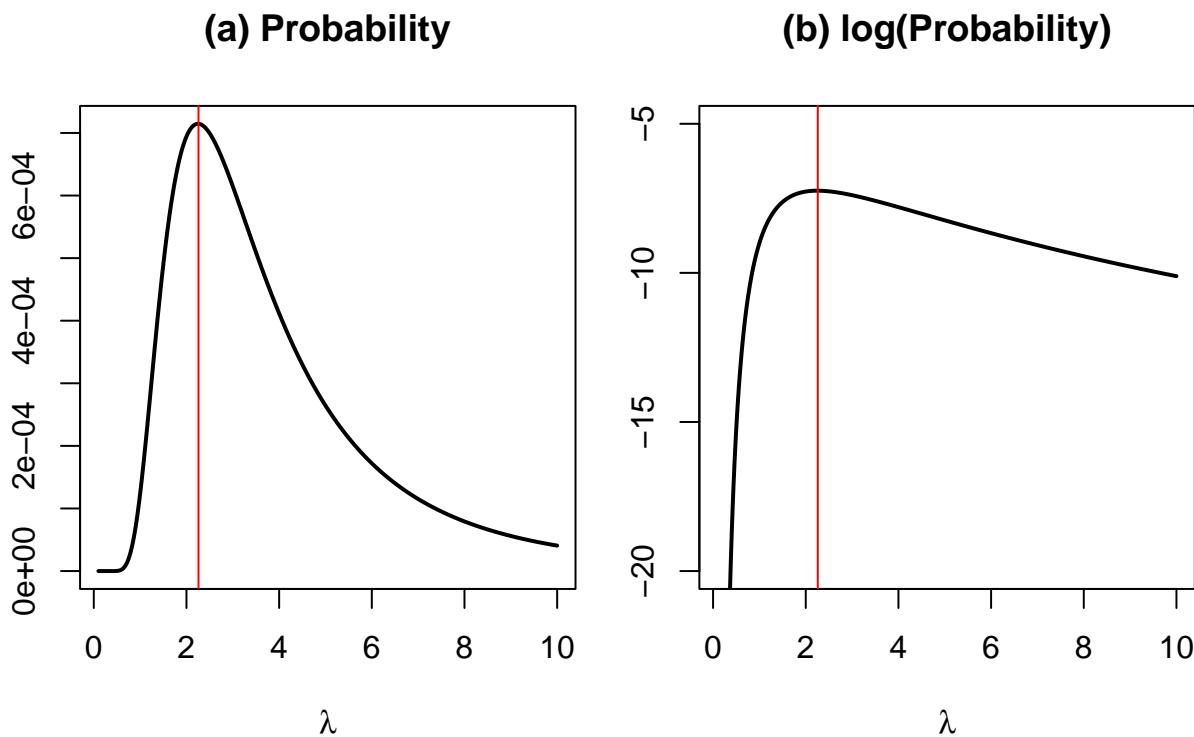


Figure 7.2: Proba. that  $y_i - \varepsilon \leq Y_i < y_i + \varepsilon$ ,  $i \in \{1, 2, 3, 4\}$ . The vertical red line indicates the maximum of the function.

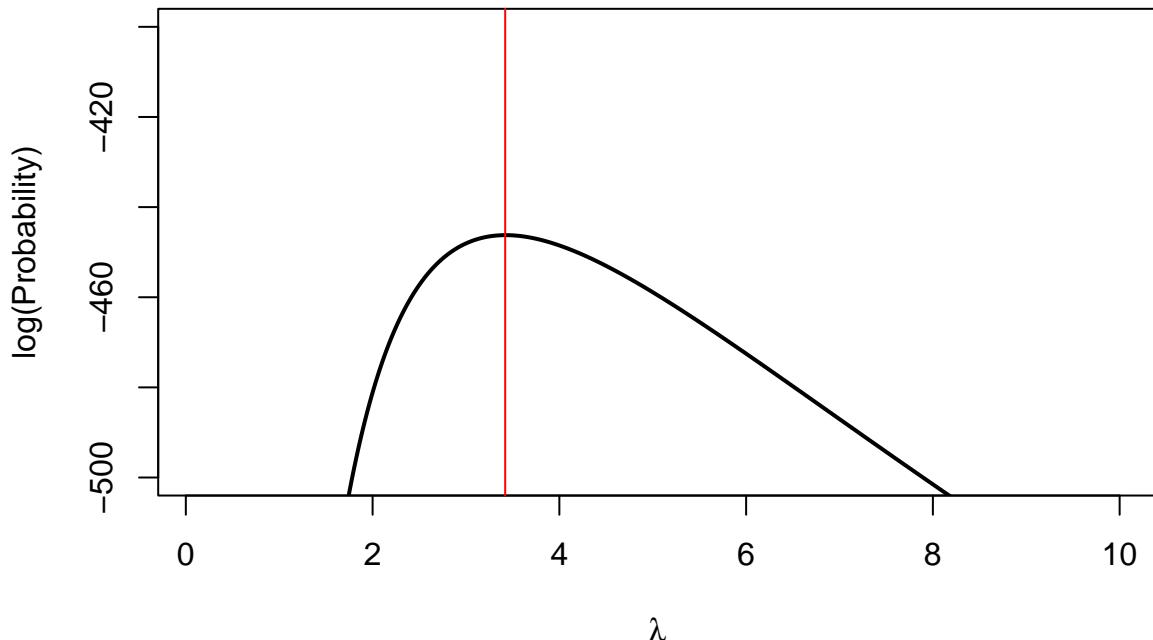


Figure 7.3: Log-likelihood function associated with the 200 i.i.d. observations. The vertical red line indicates the maximum of the function.

### 7.2.2 Definition and properties

$f(y; \theta)$  denotes the probability density function (p.d.f.) of a random variable  $Y$  which depends on a set of parameters  $\theta$ . The density of  $n$  independent and identically distributed (i.i.d.) observations of  $Y$  is given by:

$$f(\mathbf{y}; \theta) = \prod_{i=1}^n f(y_i; \theta),$$

where  $\mathbf{y}$  denotes the vector of observations;  $\mathbf{y} = \{y_1, \dots, y_n\}$ .

**Definition 7.2** (Likelihood function). The likelihood function is:

$$\mathcal{L} : \theta \rightarrow \mathcal{L}(\theta; \mathbf{y}) = f(\mathbf{y}; \theta) = f(y_1, \dots, y_n; \theta).$$

We often work with  $\log \mathcal{L}$ , the **log-likelihood function**.

**Example 7.1** (Gaussian distribution). If  $y_i \sim \mathcal{N}(\mu, \sigma^2)$ , then

$$\log \mathcal{L}(\theta; \mathbf{y}) = -\frac{1}{2} \sum_{i=1}^n \left( \log \sigma^2 + \log 2\pi + \frac{(y_i - \mu)^2}{\sigma^2} \right).$$

**Definition 7.3** (Score). The score  $S(y; \theta)$  is given by  $\frac{\partial \log f(y; \theta)}{\partial \theta}$ .

If  $y_i \sim \mathcal{N}(\mu, \sigma^2)$  (Example 7.1), then

$$\frac{\partial \log f(y; \theta)}{\partial \theta} = \begin{bmatrix} \frac{\partial \log f(y; \theta)}{\partial \mu} \\ \frac{\partial \log f(y; \theta)}{\partial \sigma^2} \end{bmatrix} = \begin{bmatrix} \frac{y - \mu}{\sigma^2} \\ \frac{1}{2\sigma^2} \left( \frac{(y - \mu)^2}{\sigma^2} - 1 \right) \end{bmatrix}.$$

**Proposition 7.1** (Score expectation). *The expectation of the score is zero.*

*Proof.* We have:

$$\begin{aligned} \mathbb{E} \left( \frac{\partial \log f(Y; \theta)}{\partial \theta} \right) &= \int \frac{\partial \log f(y; \theta)}{\partial \theta} f(y; \theta) dy \\ &= \int \frac{\partial f(y; \theta)/\partial \theta}{f(y; \theta)} f(y; \theta) dy = \frac{\partial}{\partial \theta} \int f(y; \theta) dy \\ &= \partial 1/\partial \theta = 0, \end{aligned}$$

which gives the result.  $\square$

**Definition 7.4** (Fisher information matrix). The information matrix is (minus) the expectation of the second derivatives of the log-likelihood function:

$$\mathcal{I}_Y(\theta) = -\mathbb{E} \left( \frac{\partial^2 \log f(Y; \theta)}{\partial \theta \partial \theta'} \right).$$

**Proposition 7.2.** *We have*

$$\mathcal{I}_Y(\theta) = \mathbb{E} \left[ \left( \frac{\partial \log f(Y; \theta)}{\partial \theta} \right) \left( \frac{\partial \log f(Y; \theta)}{\partial \theta} \right)' \right] = \text{Var}[S(Y; \theta)].$$

*Proof.* We have  $\frac{\partial^2 \log f(Y; \theta)}{\partial \theta \partial \theta'} = \frac{\partial^2 f(Y; \theta)}{\partial \theta \partial \theta'} \frac{1}{f(Y; \theta)} - \frac{\partial \log f(Y; \theta)}{\partial \theta} \frac{\partial \log f(Y; \theta)}{\partial \theta'}$ . The expectation of the first right-hand side term is  $\partial^2 1/(\partial \theta \partial \theta') = \mathbf{0}$ , which gives the result.  $\square$

**Example 7.2.** If  $y_i \sim i.i.d. \mathcal{N}(\mu, \sigma^2)$ , let  $\theta = [\mu, \sigma^2]'$  then

$$\frac{\partial \log f(y; \theta)}{\partial \theta} = \begin{bmatrix} \frac{y - \mu}{\sigma^2} & \frac{1}{2\sigma^2} \left( \frac{(y - \mu)^2}{\sigma^2} - 1 \right) \end{bmatrix}',$$

and

$$\mathcal{I}_Y(\theta) = \mathbb{E} \left( \frac{1}{\sigma^4} \begin{bmatrix} \sigma^2 & \frac{y - \mu}{\sigma^2} \\ y - \mu & \frac{(y - \mu)^2}{\sigma^2} - \frac{1}{2} \end{bmatrix} \right) = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{bmatrix}.$$

**Proposition 7.3** (Additive property of the Information matrix). *The information matrix resulting from two independent experiments is the sum of the information matrices:*

$$\mathcal{J}_{X,Y}(\theta) = \mathcal{J}_X(\theta) + \mathcal{J}_Y(\theta).$$

*Proof.* Directly deduced from the definition of the information matrix (Def. 7.4), using that the expectation of a product of independent variables is the product of the expectations.  $\square$

**Theorem 7.1** (Frechet-Darmois-Cramer-Rao bound). *Consider an unbiased estimator of  $\theta$  denoted by  $\hat{\theta}(Y)$ . The variance of the random variable  $\omega' \hat{\theta}$  (which is a linear combination of the components of  $\hat{\theta}$ ) is larger than:*

$$(\omega' \omega)^2 / (\omega' \mathcal{J}_Y(\theta) \omega).$$

*Proof.* The Cauchy-Schwarz inequality implies that  $\sqrt{\text{Var}(\omega' \hat{\theta}(Y)) \text{Var}(\omega' S(Y; \theta))} \geq |\omega' \text{Cov}[\hat{\theta}(Y), S(Y; \theta)] \omega|$ . Now,  $\text{Cov}[\hat{\theta}(Y), S(Y; \theta)] = \int_y \hat{\theta}(y) \frac{\partial \log f(y; \theta)}{\partial \theta} f(y; \theta) dy = \frac{\partial}{\partial \theta} \int_y \hat{\theta}(y) f(y; \theta) dy = \mathbf{I}$  because  $\hat{\theta}$  is unbiased. Therefore  $\text{Var}(\omega' \hat{\theta}(Y)) \geq \text{Var}(\omega' S(Y; \theta))^{-1} (\omega' \omega)^2$ . Prop. 7.2 leads to the result.  $\square$

**Definition 7.5** (Identifiability). The vector of parameters  $\theta$  is identifiable if, for any other vector  $\theta^*$ :

$$\theta^* \neq \theta \Rightarrow \mathcal{L}(\theta^*; \mathbf{y}) \neq \mathcal{L}(\theta; \mathbf{y}).$$

**Definition 7.6** (Maximum Likelihood Estimator (MLE)). The maximum likelihood estimator (MLE) is the vector  $\theta$  that maximizes the likelihood function. Formally:

$$\theta_{MLE} = \arg \max_{\theta} \mathcal{L}(\theta; \mathbf{y}) = \arg \max_{\theta} \log \mathcal{L}(\theta; \mathbf{y}). \quad (7.8)$$

**Definition 7.7** (Likelihood equation). A necessary condition for maximizing the likelihood function (under regularity assumption, see Hypotheses 7.1) is:

$$\frac{\partial \log \mathcal{L}(\theta; \mathbf{y})}{\partial \theta} = \mathbf{0}. \quad (7.9)$$

**Hypothesis 7.1** (Regularity assumptions). We have:

- i.  $\theta \in \Theta$  where  $\Theta$  is compact.
- ii.  $\theta_0$  is identified.
- iii. The log-likelihood function is continuous in  $\theta$ .
- iv.  $\mathbb{E}_{\theta_0}(\log f(Y; \theta))$  exists.
- v. The log-likelihood function is such that  $(1/n) \log \mathcal{L}(\theta; \mathbf{y})$  converges almost surely to  $\mathbb{E}_{\theta_0}(\log f(Y; \theta))$ , uniformly in  $\theta \in \Theta$ .
- vi. The log-likelihood function is twice continuously differentiable in an open neighborhood of  $\theta_0$ .
- vii. The matrix  $\mathbf{I}(\theta_0) = -\mathbb{E}_0 \left( \frac{\partial^2 \log \mathcal{L}(\theta; \mathbf{y})}{\partial \theta \partial \theta'} \right)$  —the Fisher Information matrix— exists and is nonsingular.

**Proposition 7.4** (Properties of MLE). *Under regularity conditions (Assumptions 7.1), the MLE is:*

- a. **Consistent:**  $\text{plim } \theta_{MLE} = \theta_0$  ( $\theta_0$  is the true vector of parameters).
- b. **Asymptotically normal:**

$$\sqrt{n}(\theta_{MLE} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \mathcal{J}_Y(\theta_0)^{-1}).$$

(7.10)

- c. **Asymptotically efficient:**  $\theta_{MLE}$  is asymptotically efficient and achieves the Freechet-Darmois-Cramer-Rao lower bound for consistent estimators.
- d. **Invariant:** The MLE of  $g(\theta_0)$  is  $g(\theta_{MLE})$  if  $g$  is a continuous and continuously differentiable function.

*Proof.* See Appendix 10.5.  $\square$

Since  $\mathcal{I}_Y(\theta_0) = \frac{1}{n}\mathbf{I}(\theta_0)$ , the asymptotic covariance matrix of the MLE is  $[\mathbf{I}(\theta_0)]^{-1}$ , that is:

$$[\mathbf{I}(\theta_0)]^{-1} = \left[ -\mathbb{E}_0 \left( \frac{\partial^2 \log \mathcal{L}(\theta; \mathbf{y})}{\partial \theta \partial \theta'} \right) \right]^{-1}.$$

A direct (analytical) evaluation of this expectation is often out of reach. It can however be estimated by, either:

$$\hat{\mathbf{I}}_1^{-1} = \left( -\frac{\partial^2 \log \mathcal{L}(\theta_{MLE}; \mathbf{y})}{\partial \theta \partial \theta'} \right)^{-1}, \quad (7.11)$$

$$\hat{\mathbf{I}}_2^{-1} = \left( \sum_{i=1}^n \frac{\partial \log \mathcal{L}(\theta_{MLE}; y_i)}{\partial \theta} \frac{\partial \log \mathcal{L}(\theta_{MLE}; y_i)}{\partial \theta'} \right)^{-1}. \quad (7.12)$$

Asymptotically, we have  $(\hat{\mathbf{I}}_1^{-1})\hat{\mathbf{I}}_2 = Id$ , that is, the two formulas provide the same result.

In case of (suspected) misspecification, one can use the so-called *sandwich estimator* of the covariance matrix.<sup>1</sup> This covariance matrix is given by:

$$\hat{\mathbf{I}}_3^{-1} = \hat{\mathbf{I}}_2^{-1} \hat{\mathbf{I}}_1 \hat{\mathbf{I}}_2^{-1}. \quad (7.13)$$

### 7.2.3 To sum up – MLE in practice

To implement MLE, we need:

- A parametric model (depending on the vector of parameters  $\theta$  whose “true” value is  $\theta_0$ ) is specified.
- i.i.d. sources of randomness are identified.
- The density associated to one observation  $y_i$  is computed analytically (as a function of  $\theta$ ):  $f(y; \theta)$ .
- The log-likelihood is  $\log \mathcal{L}(\theta; \mathbf{y}) = \sum_i \log f(y_i; \theta)$ .
- The MLE estimator results from the optimization problem (this is Eq. (7.8)):

$$\theta_{MLE} = \arg \max_{\theta} \log \mathcal{L}(\theta; \mathbf{y}). \quad (7.14)$$

- We have:  $\theta_{MLE} \sim \mathcal{N}(\theta_0, \mathbf{I}(\theta_0)^{-1})$ , where  $\mathbf{I}(\theta_0)^{-1}$  is estimated by means of Eq. (7.11), Eq. (7.12), or Eq. (7.13). Most of the time, this computation is numerical.

### 7.2.4 Example: MLE estimation of a mixture of Gaussian distribution

Consider the returns of the Swiss Market Index (SMI). Assume that these returns are independently drawn from a mixture of Gaussian distributions. The p.d.f.  $f(x; \theta)$ , with  $\theta = [\mu_1, \sigma_1, \mu_2, \sigma_2, p]'$ , is given by:

$$p \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp \left( -\frac{(x - \mu_1)^2}{2\sigma_1^2} \right) + (1-p) \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp \left( -\frac{(x - \mu_2)^2}{2\sigma_2^2} \right).$$

(See p.d.f. of mixtures of Gaussian distributions.)

```
library(AEC); data(smi)
T <- dim(smi)[1]
h <- 5 # holding period (one week)
smi$r <- c(rep(NaN,h),
           100*c(log(smi$Close[(1+h):T]/smi$Close[1:(T-h)])))
```

<sup>1</sup>For more details, see, e.g., Charles Geyer’s lectures notes.

```

indic.dates <- seq(1,T,by=5) # weekly returns
smi <- smi[indic.dates,]
smi <- smi[complete.cases(smi),]
par(mfrow=c(1,1));par(plt=c(.15,.95,.1,.95))
plot(smi$Date,smi$r,type="l",xlab="",ylab="in percent")
abline(h=0,col="blue")
abline(h=mean(smi$r,na.rm = TRUE)+2*sd(smi$r,na.rm = TRUE),lty=3,col="blue")
abline(h=mean(smi$r,na.rm = TRUE)-2*sd(smi$r,na.rm = TRUE),lty=3,col="blue")

```

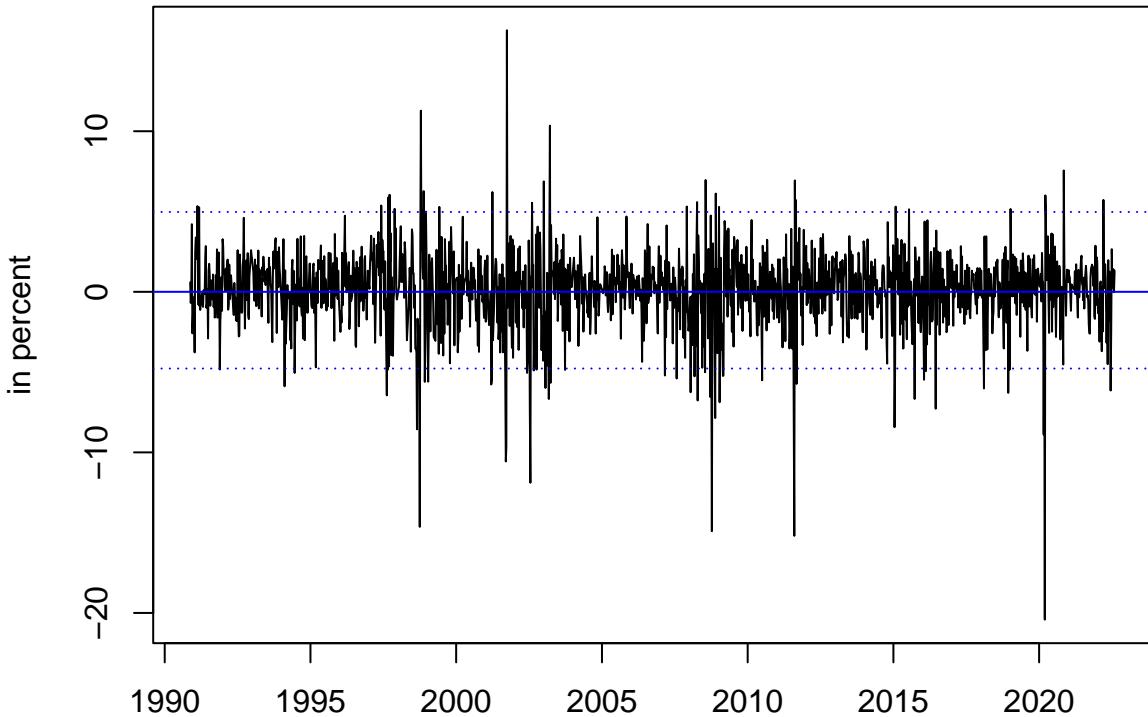


Figure 7.4: Time series of SMI weekly returns (source: Yahoo Finance).

Build the log-likelihood function (function `log.f`), and use the numerical BFGS algorithm to maximize it (using the `optim` wrapper):

```

f <- function(theta,y){ # Likelihood function
  mu.1 <- theta[1]; mu.2 <- theta[2]
  sigma.1 <- theta[3]; sigma.2 <- theta[4]
  p <- exp(theta[5])/(1+exp(theta[5]))
  res <- p*1/sqrt(2*pi*sigma.1^2)*exp(-(y-mu.1)^2/(2*sigma.1^2)) +
    (1-p)*1/sqrt(2*pi*sigma.2^2)*exp(-(y-mu.2)^2/(2*sigma.2^2))
  return(res)
}
log.f <- function(theta,y){ #log-Likelihood function
  return(-sum(log(f(theta,y))))
}
res.optim <- optim(c(0,0,0.5,1.5,.5),
  log.f,
  y=smi$r,
  method="BFGS", # could be "Nelder-Mead"
  control=list(trace=FALSE,maxit=100),hessian=TRUE)
theta <- res.optim$par
theta

## [1] 0.3012379 -1.3167476 1.7715072 4.8197596 1.9454889

```

Next, compute estimates of the covariance matrix of the MLE (using Eqs. (7.11), (7.12), and (7.13)), and compare the three sets of resulting standard deviations for the five estimated parameters:

```

# Hessian approach:
I.1 <- solve(res.optim$hessian)
# Outer-product of gradient approach:
log.f.0 <- log(f(theta,smi$r))
epsilon <- .00000001
d.log.f <- NULL
for(i in 1:length(theta)){
  theta.i <- theta
  theta.i[i] <- theta.i[i] + epsilon
  log.f.i <- log(f(theta.i,smi$r))
  d.log.f <- cbind(d.log.f,
                    (log.f.i - log.f.0)/epsilon)
}
I.2 <- solve(t(d.log.f) %*% d.log.f)
# Misspecification-robust approach (sandwich formula):
I.3 <- I.1 %*% solve(I.2) %*% I.1
cbind(diag(I.1),diag(I.2),diag(I.3))

```

```

##          [,1]      [,2]      [,3]
## [1,] 0.003683422 0.003199481 0.00586160
## [2,] 0.226892824 0.194283391 0.38653389
## [3,] 0.005764271 0.002769579 0.01712255
## [4,] 0.194081311 0.047466419 0.83130838
## [5,] 0.092114437 0.040366005 0.31347858

```

According to the first (respectively third) type of estimate for the covariance matrix, a 95% confidence interval for  $\mu_1$  is [0.182, 0.42] (resp. [0.151, 0.451]).

Note that we have not directly estimated parameter  $p$  but  $\nu = \log(p/(1-p))$  (in such a way that  $p = \exp(\nu)/(1+\exp(\nu))$ ). In order to get an estimate of the standard deviation of our estimate of  $p$ , we can implement the **Delta method**. This method is based on the fact that, for a function  $g$  that is continuous in the neighborhood of  $\theta_0$  and for large  $n$ , we have:

$$\text{Var}(g(\hat{\theta}_n)) \approx \frac{\partial g(\hat{\theta}_n)}{\partial \theta'} \text{Var}(\hat{\theta}_n) \frac{\partial g(\hat{\theta}_n)'}{\partial \theta}. \quad (7.15)$$

```

g <- function(theta){
  mu.1 <- theta[1]; mu.2 <- theta[2]
  sigma.1 <- theta[3]; sigma.2 <- theta[4]
  p <- exp(theta[5])/(1+exp(theta[5]))
  return(c(mu.1,mu.2,sigma.1,sigma.2,p))
}
# Computation of g's gradient around estimated theta:
eps <- .00001
g.theta <- g(theta)
g.gradient <- NULL
for(i in 1:5){
  theta.perturb <- theta
  theta.perturb[i] <- theta[i] + eps
  g.gradient <- cbind(g.gradient,(g(theta.perturb)-g.theta)/eps)
}
Var <- g.gradient %*% I.3 %*% t(g.gradient)
stdv.g.theta <- sqrt(diag(Var))
stdv.theta <- sqrt(diag(I.3))
cbind(theta,stdv.theta,g.theta,stdv.g.theta)

```

```

##          theta stdv.theta      g.theta stdv.g.theta
## [1,] 0.3012379 0.07656108 0.3012379 0.07656108
## [2,] -1.3167476 0.62171850 -1.3167476 0.62171850

```

```
## [3,] 1.7715072 0.13085316 1.7715072 0.13085316
## [4,] 4.8197596 0.91176114 4.8197596 0.91176114
## [5,] 1.9454889 0.55989158 0.8749539 0.06125726
```

The previous results show that the MLE estimate of  $p$  is 0.8749539, and its standard deviation is approximately equal to 0.0612573.

To finish with, let us draw the estimated parametric p.d.f. (the mixture of Gaussian distribution), and compare it to a non-parametric (kernel-based) estimate of this p.d.f. (using function `density`):

```
x <- seq(-5,5,by=.01)
par(plt=c(.1,.95,.1,.95))
plot(x,f(theta,x),type="l",lwd=2,xlab="returns, in percent",ylab="",
      ylim=c(0,1.4*max(f(theta,x))))
lines(density(smi$r),type="l",lwd=2,lty=3)
lines(x,dnorm(x,mean=mean(smi$r),sd = sd(smi$r)),col="red",lty=2,lwd=2)
rug(smi$r,col="blue")
legend("topleft",
      c("Kernel estimate (non-parametric)",
        "Estimated mixture of Gaussian distr. (MLE, parametric)",
        "Normal distribution"),
      lty=c(3,1,2),lwd=c(2), # line width
      col=c("black","black","red"),pt.bg=c(1),pt.cex = c(1),
      bg="white",seg.len = 4)
```

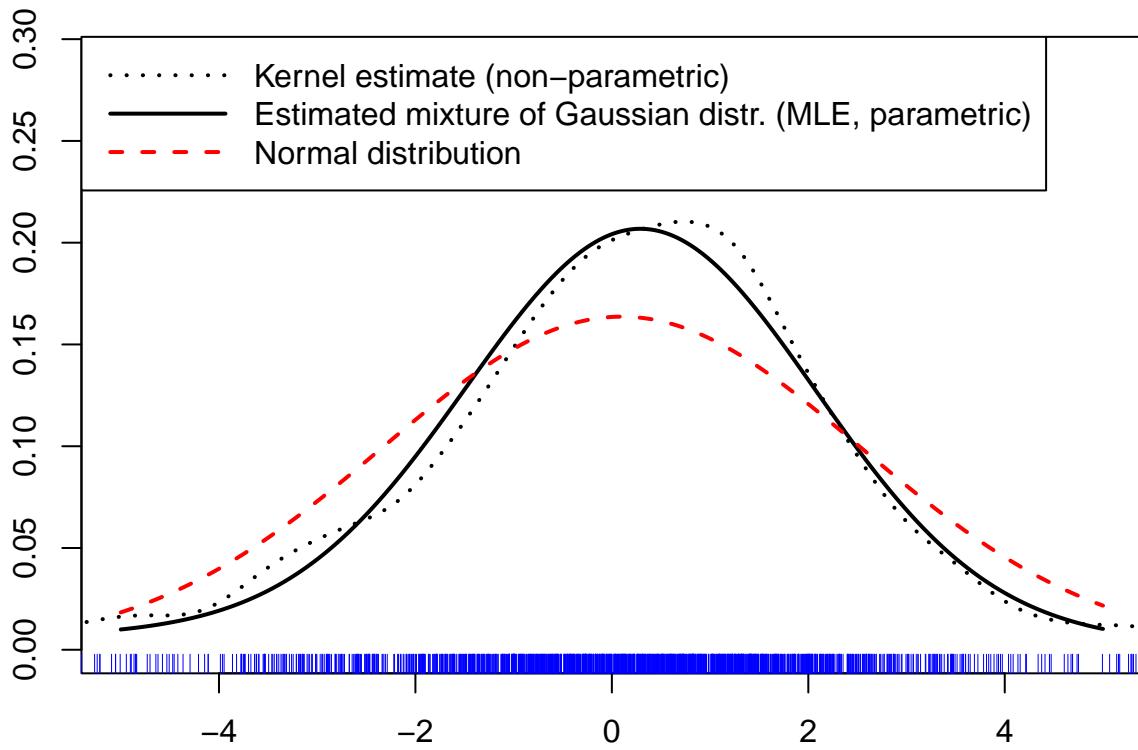


Figure 7.5: Comparison of different estimates of the distribution of returns.

### 7.2.5 Test procedures

Suppose we want to test the following parameter restrictions:

$$H_0 : \underbrace{h(\theta)}_{r \times 1} = 0. \quad (7.16)$$

In the context of MLE, three tests are largely used:

- Likelihood Ratio (LR) test,

- Wald (W) test,
- Lagrange Multiplier (LM) test.

Here is the rationale behind these three tests:<sup>2</sup>

- LR: If  $h(\theta) = 0$ , then imposing this restriction during the estimation (restricted estimator) should not result in a large decrease in the likelihood function (w.r.t the unrestricted estimation).
- Wald: If  $h(\theta) = 0$ , then  $h(\hat{\theta})$  should not be far from 0 (even if these restrictions are not imposed during the MLE).
- LM: If  $h(\theta) = 0$ , then the gradient of the likelihood function should be small when evaluated at the restricted estimator.

In terms of implementation, while the LR necessitates to estimate both restricted and unrestricted models, the Wald test requires the estimation of the unrestricted model only, and the LM tests requires the estimation of the restricted model only.

As shown below, the three test statistics associated with these three tests coincide asymptotically. (Therefore, they naturally have the same asymptotic distribution, that are  $\chi^2$ .)

**Proposition 7.5** (Asymptotic distribution of the Wald statistic). *Under regularity conditions (Assumptions 7.1) and under  $H_0 : h(\theta) = 0$ , the Wald statistic, defined by:*

$$\boxed{\xi^W = h(\hat{\theta})' \text{Var}[h(\hat{\theta})]^{-1} h(\hat{\theta}),}$$

where

$$\text{Var}[h(\hat{\theta})] = \left( \frac{\partial h(\hat{\theta})}{\partial \theta'} \right) \text{Var}[\hat{\theta}] \left( \frac{\partial h(\hat{\theta})'}{\partial \theta} \right), \quad (7.17)$$

is asymptotically  $\chi^2(r)$ , where the number of degrees of freedom  $r$  corresponds to the dimension of  $h(\theta)$ . (Note that Eq. (7.17) is the same as the one used in the Delta method, see Eq. (7.15).)

The Wald test, defined by the critical region

$$\{\xi^W \geq \chi_{1-\alpha}^2(r)\},$$

where  $\chi_{1-\alpha}^2(r)$  denotes the quantile of level  $1 - \alpha$  of the  $\chi^2(r)$  distribution, has asymptotic level  $\alpha$  and is consistent.<sup>3</sup>

*Proof.* See Appendix 10.5. □

In practice, in Eq. (7.17),  $\text{Var}[\hat{\theta}]$  is replaced by an estimate given, e.g., by Eq. (7.11), Eq. (7.12), or Eq. (7.13).

**Proposition 7.6** (Asymptotic distribution of the LM test statistic). *Under regularity conditions (Assumptions 7.1) and under  $H_0 : h(\theta) = 0$ , the LM statistic*

$$\boxed{\xi^{LM} = \left( \frac{\partial \log \mathcal{L}(\theta)}{\partial \theta'} \Big|_{\theta=\hat{\theta}^0} \right) [\mathbf{I}(\hat{\theta}^0)]^{-1} \left( \frac{\partial \log \mathcal{L}(\theta)}{\partial \theta} \Big|_{\theta=\hat{\theta}^0} \right)}, \quad (7.18)$$

(where  $\hat{\theta}^0$  is the restricted MLE estimator) is  $\chi^2(r)$ .

The test defined by the critical region:

$$\{\xi^{LM} \geq \chi_{1-\alpha}^2(r)\}$$

has asymptotic level  $\alpha$  and is consistent (see Defs. 10.7 and 10.8). This test is called Score or Lagrange Multiplier (LM) test.

<sup>2</sup>An interesting graphical presentation of the tests is proposed in Buse (1982).

<sup>3</sup>See Defs. 10.7 and 10.8 for definitions of the asymptotic levels and consistency of tests.

*Proof.* See Appendix 10.5. □

**Definition 7.8** (Likelihood Ratio test statistics). The likelihood ratio associated to a restriction of the form  $H_0 : h(\theta) = 0$  is given by:

$$LR = \frac{\mathcal{L}_R(\theta; \mathbf{y})}{\mathcal{L}_U(\theta; \mathbf{y})} \quad (\in [0, 1]),$$

where  $\mathcal{L}_R$  (respectively  $\mathcal{L}_U$ ) is the likelihood function that imposes (resp. that does not impose) the restriction. The likelihood ratio test statistic is given by  $-2 \log(LR)$ , that is:

$$\xi^{LR} = 2(\log \mathcal{L}_U(\theta; \mathbf{y}) - \log \mathcal{L}_R(\theta; \mathbf{y})).$$

**Proposition 7.7** (Asymptotic equivalence of LR, LM, and Wald tests). *Under the null hypothesis  $H_0$ , we have, asymptotically:*

$$\xi^{LM} = \xi^{LR} = \xi^W.$$

*Proof.* See Appendix 10.5. □

## 7.3 Bayesian approach

### 7.3.1 Introduction

An excellent introduction to Bayesian methods is proposed by Martin Haugh, 2017.

As suggested by the name of this approach, the starting point is the Bayes formula:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \& B)}{\mathbb{P}(B)},$$

where  $A$  and  $B$  are two “events”. For instance,  $A$  may be: parameter  $\alpha$  (conceived as something stochastic) lies in interval  $[a, b]$ . Assume that you are interested in the probability of occurrence of  $A$ . Without any specific information (or “unconditionally”), this probability is  $\mathbb{P}(A)$ . Your evaluation of this probability can only be better if you are provided with any additional form of information. Typically, if the event  $B$  tends to occur simultaneously with  $A$ , then knowledge of  $B$  can be useful. The Bayes formula says how this additional information (on  $B$ ) can be used to “update” the probability of event  $A$ .

In our case, this intuition will work as follows: assume that you know the form of the data-generating process (DGP). That is, you know the structure of the model used to draw some stochastic data; you also know the type of distributions used to generate these data. However, you do not know the numerical values of all the parameters characterizing the DGP. Let us denote by  $\theta$  the vector of unknown parameters. While these parameters are not known exactly, assume that we have –even without having observed any data– some **priors** on their distribution. Then, as was the case in the example above (with  $A$  and  $B$ ), the observation of data generated by the model can only reduce the uncertainty associated with  $\theta$ . Loosely speaking, combining the priors and the observations of data generated by the model should result in “thinner” distributions for the components of  $\theta$ . The latter distributions are called the **posterior distributions**.<sup>4</sup>

Let us formalize this intuition. Define the prior by  $f_\theta(\theta)$  and the model realizations (the “data”) by vector  $\mathbf{y}$ . The joint distribution of  $(\mathbf{y}, \theta)$  is given by:

$$f_{Y,\theta}(\mathbf{y}, \theta) = f_{Y|\theta}(\mathbf{y}, \theta)f_\theta(\theta),$$

and, symmetrically, by

$$f_{Y,\theta}(\mathbf{y}, \theta) = f_{\theta|Y}(\theta, \mathbf{y})f_Y(\mathbf{y}),$$

---

<sup>4</sup>The output of the Bayesian approach will be the (posterior) distribution of the vector of parameters  $(\theta)$ . When we speak about the *distributions of the components of  $\theta$* , we mean the marginal distributions of each component of the vector.

where  $f_{\theta|Y}(\cdot, \mathbf{y})$ , the distribution of the parameters conditional on the observations, is the **posterior** distribution.

The last two equations imply that:

$$f_{\theta|Y}(\theta, \mathbf{y}) = \frac{f_{Y|\theta}(\mathbf{y}, \theta) f_\theta(\theta)}{f_Y(\mathbf{y})}. \quad (7.19)$$

Note that  $f_Y$  is the marginal (or unconditional) distribution of  $\mathbf{y}$ , that can be written:

$$f_Y(\mathbf{y}) = \int f_{Y|\theta}(\mathbf{y}, \theta) f_\theta(\theta) d\theta. \quad (7.20)$$

Eq. (7.19) is sometimes rewritten as follows:

$$f_{\theta|Y}(\theta, \mathbf{y}) \propto f_{\theta,Y}(\theta, \mathbf{y}) := f_{Y|\theta}(\mathbf{y}, \theta) f_\theta(\theta), \quad (7.21)$$

where  $\propto$  means, loosely speaking, “*proportional to*”. In rare instances, starting from given priors, one can analytically compute the posterior distribution  $f_\theta(\theta, \mathbf{y})$ . However, in most cases, this is out of reach. One then has to resort to numerical approaches to compute the posterior distribution. Monte Carlo Markov Chains (MCMC) is one of them.

According to the Bernstein-von Mises Theorem, Bayesian and MLE estimators have the same large sample properties. (In particular, the Bayesian approach also achieve the FDCR bound, see Theorem 7.1.) The intuition behind this result is that the influence of the prior diminishes with increasing sample sizes.

### 7.3.2 Monte-Carlo Markov Chains

MCMC techniques aim at using simulations to approach a distribution whose distribution is difficult to obtain analytically. Indeed, in some circumstances, one can draw in a distribution even if we do not know its analytical expression.

**Definition 7.9** (Markov Chain). The sequence  $\{z_i\}$  is said to be a (first-order) Markovian process if it satisfies:

$$f(z_i | z_{i-1}, z_{i-2}, \dots) = f(z_i | z_{i-1}).$$

The Metropolis-Hastings (MH) algorithm is a specific MCMC approach that allows to generate samples of  $\theta$ 's whose distribution approximately corresponds to the posterior distribution of Eq. (7.19).

The MH algorithm is a recursive algorithm. That is, one can draw the  $i^{th}$  value of  $\theta$ , denoted by  $\theta_i$ , if one has already drawn  $\theta_{i-1}$ . Assume we have  $\theta_{i-1}$ . We obtain a value for  $\theta_i$  by implementing the following steps:

1. Draw  $\tilde{\theta}_i$  from the conditional distribution  $Q_{\tilde{\theta}|\theta}(\cdot, \theta_{i-1})$ , called **proposal distribution**.
2. Draw  $u$  in a uniform distribution on  $[0, 1]$ .
3. Compute

$$\alpha(\tilde{\theta}_i, \theta_{i-1}) := \min \left( \frac{f_{\theta,Y}(\tilde{\theta}_i, \mathbf{y})}{f_{\theta,Y}(\theta_{i-1}, \mathbf{y})} \times \frac{Q_{\tilde{\theta}|\theta}(\theta_{i-1}, \tilde{\theta}_i)}{Q_{\tilde{\theta}|\theta}(\tilde{\theta}_i, \theta_{i-1})}, 1 \right), \quad (7.22)$$

where  $f_{\theta,Y}$  is given in Eq. (7.21).

4. If  $u < \alpha(\tilde{\theta}_i, \theta_{i-1})$ , then take  $\theta_i = \tilde{\theta}_i$ , otherwise we leave  $\theta_i$  equal to  $\theta_{i-1}$ .

It can be shown that, the distribution of the draws converges to the posterior distribution. That is, after a sufficiently large number of iterations, the draws can be considered to be drawn from the posterior distribution.<sup>5</sup>

To get some insights into the algorithm, consider the case of a **symmetric proposal distribution**, that is:

$$Q_{\tilde{\theta}|\theta}(\tilde{\theta}_i, \theta_{i-1}) = Q_{\tilde{\theta}|\theta}(\theta_{i-1}, \tilde{\theta}_i). \quad (7.23)$$

---

<sup>5</sup>The proof of this claim is based on the fact that, if  $\theta_{i-1}$  is drawn from the posterior distribution, then it is also the case for  $\theta_i$ .

We then have:

$$\alpha(\tilde{\theta}, \theta_{i-1}) = \min \left( \frac{q(\tilde{\theta}, y)}{q(\theta_{i-1}, y)}, 1 \right). \quad (7.24)$$

Remember that, up to the marginal distribution of the data ( $f_Y(\mathbf{y})$ ),  $f_{\theta, Y}(\tilde{\theta}, \mathbf{y})$  is the probability of observing  $\mathbf{y}$  conditional on having a model parameterized by  $\tilde{\theta}$ . Then, under Eq. (7.24), it appears that if this probability is larger for  $\tilde{\theta}$  than for  $\theta_{i-1}$  (in which case  $\tilde{\theta}$  seems “more consistent with the observations  $\mathbf{y}$ ” than  $\theta_{i-1}$ ), we accept  $\tilde{\theta}$ . By contrast, if  $f_{\theta, Y}(\tilde{\theta}, \mathbf{y}) < f_{\theta, Y}(\theta_{i-1}, \mathbf{y})$ , then we do not necessarily accept the proposed value  $\tilde{\theta}$ , especially if  $f_{\theta, Y}(\tilde{\theta}, \mathbf{y}) \ll f_{\theta, Y}(\theta_{i-1}, \mathbf{y})$  (in which case  $\tilde{\theta}$  seems far less consistent with the observations  $\mathbf{y}$  than  $\theta_{i-1}$ , and, accordingly, the acceptance probability, namely  $\alpha(\tilde{\theta}, \theta_{i-1})$ , is small).

The choice of the **proposal distribution**  $Q_{\tilde{\theta}|\theta}$  is crucial to get a rapid convergence of the algorithm. Looking at Eq. (7.22), it is easily seen that the optimal choice would be  $Q_{\tilde{\theta}|\theta}(\cdot, \theta_i) = f_{\theta|Y}(\cdot, \mathbf{y})$ . In that case, we would have  $\alpha(\tilde{\theta}_i, \theta_{i-1}) \equiv 1$  (see Eq. (7.22)). We would then accept all draws from the proposal distribution, as this distribution would directly be the posterior distribution. Of course, this situation is not realistic as the objective of the algorithm is precisely to approximate the posterior distribution.

A common choice for  $Q$  is a multivariate normal distribution. If  $\theta$  is of dimension  $K$ , we can for instance use:

$$Q(\tilde{\theta}, \theta) = \frac{1}{(\sqrt{2\pi\sigma^2})^K} \exp \left( -\frac{1}{2} \sum_{j=1}^K \frac{(\tilde{\theta}_j - \theta_j)^2}{\sigma^2} \right),$$

which is an example of symmetric proposal distribution (see Eq. (7.23)). Equivalently, we then have:

$$\tilde{\theta} = \theta + \varepsilon,$$

where  $\varepsilon$  is a  $K$ -dimensional vector of independent zero-mean normal disturbances of variance  $\sigma^2$ .<sup>6</sup> One then has to determine an appropriate value for  $\sigma$ . If it is too low, then  $\alpha$  will be close to 1 (as  $\tilde{\theta}_i$  will be close to  $\theta_{i-1}$ ), and we will accept very often the proposed value ( $\tilde{\theta}_i$ ). This seems to be a favourable situation. But it may not be. Indeed, it means that it will take a large number of iterations to explore the whole distribution of  $\theta$ . What if  $\sigma$  is very large? In this case, it is likely that the proposed values ( $\tilde{\theta}_i$ ) will often result in poor likelihoods; The probability of acceptance will then be low and the Markov chain may be blocked at its initial value. Therefore, intermediate values of  $\sigma^2$  have to be determined. The acceptance rate (i.e., the average value of  $\alpha(\tilde{\theta}, \theta_{i-1})$ ) can be used as a guide for that. Indeed, a literature explores the optimal values for such acceptance rate (in order to obtain the best possible fit of the posterior for a minimum number of algorithm iterations). In particular, following Roberts et al. (1997), people often target acceptance rate of the order of magnitude of 20%.

It is important to note that, to implement this approach, one only has to be able to compute the joint p.d.f.  $q(\theta, \mathbf{y}) = f_{Y|\theta}(\mathbf{y}, \theta) f_\theta(\theta)$  (Eq. (7.21)). That is, as soon as one can evaluate the likelihood ( $f_{Y|\theta}(\mathbf{y}, \theta)$ ) and the prior ( $f_\theta(\theta)$ ), we can employ this methodology.

### 7.3.3 Example: AR(1) specification

In the following example, we employ MCMC in order to estimate the posterior distributions of the three parameters defining an AR(1) model (see Section 9.2.2). The specification is as follows:

$$y_t = \mu + \rho y_{t-1} + \sigma \varepsilon_t, \quad \varepsilon_t \sim i.i.d. \mathcal{N}(0, 1).$$

Hence, we have  $\theta = [\mu, \rho, \sigma]$ . Let us first simulate the process on  $T$  periods:

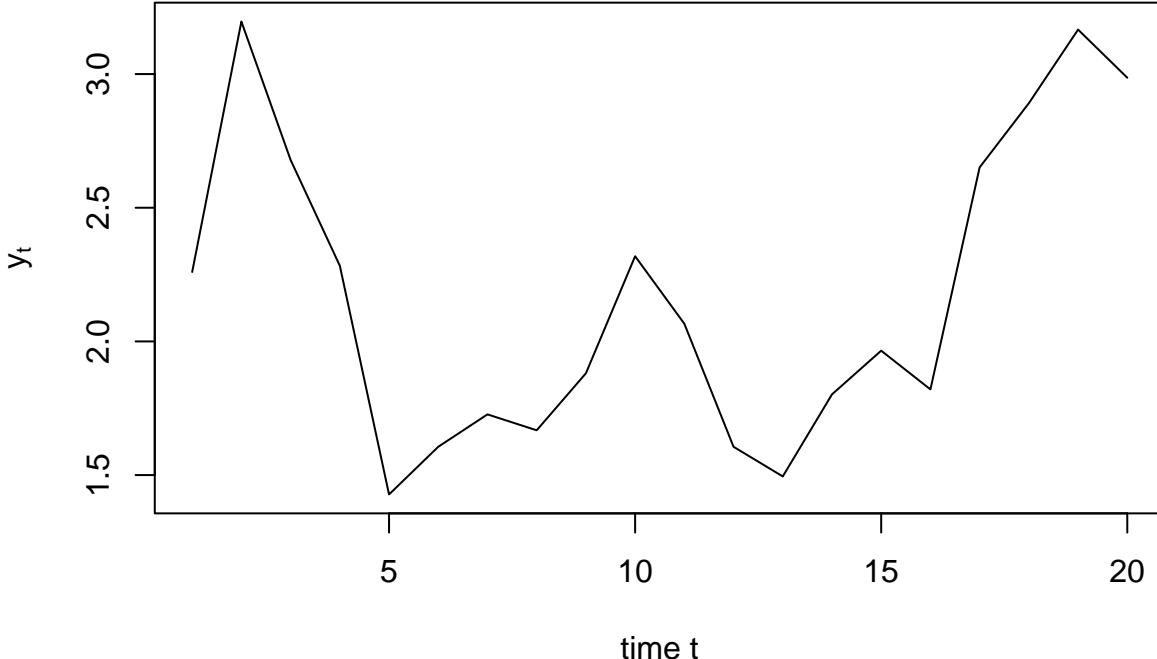
```
mu <- .6; rho <- .8; sigma <- .5 # true model specification
T <- 20 # number of observations
y0 <- mu/(1-rho)
```

<sup>6</sup>We could also have different variances for the different components of  $\theta$ . However, this may lead to complicated settings. A useful practice consists in looking for model (re)parametrization –based, e.g., on exponential and/or logistic functions– that are such that the components of  $\theta$  are all expected to be of the order of magnitude of the unity.

```

Y <- NULL
for(t in 1:T){
  if(t==1){y <- y0}
  y <- mu + rho*y + sigma * rnorm(1)
  Y <- c(Y,y)}
plot(Y,type="l",xlab="time t",ylab=expression(y[t]))

```



Next, let us write the likelihood function, i.e.  $f_{Y|\theta}(y, \theta)$ . For  $\rho$ , which is expected to be between 0 and 1, we use a logistic transformation. For  $\sigma$ , that is expected to be positive, we use an exponential transformation.

```

likelihood <- function(param,Y){
  mu <- param[1]
  rho <- exp(param[2])/(1+exp(param[2]))
  sigma <- exp(param[3])
  MU <- mu/(1-rho)
  SIGMA2 <- sigma^2/(1-rho^2)
  L <- 1/sqrt(2*pi*SIGMA2)*exp(-(Y[1]-MU)^2/(2*SIGMA2))
  Y1 <- Y[2:length(Y)]
  Y0 <- Y[1:(length(Y)-1)]
  aux <- 1/sqrt(2*pi*sigma^2)*exp(-(Y1-mu-rho*Y0)^2/(2*sigma^2))
  L <- L * prod(aux)
  return(L)
}

```

Next define function  $rQ$  that draws from the (Gaussian) proposal distribution, as well as function  $Q$ , that computes  $Q_{\tilde{\theta}|\theta}(\tilde{\theta}, \theta)$ :

```

rQ <- function(x,a){
  n <- length(x)
  y <- x + a * rnorm(n)
  return(y)}
Q <- function(y,x,a){
  q <- 1/sqrt(2*pi*a^2)*exp(-(y - x)^2/(2*a^2))
  return(prod(q))}

```

We consider Gaussian priors:

```

prior <- function(param,means_prior,stdv_prior){
  f <- 1/sqrt(2*pi*stdv_prior^2)*exp(-(param -
                                             means_prior)^2/(2*stdv_prior^2))
  return(prod(f))}

```

Function `p_tilde` corresponds to  $f_{\theta,Y}$ :

```
p_tilde <- function(param,Y,means_prior,stdv_prior){
  p <- likelihood(param,Y) * prior(param,means_prior,stdv_prior)
  return(p)}
```

We can now define function  $\alpha$  (Eq. (7.22)):

```
alpha <- function(y,x,means_prior,stdv_prior,a){
  aux <- p_tilde(y,Y,means_prior,stdv_prior)/
    p_tilde(x,Y,means_prior,stdv_prior) * Q(y,x,a)/Q(x,y,a)
  alpha_proba <- min(aux,1)
  return(alpha_proba)}
```

Now, all is set for us to write the MCMC function:

```
MCMC <- function(Y,means_prior,stdv_prior,a,N){
  x <- means_prior
  all_theta <- NULL
  count_accept <- 0
  for(i in 1:N){
    y <- rQ(x,a)
    alph <- alpha(y,x,means_prior,stdv_prior,a)
    #print(alph)
    u <- runif(1)
    if(u < alph){
      count_accept <- count_accept + 1
      x <- y}
    all_theta <- rbind(all_theta,x)}
  print(paste("Acceptance rate:",toString(round(count_accept/N,3))))
  return(all_theta)}
```

Specify the Gaussian priors:

```
true_values <- c(mu=log(rho/(1-rho)),log(sigma))
means_prior <- c(1,0,0) # as if we did not know the true values
stdv_prior <- rep(2,3)
resultMCMC <- MCMC(Y,means_prior,stdv_prior,a=.45,N=20000)

## [1] "Acceptance rate: 0.098"

par(mfrow=c(2,3))
for(i in 1:length(means_prior)){
  m <- means_prior[i]
  s <- stdv_prior[i]
  x <- seq(m-3*s,m+3*s,length.out = 100)
  par(mfg=c(1,i))
  aux <- density(resultMCMC[,i])
  par(plt=c(.15,.95,.15,.85))
  plot(x,dnorm(x,m,s),type="l",xlab="",ylab="",main=paste("Parameter",i),
    ylim=c(0,max(aux$y)))
  lines(aux$x,aux$y,col="red",lwd=2)
  abline(v=true_values[i],lty=2,col="blue")
  par(mfg=c(2,i))
  plot(resultMCMC[,i],1:length(resultMCMC[,i]),xlim=c(min(x),max(x)),
    type="l",xlab="",ylab="")}
```

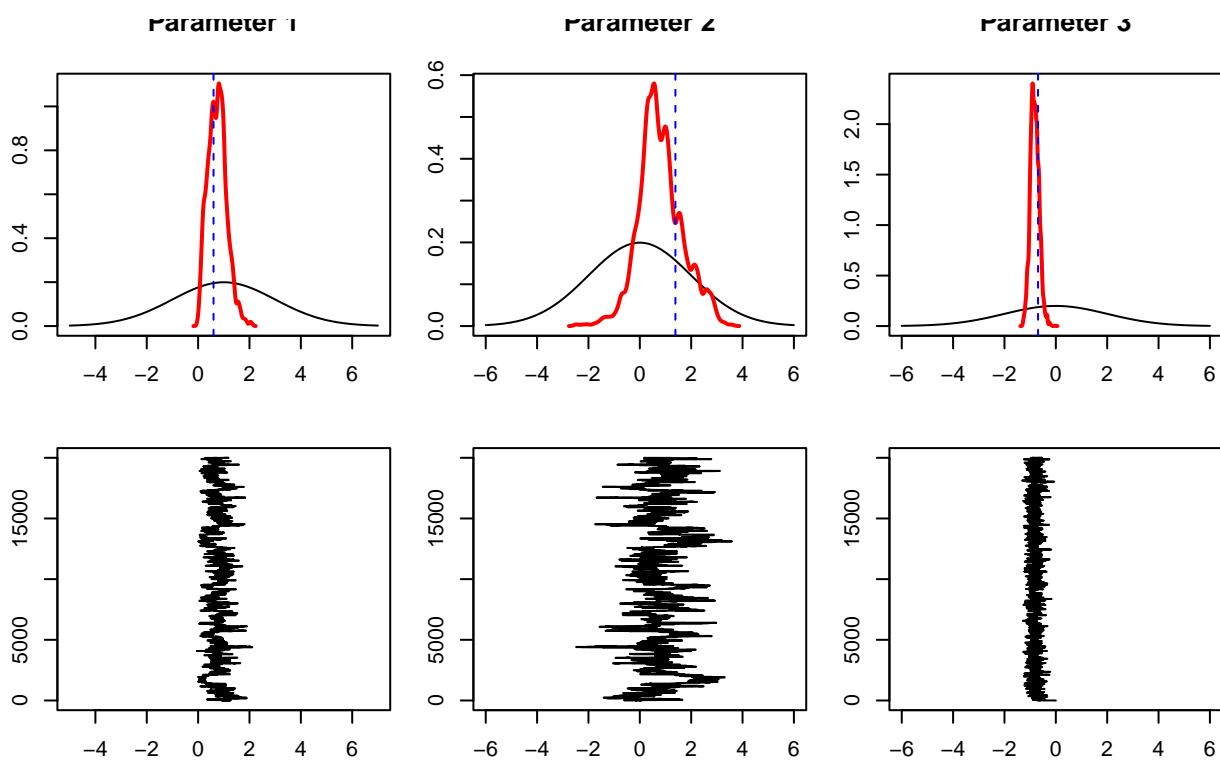


Figure 7.6: The upper line of plot compares prior (black) and posterior (red) distributions. The vertical dashed blue lines indicate the true values of the parameters. The second row of plots show the sequence of  $\theta_i$ 's generated by the MCMC algorithm. These sequences are the ones used to produce the posterior distributions (red lines) in the upper plots.

# Chapter 8

## Microeometrics

In microeconomic models, the variables of interest often feature restricted distributions—for instance with discontinuous support—which necessitates specific models. Typical examples are discrete-choice models (binary, multinomial, ordered outcomes), sample selection models (censored or truncated outcomes), and count-data models (integer outcomes). This chapter describes the estimation and interpretation of these models. It also shows how the discrete-choice models can emerge from (structural) random-utility frameworks.

### 8.1 Binary-choice models

In many instances, the variables to be explained (the  $y_i$ 's) have only two possible values (0 and 1, say). That is, they are binary variables. The probability for them to be equal to either 0 or 1 may depend on independent variables, gathered in vectors  $\mathbf{x}_i$  ( $K \times 1$ ).

The spectrum of applications is wide:

- Binary decisions (e.g. in referendums, being owner or renter, living in the city or in the countryside, in/out of the labour force,...),
- Contamination (disease or default),
- Success/failure (exams).

Without loss of generality, the model reads:

$$y_i | \mathbf{X} \sim \mathcal{B}(g(\mathbf{x}_i; \theta)), \quad (8.1)$$

where  $g(\mathbf{x}_i; \theta)$  is the parameter of the Bernoulli distribution. In other words, conditionally on  $\mathbf{X}$ :

$$y_i = \begin{cases} 1 & \text{with probability } g(\mathbf{x}_i; \theta) \\ 0 & \text{with probability } 1 - g(\mathbf{x}_i; \theta), \end{cases} \quad (8.2)$$

where  $\theta$  is a vector of parameters to be estimated.

An estimation strategy is to assume that  $g(\mathbf{x}_i; \theta)$  can be proxied by  $\tilde{\theta}' \mathbf{x}_i$  and to run a linear regression to estimate  $\tilde{\theta}$  (a situation called **Linear Probability Model, LPM**):

$$y_i = \tilde{\theta}' \mathbf{x}_i + \varepsilon_i.$$

Notwithstanding the fact that this specification does not exclude negative probabilities or probabilities greater than one, it could be compatible with the *assumption of zero conditional mean* (Hypothesis 5.2) and with the *assumption of non-correlated residuals* (Hypothesis 5.4), but more difficultly with the *homoskedasticity assumption* (Hypothesis 5.3). Moreover, the  $\varepsilon_i$ 's cannot be Gaussian (because  $y_i \in \{0, 1\}$ ). Hence, using a linear regression to study the relationship between  $\mathbf{x}_i$  and  $y_i$  can be consistent but it is inefficient.

Figure 8.1 illustrates the fit resulting from an application of the LPM model to binary (dependent) variables.

Except for its last row (LPM case), Table 8.1 provides examples of functions  $g$  valued in  $[0, 1]$ , and that can therefore be used in models of the type:  $\mathbb{P}(y_i = 1 | \mathbf{x}_i; \theta) = g(\theta' \mathbf{x}_i)$  (see

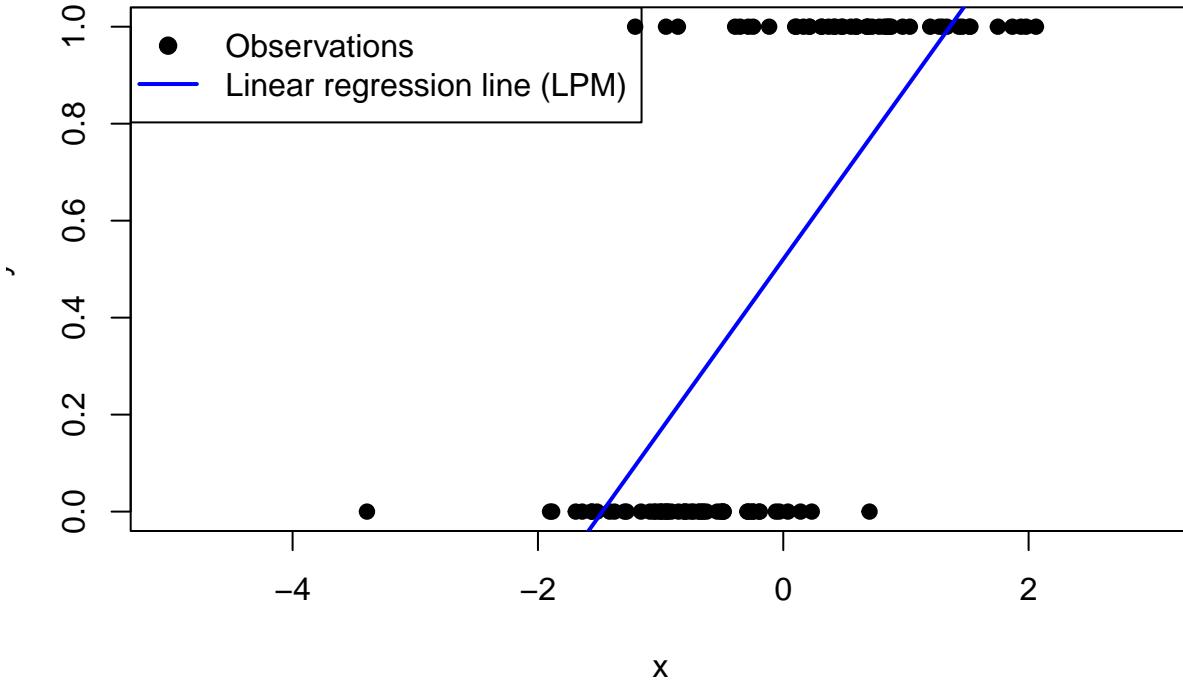


Figure 8.1: Fitting a binary variable with a linear model (Linear Probability Model, LPM). The model is  $\mathbb{P}(y_i = 1|x_i) = \Phi(0.5 + 2x_i)$ , where  $\Phi$  is the c.d.f. of the normal distribution and where  $x_i \sim i.i.d. \mathcal{N}(0, 1)$ .

Eq. (8.2)). The “linear” case is given for comparison, but note that it does not satisfy  $g(\theta' \mathbf{x}_i) \in [0, 1]$  for any value of  $\theta' \mathbf{x}_i$ .

Table 8.1: This table provides examples of function  $g$ , s.t.  $\mathbb{P}(y_i = 1|\mathbf{x}_i; \text{heta}) = g(\theta' \mathbf{x}_i)$ . The LPM case (last row) is given for comparison but, again, it does not satisfy  $g(\theta' \mathbf{x}_i) \in [0, 1]$  for any value of  $\theta' \mathbf{x}_i$ .

Model	Function $g$	Derivative
Probit	$\Phi$	$\phi$
Logit	$\frac{\exp(x)}{1 + \exp(x)}$	$\frac{\exp(x)}{(1 + \exp(x))^2}$
log-log	$1 - \exp(-\exp(x))$	$\exp(-\exp(x)) \exp(x)$
linear (LPM)	$x$	1

Figure 8.2 displays the first three  $g$  functions appearing in Table 8.1.

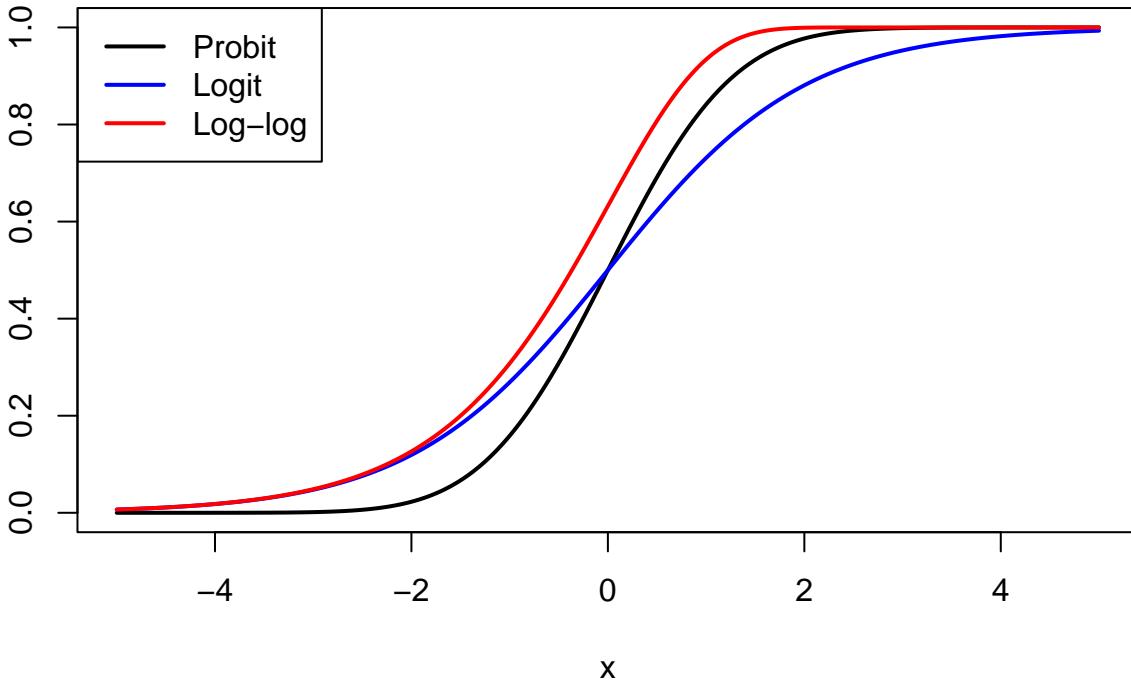


Figure 8.2: Probit, Logit, and Log-log functions.

The **probit** and the **logit** models are popular binary-choice models. In the probit model, we have:

$$g(z) = \Phi(z), \quad (8.3)$$

where  $\Phi$  is the c.d.f. of the normal distribution. And for the logit model:

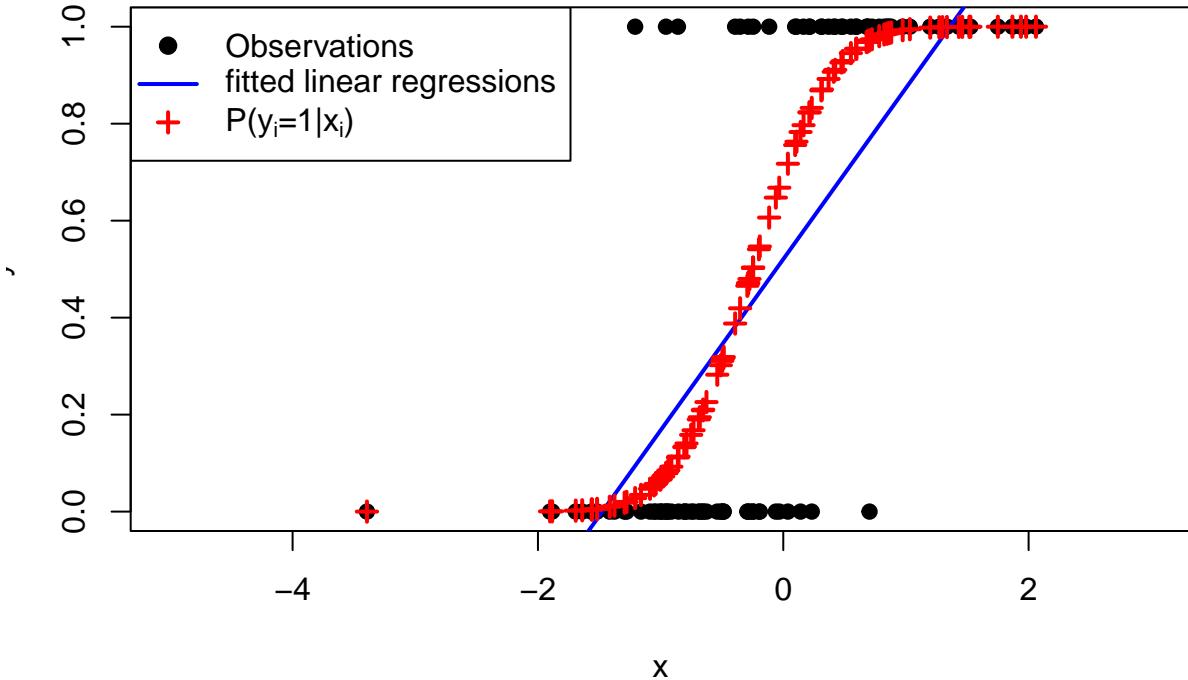


Figure 8.3: The model is  $\mathbb{P}(y_i = 1|x_i) = \Phi(0.5 + 2x_i)$ , where  $\Phi$  is the c.d.f. of the normal distribution and where  $x_i \sim i.i.d. \mathcal{N}(0, 1)$ . Crosses give the model-implied probabilities of having  $y_i = 1$  (conditional on  $x_i$ ).

In the probit model, we have:

$$\mathbb{P}(y_i = 1|\mathbf{x}_i; \theta) = \Phi(\theta' \mathbf{x}_i) = \mathbb{P}(-\varepsilon_i < \theta' \mathbf{x}_i),$$

where  $\varepsilon_i \sim \mathcal{N}(0, 1)$ . That is:

$$\mathbb{P}(y_i = 1|\mathbf{x}_i; \theta) = \mathbb{P}(0 < y_i^*),$$

where  $y_i^* = \theta' \mathbf{x}_i + \varepsilon_i$ , with  $\varepsilon_i \sim \mathcal{N}(0, 1)$ . Variable  $y_i^*$  can be interpreted as a (latent) variable that determines  $y_i$  (since  $y_i = \mathbb{I}_{\{y_i^* > 0\}}$ ).

Figure 8.4 illustrates this situation.

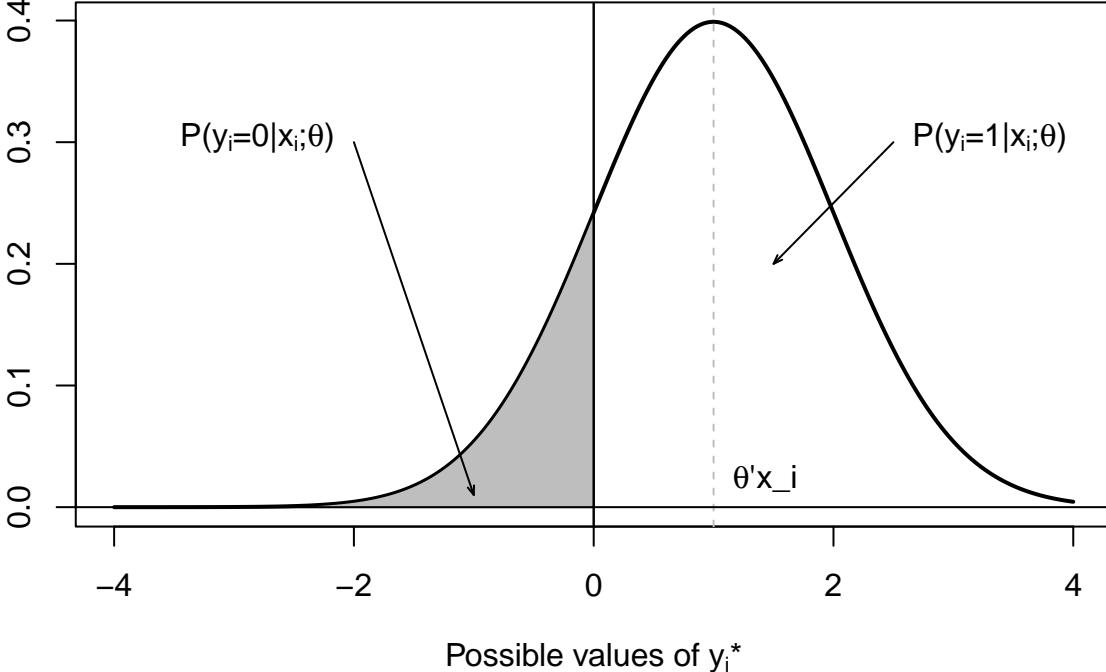


Figure 8.4: Distribution of  $y_i^*$  conditional on  $\mathbf{x}_i$ .

Assume that agent ( $i$ ) chooses  $y_i = 1$  if the utility associated with this choice ( $U_{i,1}$ ) is higher than the one associated with  $y_i = 0$  (that is  $U_{i,0}$ ). Assume further that the utility of agent  $i$ , if she chooses outcome  $j$  ( $\in \{0, 1\}$ ), is given by

$$U_{i,j} = V_{i,j} + \varepsilon_{i,j},$$

where  $V_{i,j}$  is the deterministic component of the utility associated with choice and where  $\varepsilon_{i,j}$  is a random (agent-specific) component. Moreover, posit  $V_{i,j} = \theta'_j \mathbf{x}_i$ . We then have:

$$\begin{aligned}\mathbb{P}(y_i = 1 | \mathbf{x}_i; \theta) &= \mathbb{P}(\theta'_1 \mathbf{x}_i + \varepsilon_{i,1} > \theta'_0 \mathbf{x}_i + \varepsilon_{i,0}) \\ &= F(\theta'_1 \mathbf{x}_i - \theta'_0 \mathbf{x}_i) = F([\theta_1 - \theta_0]'\mathbf{x}_i),\end{aligned}\quad (8.5)$$

where  $F$  is the c.d.f. of  $\varepsilon_{i,0} - \varepsilon_{i,1}$ .

Note that only the difference  $\theta_1 - \theta_0$  is identifiable (as opposed to  $\theta_1$  and  $\theta_0$ ). Indeed, replacing  $U$  with  $aU$  ( $a > 0$ ) gives the same model. This *scaling* issue can be solved by fixing the variance of  $\varepsilon_{i,0} - \varepsilon_{i,1}$ .

**Example 8.1** (Migration and income). The RUM approach has been used by Nakosteen and Zimmer (1980) to study migration choices. Their model is based on the comparison of marginal costs and benefits associated with migration. The main ingredients of their approach are as follows:

- Wage that can be earned at the present location:  $y_p^* = \theta'_p \mathbf{x}_p + \varepsilon_p$ .
- Migration cost:  $C^* = \theta'_c \mathbf{x}_c + \varepsilon_c$ .
- Wage earned elsewhere:  $y_m^* = \theta'_m \mathbf{x}_m + \varepsilon_m$ .

In this context, agents decision to migrate if  $y_m^* > y_p^* + C^*$ , i.e. if

$$y^* = y_m^* - y_p^* - C^* = \theta' \mathbf{x} + \underbrace{\varepsilon}_{=\varepsilon_m - \varepsilon_c - \varepsilon_p} > 0,$$

where  $\mathbf{x}$  is the union of the  $\mathbf{x}_i$ s, for  $i \in \{p, m, c\}$ .

### 8.1.2 Alternative-Varying Regressors

In some cases, regressors may depend on the considered alternative (0 or 1). For instance:

- When modeling the decision to participate in the labour force (or not), the wage depends on the alternative. Typically, it is zero if the considered agent has decided not to work (and strictly positive otherwise).
- In the context of the choice of transportation mode, “time cost” depends on the considered transportation mode.

In terms of utility, we then have:

$$V_{i,j} = \theta_j^{(u)'} \mathbf{u}_{i,j} + \theta_j^{(v)'} \mathbf{v}_i,$$

where the  $\mathbf{u}_{i,j}$ 's are regressors associated with agent  $i$ , but taking different values for the different choices ( $j = 0$  or  $j = 1$ ). In that case, Eq. (8.5) becomes:

$$\mathbb{P}(y_i = 1 | \mathbf{x}_i; \theta) = F\left(\theta_1^{(u)'} \mathbf{u}_{i,1} - \theta_0^{(u)'} \mathbf{u}_{i,0} + [\theta_1^{(v)} - \theta_0^{(v)}]'\mathbf{v}_i\right), \quad (8.6)$$

and, if  $\theta_1^{(u)} = \theta_0^{(u)} = \theta^{(u)}$  —as is customary— we get:

$$\mathbb{P}(y_i = 1 | \mathbf{x}_i; \theta) = F\left(\theta_1^{(u)'} (\mathbf{u}_{i,1} - \mathbf{u}_{i,0}) + [\theta_1^{(v)} - \theta_0^{(v)}]'\mathbf{v}_i\right). \quad (8.7)$$

**Example 8.2** (Fishing-mode dataset). The fishing-mode dataset used in Cameron and Trivedi (2005) (Chapters 14 and 15) contains alternative-specific variables. Specifically, for each individual, the price and catch rate depend on the fishing model. In the table reported below, lines `price` and `catch` correspond to the prices and catch rates associated with the chosen alternative.

```

library(mlogit)
data("Fishing", package="mlogit")
stargazer::stargazer(Fishing, type="text")

## 
## =====
## Statistic      N     Mean   St. Dev.    Min    Max
## -----
## price.beach  1,182 103.422 103.641  1.290 843.186
## price.pier   1,182 103.422 103.641  1.290 843.186
## price.boat   1,182  55.257  62.713  2.290 666.110
## price.charter 1,182  84.379  63.545  27.290 691.110
## catch.beach  1,182  0.241   0.191  0.068  0.533
## catch.pier   1,182  0.162   0.160  0.001  0.452
## catch.boat   1,182  0.171   0.210  0.0002 0.737
## catch.charter 1,182  0.629   0.706  0.002  2.310
## income        1,182 4,099.337 2,461.964 416.667 12,500.000
## -----

```

### 8.1.3 Estimation

These models can be estimated by Maximum Likelihood approaches (see Section 7.2).

To simplify the exposition, we consider the  $\mathbf{x}_i$  vectors of covariates to be deterministic. Moreover, we assume that the r.v. are independent across entities  $i$ . How to write the likelihood in that case? It is easily checked that:

$$f(y_i|\mathbf{x}_i; \theta) = g(\theta' \mathbf{x}_i)^{y_i} (1 - g(\theta' \mathbf{x}_i))^{1-y_i}.$$

Therefore, if the observations  $(\mathbf{x}_i, y_i)$  are independent across entities  $i$ , we obtain:

$$\log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X}) = \sum_{i=1}^n y_i \log[g(\theta' \mathbf{x}_i)] + (1 - y_i) \log[1 - g(\theta' \mathbf{x}_i)].$$

The likelihood equation reads (FOC of the optimization program, see Def. 7.7):

$$\frac{\partial \log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X})}{\partial \theta} = \mathbf{0},$$

that is:

$$\sum_{i=1}^n y_i \mathbf{x}_i \frac{g'(\theta' \mathbf{x}_i)}{g(\theta' \mathbf{x}_i)} - (1 - y_i) \mathbf{x}_i \frac{g'(\theta' \mathbf{x}_i)}{1 - g(\theta' \mathbf{x}_i)} = \mathbf{0}.$$

This is a nonlinear (multivariate) equation that can be solved numerically. Under regularity conditions (Hypotheses 7.1), we approximately have (Prop. 7.4):

$$\theta_{MLE} \sim \mathcal{N}(\theta_0, \mathbf{I}(\theta_0)^{-1}),$$

where

$$\mathbf{I}(\theta_0) = -\mathbb{E}_0 \left( \frac{\partial^2 \log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X})}{\partial \theta \partial \theta'} \right) = n \mathcal{J}_Y(\theta_0).$$

For finite samples, we can e.g. approximate  $\mathbf{I}(\theta_0)^{-1}$  by Eq. (7.11):

$$\mathbf{I}(\theta_0)^{-1} \approx - \left( \frac{\partial^2 \log \mathcal{L}(\theta_{MLE}; \mathbf{y}, \mathbf{X})}{\partial \theta \partial \theta'} \right)^{-1}.$$

In the Probit case (see Table 8.1), it can be shown that we have:

$$\begin{aligned} \frac{\partial^2 \log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X})}{\partial \theta \partial \theta'} &= - \sum_{i=1}^n g'(\theta' \mathbf{x}_i) [\mathbf{x}_i \mathbf{x}_i'] \times \\ &\left[ y_i \frac{g'(\theta' \mathbf{x}_i) + \theta' \mathbf{x}_i g(\theta' \mathbf{x}_i)}{g(\theta' \mathbf{x}_i)^2} + (1 - y_i) \frac{g'(\theta' \mathbf{x}_i) - \theta' \mathbf{x}_i (1 - g(\theta' \mathbf{x}_i))}{(1 - g(\theta' \mathbf{x}_i))^2} \right]. \end{aligned}$$

In the Logit case (see Table 8.1), it can be shown that we have:

$$\frac{\partial^2 \log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X})}{\partial \theta \partial \theta'} = - \sum_{i=1}^n g'(\theta' \mathbf{x}_i) \mathbf{x}_i \mathbf{x}_i'$$

where  $g'(x) = \frac{\exp(-x)}{(1 + \exp(-x))^2}$ .

Remark that, since  $g'(x) > 0$ ,  $-\partial^2 \log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X}) / \partial \theta \partial \theta'$  is positive definite.

### 8.1.4 Marginal effects

How to measure marginal effects, i.e. the effect on the probability that  $y_i = 1$  of a marginal increase of  $x_{i,k}$ ? This object is given by:

$$\frac{\partial \mathbb{P}(y_i = 1 | \mathbf{x}_i; \theta)}{\partial x_{i,k}} = \underbrace{g'(\theta' \mathbf{x}_i)}_{>0} \theta_k,$$

which is of the same sign as  $\theta_k$  if function  $g$  is monotonously increasing.

For agent  $i$ , this marginal effect is consistently estimated by  $g'(\hat{\theta}'_{MLE} \mathbf{x}_i) \hat{\theta}_{MLE,k}$ . It is important to see that the marginal effect depends on  $\mathbf{x}_i$ : respective increases by 1 unit of  $x_{i,k}$  (entity  $i$ ) and of  $x_{j,k}$  (entity  $j$ ) do not necessarily have the same effect on  $\mathbb{P}(y_i = 1 | \mathbf{x}_i; \theta)$  as on  $\mathbb{P}(y_j = 1 | \mathbf{x}_j; \theta)$ . To address this issue, one can compute some measures of “average” marginal effect. There are two main solutions. For each explanatory variable  $k$ :

- i. Denoting by  $\hat{\mathbf{x}}$  the sample average of the  $\mathbf{x}_i$ s, compute  $g'(\hat{\theta}'_{MLE} \hat{\mathbf{x}}) \hat{\theta}_{MLE,k}$ .
- ii. Compute the average (across  $i$ ) of  $g'(\hat{\theta}'_{MLE} \mathbf{x}_i) \hat{\theta}_{MLE,k}$ .

### 8.1.5 Goodness of fit

There is no obvious version of “ $R^2$ ” for binary-choice models. Existing measures are called **pseudo- $R^2$  measures**.

Denoting by  $\log \mathcal{L}_0(\mathbf{y})$  the (maximum) log-likelihood that would be obtained for a model containing only a constant term (i.e. with  $\mathbf{x}_i = 1$  for all  $i$ ), the McFadden’s pseudo- $R^2$  is given by:

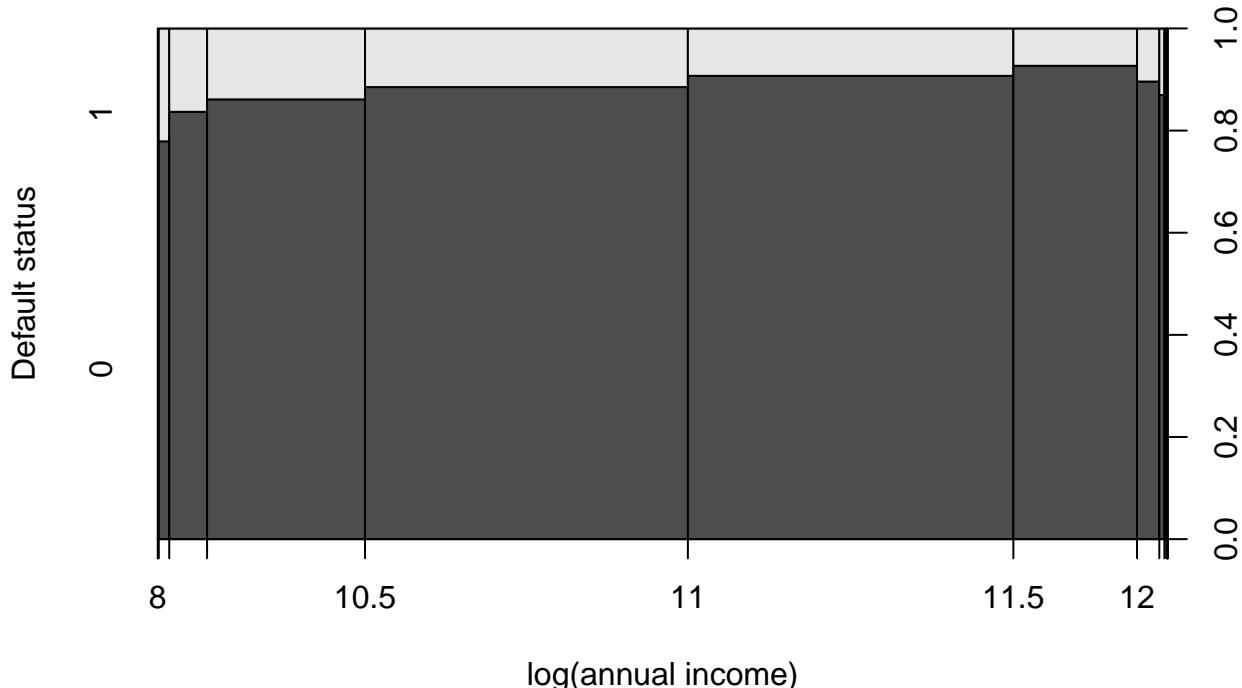
$$R^2_{MF} = 1 - \frac{\log \mathcal{L}(\theta; \mathbf{y})}{\log \mathcal{L}_0(\mathbf{y})}.$$

Intuitively,  $R^2_{MF} = 0$  if the explanatory variables do not convey any information on the outcome  $y$ . Indeed, in this case, the model is not better than the reference model, that simply captures the fraction of  $y_i$ ’s that are equal to 1.

**Example 8.3** (Credit and defaults (Lending-club dataset)). This example makes use of the `credit` data of package `AEC`. The objective is to model the default probabilities of borrowers.

Let us first represent the relationship between the fraction of households that have defaulted on their loan and their annual income:

```
library(AEC)
credit$Default <- 0
credit$Default[credit$loan_status == "Charged Off"] <- 1
credit$Default[credit$loan_status ==
  "Does not meet the credit policy. Status:Charged Off"] <- 1
credit$amt2income <- credit$loan_amnt/credit$annual_inc
plot(as.factor(credit$Default)~log(credit$annual_inc),
  ylevels=2:1, ylab="Default status", xlab="log(annual income)")
```



The previous figure suggests that the effect of annual income on the probability of default is non-monotonous. We will therefore include a quadratic term in one of our specification (namely `eq1` below).

We consider three specifications. The first one (`eq0`), with no explanatory variables, is trivial. It will just be used to compute the pseudo- $R^2$ . In the second (`eq1`), we consider a few covariates (loan amount, the ratio between the amount and annual income, The number of more-than-30 days past-due incidences of delinquency in the borrower's credit file for the past 2 years, and a quadratic function of annual income). In the third model (`eq2`), we add a credit rating.

```
eq0 <- glm(Default ~ 1, data=credit, family=binomial(link="probit"))
eq1 <- glm(Default ~ log(loan_amnt) + amt2income + delinq_2yrs +
            log(annual_inc)+ I(log(annual_inc)^2),
            data=credit, family=binomial(link="probit"))
eq2 <- glm(Default ~ grade + log(loan_amnt) + amt2income + delinq_2yrs +
            log(annual_inc)+ I(log(annual_inc)^2),
            data=credit, family=binomial(link="probit"))
stargazer::stargazer(eq0, eq1, eq2, type="text", no.space = TRUE)
```

```
##
## =====
##             Dependent variable:
## -----
##                  Default
##          (1)      (2)      (3)
## -----
## gradeB                      0.400***  

##                               (0.055)  

## gradeC                      0.587***  

##                               (0.057)  

## gradeD                      0.820***  

##                               (0.061)  

## gradeE                      0.874***  

##                               (0.091)  

## gradeF                      1.230***  

##                               (0.147)  

## gradeG                      1.439***  

##                               (0.227)  

## log(loan_amnt)           -0.149** -0.194***  

##                               (0.060)   (0.061)
```

```

## amt2income           1.266***   1.222*** 
##                               (0.383)    (0.393)
## delinq_2yrs          0.096***   0.009 
##                               (0.034)    (0.035)
## log(annual_inc)      -1.444**   -0.874 
##                               (0.569)    (0.586)
## I(log(annual_inc)^2)  0.064**   0.038 
##                               (0.025)    (0.026)
## Constant             -1.231***  7.937***   4.749 
##                               (0.017)    (3.060)    (3.154)
## -----
## Observations          9,156      9,156      9,156
## Log Likelihood        -3,157.696 -3,120.625 -2,981.343
## Akaike Inf. Crit.    6,317.392  6,253.250  5,986.686
## =====
## Note:                *p<0.1; **p<0.05; ***p<0.01

```

Let us compute the pseudo R<sup>2</sup> for the last two models:

```

logL0 <- logLik(eq0); logL1 <- logLik(eq1); logL2 <- logLik(eq2)
pseudoR2_eq1 <- 1 - logL1/logL0 # pseudo R2
pseudoR2_eq2 <- 1 - logL2/logL0 # pseudo R2
c(pseudoR2_eq1,pseudoR2_eq2)

```

```
## [1] 0.01173993 0.05584870
```

Let us now compute the (average) marginal effects, using method ii of Section 8.1.4:

```
mean(dnorm(predict(eq2)),na.rm=TRUE)*eq2$coefficients
```

```

## (Intercept)           gradeB           gradeC
## 0.840731198         0.070747353       0.103944305
## gradeD               gradeE           gradeF
## 0.145089219         0.154773742       0.217702041
## gradeG               log(loan_amnt)     amt2income
## 0.254722161         -0.034289921      0.216251992
## delinq_2yrs          log(annual_inc)  I(log(annual_inc)^2)
## 0.001574178         -0.154701321      0.006813694

```

There is an issue for the `annual_inc` variable. Indeed, the previous computation does not realize that this variable appears twice among the explanatory variables (through `log(annual_inc)` and `I(log(annual_inc)^2)`). To address this, one can proceed as follows: (1) we construct a new counterfactual dataset where annual incomes are increased by 1%, (2) we use the model to compute model-implied probabilities of default on this new dataset and (3), we subtract the probabilities resulting from the original dataset from these counterfactual probabilities:

```

new_credit <- credit
new_credit$annual_inc <- 1.01 * new_credit$annual_inc
bas_predict_eq2 <- predict(eq2, newdata = credit, type = "response")
# This is equivalent to pnorm(predict(eq2, newdata = credit))
new_predict_eq2 <- predict(eq2, newdata = new_credit, type = "response")
mean(new_predict_eq2 - bas_predict_eq2)

```

```
## [1] -6.562126e-05
```

The negative sign means that, on average across the entities considered in the analysis, a 1% increase in annual income results in a decrease in the default probability. This average effect is however pretty low. To get an economic sense of the size of this effect, let us compute the average effect associated with a unit increase in the number of delinquencies:

```
new_credit <- credit
new_credit$delinq_2yrs <- credit$delinq_2yrs + 1
new_predict_eq2 <- predict(eq2, newdata = new_credit, type = "response")
mean(new_predict_eq2 - bas_predict_eq2)

## [1] 0.001582332
```

We can employ a likelihood ratio test (see Def. 7.8) to see if the two variables associated with annual income are jointly statistically significant (in the context of eq1):

```
eq1restr <- glm(Default ~ log(loan_amnt) + amt2income + delinq_2yrs,
                  data=credit,family=binomial(link="probit"))
LRstat <- 2*(logL1 - logLik(eq1restr))
pvalue <- 1 - c(pchisq(LRstat,df=2))
```

The computation gives a p-value of 0.0436.

**Example 8.4** (Replicating Table 14.2 of Cameron and Trivedi (2005)). The following lines of codes replicate Table 14.2 of Cameron and Trivedi (2005) (see Example 8.2).

```
data.reduced <- subset(Fishing, mode %in% c("charter", "pier"))
data.reduced$lnrelp <- log(data.reduced$price.charter / data.reduced$price.pier)
data.reduced$y <- 1 * (data.reduced$mode == "charter")
# check first line of Table 14.1:
price.charter.y0 <- mean(data.reduced$pcharter[data.reduced$y == 0])
price.charter.y1 <- mean(data.reduced$pcharter[data.reduced$y == 1])
price.charter <- mean(data.reduced$pcharter)
# Run probit regression:
reg.probit <- glm(y ~ lnrelp,
                     data = data.reduced,
                     family = binomial(link = "probit"))
# Run Logit regression:
reg.logit <- glm(y ~ lnrelp,
                     data = data.reduced,
                     family = binomial(link = "logit"))
# Run OLS regression:
reg.OLS <- lm(y ~ lnrelp,
                     data = data.reduced)
# Replicates Table 14.2 of Cameron and Trivedi:
stargazer::stargazer(reg.logit, reg.probit, reg.OLS, no.space = TRUE,
                      type = "text")
```

```
##
## =====
##                               Dependent variable:
## -----
##                                     y
## logistic    probit          OLS
##           (1)      (2)      (3)
## -----
## lnrelp        -1.823***   -1.056***    -0.243***
##                   (0.145)     (0.075)      (0.010)
## Constant      2.053***    1.194***    0.784***
```

```

##                               (0.169)   (0.088)   (0.013)
## -----
## Observations             630      630      630
## R2                      0.463
## Adjusted R2              0.462
## Log Likelihood        -206.827 -204.411
## Akaike Inf. Crit.       417.654  412.822
## Residual Std. Error          0.330 (df = 628)
## F Statistic                  542.123*** (df = 1; 628)
## -----
## Note: *p<0.1; **p<0.05; ***p<0.01

```

### 8.1.6 Predictions and ROC curves

How to compute model-implied predicted outcomes? As is the case for  $y_i$ , predicted outcomes  $\hat{y}_i$  need to be valued in  $\{0, 1\}$ . A natural choice consists in considering that  $\hat{y}_i = 1$  if  $\mathbb{P}(y_i = 1 | \mathbf{x}_i; \theta) > 0.5$ , i.e., in taking a cutoff of  $c = 0.5$ . There exist, though, situations where doing so is not relevant. For instance, we may have some models where all predicted probabilities are small, but some less than others. In this context, a model-implied probability of 10% (say) could characterize a “high-risk” entity. However, using a cutoff of 50% would not identify this level of riskiness.

The **receiver operating characteristics (ROC)** curve constitutes a more general approach. The idea is to remain agnostic and to consider all possible values of the cutoff  $c$ . It works as follows. For each potential cutoff  $c \in [0, 1]$ , compute (and plot):

- The fraction of  $y = 1$  values correctly classified (*True Positive Rate*) against
- The fraction of  $y = 0$  values incorrectly specified (*False Positive Rate*).

Such a curve mechanically starts at  $(0,0)$  —which corresponds to  $c = 1$ — and terminates at  $(1,1)$  —situation when  $c = 0$ .

In the case of no predictive ability (worst situation), the ROC curve is a straight line between  $(0,0)$  and  $(1,1)$ .

**Example 8.5** (ROC with the fishing-mode dataset). Figure 8.5 shows the ROC curve associated with the probit model estimated in Example 8.4.

```

library(pROC)
predict_model <- predict.glm(reg.probit, type = "response")
roc(data.reduced$y, predict_model, percent=T,
  boot.n=1000, ci.alpha=0.9, stratified=T, plot=TRUE, grid=TRUE,
  show.thres=TRUE, legacy.axes = TRUE, reuse.auc = TRUE,
  print.auc = TRUE, print.thres.col = "blue", ci=TRUE,
  ci.type="bars", print.thres.cex = 0.7, col = 'red',
  main = paste("ROC curve using", "(N = ",nrow(data.reduced), ")"))

```

## 8.2 Multiple Choice Models

We will now consider cases where the number of possible outcomes (or alternatives) is larger than two. Let us denote by  $J$  this number. We have  $y_j \in \{1, \dots, J\}$ . This situation arise for instance when the outcome variable reflects:

- Opinions: strongly opposed / opposed / neutral / support (ranked choices),
- Occupational field: lawyer / farmer / engineer / doctor / ...,
- Alternative shopping areas,
- Transportation types.

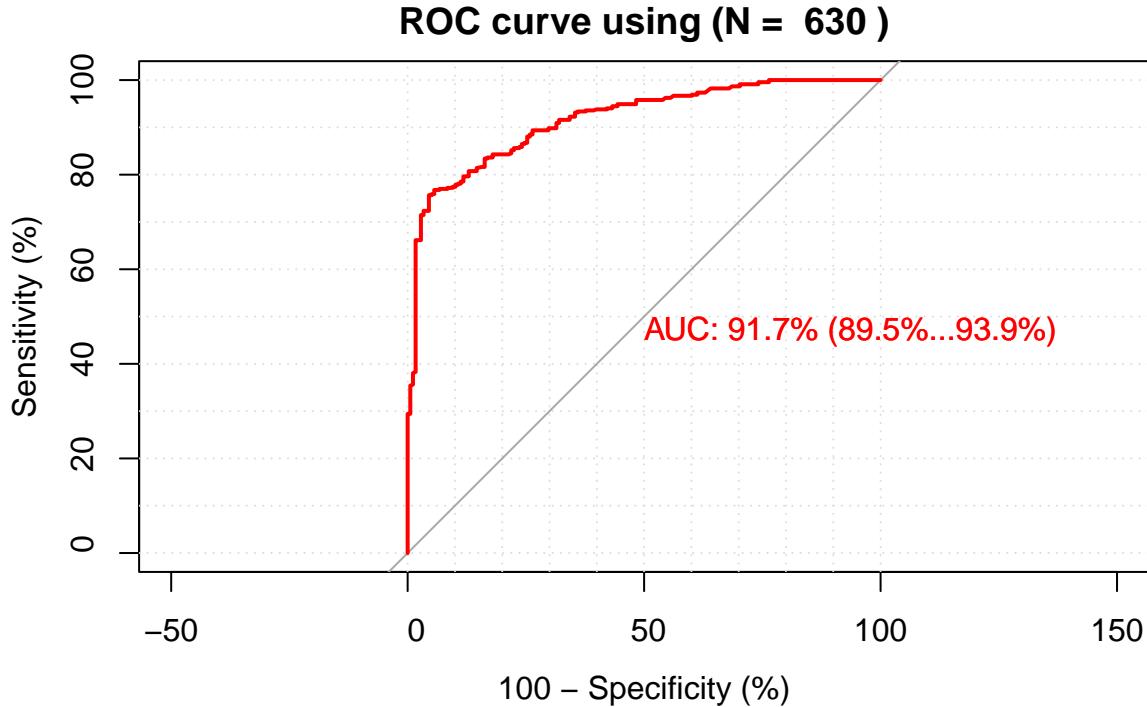


Figure 8.5: Application of the ROC methodology on the fishing-mode dataset.

In a few cases, the values associated with the choices will themselves be meaningful, for example, number of accidents per day:  $y = 0, 1, 2, \dots$  (count data). In most cases, the values are meaningless.

We assume the existence of covariates, gathered in vector  $\mathbf{x}_i$  ( $K \times 1$ ), that are suspected to influence for the probabilities of obtaining the different outcomes ( $y_i = j$ ,  $j \in \{1, \dots, J\}$ ).

In what follows, we will assume that the  $y_i$ 's are assumed to be independently distributed, with:

$$y_i = \begin{cases} 1 & \text{with probability } g_1(\mathbf{x}_i; \theta) \\ \vdots \\ J & \text{with probability } g_J(\mathbf{x}_i; \theta). \end{cases} \quad (8.8)$$

(Of course, for all entities  $(i)$ , we must have  $\sum_{j=1}^J g_j(\mathbf{x}_i; \theta) = 1$ .) Our objective is to estimate the vector of population parameters  $\theta$  given functional forms for the  $g_j$ 's.

### 8.2.1 Ordered case

Sometimes, there exists a natural order for the different alternatives. This is typically the case where respondents have to choose a level of agreement to a statement, e.g.: (1) Strongly disagree; (2) Disagree; (3) Neither agree nor disagree; (4) Agree; (5) Strongly agree. Another standard case is that of ratings (from A to F, say).

The ordered probit model consists in extending the binary case, considering the latent-variable view of the latter (see Section 8.1.1). Formally, the model is as follows:

$$\mathbb{P}(y_i = j|\mathbf{x}_i) = \mathbb{P}(\alpha_{j-1} < y_i^* < \alpha_j|\mathbf{x}_i), \quad (8.9)$$

where

$$y_i^* = \theta' \mathbf{x}_i + \varepsilon_i,$$

with  $\varepsilon_i \sim i.i.d. \mathcal{N}(0, 1)$ . The  $\alpha_j$ 's,  $j \in \{1, \dots, J-1\}$ , are (new) parameters that have to be estimated, on top of  $\theta$ . Naturally, we have  $\alpha_1 < \alpha_2 < \dots < \alpha_{J-1}$ . Moreover  $\alpha_0$  is  $-\infty$  and  $\alpha_J$  is  $+\infty$ , so that Eq. (8.9) is valid for any  $j \in \{1, \dots, J\}$  (including 1 and  $J$ ).

We have:

$$\begin{aligned} g_j(\mathbf{x}_i; \theta, \alpha) &= \mathbb{P}(y_i = j|\mathbf{x}_i) = \mathbb{P}(\alpha_{j-1} < y_i^* < \alpha_j|\mathbf{x}_i) \\ &= \mathbb{P}(\alpha_{j-1} - \theta' \mathbf{x}_i < \varepsilon_i < \alpha_j - \theta' \mathbf{x}_i) \\ &= \Phi(\alpha_j - \theta' \mathbf{x}_i) - \Phi(\alpha_{j-1} - \theta' \mathbf{x}_i), \end{aligned}$$

where  $\Phi$  is the c.d.f. of  $\mathcal{N}(0, 1)$ .

If, for all  $i$ , one of the components of  $\mathbf{x}_i$  is equal to 1 (which is what is done in linear regression to introduce an intercept in the specification), then one of the  $\alpha_j$  ( $j \in \{1, \dots, J - 1\}$ ) is not identified. One can then arbitrarily set  $\alpha_1 = 0$ . This is what is done in the binary logit/probit cases.

This model can be estimated by maximizing the likelihood function (see Section 7.2). This function is given by:

$$\log \mathcal{L}(\theta, \alpha; \mathbf{y}, \mathbf{X}) = \sum_{i=1}^n \sum_{j=1}^J \mathbb{I}_{\{y_i=j\}} \log(g_j(\mathbf{x}_i; \theta, \alpha)). \quad (8.10)$$

Let us stress that we have two types of parameters to estimate: those included in vector  $\theta$ , and the  $\alpha_j$ 's, gathered in vector  $\alpha$ .

The estimated values of the  $\theta_j$ 's are slightly more complicated to interpret (at least in term of sign) than in the binary case. Indeed, we have:

$$\mathbb{P}(y_i \leq j | \mathbf{x}_i) = \Phi(\alpha_j - \theta' \mathbf{x}_i) \Rightarrow \frac{\partial \mathbb{P}(y_i \leq j | \mathbf{x}_i)}{\mathbf{x}_i} = -\underbrace{\Phi'(\alpha_j - \theta' \mathbf{x}_i)}_{>0} \theta.$$

Hence the sign of  $\theta_k$  indicates whether  $\mathbb{P}(y_i \leq j | \mathbf{x}_i)$  increases or decreases w.r.t.  $x_{i,k}$  (the  $k^{th}$  component of  $\mathbf{x}_i$ ). By contrast:

$$\frac{\partial \mathbb{P}(y_i = j | \mathbf{x}_i)}{\mathbf{x}_i} = \underbrace{(-F'(\alpha_j + \theta' \mathbf{x}_i) + F'(\alpha_{j-1} + \theta' \mathbf{x}_i))}_{A} \theta.$$

Therefore the signs of the components of  $\theta$  are not necessarily those of the marginal effects. (For the sign of  $A$  is a priori unknown.)

**Example 8.6** (Predicting credit ratings (Lending-club dataset)). Let us use credit dataset again (see Example 8.3), and let use try and model the ratings attributed by the lending-club:

```
library(AEC)
library(MASS)
credit$emp_length_low5y <- credit$emp_length %in%
  c("< 1 year", "1 year", "2 years", "3 years", "4 years")
credit$emp_length_high10y <- credit$emp_length=="10+ years"
credit$annual_inc <- credit$annual_inc/1000
credit$loan_amnt <- credit$loan_amnt/1000
credit$income2loan <- credit$annual_inc/credit$loan_amnt
training <- credit[1:20000,] # sample is reduced
training <- subset(training, grade!=c("E", "F", "G"))
training <- droplevels(training)
training$grade.ordered <- factor(training$grade, ordered=TRUE,
                                    levels = c("D", "C", "B", "A"))
model1 <- polr(grade.ordered ~ log(loan_amnt) + log(income2loan) + delinq_2yrs,
               data=training, Hess=TRUE, method="probit")
model2 <- polr(grade.ordered ~ log(loan_amnt) + log(income2loan) + delinq_2yrs +
               emp_length_low5y + emp_length_high10y,
               data=training, Hess=TRUE, method="probit")
stargazer::stargazer(model1, model2, ord.intercepts = TRUE, type="text",
                      no.space = TRUE)
```

```
##
## =====
##             Dependent variable:
## -----
##                  grade.ordered
##                   (1)          (2)
```

```

## -----
## log(loan_amnt)      -0.014      -0.040*
##                               (0.022)    (0.022)
## log(income2loan)    0.115***   0.092*** 
##                               (0.022)    (0.022)
## delinq_2yrs        -0.399***  -0.404*** 
##                               (0.025)    (0.025)
## emp_length_low5y          -0.096*** 
##                               (0.027)
## emp_length_high10y       0.088** 
##                               (0.035)
## D| C                  -0.937***  -1.073*** 
##                               (0.082)    (0.086)
## C| B                  -0.160**   -0.295*** 
##                               (0.082)    (0.085)
## B| A                  0.696***   0.564*** 
##                               (0.082)    (0.086)
## -----
## Observations          8,695      8,695
## -----
## Note:                 *p<0.1; **p<0.05; ***p<0.01

```

Predicted ratings (and probabilities of being given a given rating) can be computed as follows:

```

pred.grade <- predict(model1,newdata = training)
# pred.grade = predicted grade, defined as the most likely according model
pred.proba <- predict(model1,newdata = training, type="probs")

```

### 8.2.2 General multinomial logit model

This section introduces the general multinomial logit model, which is the natural extension of the binary logit model (see Table 8.1). Its general formulation is as follows:

$$g_j(\mathbf{x}_i; \theta) = \frac{\exp(\theta'_j \mathbf{x}_i)}{\sum_{k=1}^J \exp(\theta'_k \mathbf{x}_i)}. \quad (8.11)$$

Note that, by construction,  $g_j(\mathbf{x}_i; \theta) \in [0, 1]$  and  $\sum_j g_j(\mathbf{x}_i; \theta) = 1$ .

The components of  $\mathbf{x}_i$  (regressors, or covariates) may be *alternative-specific* or *alternative invariant* (see also Section 8.1.2). We may, e.g., organize  $\mathbf{x}_i$  as follows:

$$\mathbf{x}_i = [\mathbf{u}'_{i,1}, \dots, \mathbf{u}'_{i,J}, \mathbf{v}'_i]', \quad (8.12)$$

where the notations are as in Section 8.1.2, that is:

- $\mathbf{u}_{i,j}$  ( $j \in \{1, \dots, J\}$ ): vector of variables associated with agent  $i$  and alternative  $j$  (alternative-specific regressors). Examples: Travel time per type of transportation (transportation choice), wage per type of work, cost per type of car.
- $\mathbf{v}_i$ : vector of variables associated with agent  $i$  but alternative-invariant. Examples: age or gender of agent  $i$ ,

When  $\mathbf{x}_i$  is as in Eq. (8.12), with obvious notations,  $\theta_j$  is of the form:

$$\theta_j = [\theta_{1,j}^{(u)'}, \dots, \theta_{J,j}^{(u)'}, \theta_j^{(v)'}]', \quad (8.13)$$

and  $\theta = [\theta'_1, \dots, \theta'_J]'$ .

The literature has considered different specific cases of the general multinomial logit model:<sup>1</sup>

---

<sup>1</sup>The labelling “CL” and “MNL”—used in the literature— are relatively *ad hoc* (see 15.4.1 in Cameron and Trivedi (2005)).

- Conditional logit (CL) with alternative-varying regressors:

$$\theta_j = [\mathbf{0}', \dots, \mathbf{0}', \underbrace{\beta'}_{j^{th} \text{ position}}, \mathbf{0}', \dots]', \quad (8.14)$$

i.e., we have  $\beta = \theta_{1,1}^{(u)} = \dots = \theta_{J,J}^{(u)}$  and  $\theta_{i,j}^{(u)} = \mathbf{0}$  for  $i \neq j$ .

- Multinomial logit (MNL) with alternative-invariant regressors:

$$\theta_j = [\mathbf{0}', \dots, \mathbf{0}', \theta_j^{(v)}']'. \quad (8.15)$$

- Mixed logit:

$$\theta_j = [\mathbf{0}', \dots, \mathbf{0}', \beta', \mathbf{0}', \dots, \mathbf{0}', \theta_j^{(v)}']'. \quad (8.16)$$

**Example 8.7** (CL and MNL with the fishing-mode dataset). The following lines replicate Table 15.2 in Cameron and Trivedi (2005) (see also Examples 8.2 and 8.4):

```
# Specify data organization:
library(mlogit)
library(stargazer)
data("Fishing", package="mlogit")
Fish <- mlogit.data(Fishing,
                     varying = c(2:9),
                     choice = "mode",
                     shape = "wide")
MNL1 <- mlogit(mode ~ price + catch, data = Fish)
MNL2 <- mlogit(mode ~ price + catch - 1, data = Fish)
MNL3 <- mlogit(mode ~ 0 | income, data = Fish)
MNL4 <- mlogit(mode ~ price + catch | income, data = Fish)
stargazer(MNL1, MNL2, MNL3, MNL4, type="text", no.space = TRUE,
           omit.stat = c("lr"))

##
## =====
##             Dependent variable:
## -----
##                               mode
##             (1)      (2)      (3)      (4)
## -----
## (Intercept):boat    0.871***      0.739***  0.527**
##                      (0.114)        (0.197)   (0.223)
## (Intercept):charter 1.499***      1.341***  1.694*** 
##                      (0.133)        (0.195)   (0.224)
## (Intercept):pier    0.307***      0.814***  0.778*** 
##                      (0.115)        (0.229)   (0.220)
## price            -0.025***  -0.020***  -0.025*** 
##                      (0.002)        (0.001)   (0.002)
## catch             0.377***  0.953***  0.358*** 
##                      (0.110)        (0.089)   (0.110)
## income:boat          0.0001**  0.0001* 
##                      (0.00004)    (0.0001)
## income:charter       -0.00003  -0.00003 
##                      (0.00004)    (0.0001)
## income:pier         -0.0001*** -0.0001** 
##                      (0.0001)    (0.0001)
## -----
## Observations       1,182      1,182      1,182      1,182
## R2                 0.178      0.014      0.189
## Log Likelihood   -1,230.784 -1,311.980 -1,477.151 -1,215.138
## =====
## Note: *p<0.1; **p<0.05; ***p<0.01
```

### ML estimation

General multinomial logit models can be estimated by Maximum Likelihood techniques (see Section 7.2). Consider the general model described in Eq. (8.8). It can be noted that:

$$f(y_i | \mathbf{x}_i; \theta) = \prod_{j=1}^J g_j(\mathbf{x}_i; \theta)^{\mathbb{I}_{\{y_i=j\}}},$$

which leads to

$$\log f(y_i | \mathbf{x}_i; \theta) = \sum_{j=1}^J \mathbb{I}_{\{y_i=j\}} \log(g_j(\mathbf{x}_i; \theta)).$$

The log-likelihood function is therefore given by:

$$\log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X}) = \sum_{i=1}^n \sum_{j=1}^J \mathbb{I}_{\{y_i=j\}} \log(g_j(\mathbf{x}_i; \theta)). \quad (8.17)$$

Numerical methods have to be employed in order to find the maximum-likelihood estimate of  $\theta$ . (Standard packages contain fast algorithms.)

### Marginal Effects

Let us consider the computation of marginal effects in the general multinomial logit model (Eq. (8.11)). Using the notation  $p_{i,j} \equiv \mathbb{P}(y_i = j | \mathbf{x}_i; \theta)$ , we have:

$$\begin{aligned} \frac{\partial p_{i,j}}{\partial x_{i,s}} &= \frac{\theta_{j,s} \exp(\theta'_j \mathbf{x}_i) \sum_{k=1}^J \exp(\theta'_k \mathbf{x}_i)}{(\sum_{k=1}^J \exp(\theta'_k \mathbf{x}_i))^2} \\ &\quad - \frac{\exp(\theta'_j \mathbf{x}_i) \sum_{k=1}^J \theta_{k,s} \exp(\theta'_k \mathbf{x}_i)}{(\sum_{k=1}^J \exp(\theta'_k \mathbf{x}_i))^2} \\ &= \theta_{j,s} p_{i,j} - \sum_{k=1}^J \theta_{k,s} p_{i,j} p_{i,k} \\ &= p_{i,j} \times \left( \theta_{j,s} - \underbrace{\sum_{k=1}^J \theta_{k,s} p_{i,k}}_{=\bar{\theta}_s^{(i)}} \right), \end{aligned}$$

where  $\bar{\theta}_s^{(i)}$  does not depend on  $j$ . Note that the sign of the marginal effect is not necessarily that of  $\theta_{j,s}$ .

### Random Utility models

The general multinomial logit model may arise as the natural specification arising in structural contexts where agents compare (random) utilities associated with  $J$  potential outcomes (see Section 8.1.1 for the binary situation).

Let's drop the  $i$  subscript for simplicity and assume that the utility derived from choosing  $j$  is given by  $U_j = V_j + \varepsilon_j$ , where  $V_j$  is deterministic (may depend on observed covariates) and  $\varepsilon_j$  is stochastic. We have (with obvious notations):

$$\begin{aligned} \mathbb{P}(y = j) &= \mathbb{P}(U_j > U_k, \forall k \neq j) \\ \mathbb{P}(y = j) &= \mathbb{P}(U_k - U_j < 0, \forall k \neq j) \\ \mathbb{P}(y = j) &= \mathbb{P}(\underbrace{\varepsilon_k - \varepsilon_j}_{=: \tilde{\varepsilon}_{k,j}} < \underbrace{V_j - V_k}_{=: -\tilde{V}_{k,j}}, \forall k \neq j). \end{aligned}$$

The last expression is an  $(J - 1)$ -variate integral. While it has, in general, no analytical solution, Prop. 8.1 shows that it is the case when employing Gumbel distributions (see Def. 8.1).

**Definition 8.1** (Gumbel distribution). The c.d.f. of the Gumbel distribution ( $\mathcal{W}$ ) is:

$$F(u) = \exp(-\exp(-u)), \quad f(u) = \exp(-u - \exp(u)).$$

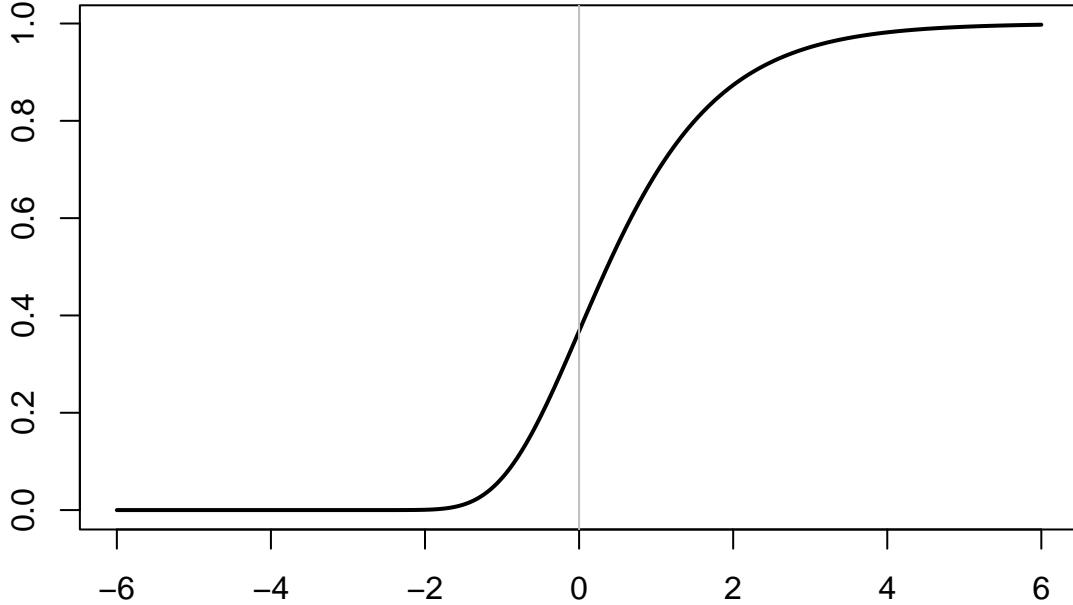


Figure 8.6: C.d.f. of the Gumbel distribution ( $F(x) = \exp(-\exp(-x))$ ).

Remark: if  $X \sim \mathcal{W}$ , then  $\mathbb{E}(X) = 0.577$  (Euler constant)<sup>2</sup> and  $\text{Var}(X) = \pi^2/6$ .

**Proposition 8.1** (Weibull). *In the context of the utility model described above, if  $\varepsilon_j \sim i.i.d. \mathcal{W}$ , then*

$$\mathbb{P}(y = j) = \frac{\exp(V_j)}{\sum_{k=1}^J \exp(V_k)}.$$

*Proof.* We have:

$$\begin{aligned} \mathbb{P}(y = j) &= \mathbb{P}(\forall k \neq j, U_k < U_j) = \mathbb{P}(\forall k \neq j, \varepsilon_k < V_j - V_k + \varepsilon_j) \\ &= \int \prod_{k \neq j} F(V_j - V_k + \varepsilon) f(\varepsilon) d\varepsilon. \end{aligned}$$

After computation, it comes that

$$\prod_{k \neq j} F(V_j - V_k + \varepsilon) f(\varepsilon) = \exp[-\varepsilon - \exp(-\varepsilon + \lambda_j)],$$

where  $\lambda_j = \log \left( 1 + \frac{\sum_{k \neq j} \exp(V_k)}{\exp(V_j)} \right)$ . We then have:

$$\begin{aligned} \mathbb{P}(y = j) &= \int \exp[-\varepsilon - \exp(-\varepsilon + \lambda_j)] d\varepsilon \\ &= \int \exp[-t - \lambda_j - \exp(-t)] d\varepsilon = \exp(-\lambda_j), \end{aligned}$$

which leads to the result.  $\square$

Some remarks on identification (see Def. 7.5) are in order.

1. We have:

$$\mathbb{P}(y = j) = \frac{\exp(V_j)}{\sum_{k=1}^J \exp(V_k)} = \frac{\exp(V_j^*)}{1 + \sum_{k=2}^J \exp(V_k^*)},$$

where  $V_j^* = V_j - V_1$ . We can therefore always assume that  $V_1 = 0$ . In the case where  $V_{i,j} = \theta'_j \mathbf{x}_i = \beta' \mathbf{u}_{i,j} + \theta_j^{(v)'} \mathbf{v}_i$  (see Eqs. (8.12) and (8.16)), we can for instance assume that:

$$\begin{aligned} (A) \quad \mathbf{u}_{i,1} &= 0, \\ (B) \quad \theta_1^{(v)} &= 0. \end{aligned}$$

If (A) does not hold, we can replace  $\mathbf{u}_{i,j}$  with  $\mathbf{u}_{i,j} - \mathbf{u}_{i,1}$ .

<sup>2</sup>The Euler constant  $\gamma$  satisfies  $\gamma = \lim_{n \rightarrow \infty} (-\ln(n) + \sum_{k=1}^n \frac{1}{k})$ .

2. If  $J = 2$  and  $j \in \{0, 1\}$  (shift by one unit), we have  $\mathbb{P}(y = 1|\mathbf{x}) = \frac{\exp(\theta' \mathbf{x})}{1 + \exp(\theta' \mathbf{x})}$ , this is the logit model (Table 8.1).

### Limitations of logit models

In a Logit model, we have:

$$\mathbb{P}(y = j|y \in \{k, j\}) = \frac{\exp(\theta'_j \mathbf{x})}{\exp(\theta'_j \mathbf{x}) + \exp(\theta'_k \mathbf{x})}. \quad (8.18)$$

This conditional probability does not depend on other alternatives (i.e., it does not depend on  $\theta_m$ ,  $m \neq j, k$ ). In particular, if  $\mathbf{x} = [\mathbf{u}'_1, \dots, \mathbf{u}'_J, \mathbf{v}']'$ , then changes in  $\mathbf{u}_m$  ( $m \neq j, k$ ) have no impact on the object shown in Eq. (8.18).

That is, a Multinomial Logit can be seen as a series of pairwise comparisons that are unaffected by the characteristics of alternatives. Such a model is said to satisfy the **independence from irrelevant alternatives (IIA)** property. That is, in these models, for any individual, the ratio of probabilities of choosing two alternatives is independent of the availability or attributes of any other alternatives. While this may not sound alarming, there are situations where you would like it not to be the case, this is for instance the case when you want to extrapolate the results of your estimated model to a situation where there is a novel outcome that is highly substitutable to one of the previous ones. This can be illustrated with the famous “red-blue bus” example:

**Example 8.8** (Red-blue bus and IIA). Assume one has a logit model capturing the decision to travel using either a car ( $y = 1$ ) or a (red) bus ( $y = 2$ ). Assume you want to augment this model to allow for a third choice ( $y = 3$ ): travel with a blue bus. If a blue bus ( $y = 3$ ) is exactly as a red bus, except for the color, then one would expect to have:

$$\mathbb{P}(y = 3|y \in \{2, 3\}) = 0.5,$$

i.e.  $\theta_2 = \theta_3$ .

Assume we had  $V_1 = V_2$ . We expect to have  $V_2 = V_3$  (hence  $p_2 = p_3$ ). A multinomial logit model would then imply  $p_1 = p_2 = p_3 = 0.33$ . It would however seem more reasonable to have  $p_1 = p_2 + p_3 = 0.5$  and  $p_2 = p_3 = 0.25$ .

### 8.2.3 Nested logits

Nested Logits are natural extensions of logit models when choices feature a nesting structure. This approach is relevant when it makes sense to group some choices into the same *nest*, also called *limbs*. Intuitively, this framework is consistent with the idea according to which, for each agent, there exist unobserved nest-specific variables.

The setup is as follows: we consider  $J$  *limbs*. For each limb  $j$ , we have  $K_j$  *branches*. Let us denote by  $y_1$  the limb choice (i.e.,  $y_1 \in \{1, \dots, J\}$ ) and by  $y_2$  the branch choice (with  $y_2 \in \{1, \dots, K_j\}$ ). The utility associated with the pair of choices  $(j, k)$  is given by

$$U_{j,k} = V_{j,k} + \varepsilon_{j,k}.$$

We have:

$$\mathbb{P}[(y_1, y_2) = (j, k)|\mathbf{x}] = \mathbb{P}(U_{j,k} > U_{l,m}, (l, m) \neq (j, k)|\mathbf{x}).$$

One usually make the following two assumptions:

- i. The deterministic part of the utility is given by  $V_{j,k} = \mathbf{u}'_j \alpha + \mathbf{v}'_{j,k} \beta_j$ , where  $\alpha$  is common to all nests and the  $\beta_j$ 's are nest-specific.
- ii. The disturbances  $\varepsilon$  follow the Generalized Extreme Value (GEV) distribution (see Def. 10.15).

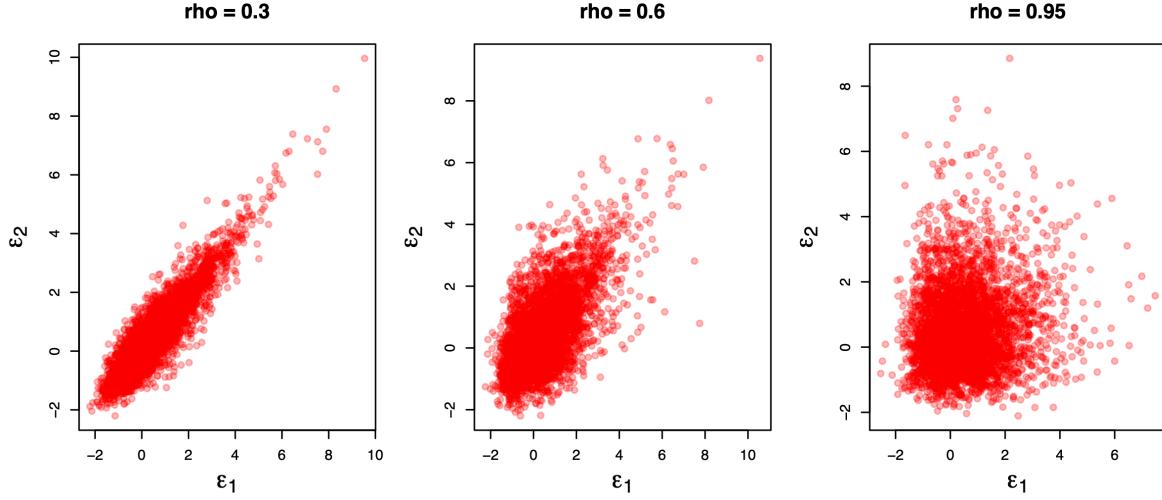


Figure 8.7: GEV simulations.

The following figure displays simulations of pairs  $(\varepsilon_1, \varepsilon_2)$  drawn from GEV distributions for different values of  $\rho$ . The simulation approach is based on Bhat. The code used to produce this chart is provided in Appendix 10.6.1.

Under (i) and (ii), we have:

$$\begin{aligned} \mathbb{P}[(y_1, y_2) = (j, k) | \mathbf{x}] &= \underbrace{\frac{\exp(\mathbf{u}'_j \alpha + \rho_j I_j)}{\sum_{m=1}^J \exp(\mathbf{u}'_m \alpha + \rho_m I_m)}}_{=\mathbb{P}[y_1=j|\mathbf{x}]} \times \\ &\quad \underbrace{\frac{\exp(\mathbf{v}'_{j,k} \beta_j / \rho_j)}{\sum_{l=1}^{K_j} \exp(\mathbf{v}'_{j,l} \beta_j / \rho_j)}}, \end{aligned} \quad (8.19)$$

where  $I_j$ 's are called inclusive values (or log sums), given by:

$$I_j = \log \left( \sum_{l=1}^{K_j} \exp(\mathbf{v}'_{j,l} \beta_j / \rho_j) \right).$$

Some remarks are in order:

- a. It can be shown that  $\rho_j = \sqrt{1 - \text{Cor}(\varepsilon_{j,k}, \varepsilon_{j,l})}$ , for  $k \neq l$ .
- b.  $\rho_j = 1$  implies that  $\varepsilon_{j,k}$  and  $\varepsilon_{j,l}$  are uncorrelated (we are then back to the multinomial logit case).
- c. When  $J = 1$ :

$$F([\varepsilon_1, \dots, \varepsilon_K]', \rho) = \exp \left( - \left( \sum_{k=1}^K \exp(-\varepsilon_k / \rho) \right)^\rho \right).$$

- d. We have:

$$I_j = \mathbb{E}(\max_k(U_{j,k})) = \mathbb{E}(\max_k(V_{j,k} + \varepsilon_{j,k})),$$

The inclusive values can therefore be seen as measures of the relative attractiveness of a nest.

This approach allows for some level of correlation across the  $\varepsilon_{j,k}$  (for a given  $j$ ). This can be interpreted as the existence of an (unobserved) *common error component* for the alternatives of a same nest. This component contributes to making the alternatives of a given nest more similar. In other words, this approach can accommodate a higher sensitivity (cross-elasticity) between the alternatives of a given nest.

Note that if the common component is reduced to zero (i.e.  $\rho_i = 1$ ), the model boils down to the multinomial logit model with no covariance of error terms among the alternatives.

Contrary to the general multinomial model, nested logits can solve the Red-Blue problem described in Section 8.2.2 (see Example 8.8). Assume you have estimated a model specifying  $U_1 = V_1 + \varepsilon_1$  (car choice) and  $U_2 = V_2 + \varepsilon_2$  (red bus choice). You can then assume that the blue-bus utility is of the form  $U_3 = V_2 + \varepsilon_3$  where  $\varepsilon_3$  is perfectly correlated to  $\varepsilon_2$ . This is done by redefining the set of choices as follows:

$$\begin{aligned} j = 1 &\Leftrightarrow (j' = 1, k = 1) \\ j = 2 &\Leftrightarrow (j' = 2, k = 1) \\ j = 3 &\Leftrightarrow (j' = 2, k = 2), \end{aligned}$$

and by setting  $\rho_2 \rightarrow 0$ .

IIA holds within a nest, but not when considering alternatives in different nests. Indeed, using Eq. (8.19):

$$\frac{\mathbb{P}[y_1 = j, y_2 = k_A | \mathbf{x}]}{\mathbb{P}[y_1 = j, y_2 = k_B | \mathbf{x}]} = \frac{\exp(\mathbf{v}'_{j,k_A} \beta_j / \rho_j)}{\exp(\mathbf{v}'_{j,k_B} \beta_j / \rho_j)},$$

i.e. we have IIA in nest  $j$ .

By contrast:

$$\begin{aligned} \frac{\mathbb{P}[y_1 = j_A, y_2 = k_A | \mathbf{x}]}{\mathbb{P}[y_1 = j_B, y_2 = k_B | \mathbf{x}]} &= \frac{\exp(\mathbf{u}'_{j_A} \alpha + \rho_{j_A} I_{j_A}) \exp(\mathbf{v}'_{j_A,k_A} \beta_{j_A} / \rho_{j_A})}{\exp(\mathbf{u}'_{j_B} \alpha + \rho_{j_B} I_{j_B}) \exp(\mathbf{v}'_{j_B,k_B} \beta_{j_B} / \rho_{j_B})} \times \\ &\quad \frac{\sum_{l=1}^{K_{j_B}} \exp(\mathbf{v}'_{j_B,l} \beta_{j_B} / \rho_{j_B})}{\sum_{l=1}^{K_{j_A}} \exp(\mathbf{v}'_{j_A,l} \beta_{j_A} / \rho_{j_A})}, \end{aligned}$$

which depends on the expected utilities of all alternatives in nest  $j_A$  and  $j_B$ . So the IIA does not hold.

**Example 8.9** (Travel-mode dataset). Let us illustrate nested logits on the travel-mode dataset used, e.g., by Hensher and Greene (2002) (see also Heiss (2002)).

```
library(mlogit)
library(stargazer)
data("TravelMode", package = "AER")
Prepared.TravelMode <- mlogit.data(TravelMode, chid.var = "individual",
                                    alt.var = "mode", choice = "choice",
                                    shape = "long")

# Fit a multinomial model:
hl <- mlogit(choice ~ wait + travel + vcost, Prepared.TravelMode,
              method = "bfgs", heterosc = TRUE, tol = 10)

## Fit a nested logit model:
TravelMode$avincome <- with(TravelMode, income * (mode == "air"))
TravelMode$time <- with(TravelMode, travel + wait)/60
TravelMode$timeair <- with(TravelMode, time * I(mode == "air"))
TravelMode$income <- with(TravelMode, income / 10)
# Hensher and Greene (2002), table 1 p.8-9 model 5
TravelMode$incomeother <- with(TravelMode,
                                 ifelse(mode %in% c('air', 'car'), income, 0))
n11 <- mlogit(choice ~ gcost + wait + incomeother, TravelMode,
               shape='long', # Indicates how the dataset is organized
               alt.var='mode', # variable that defines the alternative choices.
               nests=list(public=c('train', 'bus'),
                          car='car', air='air'), # defines the "limbs".
               un.nest.el = TRUE)
n12 <- mlogit(choice ~ gcost + wait + time, TravelMode,
               shape='long', # Indicates how the dataset is organized
               alt.var='mode', # variable that defines the alternative choices.
```

```

nestes=list(public=c('train', 'bus'),
            car='car',air='air'), # defines the "limbs".
un.nest.el = TRUE)
stargazer(n11,n12,type="text",no.space = TRUE)

##
## =====
##             Dependent variable:
## -----
##               choice
##       (1)      (2)
## -----
## (Intercept):train   -0.211    -0.284
##                      (0.562)   (0.551)
## (Intercept):bus     -0.824    -0.712
##                      (0.708)   (0.690)
## (Intercept):car     -5.237***  -3.845***
##                      (0.785)   (0.844)
## gcost                -0.013***  -0.004
##                      (0.004)   (0.006)
## wait                 -0.088***  -0.089***
##                      (0.011)   (0.011)
## incomeother          0.430*** 
##                      (0.113)
## time                  -0.202*** 
##                      (0.060)
## iv                    0.835***  0.877*** 
##                      (0.192)   (0.198)
## -----
## Observations         210      210
## R2                   0.328    0.313
## Log Likelihood      -190.779  -194.841
## LR Test (df = 7)    185.959*** 177.836*** 
## =====
## Note: *p<0.1; **p<0.05; ***p<0.01

```

### 8.3 Tobit models

In some situations, the dependent variable is incompletely observed, which may result in a non-representative sample. Typically, in some cases, observations of the dependent variable can have a lower and/or an upper limit, while the “true”, underlying, dependent variable has not. In this case, OLS regression may lead to inconsistent parameter estimates.

Tobit models have been designed to address some of these situations. This approach has been named after James Tobin, who developed this model in the late 50s (see Tobin (1956)).

Figure 8.8 illustrates the situation. The dots (white and black) represent the “true” observations. Now, assume that only the black are observed. If ones uses these observations in an OLS regression to estimate the relationship between  $x$  and  $y$ , then one gets the red line. It is clear that the sensitivity of  $y$  to  $x$  is then underestimated. The blue line is the line one would obtain if white dots were also observed; the grey line represents the model used to generate the data ( $y_i = x_i + \varepsilon_i$ ).

Assume that the (partially) observed dependent variable follows:

$$y^* = \beta' \mathbf{x} + \varepsilon,$$

with  $\varepsilon$  is drawn from a distribution characterized by a p.d.f. denoted by  $f_\gamma^*$  and a c.d.f. denoted by  $F_\gamma^*$ ; these functions depend on a vector of parameters  $\gamma$ .

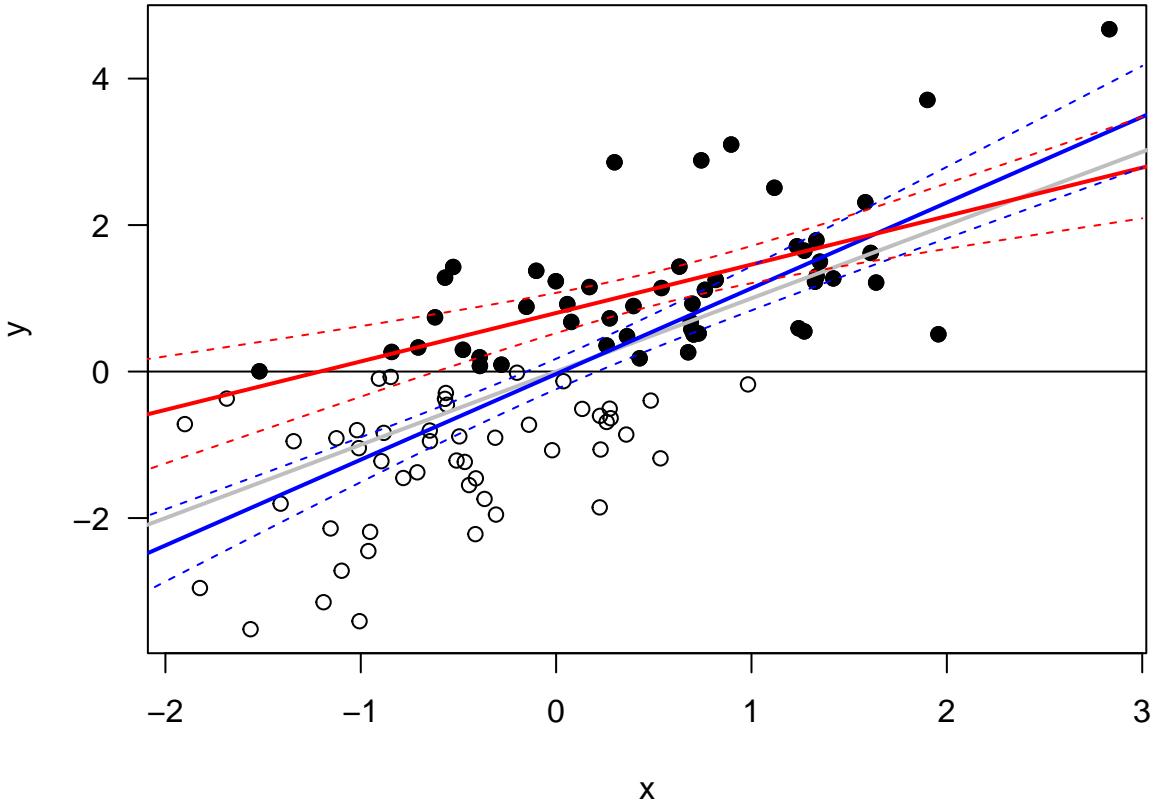


Figure 8.8: Bias in the case of sample selection. The grey line represents the population regression line. The model is  $y_i = x_i + \varepsilon_i$ , with  $\varepsilon_{i,t} \sim \mathcal{N}(0, 1)$ . The red line is the OLS regression line based on black dots only.

The observed dependent variable is:

$$\begin{aligned} \text{Censored case: } y &= \begin{cases} y^* & \text{if } y^* > L \\ L & \text{if } y^* \leq L, \end{cases} \\ \text{Truncated case: } y &= \begin{cases} y^* & \text{if } y^* > L \\ - & \text{if } y^* \leq L, \end{cases} \end{aligned}$$

where “–” stands for missing observations.

This formulation is easily extended to censoring from above ( $L \rightarrow U$ ), or censoring from both below and above.

The model parameters are gathered in vector  $\theta = [\beta', \gamma']'$ . Let us write the conditional p.d.f. of the observed variable:

$$\begin{aligned} \text{Censored case: } f(y|\mathbf{x}; \theta) &= \begin{cases} f_\gamma^*(y - \beta' \mathbf{x}) & \text{if } y > L \\ F_\gamma^*(L - \beta' \mathbf{x}) & \text{if } y = L, \end{cases} \\ \text{Truncated case: } f(y|\mathbf{x}; \theta) &= \frac{f_\gamma^*(y - \beta' \mathbf{x})}{1 - F_\gamma^*(L - \beta' \mathbf{x})} \quad \text{with } y > L. \end{aligned}$$

The (conditional) log-likelihood function is then given by:

$$\log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X}) = \sum_{i=1}^n \log f(y_i|\mathbf{x}_i; \theta).$$

In the censored case, we have:

$$\begin{aligned} \log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X}) &= \sum_{i=1}^n \left\{ \mathbb{I}_{\{y_i=L\}} \log [F_\gamma^*(L - \beta' \mathbf{x}_i)] + \right. \\ &\quad \left. \mathbb{I}_{\{y_i>0\}} \log [f_\gamma^*(y_i - \beta' \mathbf{x}_i)] \right\}. \end{aligned}$$

The Tobit, or censored/truncated normal regression model, corresponds to the case described above, but with Gaussian errors  $\varepsilon$ . Specifically:

$$y^* = \beta' \mathbf{x} + \varepsilon,$$

with  $\varepsilon \sim i.i.d. \mathcal{N}(0, \sigma^2)$  ( $\Rightarrow \gamma = \sigma^2$ ).

Without loss of generality, we can assume that  $L = 0$ . (One can shift observed data if necessary.)

- The censored density (with  $L = 0$ ) is given by:

$$f(y) = \left[ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y - \beta' \mathbf{x})^2\right) \right]^{\mathbb{I}_{\{y>0\}}} \left[ 1 - \Phi\left(\frac{\beta' \mathbf{x}}{\sigma}\right) \right]^{\mathbb{I}_{\{y=0\}}}.$$

- The truncated density (with  $L = 0$ ) is given by:

$$f(y) = \frac{1}{\Phi(\beta' \mathbf{x})} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y - \beta' \mathbf{x})^2\right).$$

Results usually heavily rely on distributional assumptions (more than in uncensored/untruncated case). The framework is easy to extend to an heteroskedastic case, for instance by setting  $\sigma_i^2 = \exp(\alpha' \mathbf{x}_i)$ . Such a situation is illustrated by Figure 8.9.

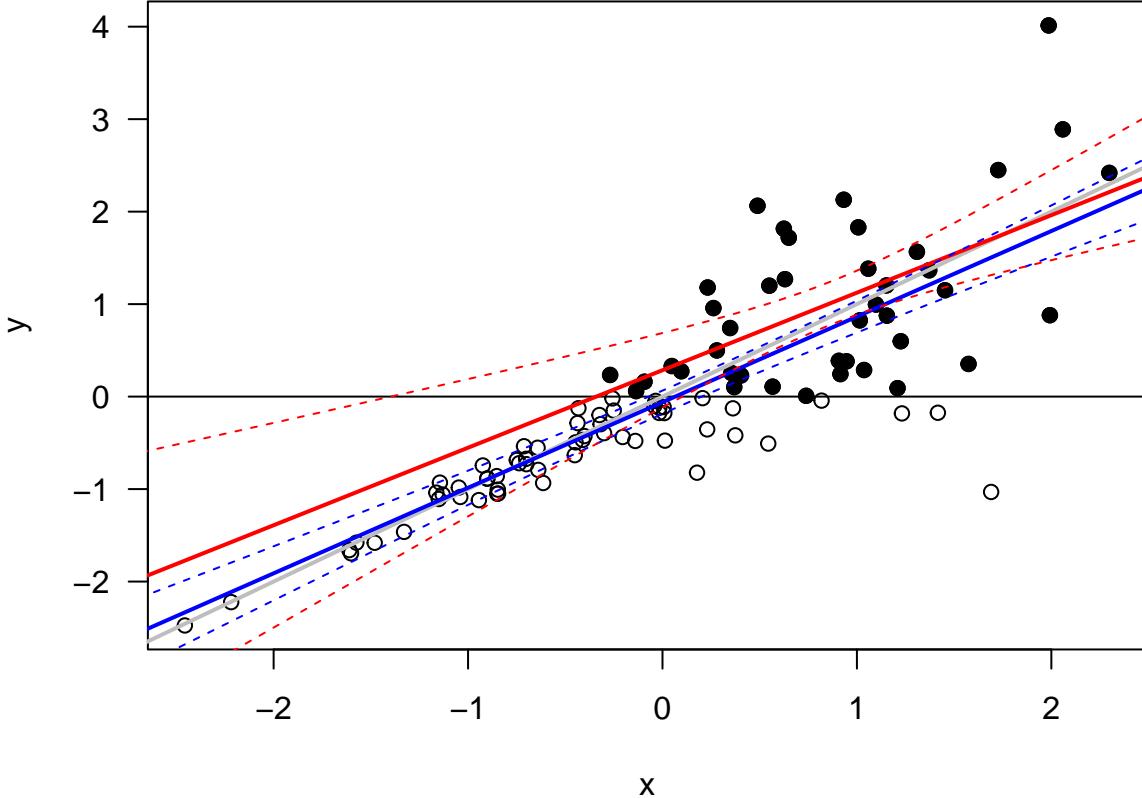


Figure 8.9: Censored dataset with heteroskedasticity. The model is  $y_i = x_i + \varepsilon_i$ , with  $\varepsilon_{i,t} \sim \mathcal{N}(0, \sigma_i^2)$  where  $\sigma_i = \exp(-1 + x_i)$ .

Let us consider the conditional means of  $y$  in the general case, i.e., for any  $\varepsilon$  distribution. Assume  $\mathbf{x}$  is observed, such that expectations are conditional on  $\mathbf{x}$ .

- For data that are left-truncated at 0, we have:

$$\mathbb{E}(y) = \mathbb{E}(y^* | y^* > 0) = \underbrace{\beta' \mathbf{x}}_{=\mathbb{E}(y^*)} + \underbrace{\mathbb{E}(\varepsilon | \varepsilon > -\beta' \mathbf{x})}_{>0} > \mathbb{E}(y^*).$$

- Consider data that are left-censored at 0. By Bayes, we have:

$$f_{y^*|y^*>0}(u) = \frac{f_{y^*}(u)}{\mathbb{P}(y^* > 0)} \mathbb{I}_{\{u>0\}}.$$

Therefore:

$$\begin{aligned} \mathbb{E}(y^* | y^* > 0) &= \frac{1}{\mathbb{P}(y^* > 0)} \int_{-\infty}^{\infty} u f_{y^*}(u) \mathbb{I}_{\{u>0\}} du \\ &= \frac{1}{\mathbb{P}(y^* > 0)} \mathbb{E}(y^* \mathbb{I}_{\{y^*>0\}}), \end{aligned}$$

and, further:

$$\begin{aligned}\mathbb{E}(y) &= \mathbb{P}(y^* > 0)\mathbb{E}(y^*|y^* > 0) \\ &> \mathbb{E}(y^*) = \mathbb{P}(y^* > 0)\mathbb{E}(y^*|y^* > 0) + \mathbb{P}(y^* < 0)\underbrace{\mathbb{E}(y^*|y^* < 0)}_{<0}.\end{aligned}$$

Now, let us come back to the Tobit (i.e., Gaussian case) case.

- For data that are left-truncated at 0:

$$\begin{aligned}\mathbb{E}(y) &= \beta' \mathbf{x} + \mathbb{E}(\varepsilon|\varepsilon > -\beta' \mathbf{x}) \\ &= \beta' \mathbf{x} + \sigma \underbrace{\frac{\phi(\beta' \mathbf{x}/\sigma)}{\Phi(\beta' \mathbf{x}/\sigma)}}_{=: \lambda(\beta' \mathbf{x}/\sigma)} = \sigma \left( \frac{\beta' \mathbf{x}}{\sigma} + \lambda \left( \frac{\beta' \mathbf{x}}{\sigma} \right) \right).\end{aligned}\quad (8.20)$$

where the penultimate line is obtained by using Eq. (10.5).

- For data that are left-censored at 0:

$$\begin{aligned}\mathbb{E}(y) &= \mathbb{P}(y^* > 0)\mathbb{E}(y^*|y^* > 0) \\ &= \Phi \left( \frac{\beta' \mathbf{x}}{\sigma} \right) \sigma \left( \frac{\beta' \mathbf{x}}{\sigma} + \lambda \left( \frac{\beta' \mathbf{x}}{\sigma} \right) \right).\end{aligned}$$

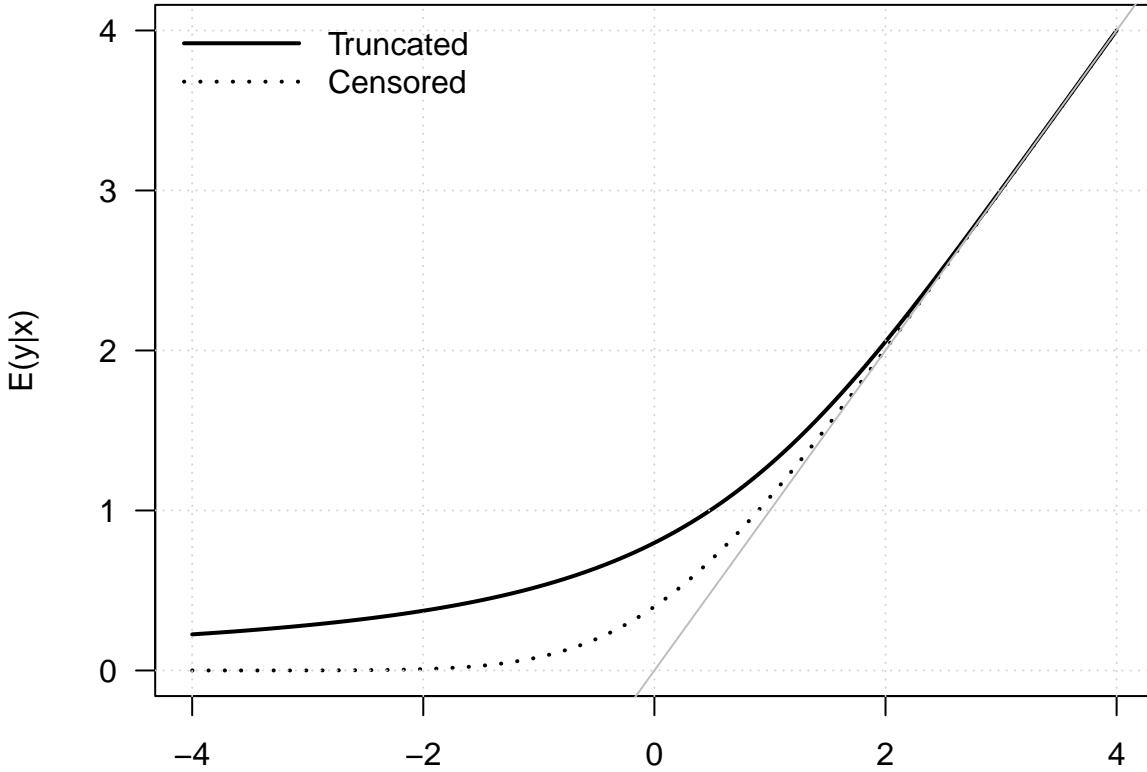


Figure 8.10: Conditional means of  $y$  in Tobit models. The model is  $y_i = x_i + \varepsilon_i$ , with  $\varepsilon_i \sim \mathcal{N}(0, 1)$ .

### Heckit regression

The previous formula (Eq. (8.20)) can in particular be used in an alternative estimation approach, namely the Heckman two-step estimation. This approach is based on two steps:<sup>3</sup>

1. Using the complete sample, fit a Probit model of  $\mathbb{I}_{\{y_i > 0\}}$  on  $\mathbf{x}$ . This provides a consistent estimate of  $\frac{\beta}{\sigma}$ , and therefore of  $\lambda(\beta' \mathbf{x}/\sigma)$ . (Indeed, if  $z_i \equiv \mathbb{I}_{\{y_i > 0\}}$ , then  $\mathbb{P}(z_i = 1 | \mathbf{x}_i; \beta/\sigma) = \Phi(\beta' \mathbf{x}_i/\sigma)$ .)

<sup>3</sup>See 16.10.2 of Cameron and Trivedi (2005) for the derivation of asymptotic standard errors of  $\beta$ .

2. Using the truncated sample only: run an OLS regression of  $\mathbf{y}$  on  $\{\mathbf{x}, \lambda(\beta' \mathbf{x}/\sigma)\}$  (having Eq. (8.20) in mind). This provides a consistent estimate of  $(\beta, \sigma)$ .

The underlying specification is of the form:

Conditional mean + disturbance.

where “Conditional mean” comes from Eq. (8.20) and “disturbance” is an error with zero conditional mean.

This approach is also applied to the case of **sample selection models** (Section 8.4).

**Example 8.10** (Wage prediction). The present example is based on the dataset used in Mroz (1987) (which is part of the **sampleSelection** package).

```
library(sampleSelection)
library(AER)
data("Mroz87")
Mroz87$lfp.yesno <- NaN
Mroz87$lfp.yesno[Mroz87$lfp==1] <- "yes"
Mroz87$lfp.yesno[Mroz87$lfp==0] <- "no"
Mroz87$lfp.yesno <- as.factor(Mroz87$lfp.yesno)
ols <- lm(wage ~ educ + exper + I(exper^2) + city, data=subset(Mroz87, lfp==1))
tobit <- tobit(wage ~ educ + exper + I(exper^2) + city,
               left = 0, right = Inf,
               data=Mroz87)
Heckit <- heckit(lfp ~ educ + exper + I(exper^2) + city, # selection equation
                  wage ~ educ + exper + I(exper^2) + city, # outcome equation
                  data=Mroz87)

stargazer(ols, Heckit, tobit, no.space = TRUE, type="text", omit.stat = "f")

##
## =====
##                               Dependent variable:
## -----
##                                wage
##          OLS      Heckman      Tobit
##          selection
##          (1)        (2)        (3)
## -----
## educ          0.481***    (0.067)    (0.270)    (0.081)
## exper         0.032       (0.062)    0.430      (0.369)    0.461***    (0.068)
## I(exper2)   -0.0003     (0.002)   -0.008      (0.008)   -0.009***    (0.002)
## city          0.449       (0.318)    0.113      (0.522)    -0.087      (0.378)
## Constant     -2.561***   (0.929)   -12.251     (8.853)   -10.395***   (1.095)
## -----
## Observations      428        753        753
## R2              0.125       0.128
## Adjusted R2     0.117       0.117
## Log Likelihood                         -1,462.700
## rho             1.063
## Inverse Mills Ratio                     5.165 (4.594)
## Residual Std. Error 3.111 (df = 423)
## Wald Test                           153.892*** (df = 4)
## =====
## Note: *p<0.1; **p<0.05; ***p<0.01
```

Figure 8.11 shows that, low wages, the OLS model tends to over-predict wages. The slope between observed and Tobit-predicted wages is closer to one (the adjustment line is closer to the 45-degree line.)

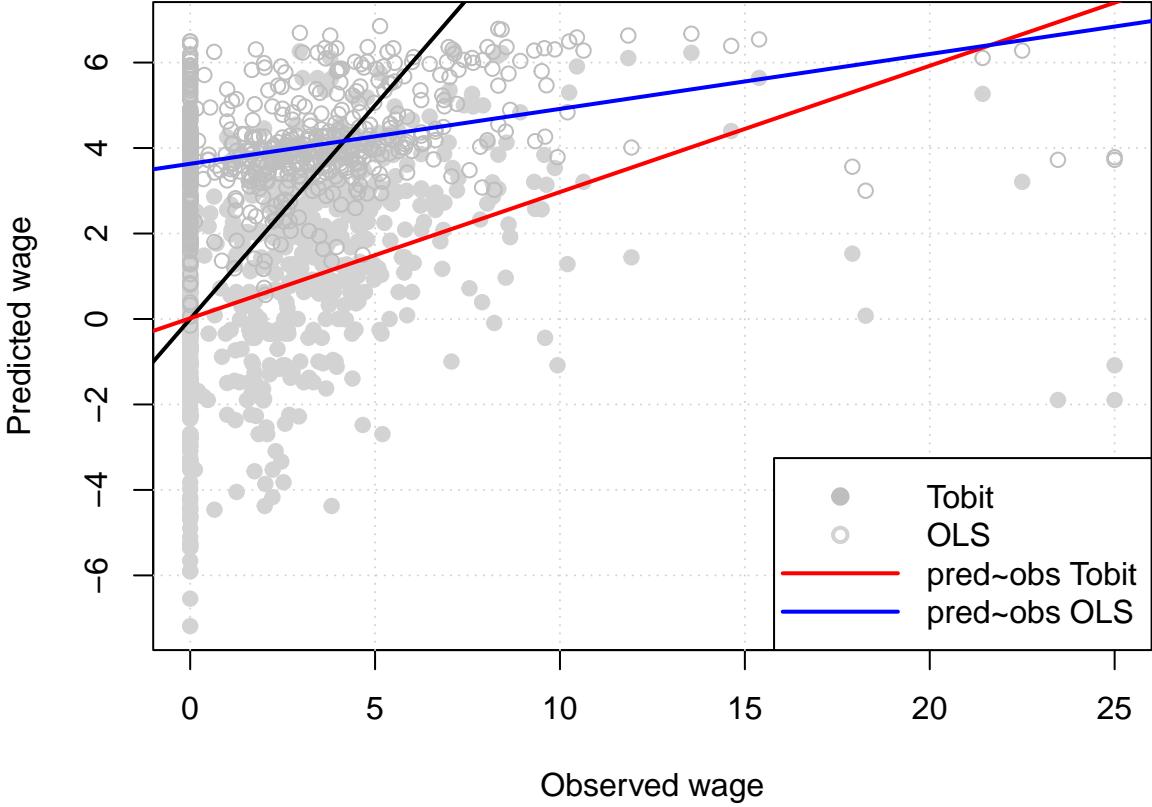


Figure 8.11: Predicted versus observed wages.

### Two-part model

In the standard Tobit framework, the model determining censored—or truncated—data *censoring mechanism* is the same as the one determining non-censored—or observed—data *outcome mechanism*. A two-part model adds flexibility by permitting the zeros and non-zeros to be generated by different densities. The second model characterizes the outcome *conditional on* the outcome being observed.

In a seminal paper, Duan et al. (1983) employ this methodology to account for individual annual hospital expenses. The two models are then as follows:

- 1<sup>st</sup> model:  $\mathbb{P}(hosp = 1|\mathbf{x}) = \Phi(\mathbf{x}'_1 \beta_1)$ ,
- 2<sup>nd</sup> model:  $Expense = \exp(\mathbf{x}'_2 \beta_2 + \eta)$ , with  $\eta \sim i.i.d. \mathcal{N}(0, \sigma^2_2)$ .

Specifically:

$$\mathbb{E}(Expense|\mathbf{x}_1, \mathbf{x}_2) = \Phi(\mathbf{x}'_1 \beta_1) \exp\left(\mathbf{x}'_2 \beta_2 + \frac{\sigma^2_2}{2}\right).$$

In sample-selection models, studied in the next section, one specifies the joint distribution for the censoring and outcome mechanisms (while the two parts are independent here).

## 8.4 Sample Selection Models

The situation tackled by sample-selection models is the following. The dependent variable of interest, denoted by  $y_2$ , depends on observed variables  $\mathbf{x}_2$ . Observing  $y_2$ , or not, depends on the value of a latent variable ( $y_1^*$ ) that is correlated to observed variables  $\mathbf{x}_1$ . The difference w.r.t. the two-part model sketched above is that, even conditionally on  $(\mathbf{x}_1, \mathbf{x}_2)$ ,  $y_1^*$  and  $y_2$  may be correlated.

As in the Tobit case, even in the simplest case of population conditional mean linear in regressors (i.e.  $y_2 = \mathbf{x}'_2 \beta_2 + \varepsilon_2$ ), OLS regression leads to inconsistent parameter estimates because the sample is not representative of the population.

There are two latent variables:  $y_1^*$  and  $y_2^*$ . We observe  $y_1$  and, if the considered entity “participates”, we also observe  $y_2$ . More specifically:

$$\begin{aligned} y_1 &= \begin{cases} 1 & \text{if } y_1^* > 0 \\ 0 & \text{if } y_1^* \leq 0 \end{cases} \quad (\text{participation equation}) \\ y_2 &= \begin{cases} y_2^* & \text{if } y_1 = 1 \\ - & \text{if } y_1 = 0 \end{cases} \quad (\text{outcome equation}). \end{aligned}$$

Moreover:

$$\begin{aligned} y_1^* &= \mathbf{x}'_1 \beta_1 + \varepsilon_1 \\ y_2^* &= \mathbf{x}'_2 \beta_2 + \varepsilon_2. \end{aligned}$$

Note that the Tobit model (Section 8.3) is the special case where  $y_1^* = y_2^*$ .

Usually:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \sim \mathcal{N} \left( \mathbf{0}, \begin{bmatrix} 1 & \rho \sigma_2 \\ \rho \sigma_2 & \sigma_2^2 \end{bmatrix} \right).$$

Let us derive the likelihood associated with this model. We have:

$$f(\underbrace{0}_{=y_1}, \underbrace{-}_{=y_2} | \mathbf{x}; \theta) = \mathbb{P}(y_1^* \leq 0) = \Phi(-\mathbf{x}'_1 \beta_1) \quad (8.21)$$

$$\begin{aligned} f(1, y_2 | \mathbf{x}; \theta) &= f(y_2^* | \mathbf{x}; \theta) \mathbb{P}(y_1^* > 0 | y_2^*, \mathbf{x}; \theta) \\ &= \frac{1}{\sigma} \phi \left( \frac{y_2 - \mathbf{x}'_2 \beta_2}{\sigma} \right) \mathbb{P}(y_1^* > 0 | y_2, \mathbf{x}; \theta). \end{aligned} \quad (8.22)$$

Let us compute  $\mathbb{P}(y_1^* > 0 | y_2, \mathbf{x}; \theta)$ . By Prop. 10.16 (in Appendix 10.4), applied to  $(\varepsilon_1, \varepsilon_2)$ , we have:

$$y_1^* | y_2 \sim \mathcal{N} \left( \mathbf{x}'_1 \beta_1 + \frac{\rho}{\sigma_2} (y_2 - \mathbf{x}'_2 \beta_2), 1 - \rho^2 \right).$$

which leads to

$$\mathbb{P}(y_1^* > 0 | y_2, \mathbf{x}; \theta) = \Phi \left( \frac{\mathbf{x}'_1 \beta_1 + \frac{\rho}{\sigma_2} (y_2 - \mathbf{x}'_2 \beta_2)}{\sqrt{1 - \rho^2}} \right). \quad (8.23)$$

Figure 8.12 displays  $\mathbb{P}(y_1^* > 0 | y_2, \mathbf{x}; \theta)$  for different values of  $y_2$  and of  $\rho$ , in the case where  $\beta_1 = \beta_2 = 0$ .

Using Eqs. (8.21), (8.22) and (8.23), one gets the log-likelihood function:

$$\begin{aligned} \log \mathcal{L}(\theta; \mathbf{y}, \mathbf{X}) &= \sum_{i=1}^n (1 - y_{1,i}) \log \Phi(-\mathbf{x}'_{1,i} \beta_1) + \\ &\quad \sum_{i=1}^n y_{1,i} \log \left( \frac{1}{\sigma} \phi \left( \frac{y_{2,i} - \mathbf{x}'_{2,i} \beta_2}{\sigma} \right) \right) + \\ &\quad \sum_{i=1}^n y_{1,i} \log \left( \Phi \left( \frac{\mathbf{x}'_{1,i} \beta_1 + \frac{\rho}{\sigma_2} (y_{2,i} - \mathbf{x}'_{2,i} \beta_2)}{\sqrt{1 - \rho^2}} \right) \right). \end{aligned}$$

We can also compute conditional expectations:

$$\begin{aligned} \mathbb{E}(y_2^* | y_1 = 1, \mathbf{x}) &= \mathbb{E}(\mathbb{E}(y_2^* | y_1^*, \mathbf{x}) | y_1 = 1, \mathbf{x}) \\ &= \mathbb{E}(\mathbf{x}'_2 \beta_2 + \rho \sigma_2 (y_1^* - \mathbf{x}'_1 \beta_1) | y_1 = 1, \mathbf{x}) \\ &= \mathbf{x}'_2 \beta_2 + \rho \sigma_2 \mathbb{E}(y_1^* - \underbrace{\mathbf{x}'_1 \beta_1}_{=\varepsilon_1 \sim \mathcal{N}(0,1)} | \varepsilon_1 > -\mathbf{x}'_1 \beta_1, \mathbf{x}) \\ &= \mathbf{x}'_2 \beta_2 + \rho \sigma_2 \frac{\phi(-\mathbf{x}'_1 \beta_1)}{1 - \Phi(-\mathbf{x}'_1 \beta_1)} \\ &= \mathbf{x}'_2 \beta_2 + \rho \sigma_2 \frac{\phi(\mathbf{x}'_1 \beta_1)}{\Phi(\mathbf{x}'_1 \beta_1)} = \mathbf{x}'_2 \beta_2 + \rho \sigma_2 \lambda(\mathbf{x}'_1 \beta_1), \end{aligned} \quad (8.24)$$

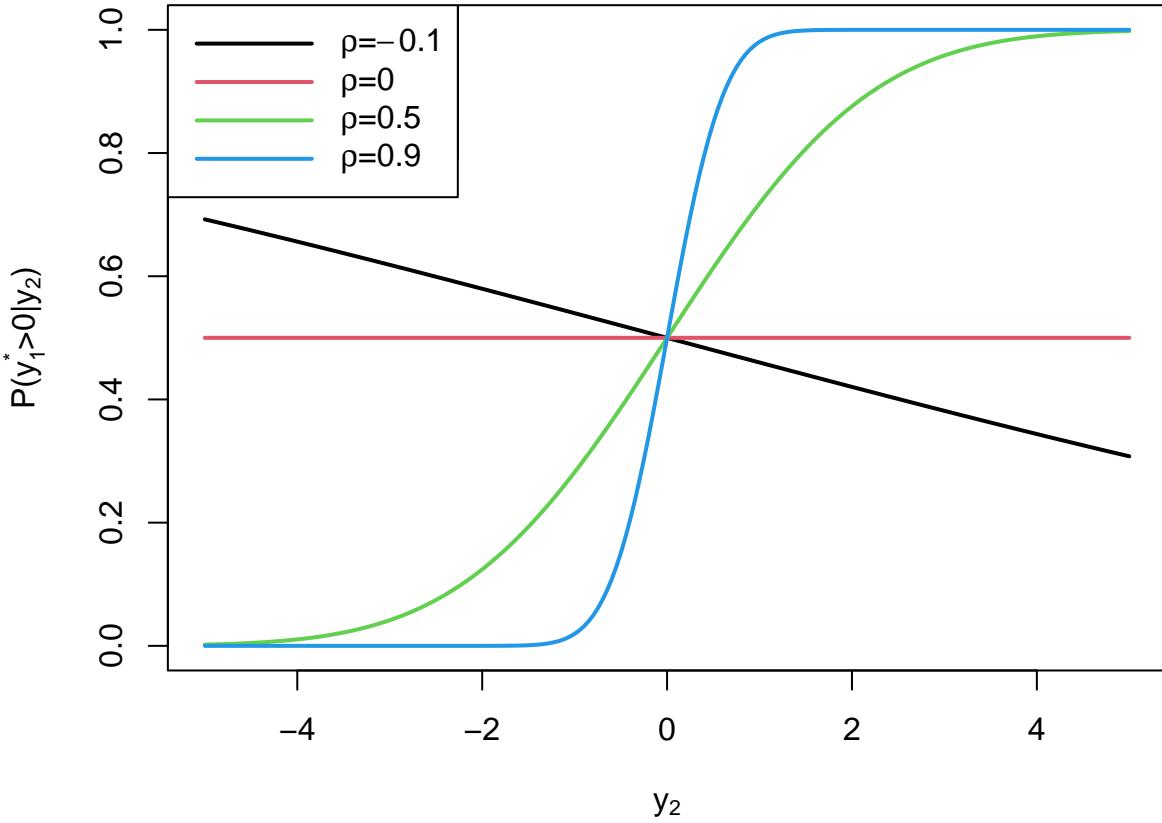


Figure 8.12: Probability of observing  $y_2$  depending on its value, for different values of conditional correlation between  $y_2$  and  $y_1^*$ .

and:

$$\begin{aligned}\mathbb{E}(y_2^*|y_1=0, \mathbf{x}) &= \mathbf{x}_2' \beta_2 + \rho \sigma_2 \mathbb{E}(y_1^* - \mathbf{x}_1' \beta_1 | \varepsilon_1 \leq -\mathbf{x}_1' \beta_1, \mathbf{x}) \\ &= \mathbf{x}_2' \beta_2 + \rho \sigma_2 \frac{\phi(-\mathbf{x}_1' \beta_1)}{1 - \Phi(-\mathbf{x}_1' \beta_1)} \\ &= \mathbf{x}_2' \beta_2 - \rho \sigma_2 \frac{\phi(-\mathbf{x}_1' \beta_1)}{\Phi(-\mathbf{x}_1' \beta_1)} = \mathbf{x}_2' \beta_2 - \rho \sigma_2 \lambda(-\mathbf{x}_1' \beta_1).\end{aligned}$$

### Heckman procedure

As for tobit models (Section 8.3), we can use the Heckman procedure to estimate this model. Eq. (8.24) shows that  $\mathbb{E}(y_2^*|y_1=1, \mathbf{x}) \neq \mathbf{x}_2' \beta_2$  when  $\rho \neq 0$ . Therefore, the OLS approach yields biased estimates based when it is employed only on the sub-sample where  $y_1 = 1$ .

The Heckman two-step procedure (or “Heckit”) consists in replacing  $\lambda(\mathbf{x}_1' \beta_1)$  appearing in Eq. (8.24) with a consistent estimate of it. More precisely:

1. Get an estimate  $\widehat{\beta}_1$  of  $\beta_1$  (probit regression of  $y_1$  on  $\mathbf{x}_1$ ).
2. Run the OLS regression (using only data associated with  $y_1 = 1$ ):

$$y_2 = \mathbf{x}_2' \beta_2 + \rho \sigma_2 \lambda(\mathbf{x}_1' \widehat{\beta}_1) + \varepsilon_2, \quad (8.25)$$

considering  $\lambda(\mathbf{x}_1' \widehat{\beta}_1)$  as a regressor.

How to estimate  $\sigma_2^2$ ? By Eq. (10.6), we have:

$$\mathbb{V}ar(y_2|y_1^* > 0, \mathbf{x}) = \mathbb{V}ar(\varepsilon_2|\varepsilon_1 > -\mathbf{x}_1' \beta_1, \mathbf{x}).$$

Using that  $\varepsilon_2$  can be decomposed as  $\rho \sigma_2 \varepsilon_1 + \xi$ , where  $\xi \sim \mathcal{N}(0, \sigma_2^2(1 - \rho^2))$  is independent from  $\varepsilon_1$ , we get:

$$\mathbb{V}ar(y_2|y_1^* > 0, \mathbf{x}) = \sigma_2^2(1 - \rho^2) + \rho^2 \sigma_2^2 \mathbb{V}ar(\varepsilon_1|\varepsilon_1 > -\mathbf{x}_1' \beta_1, \mathbf{x}).$$

Using Eq. (10.7), we get:

$$\mathbb{V}ar(\varepsilon_1|\varepsilon_1 > -\mathbf{x}_1' \beta_1, \mathbf{x}) = 1 - \mathbf{x}_1' \beta_1 \lambda(\mathbf{x}_1' \beta_1) - \lambda(\mathbf{x}_1' \beta_1)^2,$$

which gives

$$\begin{aligned}\mathbb{V}ar(y_2|y_1^* > 0, \mathbf{x}) &= \sigma_2^2(1 - \rho^2) + \rho^2\sigma_2^2(1 - \mathbf{x}'_1\beta_1\lambda(\mathbf{x}'_1\beta_1) - \lambda(\mathbf{x}'_1\beta_1)^2) \\ &= \sigma_2^2 - \rho^2\sigma_2^2(\mathbf{x}'_1\beta_1\lambda(\mathbf{x}'_1\beta_1) + \lambda(\mathbf{x}'_1\beta_1)^2),\end{aligned}$$

and, finally:

$$\sigma_2^2 \approx \widehat{\mathbb{V}ar}(y_2|y_1^* > 0, \mathbf{x}) + \widehat{\rho\sigma_2}^2 (\widehat{\mathbf{x}'_1\beta_1}\lambda(\widehat{\mathbf{x}'_1\beta_1}) + \lambda(\widehat{\mathbf{x}'_1\beta_1})^2).$$

The Heckman procedure is computationally simple. Although computational costs are no longer an issue, the two-step solution allows certain generalisations more easily than ML, and is more robust in certain circumstances. The computation of parameter standard errors is fairly complicated because of the two steps (see Cameron and Trivedi (2005), Subsection 16.10.2). Bootstrap can be resorted to.

**Example 8.11** (Wage prediction). As in Example 8.10, let us use the Mroz (1987) dataset again, with the objective of explaining wage setting.

```
library(sampleSelection)
library(AER)
data("Mroz87")
Mroz87$lfp.yesno <- NaN
Mroz87$lfp.yesno[Mroz87$lfp==1] <- "yes"
Mroz87$lfp.yesno[Mroz87$lfp==0] <- "no"
Mroz87$lfp.yesno <- as.factor(Mroz87$lfp.yesno)
# Logit & Probit (selection equation)
logitW <- glm(lfp ~ age + I( age^2 ) + kids5 + huswage + educ,
               family = binomial(link = "logit"), data = Mroz87)
probitW <- glm(lfp ~ age + I( age^2 ) + kids5 + huswage + educ,
                family = binomial(link = "probit"), data = Mroz87)
# OLS for outcome:
ols1 <- lm(log(wage) ~ educ+exper+I( exper^2 )+city, data=subset(Mroz87,lfp==1))
# Two-step Heckman estimation
heckvan <-
  heckit( lfp ~ age + I( age^2 ) + kids5 + huswage + educ, # selection equation
          log(wage) ~ educ + exper + I( exper^2 ) + city, # outcome equation
          data=Mroz87 )
# Maximum likelihood estimation of selection model:
ml <- selection(lfp~age+I(age^2)+kids5+huswage+educ,
                 log(wage)~educ+exper+I(exper^2)+city, data = Mroz87)
# Print selection-equation estimates:
stargazer(logitW,probitW,heckvan,ml,type = "text",no.space = TRUE,
           selection.equation = TRUE)
```

```
##
## =====
##                               Dependent variable:
## -----
##                                     lfp
## logistic      probit      Heckman      selection
##                                     selection
## (1)          (2)          (3)          (4)
## -----
## age          0.012       0.010       0.010       0.010
##             (0.114)     (0.069)     (0.069)     (0.069)
## I(age2)    -0.001      -0.0005     -0.0005     -0.0005
##             (0.001)     (0.001)     (0.001)     (0.001)
## kids5      -1.409***   -0.855***   -0.855***   -0.854*** 
##             (0.198)     (0.116)     (0.115)     (0.116)
## huswage    -0.069***   -0.042***   -0.042***   -0.042***
```

```

##                               (0.020)   (0.012)   (0.012)   (0.013)
## educ                  0.244***  0.148***  0.148***  0.148***
##                               (0.040)   (0.024)   (0.023)   (0.024)
## Constant                -0.938    -0.620    -0.620    -0.615
##                               (2.508)   (1.506)   (1.516)   (1.518)
## -----
## Observations            753      753      753      753
## R2                      0.158
## Adjusted R2              0.148
## Log Likelihood          -459.955  -459.901  -891.177
## Akaike Inf. Crit.       931.910  931.802
## rho                     0.018     0.014 (0.203)
## Inverse Mills Ratio     0.012 (0.152)
## -----
## Note:                   *p<0.1; **p<0.05; ***p<0.01

```

```

# Print outcome-equation estimates:
stargazer(ols1,heckvan,ml,type = "text",no.space = TRUE,omit.stat = "f")

```

```

## -----
##                               log(wage)
##                               OLS        Heckman    selection
##                               selection
##                               (1)        (2)        (3)
## -----
## educ                  0.106***   0.106***  0.106*** 
##                               (0.014)    (0.017)    (0.017)
## exper                 0.041***   0.041***  0.041*** 
##                               (0.013)    (0.013)    (0.013)
## I(exper2)             -0.001**   -0.001**  -0.001** 
##                               (0.0004)   (0.0004)   (0.0004)
## city                  0.054       0.053      0.053
##                               (0.068)    (0.069)    (0.069)
## Constant               -0.531***  -0.547*   -0.544** 
##                               (0.199)    (0.289)    (0.272)
## -----
## Observations           428       753       753
## R2                      0.158
## Adjusted R2              0.150      0.148
## Log Likelihood          -891.177
## rho                     0.018     0.014 (0.203)
## Inverse Mills Ratio     0.012 (0.152)
## Residual Std. Error 0.667 (df = 423)
## -----
## Note:                   *p<0.1; **p<0.05; ***p<0.01

```

## 8.5 Models of Count Data

Count-data models aim at explaining dependent variables  $y_i$  that take integer values. Typically, one may want to account for the number of doctor visits, of customers, of hospital stays, of borrowers' defaults, of recreational trips, of accidents. Quite often, these data feature large proportion of zeros (see, e.g., Table 20.1 in Cameron and Trivedi (2005)), and/or are skewed to the right.

### 8.5.1 Poisson model

The most basic count-data model is the Poisson model. In this model, we have  $y \sim \mathcal{P}(\mu)$ , i.e.

$$\mathbb{P}(y = k) = \frac{\mu^k e^{-\mu}}{k!},$$

implying  $\mathbb{E}(y) = \text{Var}(y) = \mu$ .

the Poisson parameter,  $\mu$ , is then assumed to depend on some observed variables, gathered in vector  $\mathbf{x}_i$  for entity  $i$ . To ensure that  $\mu_i \geq 0$ , it is common to take  $\mu_i = \exp(\beta' \mathbf{x}_i)$ , which gives:

$$y_i \sim \mathcal{P}(\exp[\beta' \mathbf{x}_i]).$$

The Poisson regression is intrinsically heteroskedastic (since  $\text{Var}(y_i) = \mu_i = \exp(\beta' \mathbf{x}_i)$ ).

Under the assumption of independence across entities, the log-likelihood is given by:

$$\log \mathcal{L}(\beta; \mathbf{y}, \mathbf{X}) = \sum_{i=1}^n (y_i \beta' \mathbf{x}_i - \exp[\beta' \mathbf{x}_i] - \ln[y_i!]).$$

The first-order condition to get the MLE is:

$$\sum_{i=1}^n (y_i - \exp[\beta' \mathbf{x}_i]) \mathbf{x}_i = \underset{K \times 1}{\mathbf{0}}. \quad (8.26)$$

Eq. (8.26) is equivalent to what would define the **Pseudo Maximum-Likelihood** estimator of  $\beta$  in the (misspecified) model

$$y_i \sim i.i.d. \mathcal{N}(\exp[\beta' \mathbf{x}_i], \sigma^2).$$

That is, Eq. (8.26) also characterizes the (true) ML estimator of  $\beta$  in the previous model.

Since  $\mathbb{E}(y_i | \mathbf{x}_i) = \exp(\beta' \mathbf{x}_i)$ , we have:

$$y_i = \exp(\beta' \mathbf{x}_i) + \varepsilon_i,$$

with  $\mathbb{E}(\varepsilon_i | \mathbf{x}_i) = 0$ . This notably implies that the (N)LS estimator of  $\beta$  is consistent.

How to interpret regression coefficients (the components of  $\beta$ )? We have:

$$\frac{\partial \mathbb{E}(y_i | \mathbf{x}_i)}{\partial x_{i,j}} = \beta_j \exp(\beta' \mathbf{x}_i),$$

which depends on the considered individual.

The average estimated response is:

$$\hat{\beta}_j \frac{1}{n} \sum_{i=1}^n \exp(\hat{\beta}' \mathbf{x}_i),$$

which is equal to  $\hat{\beta}_j \bar{y}$  if the model includes a constant (e.g., if  $x_{1,i} = 1$  for all entities  $i$ ).

The limitation of the standard Poisson model is that the distribution of  $y_i$  conditional on  $\mathbf{x}_i$  depends on a single parameter ( $\mu_i$ ). Besides, there is often a tension between fitting the fraction of zeros, i.e.  $\mathbb{P}(y_i = 0 | \mathbf{x}_i) = \exp[-\exp(\beta' \mathbf{x}_i)]$ , and the distribution of  $y_i | \mathbf{x}_i, y_i > 0$ . The following models (negative binomial, or NB model, the Hurdle model, and the Zero-Inflated model) have been designed to address these points.

### 8.5.2 Negative binomial model

In the negative binomial model, we have:

$$y_i | \lambda_i \sim \mathcal{P}(\lambda_i),$$

but  $\lambda_i$  is now random. Specifically, it takes the form:

$$\lambda_i = \nu_i \times \exp(\beta' \mathbf{x}_i),$$

where  $\nu_i \sim i.i.d. \Gamma(\frac{\delta}{\text{shape}}, \frac{1}{\text{scale}})$ . That is, the p.d.f. of  $\nu_i$  is:

$$g(\nu) = \frac{\nu^{\delta-1} e^{-\nu\delta} \delta^\delta}{\Gamma(\delta)},$$

where  $\Gamma : z \mapsto \int_0^{+\infty} t^{z-1} e^{-t} dt$  (and  $\Gamma(k+1) = k!$ ).

This notably implies that:

$$\mathbb{E}(\nu_i) = 1 \quad \text{and} \quad \text{Var}(\nu) = \frac{1}{\delta}.$$

Hence, the p.d.f. of  $y_i$  conditional on  $\mu$  and  $\delta$  (with  $\mu = \exp(\beta' \mathbf{x}_i)$ ) is obtained as a mixture of densities:

$$\mathbb{P}(y_i = k | \exp(\beta' \mathbf{x}_i) = \mu; \delta) = \int_0^{\infty} \frac{e^{-\mu\nu} (\mu\nu)^k}{k!} \frac{\nu^{\delta-1} e^{-\nu\delta} \delta^\delta}{\Gamma(\delta)} d\nu.$$

It can be shown that:

$$\mathbb{E}(y|\mathbf{x}) = \mu \quad \text{and} \quad \text{Var}(y|\mathbf{x}) = \mu(1 + \alpha\mu),$$

where  $\exp(\beta' \mathbf{x}_i) = \mu$  and  $\alpha = 1/\delta$ .

We have one additional degree of freedom w.r.t. the Poisson model ( $\alpha$ ).

Note that  $\text{Var}(y|\mathbf{x}) > \mathbb{E}(y|\mathbf{x})$  (which is often called for by the data). Moreover, the conditional variance is quadratic in the mean:

$$\text{Var}(y|\mathbf{x}) = \mu + \alpha\mu^2.$$

The previous expression is the basis of the so-called **NB2** specification. If  $\delta$  is replaced with  $\mu/\gamma$ , then we get the **NB1** model:

$$\text{Var}(y|\mathbf{x}) = \mu(1 + \gamma).$$

**Example 8.12** (Number of doctor visits). The following example compares different specifications, namely a linear regression model, a Poisson model, and a NB model, to account for the number of doctor visits. The dataset ('randdata') is the one used in Chapter 20 of Cameron and Trivedi (2005) (available on that page).

```
library(AEC)
library(COUNT)
library(pscl) # for predprob function and hurdle model
par(plt=c(.15,.95,.1,.95))
plot(table(randdata$mdvis))

randdata$LC <- log(1 + randdata$coins)
model.OLS <- lm(mdvis ~ LC + idp + lpi + fmde + physlm + disea + hlthg +
                  hlthf + hlthp - 1, data=randdata)
model.poisson <- glm(mdvis ~ LC + idp + lpi + fmde + physlm + disea +
                      hlthg + hlthf + hlthp - 1, data=randdata, family = poisson)
model.neg.bin <- glm.nb(mdvis ~ LC + idp + lpi + fmde + physlm + disea +
                        hlthg + hlthf + hlthp - 1, data=randdata)
model.neg.bin.with.intercept <-
  glm.nb(mdvis ~ LC + idp + lpi + fmde + physlm + disea + hlthg +
         hlthf + hlthp, data=randdata)
stargazer::stargazer(model.OLS, model.poisson, model.neg.bin,
                      model.neg.bin.with.intercept, type="text",
                      no.space = TRUE, omit.stat = c("f", "ser"))
```

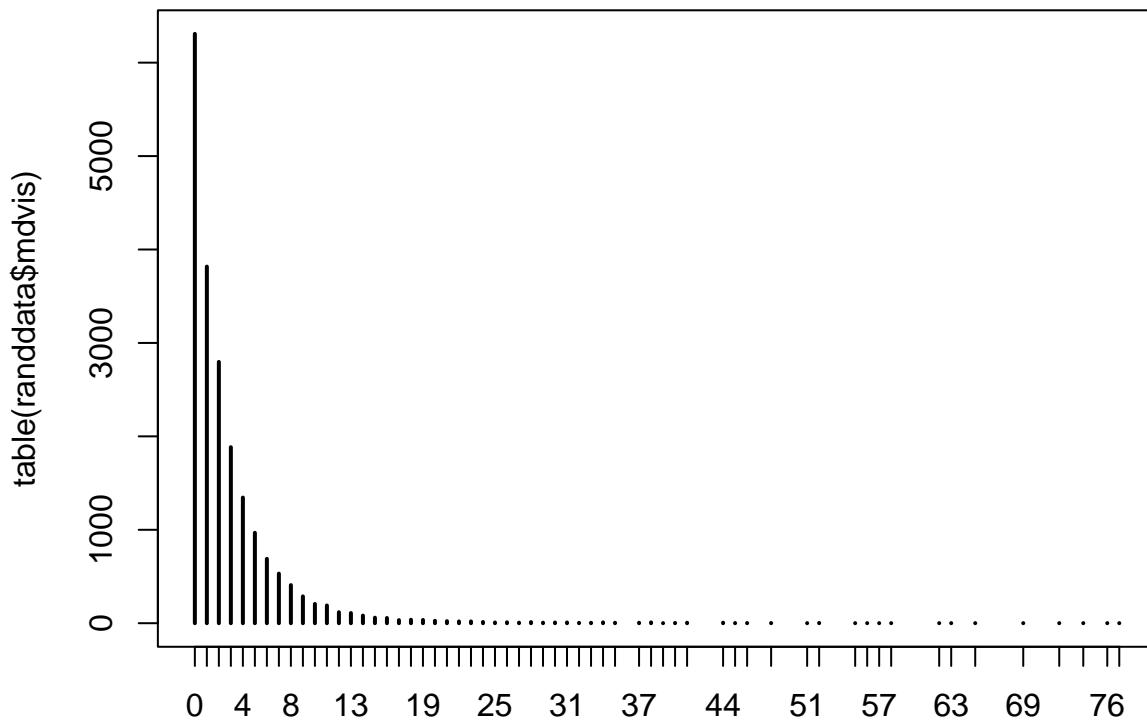


Figure 8.13: Distribution of the number of doctor visits.

```
##  
## ======  
## Dependent variable:  
## -----  
##  
##          OLS      Poisson      negative  
##                   binomial  
##          (1)       (2)       (3)       (4)  
## -----  
## LC        -0.155***   -0.051***   -0.057***   -0.058***  
##           (0.020)     (0.003)     (0.006)     (0.006)  
## idp       -0.546***   -0.183***   -0.212***   -0.268***  
##           (0.075)     (0.011)     (0.023)     (0.023)  
## lpi        0.230***   0.095***   0.088***   0.041***  
##           (0.012)     (0.002)     (0.004)     (0.004)  
## fmde      -0.073***   -0.029***   -0.030***   -0.038***  
##           (0.012)     (0.002)     (0.004)     (0.003)  
## physlm    0.945***   0.217***   0.229***   0.269***  
##           (0.104)     (0.013)     (0.031)     (0.030)  
## disea     0.177***   0.050***   0.062***   0.038***  
##           (0.004)     (0.0005)    (0.001)     (0.001)  
## hlthg     0.270***   0.126***   0.068***   -0.044**  
##           (0.066)     (0.009)     (0.020)     (0.020)  
## hlthf     0.455***   0.149***   0.084**    0.017  
##           (0.123)     (0.016)     (0.037)     (0.036)  
## hlthp     1.537***   0.197***   0.185**   0.178**  
##           (0.263)     (0.027)     (0.076)     (0.074)  
## Constant          0.664***  
##                   (0.025)  
## -----  
## Observations    20,190      20,190      20,190      20,190  
## R2              0.322  
## Adjusted R2     0.322  
## Log Likelihood -64,221.340   -43,745.860   -43,384.660  
## theta           0.732*** (0.010)  0.773*** (0.011)  
## Akaike Inf. Crit. 128,460.700   87,509.710   86,789.320  
## ======  
## Note: *p<0.1; **p<0.05; ***p<0.01
```

Models' predictions can be obtained as follows:

```
# prediction of beta'x, equivalent to "model.poisson$fitted.values":
predict_poisson.beta.x <- predict(model.poisson)
# prediction of the number of events (exp(beta'x)):
predict_poisson <- predict(model.poisson,type="response")
predict_NB <- model.neg.bin$fitted.values
```

Let us now compute the model-implied probabilities, and let's compare them with the frequencies observed in the data.

```
prop.table.data <- prop.table(table(randdata$mdvis))
predprob.poisson <- predprob(model.poisson) # part of pscl package
predprob.nb      <- predprob(model.neg.bin)
print(rbind(prop.table.data[1:6],
            apply(predprob.poisson[,1:6], 2, mean),
            apply(predprob.nb[,1:6], 2, mean)))
```

```
##          0         1         2         3         4         5
## [1,] 0.3124319 0.1890540 0.1385339 0.09331352 0.06661714 0.04794453
## [2,] 0.1220592 0.2230328 0.2271621 0.17173478 0.10888940 0.06244353
## [3,] 0.3486824 0.1884640 0.1219340 0.08385899 0.05968603 0.04350209
```

It appears that the NB model is better at capturing the relatively large number of zeros than the Poisson model. This will also be the case for the Hurdle and Zero-Inflation models:

### 8.5.3 Hurdle model

The main objective of this model, w.r.t. the Poisson model, is to generate more zeros in the data than predicted by the previous count models. The idea is to separate the modeling of the number of zeros and that of the number of non-zero counts. Specifically, the frequency of zeros is determined by  $f_1$ , the (relative) frequencies of non-zero counts are determined by  $f_2$ :

$$f(y) = \begin{cases} f_1(0) & \text{if } y = 0, \\ \frac{1 - f_1(0)}{1 - f_2(0)} f_2(y) & \text{if } y > 0. \end{cases}$$

Note that we are back to the standard Poisson model if  $f_1 \equiv f_2$ . This model is straightforwardly estimated by ML.

### 8.5.4 Zero-inflated model

The objective is the same as for the Hurdle model, the modeling is slightly different. It is based on a mixture of a binary process  $B$  (p.d.f.  $f_1$ ) and a process  $Z$  (p.d.f.  $f_2$ ).  $B$  and  $Z$  are independent. Formally:

$$y = BZ,$$

implying:

$$f(y) = \begin{cases} f_1(0) + (1 - f_1(0))f_2(0) & \text{if } y = 0, \\ (1 - f_1(0))f_2(y) & \text{if } y > 0. \end{cases}$$

Typically,  $f_1$  corresponds to a logit model and  $f_2$  is Poisson or negative binomial density. This model is easily estimated by ML techniques.

**Example 8.13** (Number of doctor visits). Let us come back to the data used in Example 8.12, and estimate Hurdle and a zero-inflation models:

```

model.hurdle <-
  hurdle(mdvis ~ LC + idp + lpi + fmde + physlm + disea + hlthg + hlthf +
    hlthp, data=randdata,
    dist = "poisson", zero.dist = "binomial", link = "logit")
model.zeroinfl <- zeroinfl(mdvis ~ LC + idp + lpi + fmde + physlm +
  disea + hlthg + hlthf + hlthp, data=randdata,
  dist = "poisson", link = "logit")
stargazer(model.hurdle,model.zeroinfl,zero.component=FALSE,
  no.space=TRUE,type="text")

```

```

##
## =====
##             Dependent variable:
## -----
##                   mdvis
##       hurdle      zero-inflated
##                   count data
##       (1)          (2)
## -----
##  LC          -0.015***   -0.015***
##              (0.003)     (0.003)
##  idp         -0.085***   -0.086***  

##              (0.011)     (0.011)
##  lpi          0.010***   0.010***  

##              (0.002)     (0.002)
##  fmde        -0.021***   -0.021***  

##              (0.002)     (0.002)
##  physlm      0.231***   0.231***  

##              (0.012)     (0.012)
##  disea        0.022***   0.022***  

##              (0.001)     (0.001)
##  hlthg        0.027***   0.026***  

##              (0.010)     (0.010)
##  hlthf        0.147***   0.146***  

##              (0.016)     (0.016)
##  hlthp        0.304***   0.303***  

##              (0.026)     (0.026)
##  Constant     1.133***   1.133***  

##              (0.012)     (0.012)
## -----
##  Observations   20,190      20,190
##  Log Likelihood -54,772.100  -54,772.550
## =====
## Note:           *p<0.1; **p<0.05; ***p<0.01

```

```

stargazer(model.hurdle,model.zeroinfl,zero.component=TRUE,
  no.space=TRUE,type="text")

```

```

##
## =====
##             Dependent variable:
## -----
##                   mdvis
##       hurdle      zero-inflated
##                   count data
##       (1)          (2)
## -----
##  LC          -0.150***   0.154***  

##              (0.012)     (0.012)
##  Observations   20,190      20,190
##  Log Likelihood -54,772.100  -54,772.550
## =====
## Note:           *p<0.1; **p<0.05; ***p<0.01

```

```

##          (0.010)      (0.011)
## idp     -0.631***    0.637*** 
##          (0.038)      (0.040)
## lpi      0.102***   -0.105*** 
##          (0.007)      (0.007)
## fmde    -0.062***    0.060*** 
##          (0.006)      (0.006)
## physlm   0.239***   -0.203*** 
##          (0.056)      (0.058)
## disea    0.062***   -0.059*** 
##          (0.003)      (0.003)
## hlthg   -0.142***    0.158*** 
##          (0.034)      (0.036)
## hlthf   -0.352***    0.396*** 
##          (0.062)      (0.064)
## hlthp    -0.181       0.233
##          (0.149)      (0.151)
## Constant 0.411***   -0.528*** 
##          (0.044)      (0.047)
## -----
## Observations   20,190      20,190
## Log Likelihood -54,772.100 -54,772.550
## =====
## Note:          *p<0.1; **p<0.05; ***p<0.01

```

Let us test the importance of LC in the model using a Wald test:

```

# Test whether LC is important in the model:
model.hurdle.reduced <- update(model.hurdle,.~.-LC)
lmtest::waldtest(model.hurdle, model.hurdle.reduced)

```

```

## Wald test
##
## Model 1: mdvis ~ LC + idp + lpi + fmde + physlm + disea + hlthg + hlthf +
##           hlthp
## Model 2: mdvis ~ idp + lpi + fmde + physlm + disea + hlthg + hlthf + hlthp
##   Res.Df Df  Chisq Pr(>Chisq)
## 1  20170
## 2  20172 -2 247.64 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

```

Finally, we compare average model-implied probabilities with the frequencies observed in the data:

```

predprob.hurdle <- predprob(model.hurdle)
predprob.zeroinfl <- predprob(model.zeroinfl)
print(rbind(prop.table.data[,1:6],
  apply(predprob.poisson[,1:6],2,mean),
  apply(predprob.nb[,1:6],2,mean),
  apply(predprob.hurdle[,1:6],2,mean),
  apply(predprob.zeroinfl[,1:6],2,mean)))

```

	0	1	2	3	4	5
[1,]	0.3124319	0.18905399	0.1385339	0.09331352	0.06661714	0.04794453
[2,]	0.1220592	0.22303277	0.2271621	0.17173478	0.10888940	0.06244353
[3,]	0.3486824	0.18846395	0.1219340	0.08385899	0.05968603	0.04350209
[4,]	0.3124319	0.06056959	0.1083120	0.13262624	0.12553899	0.09847017
[5,]	0.3124684	0.06053026	0.1082799	0.13262562	0.12556531	0.09850218



# Chapter 9

## Time Series

### 9.1 Introduction to time series

A time series is an infinite sequence of random variables indexed by time:  $\{y_t\}_{t=-\infty}^{+\infty} = \{\dots, y_{-2}, y_{-1}, y_0, y_1, \dots, y_t, \dots\}$ ,  $y_i \in \mathbb{R}^k$ . In practice, we only observe samples, typically:  $\{y_1, \dots, y_T\}$ .

Standard time series models are built using **shocks** that we will often denote by  $\varepsilon_t$ . Typically,  $\mathbb{E}(\varepsilon_t) = 0$ . In many models, the shocks are supposed to be i.i.d., but there exist other (less restrictive) notions of shocks. In particular, the definition of many processes is based on white noises:

**Definition 9.1** (White noise). The process  $\{\varepsilon_t\}_{t \in ]-\infty, +\infty[}$  is a white noise if, for all  $t$ :

- a.  $\mathbb{E}(\varepsilon_t) = 0$ ,
- b.  $\mathbb{E}(\varepsilon_t^2) = \sigma^2 < \infty$  and
- c. for all  $s \neq t$ ,  $\mathbb{E}(\varepsilon_t \varepsilon_s) = 0$ .

Another type of shocks that are commonly used are Martingale Difference Sequences:

**Definition 9.2** (Martingale Difference Sequence). The process  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  is a martingale difference sequence (MDS) if  $\mathbb{E}(|\varepsilon_t|) < \infty$  and if, for all  $t$ ,

$$\underbrace{\mathbb{E}_{t-1}(\varepsilon_t)}_{\text{Expectation conditional on the past}} = 0.$$

By definition, if  $y_t$  is a martingale, then  $y_t - y_{t-1}$  is a MDS.

**Example 9.1** (ARCH process). The Autoregressive conditional heteroskedasticity (ARCH) process is an example of shock that satisfies the MDS definition but that is not i.i.d.:

$$\varepsilon_t = \sigma_t \times z_t,$$

where  $z_t \sim i.i.d. \mathcal{N}(0, 1)$  and  $\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2$ .

**Example 9.2.** A white noise process is not necessarily a MDS. This is for instance the following process:

$$\varepsilon_t = z_t + z_{t-1} z_{t-2},$$

where  $z_t \sim i.i.d. \mathcal{N}(0, 1)$ .

Let us now introduce the lag operator. The lag operator, denoted by  $L$ , is defined on the time series space and is defined by:

$$L : \{y_t\}_{t=-\infty}^{+\infty} \rightarrow \{w_t\}_{t=-\infty}^{+\infty} \quad \text{with} \quad w_t = y_{t-1}. \tag{9.1}$$

We have:  $L^2 y_t = y_{t-2}$  and, more generally,  $L^k y_t = y_{t-k}$ .

Consider a time series  $y_t$  defined by  $y_t = \mu + \phi y_{t-1} + \varepsilon_t$ , where the  $\varepsilon_t$ 's are i.i.d.  $\mathcal{N}(0, \sigma^2)$ . Using the lag operator, the dynamics of  $y_t$  can be expressed as follows:

$$(1 - \phi L)y_t = \mu + \varepsilon_t.$$

It is easily checked that we have  $L^2 y_t = y_{t-2}$  and, generally,  $L^k y_t = y_{t-k}$ .

If it exists, the **unconditional (or marginal) mean** of the random variable  $y_t$  is given by:

$$\mu_t := \mathbb{E}(y_t) = \int_{-\infty}^{\infty} y_t f_{Y_t}(y_t) dy_t,$$

where  $f_{Y_t}$  is the unconditional (or marginal) density of  $y_t$ . Similarly, if it exists, the **unconditional (or marginal) variance** of the random variable  $y_t$  is:

$$\text{Var}(y_t) = \int_{-\infty}^{\infty} (y_t - \mathbb{E}(y_t))^2 f_{Y_t}(y_t) dy_t.$$

**Definition 9.3** (Autocovariance). The  $j^{th}$  autocovariance of  $y_t$  is given by:

$$\begin{aligned} \gamma_{j,t} &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} [y_t - \mathbb{E}(y_t)][y_{t-j} - \mathbb{E}(y_{t-j})] \times \\ &\quad f_{Y_t, Y_{t-1}, \dots, Y_{t-j}}(y_t, y_{t-1}, \dots, y_{t-j}) dy_t dy_{t-1} \dots dy_{t-j} \\ &= \mathbb{E}([y_t - \mathbb{E}(y_t)][y_{t-j} - \mathbb{E}(y_{t-j})]), \end{aligned}$$

where  $f_{Y_t, Y_{t-1}, \dots, Y_{t-j}}(y_t, y_{t-1}, \dots, y_{t-j})$  is the joint distribution of  $y_t, y_{t-1}, \dots, y_{t-j}$ .

In particular,  $\gamma_{0,t} = \text{Var}(y_t)$ .

**Definition 9.4** (Covariance stationarity). The process  $y_t$  is covariance stationary —or weakly stationary— if, for all  $t$  and  $j$ ,

$$\mathbb{E}(y_t) = \mu \quad \text{and} \quad \mathbb{E}\{(y_t - \mu)(y_{t-j} - \mu)\} = \gamma_j.$$

Figure 9.1 displays the simulation of a process that is not covariance stationary. This process follows  $y_t = 0.1t + \varepsilon_t$ , where  $\varepsilon_t \sim \text{i.i.d. } \mathcal{N}(0, 1)$ . Indeed, for such a process, we have:  $\mathbb{E}(y_t) = 0.1t$ , which depends on  $t$ .

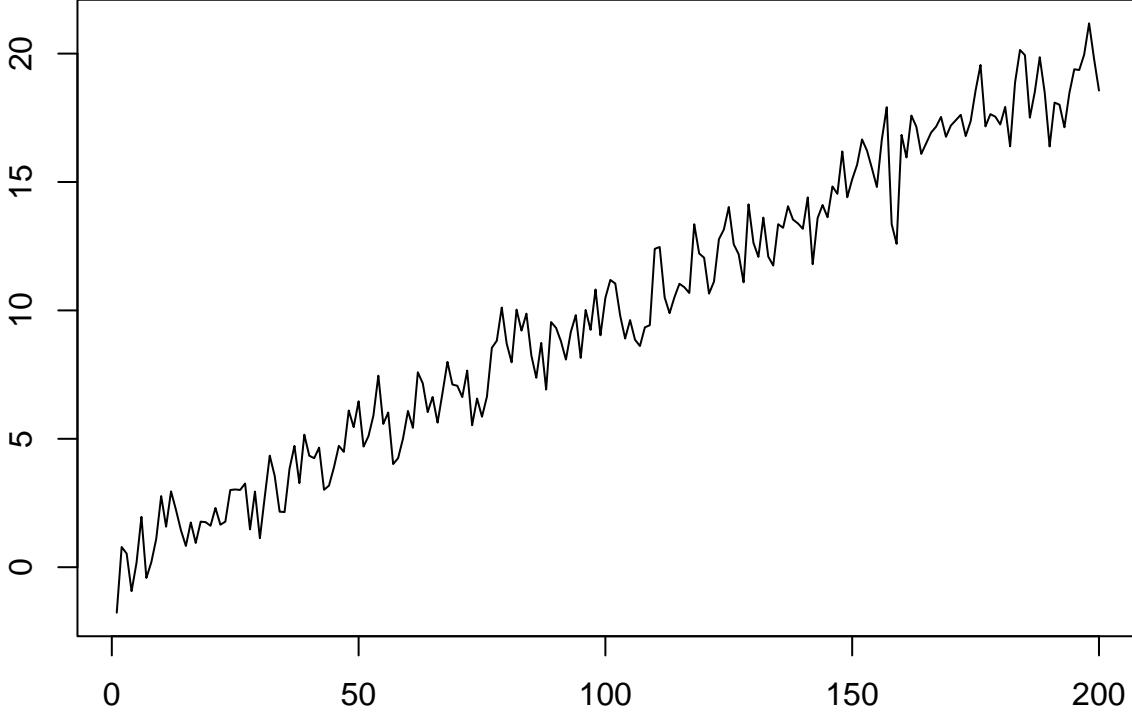


Figure 9.1: Example of a process that is not covariance stationary ( $y_t = 0.1t + \varepsilon_t$ , where  $\varepsilon_t \sim \mathcal{N}(0, 1)$ ).

**Definition 9.5** (Strict stationarity). The process  $y_t$  is strictly stationary if, for all  $t$  and all sets of integers  $J = \{j_1, \dots, j_n\}$ , the distribution of  $(y_t, y_{t+j_1}, \dots, y_{t+j_n})$  depends on  $J$  but not on  $t$ .

The following process is covariance stationary but not strictly stationary:

$$y_t = \mathbb{I}_{\{t < 1000\}} \varepsilon_{1,t} + \mathbb{I}_{\{t \geq 1000\}} \varepsilon_{2,t},$$

where  $\varepsilon_{1,t} \sim \mathcal{N}(0, 1)$  and  $\varepsilon_{2,t} \sim \sqrt{\frac{\nu-2}{\nu}} t(\nu)$  and  $\nu = 4$ .

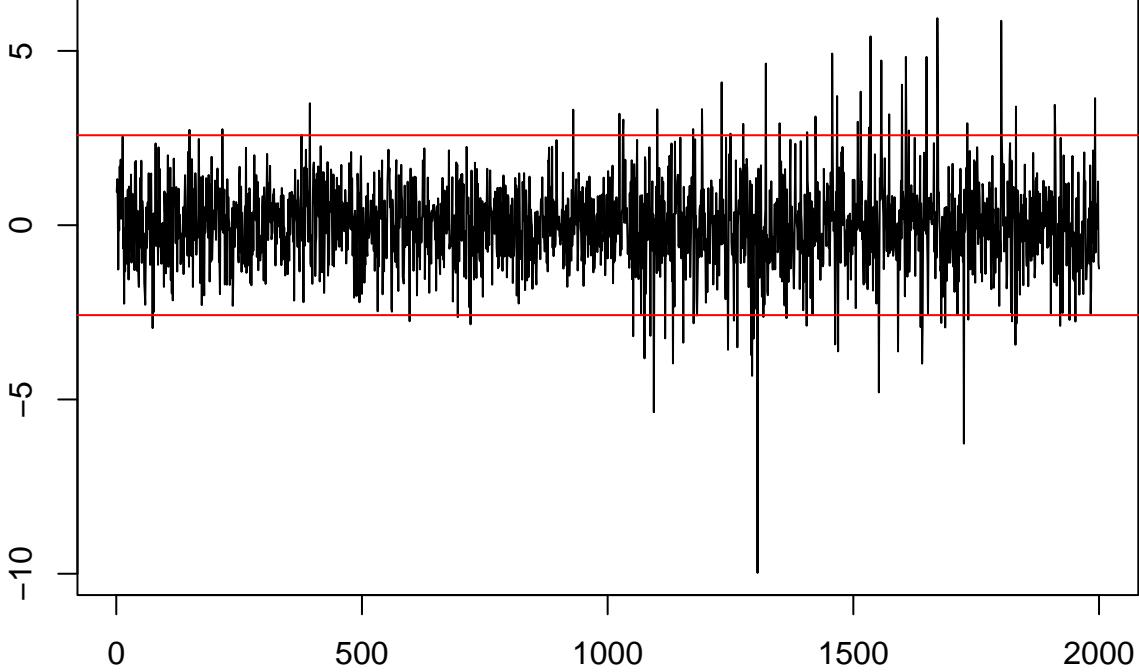


Figure 9.2: Example of a process that is covariance stationary but not strictly stationary. The red lines delineate the 99% confidence interval of the standard normal distribution ( $\pm 2.58$ ).

**Proposition 9.1.** If  $y_t$  is covariance stationary, then  $\gamma_j = \gamma_{-j}$ .

*Proof.* Since  $y_t$  is covariance stationary, the covariance between  $y_t$  and  $y_{t-j}$  (i.e.  $\gamma_j$ ) is the same as that between  $y_{t+j}$  and  $y_{t+j-j}$  (i.e.  $\gamma_{-j}$ ).  $\square$

**Definition 9.6** (Auto-correlation). The  $j^{th}$  auto-correlation of a covariance-stationary process is:

$$\rho_j = \frac{\gamma_j}{\gamma_0}.$$

Consider a long historical time series of the Swiss GDP growth, taken from the Jordà et al. (2017) dataset.<sup>1</sup>

**Definition 9.7** (Mean ergodicity). The covariance-stationary process  $y_t$  is ergodic for the mean if:

$$\text{plim}_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T y_t = \mathbb{E}(y_t).$$

**Definition 9.8** (Second-moment ergodicity). The covariance-stationary process  $y_t$  is ergodic for second moments if, for all  $j$ :

$$\text{plim}_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T (y_t - \mu)(y_{t-j} - \mu) = \gamma_j.$$

It should be noted that ergodicity and stationarity are different properties. Typically if the process  $\{x_t\}$  is such that,  $\forall t$ ,  $x_t \equiv y$ , where  $y \sim \mathcal{N}(0, 1)$  (say), then  $\{x_t\}$  is stationary but not ergodic.

<sup>1</sup>Version 6 of the dataset, available on this website.

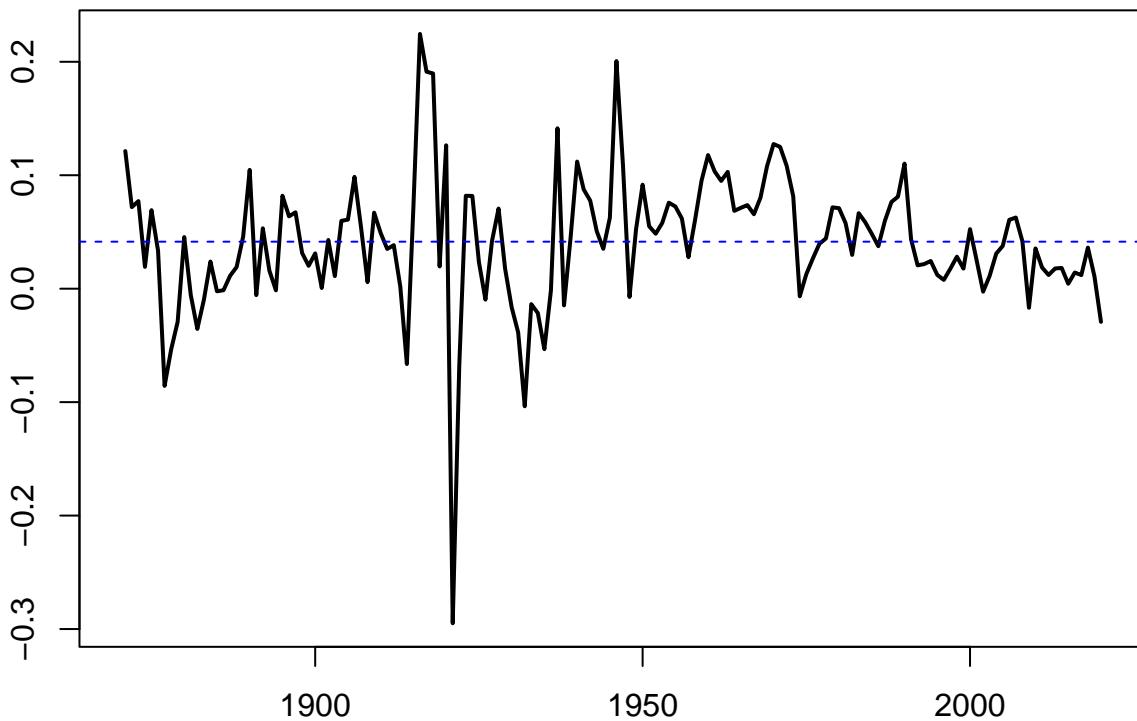


Figure 9.3: Annual growth rate of Swiss GDP, based on the Jorda-Schularick-Taylor Macro-history Database.

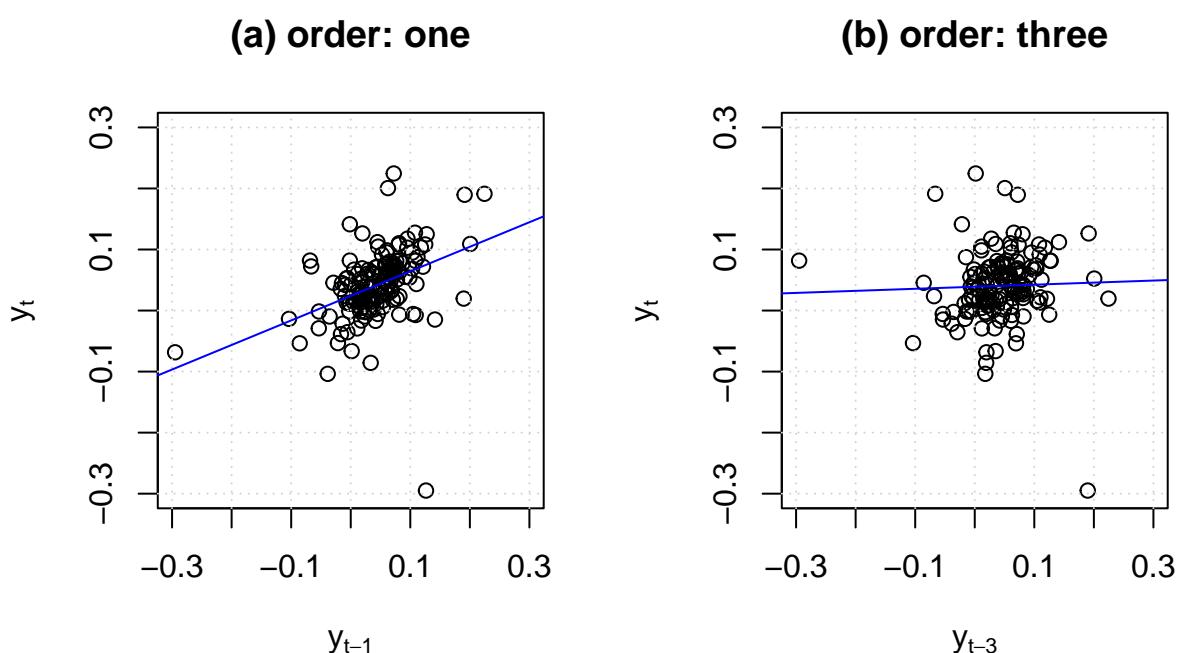


Figure 9.4: For order  $j$ , the slope of the blue line is, approximately,  $\hat{\gamma}_j / \widehat{\text{Var}}(y_t)$ , where hats indicate sample moments.

**Theorem 9.1** (Central Limit Theorem for covariance-stationary processes). *If process  $y_t$  is covariance stationary and if the series of autocovariances is absolutely summable ( $\sum_{j=-\infty}^{+\infty} |\gamma_j| < \infty$ ), then:*

$$\bar{y}_T \xrightarrow{m.s.} \mu = \mathbb{E}(y_t) \quad (9.2)$$

$$\lim_{T \rightarrow +\infty} T \mathbb{E}[(\bar{y}_T - \mu)^2] = \sum_{j=-\infty}^{+\infty} \gamma_j \quad (9.3)$$

$$\sqrt{T}(\bar{y}_T - \mu) \xrightarrow{d} \mathcal{N}\left(0, \sum_{j=-\infty}^{+\infty} \gamma_j\right). \quad (9.4)$$

[Mean square (m.s.) and distribution (d.) convergences: see Definitions 10.19 and 10.17.]

*Proof.* By Proposition 10.8, Eq. (9.3) implies Eq. (9.2). For Eq. (9.3), see Appendix 10.5. For Eq. (9.4), see Anderson (1971), p. 429.  $\square$

**Definition 9.9** (Long-run variance). Under the assumptions of Theorem 9.1, the limit appearing in Eq. (9.3) exists and is called **long-run variance**. It is denoted by  $S$ , i.e.:

$$S = \sum_{j=-\infty}^{+\infty} \gamma_j = \lim_{T \rightarrow +\infty} T \mathbb{E}[(\bar{y}_T - \mu)^2].$$

If  $y_t$  is ergodic for second moments (see Def. 9.8), a natural estimator of  $S$  is:

$$\hat{\gamma}_0 + 2 \sum_{\nu=1}^q \hat{\gamma}_{\nu}, \quad (9.5)$$

where  $\hat{\gamma}_{\nu} = \frac{1}{T} \sum_{t=1}^T (y_t - \bar{y})(y_{t-\nu} - \bar{y})$ .

However, for small samples, Eq. (9.5) does not necessarily result in a positive definite matrix. Newey and West (1987) have proposed an estimator that does not have this defect. Their estimator is given by:

$$S^{NW} = \hat{\gamma}_0 + 2 \sum_{\nu=1}^q \left(1 - \frac{\nu}{q+1}\right) \hat{\gamma}_{\nu}. \quad (9.6)$$

Loosely speaking, Theorem 9.1 says that, for a given sample size, the higher the “persistency” of a process, the lower the accuracy of the sample mean as an estimate of the population mean. To illustrate, consider three processes that feature the same marginal variance (equal to one, say), but different autocorrelations: 0%, 70%, and 99.9%. Figure 9.5 displays simulated paths of such three processes. It indeed appears that, the larger the autocorrelation of the process, the further the sample mean (dashed red line) from the population mean (red solid line).

The same type of simulations can be performed using this ShinyApp (use panel “AR(1)”).

## 9.2 Univariate processes

### 9.2.1 Moving Average (MA) processes

**Definition 9.10.** Consider a white noise process  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  (Def. 9.1). Then  $y_t$  is a first-order moving average process if, for all  $t$ :

$$y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}.$$

If  $\mathbb{E}(\varepsilon_t^2) = \sigma^2$ , it is easily obtained that the unconditional mean and variances of  $y_t$  are:

$$\mathbb{E}(y_t) = \mu, \quad \text{Var}(y_t) = (1 + \theta^2)\sigma^2.$$

The first auto-covariance is:

$$\gamma_1 = \mathbb{E}\{(y_t - \mu)(y_{t-1} - \mu)\} = \theta\sigma^2.$$

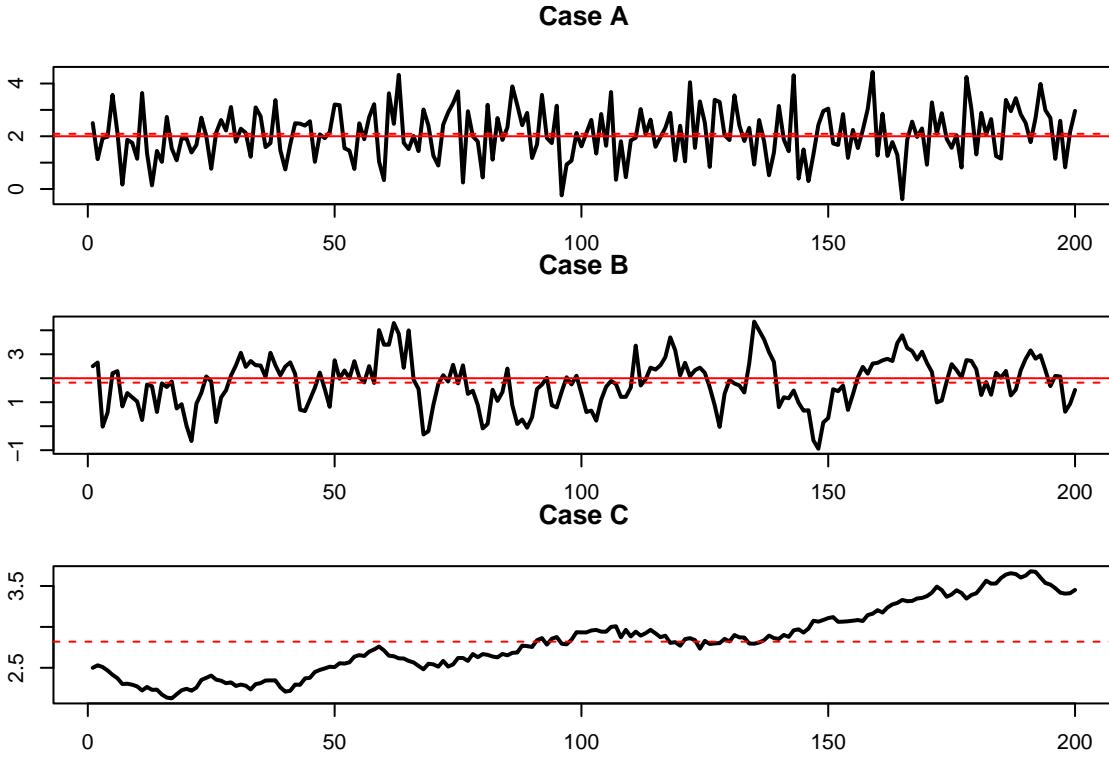


Figure 9.5: The three samples have been simulated using the following data generating process:  $x_t = \mu + \rho(x_{t-1} - \mu) + \sqrt{1 - \rho^2}\varepsilon_t$ , where  $\varepsilon_t \sim \mathcal{N}(0, 1)$ . Case A:  $\rho = 0$ ; Case B:  $\rho = 0.7$ ; Case C:  $\rho = 0.999$ . In the three cases,  $\mathbb{E}(x_t) = \mu = 2$  and  $\text{Var}(x_t) = 1$ .

Higher-order auto-covariances are zero ( $\gamma_j = 0$  for  $j > 1$ ). Therefore: An MA(1) process is covariance-stationary (Def. 9.4).

For a MA(1) process, the autocorrelation of order  $j$  (see Def. 9.6) is given by:

$$\rho_j = \begin{cases} 1 & \text{if } j = 0, \\ \theta/(1 + \theta^2) & \text{if } j = 1 \\ 0 & \text{if } j > 1. \end{cases}$$

Notice that process  $y_t$  defined through:

$$y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1},$$

where  $\text{Var}(\varepsilon_t) = \sigma^2$ , has the same mean and autocovariances as

$$y_t = \mu + \varepsilon_t^* + \frac{1}{\theta}\varepsilon_{t-1}^*,$$

where  $\text{Var}(\varepsilon_t^*) = \theta^2\sigma^2$ . That is, even if we perfectly know the mean and auto-covariances of this process, it is not possible to identify which specification is the one that has been used to generate the data. Only one of these two specifications is said to be *fundamental*, that is the one that satisfies  $|\theta_1| < 1$  (see Eq. (9.32)).

**Definition 9.11** (MA(q) process). A  $q^{th}$  order Moving Average process is defined through:

$$y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q}.$$

where  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  is a white noise process (Def. 9.1).

**Proposition 9.2** (Covariance-stationarity of an MA(q) process). *Finite-order Moving Average processes are covariance-stationary.*

Moreover, the autocovariances of an MA( $q$ ) process (as defined in Def. 9.11) are given by:

$$\gamma_j = \begin{cases} \sigma^2(\theta_j\theta_0 + \theta_{j+1}\theta_1 + \dots + \theta_q\theta_{q-j}) & \text{for } j \in \{0, \dots, q\} \\ 0 & \text{for } j > q, \end{cases} \quad (9.7)$$

where we use the notation  $\theta_0 = 1$ , and  $\text{Var}(\varepsilon_t) = \sigma^2$ .

*Proof.* The unconditional expectation of  $y_t$  does not depend on time, since  $\mathbb{E}(y_t) = \mu$ . Let's turn to autocovariances. We can extend the series of the  $\theta_j$ 's by setting  $\theta_j = 0$  for  $j > q$ . We then have:

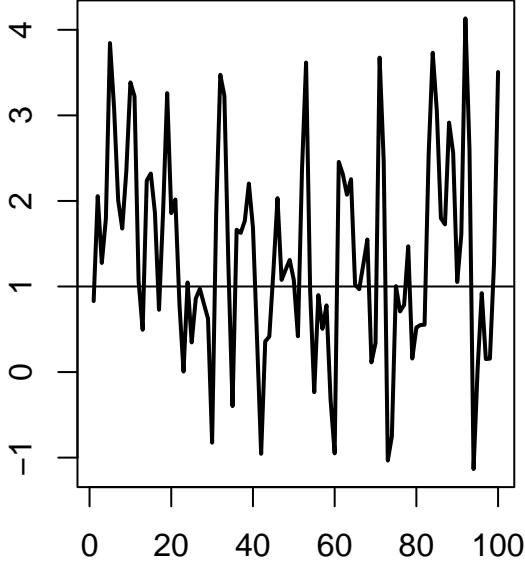
$$\begin{aligned}\mathbb{E}((y_t - \mu)(y_{t-j} - \mu)) &= \mathbb{E}[(\theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_j \varepsilon_{t-j} + \theta_{j+1} \varepsilon_{t-j-1} + \dots) \times \\ &\quad (\theta_0 \varepsilon_{t-j} + \theta_1 \varepsilon_{t-j-1} + \dots)].\end{aligned}$$

Then use the fact that  $\mathbb{E}(\varepsilon_t \varepsilon_s) = 0$  if  $t \neq s$  (because  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  is a white noise process).  $\square$

Figure 9.6 displays simulated paths of two MA processes (an MA(1) and an MA(4)). Such simulations can be produced by using panel “ARMA(p,q)” of this web interface.

```
library(AEC)
T <- 100;nb.sim <- 1
y.0 <- c(0)
c <- 1;phi <- c(0);sigma <- 1
theta <- c(1,1) # MA(1) specification
y.sim <- sim.arma(c,phi,theta,sigma,T,y.0,nb.sim)
par(mfrow=c(1,2))
par(plt=c(.2,.9,.2,.85))
plot(y.sim[,1],xlab="",ylab="",type="l",lwd=2,
      main=expression(paste(theta[0],"=1, ",theta[1],"=1",sep="")))
abline(h=c)
theta <- c(1,1,1,1,1) # MA(4) specification
y.sim <- sim.arma(c,phi,theta,sigma,T,y.0,nb.sim)
plot(y.sim[,1],xlab="",ylab="",type="l",lwd=2,
      main=expression(paste(theta[0],"...=",theta[4],"=1",sep="")))
abline(h=c)
```

$\theta_0=1, \theta_1=1$



$\theta_0=\dots=\theta_4=1$

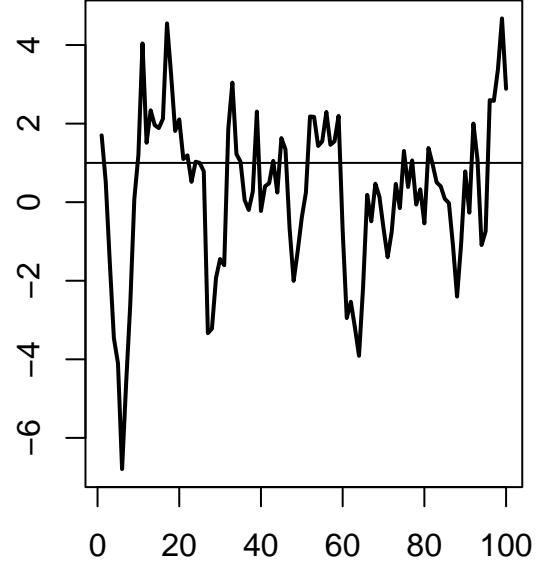


Figure 9.6: Simulation of MA processes.

What if the order  $q$  of an MA( $q$ ) process gets infinite? The notion of **infinite-order Moving Average process** exists and is important in time series analysis. The (infinite) sequence of  $\theta_j$  has to satisfy some conditions for such a process to be well-defined (see Theorem 9.2 below). These conditions relate to the “summability” of  $\{\theta_i\}_{i \in \mathbb{N}}$  (see Definition 9.12).

**Definition 9.12** (Absolute and square summability). The sequence  $\{\theta_i\}_{i \in \mathbb{N}}$  is absolutely summable if  $\sum_{i=0}^{\infty} |\theta_i| < +\infty$ , and it is square summable if  $\sum_{i=0}^{\infty} \theta_i^2 < +\infty$ .

According to Prop. 10.8, absolute summability implies square summability.

**Theorem 9.2** (Existence condition for an infinite MA process). *If  $\{\theta_i\}_{i \in \mathbb{N}}$  is square summable (see Def. 9.12) and if  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  is a white noise process (see Def. 9.1), then*

$$\mu + \sum_{i=0}^{+\infty} \theta_i \varepsilon_{t-i}$$

*defines a well-behaved [covariance-stationary] process, called infinite-order MA process (MA( $\infty$ )).*

*Proof.* See Appendix 3.A in Hamilton. “Well behaved” means that  $\sum_{i=0}^T \theta_{t-i} \varepsilon_{t-i}$  converges in mean square (Def. 10.17) to some random variable  $Z_t$ . The proof makes use of the fact that:

$$\mathbb{E} \left[ \left( \sum_{i=N}^M \theta_i \varepsilon_{t-i} \right)^2 \right] = \sum_{i=N}^M |\theta_i|^2 \sigma^2,$$

and that, when  $\{\theta_i\}$  is square summable,  $\forall \eta > 0$ ,  $\exists N$  s.t. the right-hand-side term in the last equation is lower than  $\eta$  for all  $M \geq N$  (static Cauchy criterion, Theorem 10.2). This implies that  $\sum_{i=0}^T \theta_i \varepsilon_{t-i}$  converges in mean square (stochastic Cauchy criterion, see Theorem 10.3).  $\square$

**Proposition 9.3** (First two moments of an infinite MA process). *If  $\{\theta_i\}_{i \in \mathbb{N}}$  is absolutely summable, i.e. if  $\sum_{i=0}^{+\infty} |\theta_i| < +\infty$ , then*

i.  $y_t = \mu + \sum_{i=0}^{+\infty} \theta_i \varepsilon_{t-i}$  exists (Theorem 9.2) and is such that:

$$\begin{aligned} \mathbb{E}(y_t) &= \mu \\ \gamma_0 = \mathbb{E}([y_t - \mu]^2) &= \sigma^2(\theta_0^2 + \theta_1^2 + \dots) \\ \gamma_j = \mathbb{E}([y_t - \mu][y_{t-j} - \mu]) &= \sigma^2(\theta_0 \theta_j + \theta_1 \theta_{j+1} + \dots). \end{aligned}$$

ii. Process  $y_t$  has absolutely summable auto-covariances, which implies that the results of Theorem 9.1 (Central Limit) apply.

*Proof.* The absolute summability of  $\{\theta_i\}$  and the fact that  $\mathbb{E}(\varepsilon^2) < \infty$  imply that the order of integration and summation is interchangeable (see Hamilton, 1994, Footnote p. 52), which proves (i). For (ii), see end of Appendix 3.A in Hamilton (1994).  $\square$

### 9.2.2 Auto-Regressive (AR) processes

**Definition 9.13** (First-order AR process (AR(1))). Consider a white noise process  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  (see Def. 9.1). Process  $y_t$  is an AR(1) process if it is defined by the following difference equation:

$$y_t = c + \phi y_{t-1} + \varepsilon_t.$$

If  $|\phi| \geq 1$ ,  $y_t$  is not stationary. Indeed, we have:

$$y_{t+k} = c + \varepsilon_{t+k} + \phi(c + \varepsilon_{t+k-1}) + \phi^2(c + \varepsilon_{t+k-2}) + \dots + \phi^{k-1}(c + \varepsilon_{t+1}) + \phi^k y_t.$$

Therefore, the conditional variance

$$\text{Var}_t(y_{t+k}) = \sigma^2(1 + \phi^2 + \phi^4 + \dots + \phi^{2(k-1)})$$

does not converge for large  $k$ 's. This implies that  $\text{Var}(y_t)$  does not exist.

By contrast, if  $|\phi| < 1$ , one can see that:

$$y_t = c + \varepsilon_t + \phi(c + \varepsilon_{t-1}) + \phi^2(c + \varepsilon_{t-2}) + \dots + \phi^k(c + \varepsilon_{t-k}) + \dots$$

Hence, if  $|\phi| < 1$ , the unconditional mean and variance of  $y_t$  are:

$$\mathbb{E}(y_t) = \frac{c}{1-\phi} =: \mu \quad \text{and} \quad \text{Var}(y_t) = \frac{\sigma^2}{1-\phi^2}.$$

Let us compute the  $j^{th}$  autocovariance of the AR(1) process:

$$\begin{aligned}\mathbb{E}([y_t - \mu][y_{t-j} - \mu]) &= \mathbb{E}([\varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \dots + \phi^j\varepsilon_{t-j} + \phi^{j+1}\varepsilon_{t-j-1} \dots] \times \\ &\quad [\varepsilon_{t-j} + \phi\varepsilon_{t-j-1} + \phi^2\varepsilon_{t-j-2} + \dots + \phi^k\varepsilon_{t-j-k} + \dots]) \\ &= \mathbb{E}(\phi^j\varepsilon_{t-j}^2 + \phi^{j+2}\varepsilon_{t-j-1}^2 + \phi^{j+4}\varepsilon_{t-j-2}^2 + \dots) \\ &= \frac{\phi^j\sigma^2}{1-\phi^2}.\end{aligned}$$

Therefore  $\rho_j = \phi^j$ .

By what precedes, we have:

**Proposition 9.4** (Covariance-stationarity of an AR(1) process). *The AR(1) process, as defined in Def. 9.13, is covariance-stationary iff  $|\phi| < 1$ .*

**Definition 9.14** (AR(p) process). Consider a white noise process  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  (see Def. 9.1). Process  $y_t$  is a  $p^{th}$ -order autoregressive process (AR(p)) if its dynamics is defined by the following difference equation (with  $\phi_p \neq 0$ ):

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t. \quad (9.8)$$

As we will see, the covariance-stationarity of process  $y_t$  hinges on matrix  $F$  defined as:

$$F = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_p \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \quad (9.9)$$

Note that this matrix  $F$  is such that if  $y_t$  follows Eq. (9.8), then process  $\mathbf{y}_t$  follows:

$$\mathbf{y}_t = \mathbf{c} + F\mathbf{y}_{t-1} + \xi_t$$

with

$$\mathbf{c} = \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \xi_t = \begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{y}_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix}.$$

**Proposition 9.5** (The eigenvalues of matrix  $F$ ). *The eigenvalues of  $F$  (defined by Eq. (9.9)) are the solutions of:*

$$\lambda^p - \phi_1\lambda^{p-1} - \dots - \phi_{p-1}\lambda - \phi_p = 0. \quad (9.10)$$

**Proposition 9.6** (Covariance-stationarity of an AR(p) process). *These four statements are equivalent:*

- i. Process  $\{y_t\}$ , defined in Def. 9.14, is covariance-stationary.
- ii. The eigenvalues of  $F$  (as defined Eq. (9.9)) lie strictly within the unit circle.
- iii. The roots of Eq. (9.11) (below) lie strictly outside the unit circle.

$$1 - \phi_1 z - \dots - \phi_{p-1} z^{p-1} - \phi_p z^p = 0. \quad (9.11)$$

- iv. The roots of Eq. (9.12) (below) lie strictly inside the unit circle.

$$\lambda^p - \phi_1\lambda^{p-1} - \dots - \phi_{p-1}\lambda - \phi_p = 0. \quad (9.12)$$

*Proof.* We consider the case where the eigenvalues of  $F$  are distinct; Jordan decomposition can be used in the general case. When the eigenvalues of  $F$  are distinct,  $F$  admits the following spectral decomposition:  $F = PDP^{-1}$ , where  $D$  is diagonal. Using the notations introduced in Eq. (9.9), we have:

$$\mathbf{y}_t = \mathbf{c} + F\mathbf{y}_{t-1} + \xi_t.$$

Let's introduce  $\mathbf{d} = P^{-1}\mathbf{c}$ ,  $\mathbf{z}_t = P^{-1}\mathbf{y}_t$  and  $\eta_t = P^{-1}\xi_t$ . We have:

$$\mathbf{z}_t = \mathbf{d} + D\mathbf{z}_{t-1} + \eta_t.$$

Because  $D$  is diagonal, the different component of  $\mathbf{z}_t$ , denoted by  $z_{i,t}$ , follow AR(1) processes. The (scalar) autoregressive parameters of these AR(1) processes are the diagonal entries of  $D$  – which also are the eigenvalues of  $F$  – that we denote by  $\lambda_i$ .

Process  $y_t$  is covariance-stationary iff  $\mathbf{y}_t$  also is covariance-stationary, which is the case iff all  $z_{i,t}$ ,  $i \in [1, p]$ , are covariance-stationary. By Prop. 9.4, process  $z_{i,t}$  is covariance-stationary iff  $|\lambda_i| < 1$ . This proves that (i) is equivalent to (ii). Prop. 9.5 further proves that (ii) is equivalent to (iv). Finally, it is easily seen that (iii) is equivalent to (iv) (as long as  $\phi_p \neq 0$ ).  $\square$

Using the lag operator (see Eq (9.1)), if  $y_t$  is a covariance-stationary AR(p) process (Def. 9.14), we can write:

$$y_t = \mu + \psi(L)\varepsilon_t,$$

where

$$\psi(L) = (1 - \phi_1 L - \dots - \phi_p L^p)^{-1}, \quad (9.13)$$

and

$$\mu = \mathbb{E}(y_t) = \frac{c}{1 - \phi_1 - \dots - \phi_p}. \quad (9.14)$$

In the following lines of codes, we compute the eigenvalues of the  $F$  matrices associated with the following processes (where  $\varepsilon_t$  is a white noise):

$$\begin{aligned} x_t &= 0.9x_{t-1} - 0.2x_{t-2} + \varepsilon_t \\ y_t &= 1.1y_{t-1} - 0.3y_{t-2} + \varepsilon_t \\ w_t &= 1.4w_{t-1} - 0.7w_{t-2} + \varepsilon_t \\ z_t &= 0.9z_{t-1} + 0.2z_{t-2} + \varepsilon_t \end{aligned}$$

```
F <- matrix(c(.9, 1, -.2, 0), 2, 2)
lambda_x <- eigen(F)$values
F[1,] <- c(1.1, -.3)
lambda_y <- eigen(F)$values
F[1,] <- c(1.4, -.7)
lambda_w <- eigen(F)$values
F[1,] <- c(.9, .2)
lambda_z <- eigen(F)$values
rbind(lambda_x, lambda_y, lambda_w, lambda_z)
```

```
##          [,1]          [,2]
## lambda_x 0.500000+0.0000000i 0.4000000+0.0000000i
## lambda_y 0.600000+0.0000000i 0.5000000+0.0000000i
## lambda_w 0.700000+0.4582576i 0.7000000-0.4582576i
## lambda_z 1.084429+0.0000000i -0.1844289+0.0000000i
```

The absolute values of the eigenvalues associated with process  $w_t$  are both equal to 0.837. Therefore, according to Proposition 9.6, processes  $x_t$ ,  $y_t$ , and  $w_t$  are covariance-stationary, but not  $z_t$  (because the absolute value of one of the eigenvalues of the  $F$  matrix associated with this process is larger than 1).

The computation of the autocovariances of  $y_t$  is based on the so-called **Yule-Walker equations** (Eq. (9.15)). Let's rewrite Eq. (9.8):

$$(y_t - \mu) = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \dots + \phi_p(y_{t-p} - \mu) + \varepsilon_t.$$

Multiplying both sides by  $y_{t-j} - \mu$  and taking expectations leads to the (Yule-Walker) equations:

$$\gamma_j = \begin{cases} \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} + \dots + \phi_p\gamma_{j-p} & \text{if } j > 0 \\ \phi_1\gamma_1 + \phi_2\gamma_2 + \dots + \phi_p\gamma_p + \sigma^2 & \text{for } j = 0. \end{cases} \quad (9.15)$$

Using  $\gamma_j = \gamma_{-j}$  (Prop. 9.1), one can express  $(\gamma_0, \gamma_1, \dots, \gamma_p)$  as functions of  $(\sigma^2, \phi_1, \dots, \phi_p)$ .

### 9.2.3 AR-MA processes

**Definition 9.15** (ARMA(p,q) process).  $\{y_t\}$  is an ARMA( $p, q$ ) process if its dynamics is described by the following equation:

$$y_t = c + \underbrace{\phi_1 y_{t-1} + \cdots + \phi_p y_{t-p}}_{\text{AR part}} + \underbrace{\varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}}_{\text{MA part}}, \quad (9.16)$$

where  $\{\varepsilon_t\}_{t \in [-\infty, +\infty[}$  is a white noise process (see Def. 9.1).

**Proposition 9.7** (Stationarity of an ARMA(p,q) process). *The ARMA( $p, q$ ) process defined in 9.15 is covariance stationary iff the roots of*

$$1 - \phi_1 z - \cdots - \phi_p z^p = 0$$

*lie strictly outside the unit circle or, equivalently, iff those of*

$$\lambda^p - \phi_1 \lambda^{p-1} - \cdots - \phi_p = 0$$

*lie strictly within the unit circle.*

*Proof.* The proof of Prop. 9.6 can be adapted to the present case.  $\square$

We can write:

$$(1 - \phi_1 L - \cdots - \phi_p L^p) y_t = c + (1 + \theta_1 L + \cdots + \theta_q L^q) \varepsilon_t.$$

If the roots of  $1 - \phi_1 z - \cdots - \phi_p z^p = 0$  lie outside the unit circle, we have:

$$y_t = \mu + \psi(L) \varepsilon_t, \quad (9.17)$$

where

$$\psi(L) = \frac{1 + \theta_1 L + \cdots + \theta_q L^q}{1 - \phi_1 L - \cdots - \phi_p L^p} \quad \text{and} \quad \mu = \frac{c}{1 - \phi_1 - \cdots - \phi_p}.$$

Eq. (9.17) is the **Wold representation** of this ARMA process (see Theorem 9.3 below).

The stationarity of the process depends only on the AR specification (or on the eigenvalues of matrix  $F$ , exactly as in Prop. 9.6). If the process is stationary, the weights in  $\psi(L)$  decay at a geometric rate.

### 9.2.4 PACF approach to identify AR/MA processes

We have seen that the  $k^{th}$ -order auto-correlation of a MA(q) process is null if  $k > q$ . This is exploited, in practice, to determine the order of a MA process. Moreover, since this is not the case for an AR process, this can be used to distinguish an AR from an MA process.

There exists an equivalent approach to determine whether a process can be modeled as an AR process; it is based on partial auto-correlations:

**Definition 9.16** (Partial auto-correlation). In a time series context, the partial auto-correlation ( $\phi_{h,h}$ ) of process  $\{y_t\}$  is defined as the partial correlation of  $y_{t+h}$  and  $y_t$  given  $y_{t+h-1}, \dots, y_{t+1}$ . (see Def. 10.5 for the definition of partial correlation.)

If  $h > p$ , the regression of  $y_{t+h}$  on  $y_{t+h-1}, \dots, y_{t+1}$  is:

$$y_{t+h} = c + \phi_1 y_{t+h-1} + \cdots + \phi_p y_{t+h-p} + \varepsilon_{t+h}.$$

The residuals of the latter regressions ( $\varepsilon_{t+h}$ ) are uncorrelated to  $y_t$ . Then the partial auto-correlation is zero for  $h > p$ .

Besides, it can be shown that  $\phi_{p,p} = \phi_p$ . Hence  $\phi_{p,p} = \phi_p$  but  $\phi_{h,h} = 0$  for  $h > p$ . This can be used to determine the order of an AR process. By contrast (importantly) if  $y_t$  follows an MA(q) process, then  $\phi_{k,k}$  asymptotically approaches zero instead of cutting off abruptly.

As illustrated below, functions `acf` and `pacf` can be conveniently used to employ the (P)ACF approach. (Note also the use of function `sim.arma` to simulate ARMA processes.)

```

library(AEC)
par(mfrow=c(3,2))
par(plt=c(.2,.9,.2,.95))
theta <- c(1,2,1);phi=0
y.sim <- sim.arma(c=0,phi,theta,sigma=1,T=1000,y.0=0,nb.sim=1)
par(mfg=c(1,1));plot(y.sim,type="l",lwd=2)
par(mfg=c(2,1));acf(y.sim)
par(mfg=c(3,1));pacf(y.sim)
theta <- c(1);phi=0.9
y.sim <- sim.arma(c=0,phi,theta,sigma=1,T=1000,y.0=0,nb.sim=1)
par(mfg=c(1,2));plot(y.sim,type="l",lwd=2)
par(mfg=c(2,2));acf(y.sim)
par(mfg=c(3,2));pacf(y.sim)

```

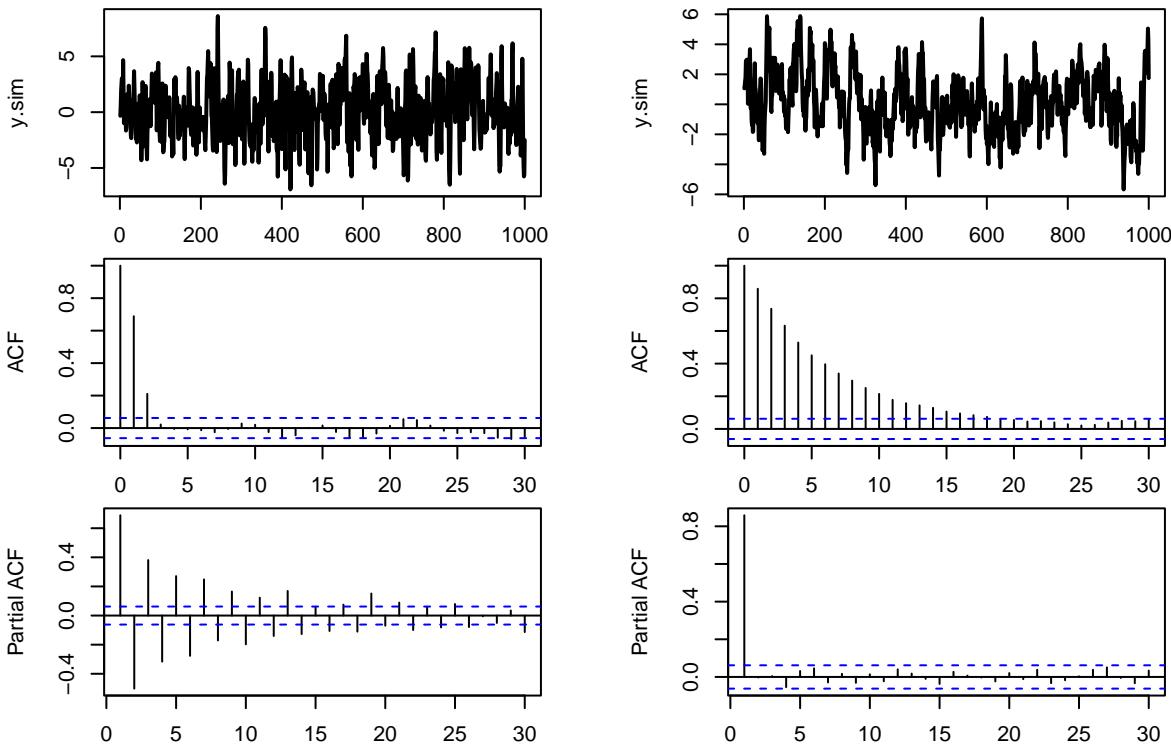


Figure 9.7: ACF/PACF analysis of two processes (MA process on the left, AR on the right).

### 9.2.5 Wold decomposition

The Wold decomposition is an important result in time series analysis:

**Theorem 9.3** (Wold decomposition). *Any covariance-stationary process admits the following representation:*

$$y_t = \mu + \sum_0^{+\infty} \theta_i \varepsilon_{t-i} + \kappa_t,$$

where

- $\theta_0 = 1$ ,  $\sum_{i=0}^{\infty} \theta_i^2 < +\infty$  (square summability, see Def. 9.12).
- $\{\varepsilon_t\}$  is a white noise (see Def. 9.1);  $\varepsilon_t$  is the error made when forecasting  $y_t$  based on a linear combination of lagged  $y_t$ 's ( $\varepsilon_t = y_t - \hat{E}[y_t|y_{t-1}, y_{t-2}, \dots]$ ).
- For any  $j \geq 1$ ,  $\kappa_t$  is not correlated with  $\varepsilon_{t-j}$ ; but  $\kappa_t$  can be perfectly forecasted based on a linear combination of lagged  $y_t$ 's (i.e.  $\kappa_t = \hat{E}(\kappa_t|y_{t-1}, y_{t-2}, \dots)$ ).  $\kappa_t$  is called the **deterministic component** of  $y_t$ .

*Proof.* See Anderson (1971). Partial proof in L. Christiano.  $\square$

For an ARMA process, the Wold representation is given by Eq. (9.17). As detailed in Prop. 9.8, it can be computed by recursively replacing the lagged  $y_t$ 's in Eq. (9.16). In this case, the deterministic component ( $\kappa$ ) is null.

### 9.2.6 Impulse Response Functions (IRFs) in ARMA models

Consider the ARMA(p,q) process defined in Def. 9.15, whose associated sequence of white noise is  $\{\varepsilon_t\}$ . Let us construct a novel (counterfactual) sequence of shocks  $\{\tilde{\varepsilon}_t^{(s)}\}$ :

$$\tilde{\varepsilon}_t^{(s)} = \begin{cases} \varepsilon_t & \text{if } t \neq s, \\ \varepsilon_t + \delta & \text{if } t = s. \end{cases}$$

We denote by  $\{\tilde{y}_t^{(s)}\}$  the process following Eq. (9.16) where  $\{\varepsilon_t\}$  is replaced with  $\{\tilde{\varepsilon}_t^{(s)}\}$ . The time series  $\{\tilde{y}_t^{(s)}\}$  is the counterfactual series  $\{y_t\}$  that would have prevailed if  $\varepsilon_t$  had been shifted by  $\delta$  on date  $s$  (and that would be the only change).

The relationship between  $\{y_t\}$  and  $\{\tilde{y}_t^{(s)}\}$  defines the **dynamics multiplier**. The latter is denoted by  $\frac{\partial y_t}{\partial \varepsilon_s}$  and is such that:

$$\tilde{y}_t^{(s)} = y_t + \frac{\partial y_t}{\partial \varepsilon_s} \delta.$$

We will see that the dynamic multipliers are closely related to the infinite MA representation (or **Wold decomposition**, Theorem 9.3) of  $y_t$ :

$$y_t = \mu + \sum_{i=0}^{+\infty} \psi_i \varepsilon_{t-i}.$$

For  $t < s$ , we have  $y_t = \tilde{y}_t^{(s)}$  (because  $\tilde{\varepsilon}_{t-i} = \varepsilon_{t-i}$  for all  $i \geq 0$  if  $t < s$ ).

For  $t \geq s$ :

$$\tilde{y}_t^{(s)} = \mu + \left( \sum_{i=0}^{t-s-1} \psi_i \varepsilon_{t-i} \right) + \psi_{t-s} (\varepsilon_s + \delta) + \left( \sum_{i=t-s+1}^{+\infty} \psi_i \varepsilon_{t-i} \right) = y_t + \frac{\partial y_t}{\partial \varepsilon_s} \delta.$$

Therefore, for  $t \geq s$ , we have:

$$\boxed{\frac{\partial y_t}{\partial \varepsilon_s} = \psi_{t-s}.}$$

That is,  $\{y_t\}$ 's dynamic multiplier of order  $k$  is the same object as the  $k^{\text{th}}$  loading  $\psi_k$  in the Wold decomposition of  $\{y_t\}$ . The sequence  $\left\{ \frac{\partial y_{t+h}}{\partial \varepsilon_t} \right\}_{h \geq 0} \equiv \{\psi_h\}_{h \geq 0}$  defines the **impulse response function (IRF)** of  $y_t$  to the shock  $\varepsilon_t$ .

For ARMA processes, the computation of the IRFs is easy:

**Proposition 9.8** (IRF of an ARMA(p,q) process). *The coefficients  $\psi_h$ , that define the IRF of process  $y_t$  to  $\varepsilon_t$ , can be computed recursively as follows:*

1. Set  $\psi_{-1} = \dots = \psi_{-p} = 0$ .
2. For  $h \geq 0$ , (recursively) apply:

$$\psi_h = \phi_1 \psi_{h-1} + \dots + \phi_p \psi_{h-p} + \theta_h,$$

where  $\theta_h = 0$  for  $h > q$ .

*Proof.* This is obtained by applying the operator  $\frac{\partial}{\partial \varepsilon_t}$  on both sides of Eq. (9.16):

$$y_{t+h} = c + \phi_1 y_{t+h-1} + \dots + \phi_p y_{t+h-p} + \varepsilon_{t+h} + \theta_1 \varepsilon_{t+h-1} + \dots + \theta_q \varepsilon_{t+h-q}.$$

□

Note that Proposition 9.8 constitutes a simple way to compute the MA( $\infty$ ) representation (or Wold representation) of an ARMA process.

One can use function `sim.arma` of package `AEC` to compute ARMA's IRFs (with the argument `make.IRF = 1`):

```

T <- 21 # number of periods for IRF
theta <- c(1,1,1);phi <- c(0);c <- 0
y.sim <- sim.arma(c,phi,theta,sigma=1,T,y.0=rep(0,length(phi)),
                   nb.sim=1,make.IRF = 1)
par(mfrow=c(1,3));par(plt=c(.25,.95,.2,.85))
plot(0:(T-1),y.sim[,1],type="l",lwd=2,
      main="(a) Process 1",xlab="Time after shock on epsilon",
      ylab="Dynamic multiplier (shock on epsilon at t=0)",col="red")
abline(h=0)
theta <- c(1,.5);phi <- c(0.6)
y.sim <- sim.arma(c,phi,theta,sigma=1,T,y.0=rep(0,length(phi)),
                   nb.sim=1,make.IRF = 1)
plot(0:(T-1),y.sim[,1],type="l",lwd=2,
      main="(b) Process 2",xlab="Time after shock on epsilon",
      ylab="",col="red")
theta <- c(1,1,1);phi <- c(0,0,.5,.4)
y.sim <- sim.arma(c,phi,theta,sigma=1,T,y.0=rep(0,length(phi)),
                   nb.sim=1,make.IRF = 1)
plot(0:(T-1),y.sim[,1],type="l",lwd=2,
      main="(c) Process 3",xlab="Time after shock on epsilon",
      ylab="",col="red")

```

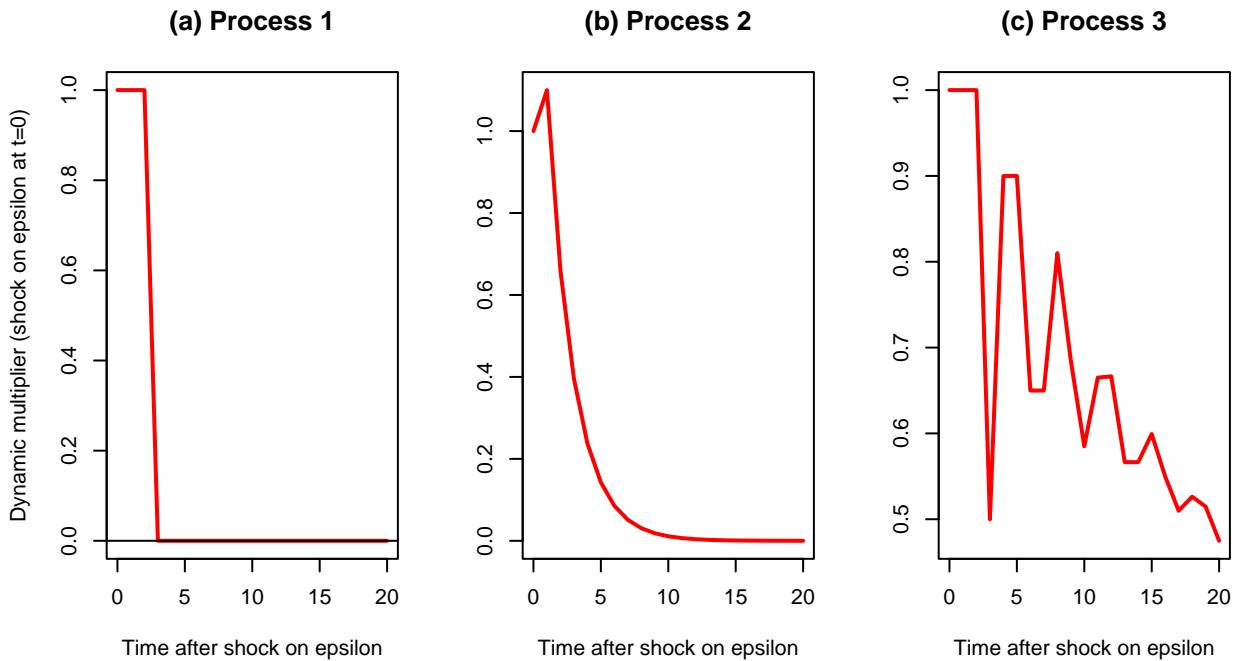


Figure 9.8: IRFs associated with the three processes. Process 1 (MA(2)):  $y_t = \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}$ . Process 2 (ARMA(1,1)):  $y_t = 0.6y_{t-1} + \varepsilon_t + 0.5\varepsilon_{t-1}$ . Process 3 (ARMA(4,2)):  $y_t = 0.5y_{t-3} + 0.4y_{t-4} + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}$ .

Consider the annual Swiss GDP growth from the JST macro-history database. Let us first determine relevant orders for AR and MA processes using the (P)ACF approach.

```

library(AEC)
data(JST)
data <- subset(JST,iso=="CHE")
par(plt=c(.1,.95,.1,.95))
T <- dim(data)[1]
growth <- log(data$gdp[2:T]/data$gdp[1:(T-1)])
par(mfrow=c(3,1))
par(plt=c(.1,.95,.15,.95))
plot(data$year[2:T],growth,type="l",xlab="",ylab="",lwd=2)
abline(h=0,lty=2)
acf(growth)
pacf(growth)

```

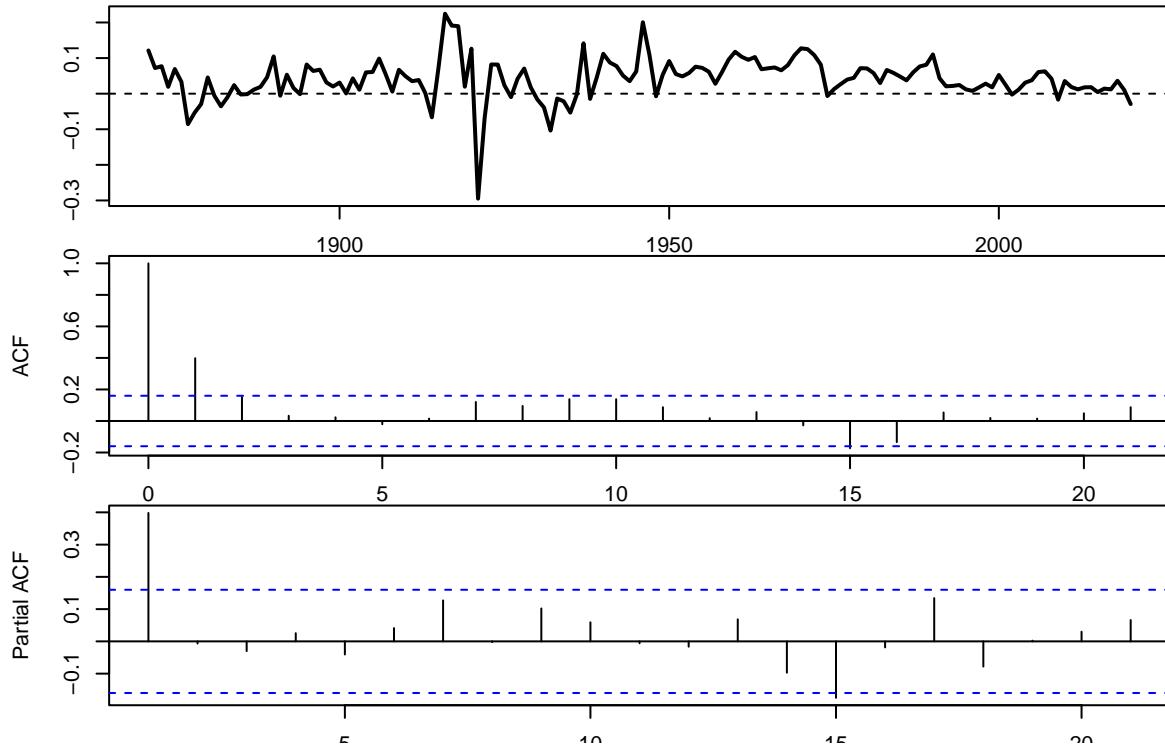


Figure 9.9: (P)ACF analysis of Swiss GDP growth.

The two bottom plots of Figure 9.9 suggest that either an MA(2) or an AR(1) could be used to model the GDP growth rate series. Figure 9.10 shows the IRFs based on these two respective specifications.

```
# Fit an AR process:
res <- arima(growth, order=c(1,0,0))
phi <- res$coef[1]
T <- 11
y.sim <- sim.arma(c=0, phi, theta=1, sigma=1, T, y.0=rep(0,length(phi)),
                    nb.sim=1, make.IRF = 1)
par(plt=c(.15,.95,.25,.95))
plot(0:(T-1), y.sim[,1], type="l", lwd=3,
      xlab="Time after shock on epsilon",
      ylab="Dynamic multiplier (shock on epsilon at t=0)", col="red")
# Fit a MA process:
res <- arima(growth, order=c(0,0,2))
phi <- 0; theta <- c(1,res$coef[1:2])
y.sim <- sim.arma(c=0, phi, theta, sigma=1, T, y.0=rep(0,length(phi)),
                    nb.sim=1, make.IRF = 1)
lines(0:(T-1), y.sim[,1], lwd=3, col="red", lty=2)
abline(h=0)
```

The same kind of algorithm can be used to compute the impact of an increase in an exogenous variable  $x_t$  within an ARMAX(p,q,r) model (see next section).

### 9.2.7 ARMA processes with exogenous variables (ARMA-X)

ARMA processes do not allow to investigate the influence of an exogenous variable (say  $x_t$ ) on the variable of interest (say  $y_t$ ). When  $x_t$  and  $y_t$  have reciprocal influences, the Vector Autoregressive (VAR) model may be used (this tools will be studied later, in Section 9.3). However, when one suspects that  $x_t$  has an “exogenous” influence on  $y_t$ , then a simple extension of the ARMA processes may be considered. Loosely speaking,  $x_t$  has an “exogenous” influence on  $y_t$  if  $y_t$  does not affect  $x_t$ . This extension is called ARMAX(p,q,r).

To begin with, let us formalize this notion of exogeneity. Consider a white noise sequence  $\{\varepsilon_t\}$  (Def. 9.1).

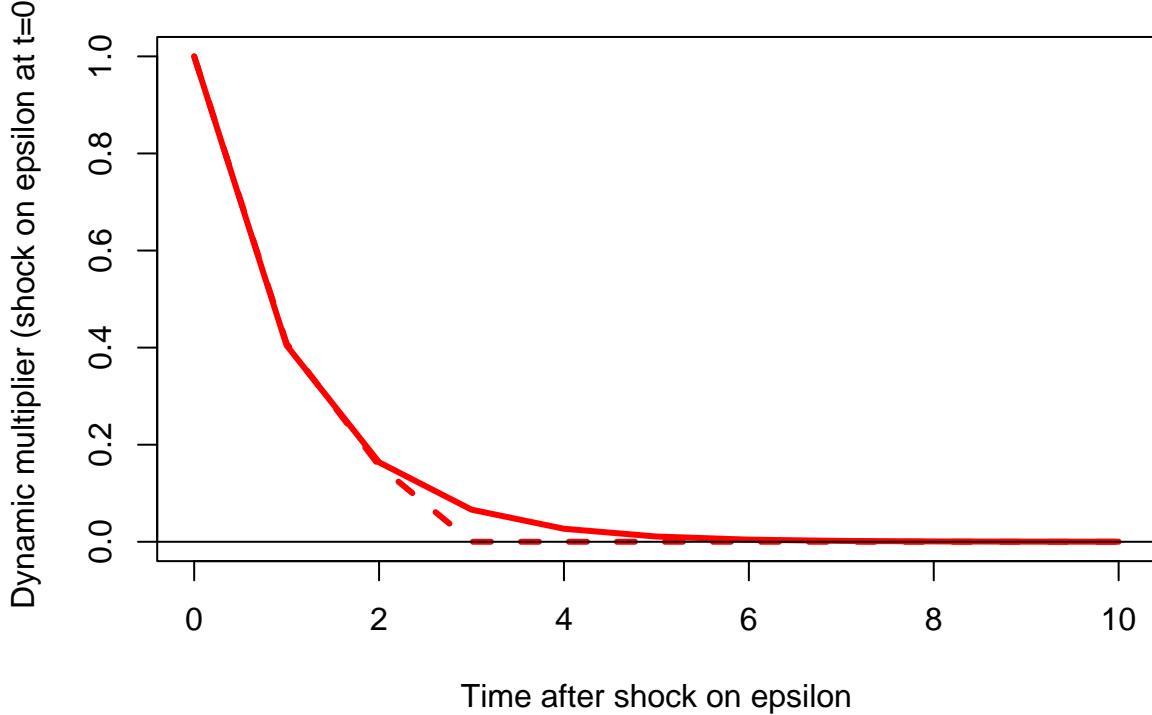


Figure 9.10: Dynamic response of Swiss annual growth to a shock on the innovation  $\varepsilon_t$  at date  $t = 0$ . The solid line corresponds to an AR(1) specification; the dashed line corresponds to a MA(2) specification.

**Definition 9.17** (Exogeneity). We say that  $x_t$  is (strictly) exogenous to  $\{\varepsilon_t\}$  if

$$\mathbb{E}(\varepsilon_t | \underbrace{\dots, x_{t+1}}_{\text{future}}, \underbrace{x_t, x_{t-1}, \dots}_{\text{present and past}}) = 0.$$

Hence, if  $\{x_t\}$  is strictly exogenous to  $\varepsilon_t$ , then past, present and future values of  $x_t$  do not allow to predict the  $\varepsilon_t$ 's.

In the following, we assume that  $\{x_t\}$  is a covariance stationary process.

**Definition 9.18** (ARMAX(p,q,r) model). The process  $\{y_t\}$  is an ARMAX(p,q,r) if it follows a difference equation:

$$y_t = \underbrace{c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p}}_{\text{AR}(p) \text{ part}} + \underbrace{\beta_0 x_t + \dots + \beta_r x_{t-r}}_{\text{X(r) part}} + \underbrace{\varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}}_{\text{MA}(q) \text{ part}} \quad (9.18)$$

where  $\{\varepsilon_t\}$  is an i.i.d. white noise sequence and  $\{x_t\}$  is exogenous to  $y_t$ .

What is the effect of a one-unit increase in  $x_t$  on  $y_t$ ? To address this question, this notion of “effect” has to be formalized. Let us introduce two related sequences of values for  $\{x\}$ . Denote the first by  $\{a\}$  and the second by  $\{\tilde{a}^t\}$ . Further, we posit  $a_s = \tilde{a}_s^t$  for all  $s \neq t$ , and  $\tilde{a}_t^t = a_t + 1$ .

With these notations, we define  $\frac{\partial y_{t+h}}{\partial x_t}$  as follows:

$$\frac{\partial y_{t+h}}{\partial x_t} := \mathbb{E}(y_{t+h} | \{x\} = \{\tilde{a}^t\}) - \mathbb{E}(y_{t+h} | \{x\} = \{a\}). \quad (9.19)$$

Under the exogeneity assumption, it is easily seen that

$$\frac{\partial y_t}{\partial x_t} = \beta_0.$$

Now, since

$$y_{t+1} = c + \phi_1 y_t + \dots + \phi_p y_{t+1-p} + \beta_0 x_{t+1} + \dots + \beta_r x_{t+1-r} + \varepsilon_{t+1} + \theta_1 \varepsilon_t + \dots + \theta_q \varepsilon_{t+1-q},$$

and using the exogeneity assumption, we obtain:

$$\frac{\partial y_{t+1}}{\partial x_t} := \phi_1 \frac{\partial y_t}{\partial x_t} + \beta_1 = \phi_1 \beta_0 + \beta_1.$$

This can be applied recursively to give  $\frac{\partial y_{t+h}}{\partial x_t}$  for any  $h \geq 0$ :

**Proposition 9.9** (Dynamic multipliers in ARMAX models). *One can recursively compute the dynamic multipliers  $\frac{\partial y_{t+h}}{\partial x_t}$  as follows:*

- i. Initialization:  $\frac{\partial y_{t+h}}{\partial x_t} = 0$  for  $h < 0$ .
- ii. For  $h \geq 0$  and assuming that the first  $h - 1$  multipliers have been computed, we have:

$$\frac{\partial y_{t+h}}{\partial x_t} = \phi_1 \frac{\partial y_{t+h-1}}{\partial x_t} + \dots + \phi_p \frac{\partial y_{t+h-p}}{\partial x_t} + \beta_h, \quad (9.20)$$

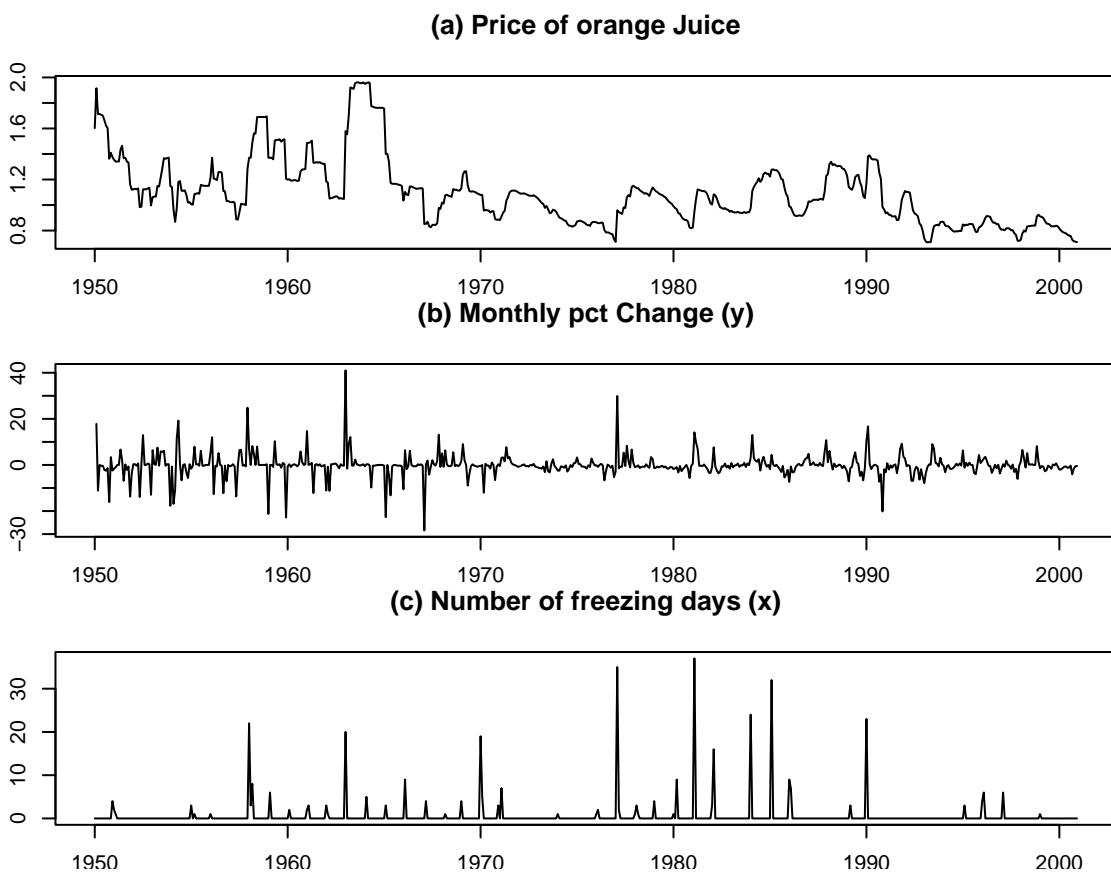
where we use the notation  $\beta_h = 0$  if  $h > r$ .

Remark that the resulting dynamic multipliers are the same as those obtained for an ARMA(p,r) model where the  $\theta_i$ 's are replaced with  $\beta_i$ 's (see Proposition 9.8 in Section 9.2.7).

It has to be stressed that the definition of the dynamic multipliers (Eq. (9.20)) does not reflect a potential persistency of the shock occurring on date  $t$  in process  $\{x\}$  itself. Going in this direction would necessitate to model the joint dynamics of  $x_t$  (for instance using a VAR model , see Section 9.3).

**Example 9.3** (Influence of the number of freezing days on the price of orange juice). This example is based on data used in Stock and Watson (2003) (Chapter 16). The objective is to study the influence of the number of freezing days on the price of orange juice. Let us first estimate a ARMAX(0,0,12) model:

```
library(AEC)
library(AER)
data("FrozenJuice")
FJ <- as.data.frame(FrozenJuice)
date <- time(FrozenJuice)
price <- FJ$price/FJ$ppi
T <- length(price)
k <- 1
dprice <- 100*log(price[(k+1):T]/price[1:(T-k)])
fdd <- FJ$fdd[(k+1):T]
par(mfrow=c(3,1))
par(plt=c(.1,.95,.15,.75))
plot(date,price,type="l",xlab="",ylab="",
      main="(a) Price of orange Juice")
plot(date,c(NaN,dprice),type="l",xlab="",ylab="",
      main="(b) Monthly pct Change (y)")
plot(date,FJ$fdd,type="l",xlab="",ylab="",
      main="(c) Number of freezing days (x)")
```



```

nb.lags <- 12
FDD <- FJ$fdd[(nb.lags+1):T]
names.FDD <- NULL
for(i in 1:nb.lags){
  FDD <- cbind(FDD,FJ$fdd[(nb.lags+1-i):(T-i)])
  names.FDD <- c(names.FDD,paste(" Lag ",toString(i),sep=""))
}
colnames(FDD) <- c(" Lag 0",names.FDD)
dprice <- dprice[(length(dprice)-dim(FDD)[1]+1):length(dprice)]
eq <- lm(dprice~FDD)

# Compute the Newey-West std errors:
var.cov.mat <- NeweyWest(eq,lag = 7, prewhite = FALSE)
robust_se <- sqrt(diag(var.cov.mat))

# Stargazer output (with and without Robust SE)
stargazer::stargazer(eq, eq, type = "text",
                      column.labels=c("(no HAC)","(HAC)"),keep.stat="n",
                      se = list(NULL,robust_se),no.space = TRUE)

```

```

## 
## =====
##             Dependent variable:
## -----
##                   dprice
##             (no HAC)      (HAC)
##             (1)          (2)
## -----
## FDD Lag 0      0.496***    0.496***
##                 (0.058)     (0.139)
## FDD Lag 1      0.150***    0.150*
##                 (0.058)     (0.087)
## FDD Lag 2      0.046       0.046
##                 (0.057)     (0.056)
## FDD Lag 3      0.062       0.062
##                 (0.057)     (0.046)
## FDD Lag 4      0.024       0.024
##                 (0.057)     (0.030)
## FDD Lag 5      0.036       0.036

```

```

##          (0.057)      (0.030)
## FDD Lag 6      0.037      0.037
##          (0.057)      (0.046)
## FDD Lag 7      0.019      0.019
##          (0.057)      (0.015)
## FDD Lag 8     -0.038     -0.038
##          (0.057)      (0.034)
## FDD Lag 9     -0.006     -0.006
##          (0.057)      (0.050)
## FDD Lag 10    -0.112*    -0.112
##          (0.057)      (0.069)
## FDD Lag 11    -0.063     -0.063
##          (0.058)      (0.052)
## FDD Lag 12   -0.140**   -0.140*
##          (0.058)      (0.078)
## Constant     -0.426*    -0.426*
##          (0.238)      (0.243)
## -----
## Observations   600       600
## =====
## Note:          *p<0.1; **p<0.05; ***p<0.01

```

Let us now use function `estim.armax`, from package AECto estimate an ARMA-X(2,0,1) model:

```

nb.lags <- 1
FDD <- FJ$fdd[(nb.lags+1):T]
names.FDD <- NULL
for(i in 1:nb.lags){
  FDD <- cbind(FDD,FJ$fdd[(nb.lags+1-i):(T-i)])
  names.FDD <- c(names.FDD,paste(" Lag ",toString(i),sep=""))
}
colnames(FDD) <- c(" Lag 0",names.FDD)
dprice <- 100*log(price[(k+1):T]/price[1:(T-k)])
dprice <- dprice[(length(dprice)-dim(FDD)[1]+1):length(dprice)]
res.armax <- estim.armax(Y = dprice,p=3,q=0,X=FDD)

## [1] =====
## [1] " ESTIMATING"
## [1] =====
## [1] " END OF ESTIMATION"
## [1] =====
## [1] ""
## [1] " RESULTS:"
## [1] " -----"
##           THETA      st.dev   t.ratio
## c        -0.46556249 0.19554352 -2.380864
## phi     t-1  0.09788977 0.04025907  2.431496
## phi     t-2  0.05049849 0.03827488  1.319364
## phi     t-3  0.07155170 0.03764750  1.900570
## sigma    4.64917949 0.13300769 34.954215
## beta    t-0  0.47015552 0.05665344  8.298800
## beta    t-1  0.10015862 0.05972526  1.676989
## [1] =====

```

Figure 9.11 shows the IRF associated with each of the two models.

```

nb.periods <- 20
IRF1 <- sim.arma(c=0,phi=c(0),theta=eq$coefficients[2:13],sigma=1,

```

```

T=nb.periods,y.0=c(0),nb.sim=1,make.IRF=1)
IRF2 <- sim.arma(c=0,phi=res.armax$phi,theta=res.armax$beta,sigma=1,
                  T=nb.periods,y.0=rep(0,length(res.armax$phi)),
                  nb.sim=1,make.IRF=1)
par(plt=c(.15,.95,.2,.95))
plot(IRF1,type="l",lwd=2,col="red",xlab="months after shock",
     ylab="Chge in price (percent)")
lines(IRF2,lwd=2,col="red",lty=2)
abline(h=0,col="grey")

```

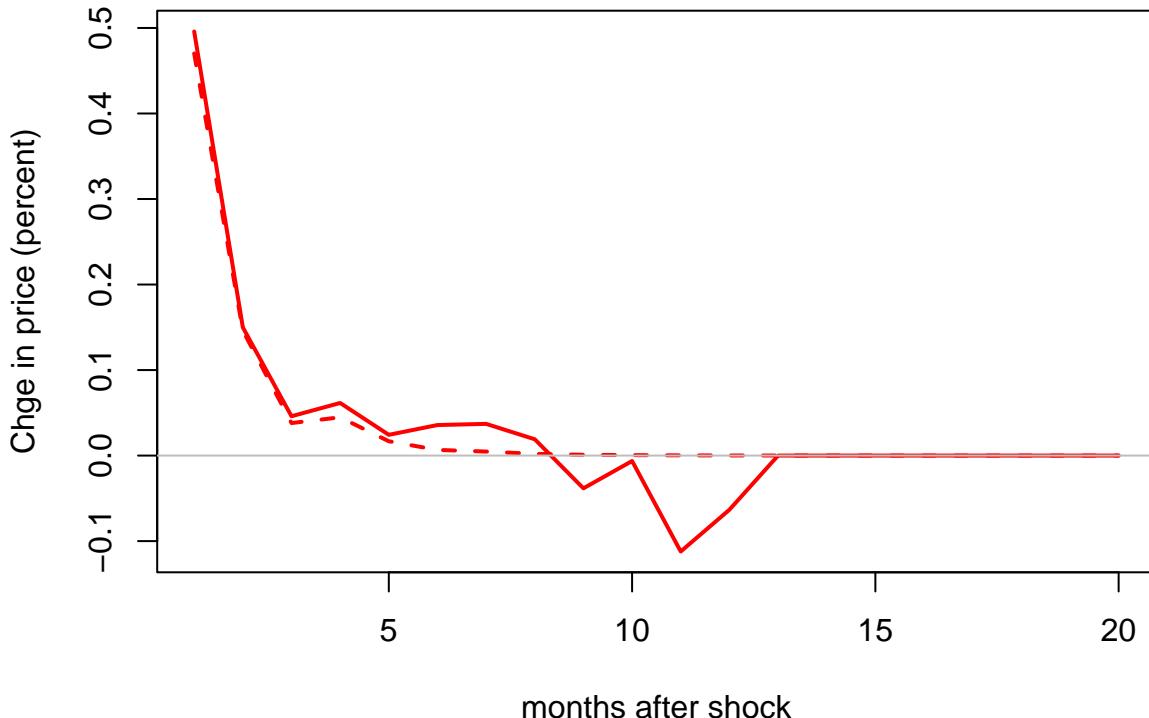


Figure 9.11: Response of changes in orange juice price (in percent) to the number of freezing days. The solid (respectively dashed) line corresponds to the ARMAX(0,0,12) (resp. ARMAX(3,0,1)) model. The first model is estimated by OLS (see above), the second by MLE.

**Example 9.4** (Real effect of a monetary policy shock). In this example, we make use of monetary shocks identified through high-frequency data (see Gertler and Karadi (2015)). This dataset comes from Valerie Ramey's website (see Ramey (2016)).

```

library(AEC)
T <- dim(Ramey)[1]
# Construct growth series:
Ramey$growth <- Ramey$LIP - c(rep(NaN,12),Ramey$LIP[1:(length(Ramey$LIP)-12)])
# Prepare matrix of exogenous variables:
vec.lags <- c(9,12,18)
Matrix.of.Exog <- NULL
shocks <- Ramey$ED2_TC
for(i in 1:length(vec.lags)){Matrix.of.Exog <-
  cbind(Matrix.of.Exog,c(rep(NaN,vec.lags[i]),shocks[1:(T-vec.lags[i])]))}
# Look for dates where data are available:
indic.good.dates <- complete.cases(Matrix.of.Exog)

# Estimate ARMAX:
p <- 1; q <- 0
x <- estim.armax(Ramey$growth[indic.good.dates],p,q,
                  X=Matrix.of.Exog[indic.good.dates,])
## [1] "====="

```

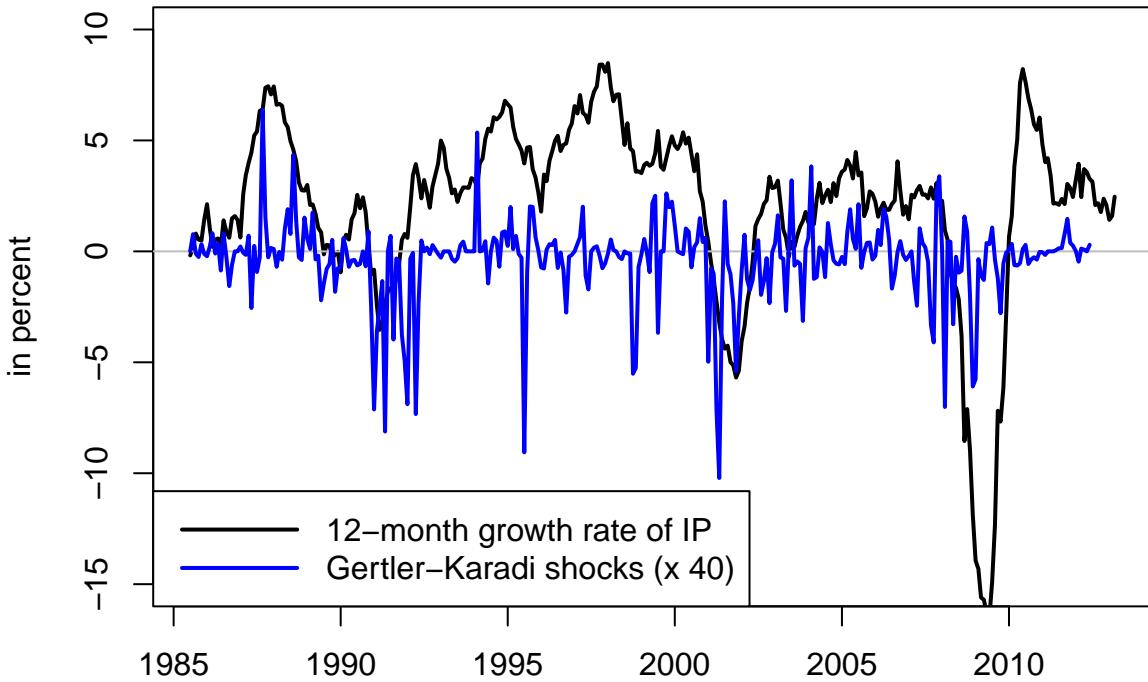


Figure 9.12: The blue line corresponds to monetary-policy shocks identified by means of the Gertler and Karadi (2015)'s approach (high-frequency change in Euro-dollar futures). The black solid line is the year-on-year growth rate of industrial production.

```
## [1] " ESTIMATING"
## [1] "=====
## [1] " END OF ESTIMATION"
## [1] "=====
## [1] ""
## [1] " RESULTS:"
## [1] " -----
##           THETA      st.dev   t.ratio
## c       -0.0001716198 0.0005845907 -0.2935726
## phi    t-1  0.9825608412 0.0120458531 81.5683897
## sigma   0.0087948724 0.0003211748 27.3834438
## beta   t-0 -0.0193570616 0.0087331529 -2.2165032
## beta   t-1 -0.0225707935 0.0086750938 -2.6017925
## beta   t-2 -0.0070131593 0.0086387440 -0.8118263
## [1] "=====

# Compute IRF:
irf <- sim.arma(0,x$phi,x$beta,x$sigma,T=60,y.0=rep(0,length(x$phi)),
                nb.sim=1,make.IRF=1,X=NaN,beta=NaN)
```

Figure 9.13 displays the resulting IRF, with a 95% confidence band. The code used to produce the confidence bands (i.e., to compute the standard deviation of the dynamic multipliers for the different horizons) is based on the Delta method (see Eq. (7.15)). The codes are available in Appendix 10.6.2.

### 9.2.8 Maximum Likelihood Estimation of ARMA processes

Consider the general case (of any time series); assume we observe a sample  $\mathbf{y} = [y_1, \dots, y_T]'$ . In order to implement ML techniques (see Section 7.2), we need to evaluate the joint p.d.f. (or “likelihood”) of  $\mathbf{y}$ , i.e.,  $\mathcal{L}(\theta; \mathbf{y})$ , where  $\theta$  is a vector of parameters that characterizes the dynamics of  $y_t$ . The Maximum Likelihood (ML) estimate of  $\theta$  is then given by:

$$\theta_{MLE} = \arg \max_{\theta} \mathcal{L}(\theta; \mathbf{y}) = \arg \max_{\theta} \log \mathcal{L}(\theta; \mathbf{y}).$$

In the time series context, if process  $y_t$  is Markovian, then there exists a useful way to rewrite the likelihood  $\mathcal{L}(\theta; \mathbf{y})$ . Let us first recall the definition of a Markovian process (see also Def. 7.9):

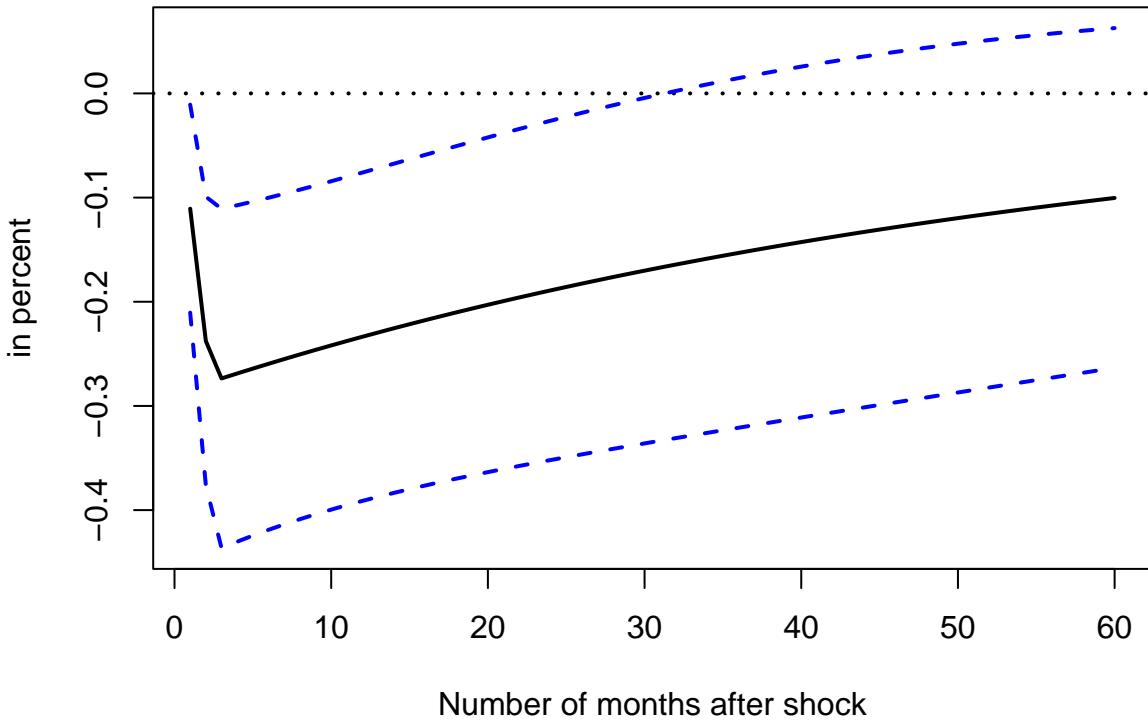


Figure 9.13: Response of industrial-production growth to monetary-policy shocks. Dashed lines correspond to the  $\pm 2$ -standard-deviation bands.

**Definition 9.19** (Markovian process). Process  $y_t$  is Markovian of order one if  $f_{Y_t|Y_{t-1}, Y_{t-2}, \dots} = f_{Y_t|Y_{t-1}}$ . More generally, it is Markovian of order  $k$  if  $f_{Y_t|Y_{t-1}, Y_{t-2}, \dots} = f_{Y_t|Y_{t-1}, \dots, Y_{t-k}}$ .

Now, remember Bayes' formula:

$$\mathbb{P}(X_2 = x, X_1 = y) = \mathbb{P}(X_2 = x|X_1 = y)\mathbb{P}(X_1 = y).$$

Using it leads to the following decomposition of our likelihood function:

$$f_{Y_T, \dots, Y_1}(y_T, \dots, y_1; \theta) = f_{Y_T|Y_{T-1}, \dots, Y_1}(y_T, \dots, y_1; \theta) \times f_{Y_{T-1}, \dots, Y_1}(y_{T-1}, \dots, y_1; \theta).$$

Using the previous expression recursively, one obtains:

$$f_{Y_T, \dots, Y_1}(y_T, \dots, y_1; \theta) = f_{Y_1}(y_1; \theta) \prod_{t=2}^T f_{Y_t|Y_{t-1}, \dots, Y_1}(y_t, \dots, y_1; \theta). \quad (9.21)$$

Let us start with the Gaussian AR(1) process (which is Markovian of order one):

$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2).$$

For  $t > 1$ :

$$f_{Y_t|Y_{t-1}, \dots, Y_1}(y_t, \dots, y_1; \theta) = f_{Y_t|Y_{t-1}}(y_t, y_{t-1}; \theta)$$

and

$$f_{Y_t|Y_{t-1}}(y_t, y_{t-1}; \theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_t - c - \phi_1 y_{t-1})^2}{2\sigma^2}\right).$$

These expressions can be plugged into Eq. (9.21). But what about  $f_{Y_1}(y_1; \theta)$ ? There exist two possibilities:

1. **Case 1:** We use the marginal distribution:  $y_1 \sim \mathcal{N}\left(\frac{c}{1 - \phi_1}, \frac{\sigma^2}{1 - \phi_1^2}\right)$ .
2. **Case 2:**  $y_1$  is considered to be deterministic. In a way, that means that the first observation is “sacrificed”.

For a Gaussian AR(1) process, we have:

1. **Case 1:** The (exact) log-likelihood is:

$$\begin{aligned}\log \mathcal{L}(\theta; \mathbf{y}) &= -\frac{T}{2} \log(2\pi) - T \log(\sigma) + \frac{1}{2} \log(1 - \phi_1^2) \\ &\quad - \frac{(y_1 - c/(1 - \phi_1))^2}{2\sigma^2/(1 - \phi_1^2)} - \sum_{t=2}^T \left[ \frac{(y_t - c - \phi_1 y_{t-1})^2}{2\sigma^2} \right].\end{aligned}\quad (9.22)$$

The Maximum Likelihood Estimator of  $\theta = [c, \phi_1, \sigma^2]$  is obtained by numerical optimization.

2. **Case 2:** The (conditional) log-likelihood is:

$$\begin{aligned}\log \mathcal{L}^*(\theta; \mathbf{y}) &= -\frac{T-1}{2} \log(2\pi) - (T-1) \log(\sigma) \\ &\quad - \sum_{t=2}^T \left[ \frac{(y_t - c - \phi_1 y_{t-1})^2}{2\sigma^2} \right].\end{aligned}\quad (9.23)$$

Exact MLE and conditional MLE have the same asymptotic (i.e. large-sample) distribution. Indeed, when the process is stationary,  $f_{Y_1}(y_1; \theta)$  makes a relatively negligible contribution to  $\log \mathcal{L}(\theta; \mathbf{y})$ .

The conditional MLE has a substantial advantage: in the Gaussian case, the conditional MLE is simply obtained by OLS. Indeed, let us introduce the notations:

$$Y = \begin{bmatrix} y_2 \\ \vdots \\ y_T \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & y_1 \\ \vdots & \vdots \\ 1 & y_{T-1} \end{bmatrix}.$$

Eq. (9.23) then rewrites:

$$\begin{aligned}\log \mathcal{L}^*(\theta; \mathbf{y}) &= -\frac{T-1}{2} \log(2\pi) - (T-1) \log(\sigma) \\ &\quad - \frac{1}{2\sigma^2} (Y - X[c, \phi_1]')' (Y - X[c, \phi_1]'),\end{aligned}\quad (9.24)$$

which is maximised for:

$$[\hat{c}, \hat{\phi}_1]' = (X'X)^{-1} X' Y \quad (9.25)$$

$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{T-1} \sum_{t=2}^T (y_t - \hat{c} - \hat{\phi}_1 y_{t-1})^2 \\ &= \frac{1}{T-1} Y' (I - X(X'X)^{-1} X') Y.\end{aligned}\quad (9.26)$$

Let us turn to the case of an AR(p) process. We have:

$$\begin{aligned}\log \mathcal{L}(\theta; \mathbf{y}) &= \log f_{Y_p, \dots, Y_1}(y_p, \dots, y_1; \theta) + \\ &\quad \underbrace{\sum_{t=p+1}^T \log f_{Y_t | Y_{t-1}, \dots, Y_{t-p}}(y_t, \dots, y_{t-p}; \theta)}_{\log \mathcal{L}^*(\theta; \mathbf{y})}.\end{aligned}$$

where  $f_{Y_p, \dots, Y_1}(y_p, \dots, y_1; \theta)$  is the marginal distribution of  $\mathbf{y}_{1:p} := [y_p, \dots, y_1]'$ . The marginal distribution of  $\mathbf{y}_{1:p}$  is Gaussian; it is therefore fully characterised by its mean and covariance matrix:

$$\begin{aligned}\mathbb{E}(\mathbf{y}_{1:p}) &= \frac{c}{1 - \phi_1 - \dots - \phi_p} \mathbf{1}_{p \times 1} \\ \mathbb{V}ar(\mathbf{y}_{1:p}) &= \begin{bmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \dots & \gamma_{p-2} \\ \vdots & & \ddots & \vdots \\ \gamma_{p-1} & \gamma_{p-2} & \dots & \gamma_0 \end{bmatrix},\end{aligned}$$

where the  $\gamma_i$ 's are computed using the Yule-Walker equations (Eq. (9.15)). Note that they depend, in a non-linear way, on the model parameters. Hence, the maximization of the exact log-likelihood necessitates numerical optimization procedures. By contrast, the maximization of the conditional log-likelihood  $\log \mathcal{L}^*(\theta; \mathbf{y})$  only requires OLS, using Eqs. (9.25) and (9.26), with:

$$Y = \begin{bmatrix} y_{p+1} \\ \vdots \\ y_T \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & y_p & \cdots & y_1 \\ \vdots & \vdots & & \vdots \\ 1 & y_{T-1} & \cdots & y_{T-p} \end{bmatrix}.$$

Again, for stationary processes, conditional and exact MLE have the same asymptotic (large-sample) distribution. In small samples, the OLS formula is however biased. Indeed, consider the regression (where  $y_t$  follows an AR(p) process):

$$y_t = \beta' \mathbf{x}_t + \varepsilon_t, \quad (9.27)$$

with  $\mathbf{x}_t = [1, y_{t-1}, \dots, y_{t-p}]'$  and  $\beta = [c, \phi_1, \dots, \phi_p]'$ .

The bias results from the fact that  $\mathbf{x}_t$  correlates to the  $\varepsilon_s$ 's for  $s < t$ . To be sure:

$$\mathbf{b} = \beta + (X'X)^{-1}X'\varepsilon, \quad (9.28)$$

and because of the specific form of  $X$ , we have non-zero correlation between  $\mathbf{x}_t$  and  $\varepsilon_s$  for  $s < t$ , therefore  $\mathbb{E}[(X'X)^{-1}X'\varepsilon] \neq 0$ . Again, asymptotically, the previous expectation goes to zero, and we have:

**Proposition 9.10** (Large-sample properties of the OLS estimator of AR(p) models). *Assume  $\{y_t\}$  follows the AR(p) process:*

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$

where  $\{\varepsilon_t\}$  is an i.i.d. white noise process. If  $\mathbf{b}$  is the OLS estimator of  $\beta$  (Eq. (9.27)), we have:

$$\sqrt{T}(\mathbf{b} - \beta) = \underbrace{\left[ \frac{1}{T} \sum_{t=p}^T \mathbf{x}_t \mathbf{x}'_t \right]}^{p \rightarrow \mathbf{Q}^{-1}} \underbrace{\sqrt{T} \left[ \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \varepsilon_t \right]}_{d \rightarrow \mathcal{N}(0, \sigma^2 \mathbf{Q})},$$

where  $\mathbf{Q} = \text{plim } \frac{1}{T} \sum_{t=p}^T \mathbf{x}_t \mathbf{x}'_t = \text{plim } \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \mathbf{x}'_t$  is given by:

$$\mathbf{Q} = \begin{bmatrix} 1 & \mu & \mu & \cdots & \mu \\ \mu & \gamma_0 + \mu^2 & \gamma_1 + \mu^2 & \cdots & \gamma_{p-1} + \mu^2 \\ \mu & \gamma_1 + \mu^2 & \gamma_0 + \mu^2 & \cdots & \gamma_{p-2} + \mu^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mu & \gamma_{p-1} + \mu^2 & \gamma_{p-2} + \mu^2 & \cdots & \gamma_0 + \mu^2 \end{bmatrix}. \quad (9.29)$$

*Proof.* Rearranging Eq. (9.47), we have:

$$\sqrt{T}(\mathbf{b} - \beta) = (X'X/T)^{-1}\sqrt{T}(X'\varepsilon/T).$$

Let us consider the autocovariances of  $\mathbf{v}_t = \mathbf{x}_t \varepsilon_t$ , denoted by  $\gamma_j^v$ . Using the fact that  $\mathbf{x}_t$  is a linear combination of past  $\varepsilon_t$ 's and that  $\varepsilon_t$  is a white noise, we get that  $\mathbb{E}(\varepsilon_t \mathbf{x}_t) = 0$ . Therefore

$$\gamma_j^v = \mathbb{E}(\varepsilon_t \varepsilon_{t-j} \mathbf{x}_t \mathbf{x}'_{t-j}).$$

If  $j > 0$ , we have

$$\begin{aligned} \mathbb{E}(\varepsilon_t \varepsilon_{t-j} \mathbf{x}_t \mathbf{x}'_{t-j}) &= \mathbb{E}(\mathbb{E}[\varepsilon_t \varepsilon_{t-j} \mathbf{x}_t \mathbf{x}'_{t-j} | \varepsilon_{t-j}, \mathbf{x}_t, \mathbf{x}_{t-j}]) \\ &= \mathbb{E}(\varepsilon_{t-j} \mathbf{x}_t \mathbf{x}'_{t-j} \mathbb{E}[\varepsilon_t | \varepsilon_{t-j}, \mathbf{x}_t, \mathbf{x}_{t-j}]) = 0. \end{aligned}$$

Note that, for  $j > 0$ , we have  $\mathbb{E}[\varepsilon_t | \varepsilon_{t-j}, \mathbf{x}_t, \mathbf{x}_{t-j}] = 0$  because  $\{\varepsilon_t\}$  is an i.i.d. white noise sequence. If  $j = 0$ , we have:

$$\gamma_0^v = \mathbb{E}(\varepsilon_t^2 \mathbf{x}_t \mathbf{x}'_t) = \mathbb{E}(\varepsilon_t^2) \mathbb{E}(\mathbf{x}_t \mathbf{x}'_t) = \sigma^2 \mathbf{Q}.$$

The convergence in distribution of  $\sqrt{T}(X'\varepsilon/T) = \sqrt{T} \frac{1}{T} \sum_{t=1}^T v_t$  results from Theorem 9.1 (applied on  $\mathbf{v}_t = \mathbf{x}_t \varepsilon_t$ ), using the  $\gamma_j^v$  computed above.  $\square$

These two cases (exact or conditional log-likelihoods) can be implemented when asking R to fit an AR process by means of function `arima`. Let us for instance use the output gap of the `US3var` dataset (US quarterly data, covering the period 1959:2 to 2015:1, used in Gouriéroux et al. (2017)).

```
library(AEC)
y <- US3var$y.gdp.gap
ar3.Case1 <- arima(y, order = c(3,0,0), method="ML")
ar3.Case2 <- arima(y, order = c(3,0,0), method="CSS")
rbind(ar3.Case1$coef,ar3.Case2$coef)

##           ar1          ar2          ar3  intercept
## [1,] 1.191267 -0.08934705 -0.1781163 -0.9226007
## [2,] 1.192003 -0.08811150 -0.1787662 -1.0341696
```

The two sets of estimated coefficients appear to be very close to each other.

Let us now turn to Moving-Average processes. Start with the MA(1):

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad \varepsilon_t \sim i.i.d.\mathcal{N}(0, \sigma^2).$$

The  $\varepsilon_t$ 's are easily computed recursively, starting with  $\varepsilon_t = y_t - \mu - \theta_1 \varepsilon_{t-1}$ . We obtain:

$$\varepsilon_t = y_t - \theta_1 y_{t-1} + \theta_1^2 y_{t-2} + \cdots + (-1)^{t-1} \theta_1^{t-1} y_1 + (-1)^t \theta_1^t \varepsilon_0.$$

Assume that one wants to recover the sequence of  $\{\varepsilon_t\}$ 's based on observed values of  $y_t$  (from date 1 to date  $t$ ). One can use the previous expression, but what value should be used for  $\varepsilon_0$ ? If one does not use the true value of  $\varepsilon_0$  but 0 (say), one does not obtain  $\varepsilon_t$ , but only an estimate of it ( $\hat{\varepsilon}_t$ , say), with:

$$\hat{\varepsilon}_t = \varepsilon_t - (-1)^t \theta_1^t \varepsilon_0.$$

Clearly, if  $|\theta_1| < 1$ , then the error becomes small for large  $t$ . Formally, when  $|\theta_1| < 1$ , we have:

$$\hat{\varepsilon}_t \xrightarrow{P} \varepsilon_t.$$

Hence, when  $|\theta_1| < 1$ , a consistent estimate of the conditional log-likelihood is given by:

$$\log \hat{\mathcal{L}}^*(\theta; \mathbf{y}) = -\frac{T}{2} \log(2\pi) - \frac{T}{2} \log(\sigma^2) - \sum_{t=1}^T \frac{\hat{\varepsilon}_t^2}{2\sigma^2}. \quad (9.30)$$

Loosely speaking, if  $|\theta_1| < 1$  and if  $T$  is sufficiently large:

$$\text{approximate conditional MLE} \approx \text{exact MLE}.$$

Note that  $\hat{\mathcal{L}}^*(\theta; \mathbf{y})$  is a complicated nonlinear function of  $\mu$  and  $\theta$ . Its maximization therefore has to be based on numerical optimization procedures.

Let us now consider the case of a Gaussian MA( $q$ ) process:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim i.i.d.\mathcal{N}(0, \sigma^2). \quad (9.31)$$

Let us assume that this process is an **invertible MA process**. That is, assume that the roots of:

$$\lambda^q + \theta_1 \lambda^{q-1} + \cdots + \theta_{q-1} \lambda + \theta_q = 0 \quad (9.32)$$

lie strictly inside of the unit circle. In this case, the polynomial form  $\Theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$  is *invertible* and Eq. (9.31) writes:

$$\varepsilon_t = \Theta(L)^{-1}(y_t - \mu),$$

which implies that, if we knew all past values of  $y_t$ , we would also know  $\varepsilon_t$ . In this case, we can consistently estimate the  $\varepsilon_t$ 's by recursively computing the  $\hat{\varepsilon}_t$ 's as follows (for  $t > 0$ ):

$$\hat{\varepsilon}_t = y_t - \mu - \theta_1 \hat{\varepsilon}_{t-1} - \cdots - \theta_q \hat{\varepsilon}_{t-q}, \quad (9.33)$$

with

$$\hat{\varepsilon}_0 = \dots = \hat{\varepsilon}_{-q+1} = 0. \quad (9.34)$$

In this context, a consistent estimate of the conditional log-likelihood is still given by Eq. (9.30), using Eqs. (9.33) and (9.34) to recursively compute the  $\hat{\varepsilon}_t$ 's.

Note that we could determine the exact likelihood of an MA process. Indeed, vector  $\mathbf{y} = [y_1, \dots, y_T]'$  is a Gaussian-distributed vector of mean  $\mu = [\mu, \dots, \mu]'$  and of variance:

$$\Omega = \begin{bmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_q & 0 & \dots & 0 \\ \gamma_1 & \gamma_0 & \gamma_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \gamma_1 & \ddots & \ddots & \ddots & \ddots & 0 \\ \gamma_q & \ddots & & & & & \gamma_q \\ 0 & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \gamma_1 & \gamma_0 & \gamma_1 \\ 0 & \dots & 0 & \gamma_q & \dots & \gamma_1 & \gamma_0 \end{bmatrix},$$

where the  $\gamma_j$ 's are given by Eq. (9.7). The p.d.f. of  $\mathbf{y}$  is then given by (see Prop. 10.18):

$$(2\pi)^{-T/2} |\Omega|^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{y} - \mu)' \Omega^{-1} (\mathbf{y} - \mu)\right).$$

For large samples, the computation of this likelihood however becomes numerically demanding.

Finally, let us consider the MLE of an ARMA( $p,q$ ) processes:

$$y_t = c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2).$$

If the MA part of this process is invertible, the log-likelihood function can be consistently approximated by its conditional counterpart (of the form of Eq. (9.30)), using consistent estimates  $\hat{\varepsilon}_t$  of the  $\varepsilon_t$ . The  $\hat{\varepsilon}_t$ 's are computed recursively as:

$$\hat{\varepsilon}_t = y_t - c - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p} - \theta_1 \hat{\varepsilon}_{t-1} - \dots - \theta_q \hat{\varepsilon}_{t-q}, \quad (9.35)$$

given some initial conditions, for instance:

- a.  $\hat{\varepsilon}_0 = \dots = \hat{\varepsilon}_{-q+1} = 0$  and  $y_0 = \dots = y_{-p+1} = \mathbb{E}(y_i) = \mu$ . (Recursions in Eq. (9.35) then start for  $t = 1$ .)
- b.  $\hat{\varepsilon}_p = \dots = \hat{\varepsilon}_{p-q+1} = 0$  and actual values of the  $y_i$ 's for  $i \in [1, p]$ . In that case, the first  $p$  observations of  $y_t$  will not be used. Recursions in Eq. (9.35) then start for  $t = p + 1$ .

### 9.2.9 Specification choice

The previous section explains how to fit a given ARMA specification. But how to choose an appropriate specification? A possibility is to employ the (P)ACF approach (see Figure 9.7). However, the previous approach leads to either an AR or a MA process (and not an ARMA process). If one wants to consider various ARMA( $p,q$ ) specifications, for  $p \in \{1, \dots, P\}$  and  $q \in \{1, \dots, Q\}$ , say, then one can resort to **information criteria**.

In general, when choosing a specification, one faces the following dilemma:

- a. Too rich a specification may lead to “overfitting”/misspecification, implying additional estimation errors (in out-of-sample forecasts).
- b. Too simple a specification may lead to potential omission of valuable information (e.g., contained in older lags).

The lag selection approach based on the so-called **information criteria** consists in maximizing the fit of the data, but adding a penalty for the “richness” of the model. More precisely, using this approach amounts to minimizing a loss function that (a) negatively depends on the fitting errors and (b) positively depends on the number of parameters in the model.

**Definition 9.20** (Information Criteria). The Akaike (AIC), Hannan-Quinn (HQ) and Schwarz information (BIC) criteria are of the form

$$c^{(i)}(k) = \frac{-2 \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y})}{T} + \underbrace{\frac{k\phi^{(i)}(T)}{T}}_{\text{decreases w.r.t. } k \text{ increases w.r.t. } k},$$

with  $(i) \in \{AIC, HQ, BIC\}$  and where  $\hat{\theta}_T(k)$  denotes the ML estimate of  $\theta_0(k)$ , which is a vector of parameters of length  $k$ .

Criterion (i)		$\phi^{(i)}(T)$
Akaike	AIC	2
Hannan-Quinn	HQ	$2 \log(\log(T))$
Schwarz	BIC	$\log(T)$

The lag suggested by criterion  $(i)$  is then given by:

$$\hat{k}^{(i)} = \operatorname{argmin}_k c^{(i)}(k).$$

In the case of an ARMA(p,q) process,  $k = 2 + p + q$ .

**Proposition 9.11** (Consistency of the criteria-based lag selection). *The lag selection procedure is consistent (see Def. 10.8) if*

$$\lim_{T \rightarrow \infty} \phi(T) = \infty \quad \text{and} \quad \lim_{T \rightarrow \infty} \phi(T)/T = 0.$$

This is notably the case of the HQ and the BIC criteria.

*Proof.* The true number of lags is denoted by  $k_0$ . We will show that  $\lim_{T \rightarrow \infty} \mathbb{P}(\hat{k}_T \neq k_0) = 0$ .

- Case  $k < k_0$ : The model with  $k$  parameter is misspecified, therefore:

$$\operatorname{plim}_{T \rightarrow \infty} \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y})/T < \operatorname{plim}_{T \rightarrow \infty} \log \mathcal{L}(\hat{\theta}_T(k_0); \mathbf{y})/T.$$

Hence, if  $\lim_{T \rightarrow \infty} \phi(T)/T = 0$ , we have:  $\lim_{T \rightarrow \infty} \mathbb{P}(c(k_0) \geq c(k)) \rightarrow 0$  and

$$\lim_{T \rightarrow \infty} \mathbb{P}(\hat{k} < k_0) \leq \lim_{T \rightarrow \infty} \mathbb{P}\{c(k_0) \geq c(k) \text{ for some } k < k_0\} = 0.$$

- Case  $k > k_0$ : under the null hypothesis, the likelihood ratio (LR) test statistic (see Def. 7.8) satisfies:

$$2 \left( \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y}) - \log \mathcal{L}(\hat{\theta}_T(k_0); \mathbf{y}) \right) \sim \chi^2(k - k_0).$$

If  $\lim_{T \rightarrow \infty} \phi(T) = \infty$ , we have:  $\operatorname{plim}_{T \rightarrow \infty} -2 \left( \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y}) - \log \mathcal{L}(\hat{\theta}_T(k_0); \mathbf{y}) \right) / \phi(T) = 0$ . Hence  $\operatorname{plim}_{T \rightarrow \infty} T[c(k_0) - c(k)]/\phi(T) \leq -1$  and  $\lim_{T \rightarrow \infty} \mathbb{P}(c(k_0) \geq c(k)) \rightarrow 0$ , which implies, in the same spirit as before, that  $\lim_{T \rightarrow \infty} \mathbb{P}(\hat{k} > k_0) = 0$ .

Therefore,  $\lim_{T \rightarrow \infty} \mathbb{P}(\hat{k} = k_0) = 1$ . □

**Example 9.5** (Linear regression). Consider a linear regression with normal disturbances:

$$y_t = \mathbf{x}'_t \beta + \varepsilon_t, \quad \varepsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2).$$

The associated log-likelihood is of the form of Eq. (9.30). In that case, we have:

$$\begin{aligned} c^{(i)}(k) &= \frac{-2 \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y})}{T} + \frac{k\phi^{(i)}(T)}{T} \\ &\approx \log(2\pi) + \log(\widehat{\sigma}^2) + \frac{1}{T} \sum_{t=1}^T \frac{\varepsilon_t^2}{\widehat{\sigma}^2} + \frac{k\phi^{(i)}(T)}{T}. \end{aligned}$$

For a large  $T$ , for all consistent estimation scheme, we have:

$$\widehat{\sigma}^2 \approx \frac{1}{T} \sum_{t=1}^T \varepsilon_t^2 = SSR/T.$$

$$\text{Hence } \hat{k}^{(i)} \approx \operatorname{argmin}_k \log(SSR/T) + \frac{k\phi^{(i)}(T)}{T}.$$

**Example 9.6** (Swiss GDP growth). Consider a long historical time series of the Swiss GDP growth (see Figure 9.3), taken from the Jordà et al. (2017) dataset. Let us look for the best ARMA specification using the AIC criteria:

```
library(AEC);data(JST)
data <- subset(JST,iso=="CHE")
T <- dim(data)[1]
y <- c(NaN,log(data$gdp[2:T]/data$gdp[1:(T-1)]))
# Use AIC criteria to look for appropriate specif:
max.p <- 3;max.q <- 3;
all.AIC <- NULL
for(p in 0:max.p){
  for(q in 0:max.q){
    res <- arima(y,order=c(p,q))
    if(res$aic<min(all.AIC)){best.p<-p;best.q<-q}
    all.AIC <- c(all.AIC,res$aic)}
  print(c(best.p,best.q))

## [1] 1 0
```

The best specification therefore is an AR(1) model. That is, although an AR(2) (say) would result in a better fit of the data, the fit improvement is not be large enough to compensate for the additional AIC cost associated with an additional parameter.

## 9.3 Multivariate models

This section presents Vector Auto-Regressive Moving-Average (SVARMA) models. These models are widely used in macroeconomic analysis. While simple and easy to estimate, they make it possible to conveniently capture the dynamics of complex multivariate systems. VAR popularity is notably due to Sims (1980)'s influential work. A nice survey if proposed by Stock and Watson (2016).

In economics, VAR models are often employed in order to identify *structural shocks*, that are independent primitive exogenous forces that drive economic variables (Ramey (2016)). They are often given a specific economic meaning (e.g., demand and supply shocks).

Working with these models (VAR and VARMA models) often is often based on two steps: in a first step, the **reduced-form** version of the model is estimated; in a second step, **structural shocks** are identified and IRFs are produced.

->

### 9.3.1 Definition of VARs (and SVARMA) models

**Definition 9.21** ((S)VAR model). Let  $y_t$  denote a  $n \times 1$  vector of random variables. Process  $y_t$  follows a  $p^{th}$ -order (S)VAR if, for all  $t$ , we have

$$\begin{aligned} VAR: \quad y_t &= c + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + \varepsilon_t, \\ SVAR: \quad y_t &= c + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + B\eta_t, \end{aligned} \tag{9.36}$$

with  $\varepsilon_t = B\eta_t$ , where  $\{\eta_t\}$  is a white noise sequence whose components are mutually and serially independent.

The first line of Eq. (9.36) corresponds to the **reduced-form** of the VAR model (**structural form** for the second line).

While the structural shocks (the components of  $\eta_t$ ) are mutually uncorrelated, this is not the case of the *innovations*, that are the components of  $\varepsilon_t$ . However, in boths cases, vectors  $\eta_t$  and  $\varepsilon_t$  are serially correlated (through time).

As was the case for univariate models, VARs can be extended with MA terms in  $\eta_t$ :

**Definition 9.22** ((S)VARMA model). Let  $y_t$  denote a  $n \times 1$  vector of random variables. Process  $y_t$  follows a VARMA model of order (p,q) if, for all  $t$ , we have

$$\begin{aligned} \text{VARMA : } y_t &= c + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \cdots + \Theta_q, \\ \text{SVARMA : } y_t &= c + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + B_0 \eta_t + B_1 \eta_{t-1} + \cdots + B_q \eta_{t-q}, \end{aligned} \quad (9.37)$$

with  $\varepsilon_t = B_0 \eta_t$  (and  $B_j = \Theta_j B_0$ , for  $j \geq 0$ ), where  $\{\eta_t\}$  is a white noise sequence whose components are mutually and serially independent.

### 9.3.2 IRFs in SVARMA

One of the main objectives of macro-econometrics is to derive IRFs, that represent the dynamic effects of structural shocks (components of  $\eta_t$ ) though the system of variables  $y_t$ .

Formally, an IRF is a difference in conditional expectations:

$$\boxed{\Psi_{i,j,h} = \mathbb{E}(y_{i,t+h} | \eta_{j,t} = 1) - \mathbb{E}(y_{i,t+h})}$$

(effect on  $y_{i,t+h}$  of a one-unit shock on  $\eta_{j,t}$ ).

If the dynamics of process  $y_t$  can be described as a VARMA model, and if  $y_t$  is covariance stationary (see Def. 9.4), then  $y_t$  admits the following infinite MA representation (MA( $\infty$ )):

$$y_t = \mu + \sum_{h=0}^{\infty} \Psi_h \eta_{t-h}. \quad (9.38)$$

This is also the Wold decomposition of process  $\{y_t\}$  (see Theorem 9.3).

Estimating IRFs amounts to estimating the  $\Psi_h$ 's. In general, there exist three main approaches for that:

- Calibrate and solve a (purely structural) Dynamic Stochastic General Equilibrium (DSGE) model at the first order (linearization). The solution takes the form of Eq. (9.38).
- Directly estimate the  $\Psi_h$  based on **projection approaches** (see Section 9.3.11).
- Approximate the infinite MA representation by estimating a parsimonious type of model, e.g. **VAR(MA) models** (see Section 9.3.4). Once a (Structural) VARMA representation is obtained, Eq. (9.38) is easily deduced. For that, one can use the same recursive algorithm as for univariate processes (see Prop. 9.8).

Typically, consider the AR(2) case. The first steps of the algorithm mentioned in the last bullet point are as follows:

$$\begin{aligned} y_t &= \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + B \eta_t \\ &= \Phi_1(\Phi_1 y_{t-2} + \Phi_2 y_{t-3} + B \eta_{t-1}) + \Phi_2 y_{t-2} + B \eta_t \\ &= B \eta_t + \Phi_1 B \eta_{t-1} + (\Phi_2 + \Phi_1^2) y_{t-2} + \Phi_1 \Phi_2 y_{t-3} \\ &= B \eta_t + \Phi_1 B \eta_{t-1} + (\Phi_2 + \Phi_1^2)(\Phi_1 y_{t-3} + \Phi_2 y_{t-4} + B \eta_{t-2}) + \Phi_1 \Phi_2 y_{t-3} \\ &= \underbrace{B}_{=\Psi_0} \eta_t + \underbrace{\Phi_1 B}_{=\Psi_1} \eta_{t-1} + \underbrace{(\Phi_2 + \Phi_1^2) B}_{=\Psi_2} \eta_{t-2} + f(y_{t-3}, y_{t-4}). \end{aligned}$$

In particular, we have  $B = \Psi_0$ . Matrix  $B$  indeed captures the contemporaneous impact of  $\eta_t$  on  $y_t$ . That is why matrix  $B$  is sometimes called *impulse matrix*.

**Example 9.7** (IRFs of an SVARMA model). Consider the following VARMA(1,1) model:

$$y_t = \underbrace{\begin{bmatrix} 0.5 & 0.3 \\ -0.4 & 0.7 \end{bmatrix}}_{\Phi_1} y_{t-1} + \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}}_B \eta_t + \underbrace{\begin{bmatrix} 2 & 0 \\ 1 & 0.5 \end{bmatrix}}_{\Theta_1} \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}}_B \eta_{t-1}. \quad (9.39)$$

We can use function `simul.VARMA` of package `AEC` to produce IRFs (using `indic.IRF=1` in the list of arguments):

```

library(AEC)
distri <- list(type=c("gaussian","gaussian"),df=c(4,4))
n <- length(distri$type) # dimension of  $y_t$ 
nb.sim <- 30
eps <- simul.distri(distri,nb.sim)
Phi <- array(NaN,c(n,n,1))
Phi[,,1] <- matrix(c(.5,-.4,.3,.7),2,2)
p <- dim(Phi)[3]
Theta <- array(NaN,c(n,n,1))
Theta[,,1] <- -matrix(c(2,1,0,.5),2,2)
q <- dim(Theta)[3]
Mu <- rep(0,n)
C <- matrix(c(1,-1,2,1),2,2)
Model <- list(
  Mu = Mu,Phi = Phi,Theta = Theta,C = C,distri = distri)
Y0 <- rep(0,n)
eta0 <- c(1,0)
res.sim.1 <- simul.VARMA(Model,nb.sim,Y0,eta0,indic.IRF=1)
eta0 <- c(0,1)
res.sim.2 <- simul.VARMA(Model,nb.sim,Y0,eta0,indic.IRF=1)
par(plt=c(.15,.95,.25,.8))
par(mfrow=c(2,2))
plot(res.sim.1$Y[1,],las=1,
  type="l",lwd=3,xlab="",ylab="",
  main=expression(paste("Response of ",y[1],"*,*",t],
    " to a one-unit increase in ",eta[1],sep="")))
abline(h=0,col="grey",lty=3)
plot(res.sim.2$Y[1,],las=1,
  type="l",lwd=3,xlab="",ylab="",
  main=expression(paste("Response of ",y[1],"*,*",t],
    " to a one-unit increase in ",eta[2],sep="")))
abline(h=0,col="grey",lty=3)
plot(res.sim.1$Y[2,],las=1,
  type="l",lwd=3,xlab="",ylab="",
  main=expression(paste("Response of ",y[2],"*,*",t],
    " to a one-unit increase in ",eta[1],sep="")))
abline(h=0,col="grey",lty=3)
plot(res.sim.2$Y[2,],las=1,
  type="l",lwd=3,xlab="",ylab="",
  main=expression(paste("Response of ",y[2],"*,*",t],
    " to a one-unit increase in ",eta[2],sep="")))
abline(h=0,col="grey",lty=3)

```

### 9.3.3 Covariance-stationary VARMA models

Let's come back to the infinite MA case (Eq. (9.38)):

$$y_t = \mu + \sum_{h=0}^{\infty} \Psi_h \eta_{t-h}.$$

For  $y_t$  to be covariance-stationary (and ergodic for the mean), it has to be the case that

$$\sum_{i=0}^{\infty} \|\Psi_i\| < \infty, \quad (9.40)$$

where  $\|A\|$  denotes a norm of the matrix  $A$  (e.g.  $\|A\| = \sqrt{\text{tr}(AA')}$ ). This notably implies that if  $y_t$  is stationary (and ergodic for the mean), then  $\|\Psi_h\| \rightarrow 0$  when  $h$  gets large.

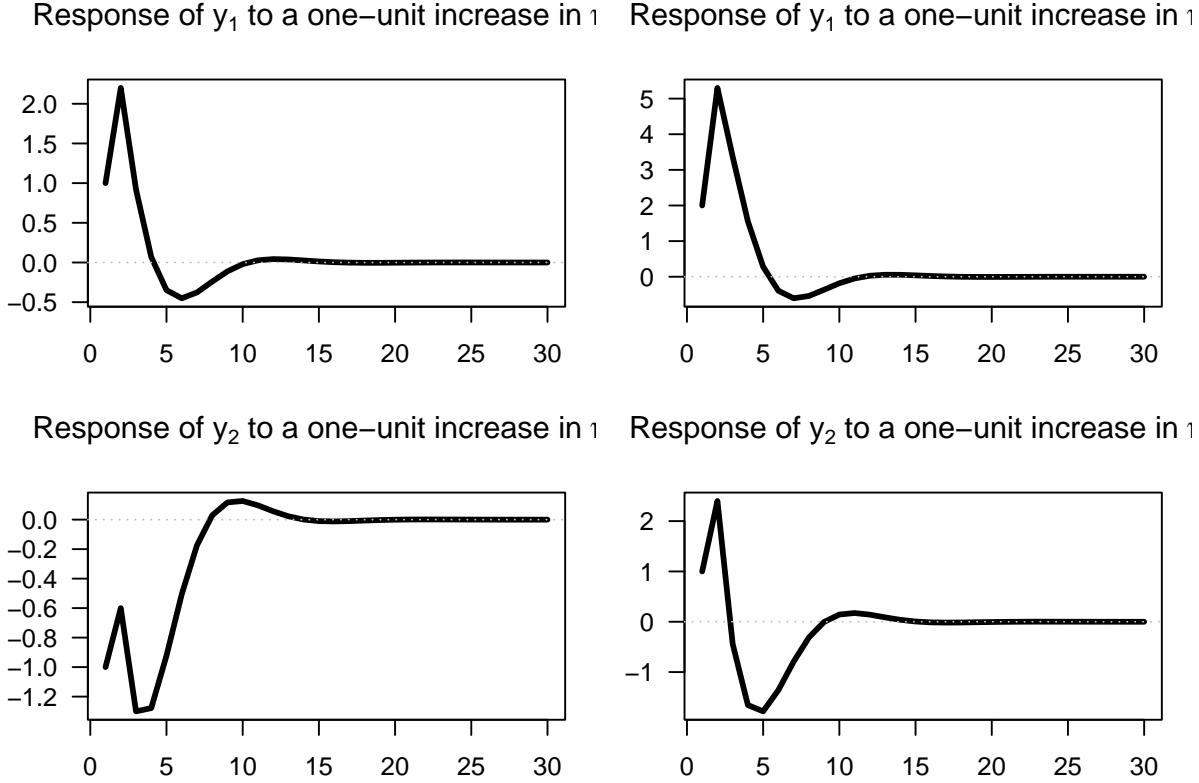


Figure 9.14: Impulse response functions

What should be satisfied by  $\Phi_k$ 's and  $\Theta_k$ 's for a VARMA-based process (Eq. (??)) to be stationary? The conditions will be similar to that we had in the univariate case (see Prop. 9.6). Let us introduce the following notations:

$$\begin{aligned} y_t &= c + \underbrace{\Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p}}_{\text{AR component}} + \\ &\quad \underbrace{B\eta_t + \Theta_1 B\eta_{t-1} + \cdots + \Theta_q B\eta_{t-q}}_{\text{MA component}} \\ \Leftrightarrow & \underbrace{(I - \Phi_1 L - \cdots - \Phi_p L^p)}_{=\Phi(L)} y_t = c + \underbrace{(I - \Theta_1 L - \cdots - \Theta_q L^q)}_{=\Theta(L)} B\eta_t. \end{aligned} \quad (9.41)$$

Process  $y_t$  is stationary iff the roots of  $\det(\Phi(z)) = 0$  are strictly outside the unit circle or, equivalently, iff the eigenvalues of

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_p \\ I & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \quad (9.42)$$

lie strictly within the unit circle. Hence, as was the case for univariate processes, the covariance-stationarity of a VARMA model depends only on the specification of its AR part.

Let's derive the first two unconditional moments of a (covariance-stationary) VARMA process.

Based on Eq. (9.41), we have  $\mathbb{E}(\Phi(L)y_t) = c$ , which gives  $\Phi(1)\mathbb{E}(y_t) = c$ , or::

$$\mathbb{E}(y_t) = (I - \Phi_1 - \cdots - \Phi_p)^{-1}c.$$

The autocovariances of  $y_t$  can be deduced from the infinite MA representation (Eq. (9.38)). We have:

$$\gamma_j \equiv \mathbb{C}ov(y_t, y_{t-j}) = \sum_{i=j}^{\infty} \Psi_i \Psi'_{i-j}.$$

(Note that this infinite sum exists as soon as Eq. (9.40) is satisfied.)

Conditional means and autocovariances can also be deduced from Eq. (9.38). For  $0 \leq h$  and  $0 \leq h_1 \leq h_2$ :

$$\begin{aligned}\mathbb{E}_t(y_{t+h}) &= \mu + \sum_{k=0}^{\infty} \Psi_{k+h} \eta_{t-k} \\ \text{Cov}_t(y_{t+1+h_1}, y_{t+1+h_2}) &= \sum_{k=0}^{h_1} \Psi_k \Psi'_{k+h_2-h_1}.\end{aligned}$$

The previous formula implies in particular that the forecasting error  $y_{t+h} - \mathbb{E}_t(y_{t+h})$  has a variance equal to:

$$\text{Var}_t(y_{t+h}) = \sum_{k=1}^h \Psi_k \Psi'_k.$$

Because the  $\eta_t$  are mutually and serially independent (and therefore uncorrelated), we have:

$$\text{Var}(\Psi_k \eta_{t-k}) = \text{Var} \left( \sum_{i=1}^n \psi_{k,i} \eta_{i,t-k} \right) = \sum_{i=1}^n \psi_{k,i} \psi'_{k,i},$$

where  $\psi_{k,i}$  denotes the  $i^{th}$  column of  $\Psi_k$ .

This suggests the following decomposition of the variance of the forecast error (called **variance decomposition**):

$$\text{Var}_t(y_{t+h}) = \sum_{i=1}^n \underbrace{\sum_{k=1}^h \psi_{k,i} \psi'_{k,i}}_{\text{Contribution of } \eta_{i,t}}.$$

Let us now turn to the estimation of VAR(MA) models.

If there is a MA component, OLS regressions yield biased estimates (even for asymptotically large samples).

Assume  $y_t$  follows a VARMA(1,1) model. We have:

$$y_{i,t} = \phi_i y_{t-1} + \varepsilon_{i,t},$$

where  $\phi_i$  is the  $i^{th}$  row of  $\Phi_1$ , and where  $\varepsilon_{i,t}$  is a linear combination of  $\eta_t$  and  $\eta_{t-1}$ .

Since  $y_{t-1}$  (the regressor) is correlated to  $\eta_{t-1}$ , it is also correlated to  $\varepsilon_{i,t}$ .

The OLS regression of  $y_{i,t}$  on  $y_{t-1}$  yields a biased estimator of  $\phi_i$ . Hence, SVARMA models cannot be consistently estimated by simple OLS regressions (contrary to VAR models, as we will see in the next section); instrumental-variable approaches can be employed to estimate SVARMA models.

### 9.3.4 VAR estimation

This section discusses the estimation of VAR models. (The estimation of SVARMA models is more challenging, see, e.g., Gouriéroux et al. (2020).) Eq. (9.36) can be written:

$$y_t = c + \Phi(L)y_{t-1} + \varepsilon_t,$$

with  $\Phi(L) = \Phi_1 + \Phi_2 L + \dots + \Phi_p L^{p-1}$ .

Consequently:

$$y_t \mid y_{t-1}, y_{t-2}, \dots, y_{-p+1} \sim \mathcal{N}(c + \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p}, \Omega).$$

Using Hamilton (1994)'s notations, denote with  $\Pi$  the matrix  $[ c \quad \Phi_1 \quad \Phi_2 \quad \dots \quad \Phi_p ]'$  and with  $x_t$  the vector  $[ 1 \quad y'_{t-1} \quad y'_{t-2} \quad \dots \quad y'_{t-p} ]'$ , we have:

$$y_t = \Pi' x_t + \varepsilon_t. \tag{9.43}$$

The previous representation is convenient to discuss the estimation of the VAR model, as parameters are gathered in two matrices only:  $\Pi$  and  $\Omega$ .

Let us start with the case where the shocks are Gaussian.

**Proposition 9.12** (MLE of a Gaussian VAR). *If  $y_t$  follows a  $VAR(p)$  (see Definition 9.21), and if  $\varepsilon_t \sim i.i.d. \mathcal{N}(0, \Omega)$ , then the ML estimate of  $\Pi$ , denoted by  $\hat{\Pi}$  (see Eq. (9.43)), is given by*

$$\hat{\Pi} = \left[ \sum_{t=1}^T x_t x_t' \right]^{-1} \left[ \sum_{t=1}^T y_t' x_t \right] = (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \mathbf{y}, \quad (9.44)$$

where  $\mathbf{X}$  is the  $T \times (np)$  matrix whose  $t^{th}$  row is  $x_t$  and where  $\mathbf{y}$  is the  $T \times n$  matrix whose  $t^{th}$  row is  $y_t'$ .

That is, the  $i^{th}$  column of  $\hat{\Pi}$  ( $b_i$ , say) is the OLS estimate of  $\beta_i$ , where:

$$y_{i,t} = \beta_i' x_t + \varepsilon_{i,t}, \quad (9.45)$$

(i.e.,  $\beta_i' = [c_i, \phi_{i,1}', \dots, \phi_{i,p}']'$ ).

The ML estimate of  $\Omega$ , denoted by  $\hat{\Omega}$ , coincides with the sample covariance matrix of the  $n$  series of the OLS residuals in Eq. (9.45), i.e.:

$$\hat{\Omega} = \frac{1}{T} \sum_{i=1}^T \hat{\varepsilon}_t \hat{\varepsilon}_t', \quad \text{with } \hat{\varepsilon}_t = y_t - \hat{\Pi}' x_t. \quad (9.46)$$

The asymptotic distributions of these estimators are the ones resulting from standard OLS formula.

*Proof.* See Appendix 10.5. □

As stated by Proposition 9.13, when the shocks are not Gaussian, then the OLS regressions still provide consistent estimates of the model parameters. However, since  $x_t$  correlates to  $\varepsilon_s$  for  $s < t$ , the OLS estimator  $\mathbf{b}_i$  of  $\beta_i$  is biased in small sample. (That is also the case for the ML estimator.)

Indeed, denoting by  $\varepsilon_i$  the  $T \times 1$  vector of  $\varepsilon_{i,t}$ 's, and using the notations of  $b_i$  and  $\beta_i$  introduced in Proposition 9.12, we have:

$$\mathbf{b}_i = \beta_i + (\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \varepsilon_i. \quad (9.47)$$

We have non-zero correlation between  $x_t$  and  $\varepsilon_{i,s}$  for  $s < t$  and, therefore,  $\mathbb{E}[(\mathbf{X}' \mathbf{X})^{-1} \mathbf{X}' \varepsilon_i] \neq 0$ .

However, when  $y_t$  is covariance stationary, then  $\frac{1}{n} \mathbf{X}' \mathbf{X}$  converges to a positive definite matrix  $\mathbf{Q}$ , and  $\frac{1}{n} \mathbf{X}' \varepsilon_i$  converges to 0. Hence  $\mathbf{b}_i \xrightarrow{p} \beta_i$ . More precisely:

**Proposition 9.13** (Asymptotic distribution of the OLS estimate of  $\beta_i$ ). *If  $y_t$  follows a VAR model, as defined in Definition 9.21, we have:*

$$\sqrt{T}(\mathbf{b}_i - \beta_i) = \underbrace{\left[ \frac{1}{T} \sum_{t=p}^T x_t x_t' \right]^{-1}}_{\xrightarrow{p} \mathbf{Q}^{-1}} \underbrace{\sqrt{T} \left[ \frac{1}{T} \sum_{t=1}^T x_t \varepsilon_{i,t} \right]}_{\xrightarrow{d} \mathcal{N}(0, \sigma_i^2 \mathbf{Q})},$$

where  $\sigma_i = \text{Var}(\varepsilon_{i,t})$  and where  $\mathbf{Q} = \text{plim} \frac{1}{T} \sum_{t=p}^T x_t x_t'$  is given by:

$$\mathbf{Q} = \begin{bmatrix} 1 & \mu' & \mu' & \dots & \mu' \\ \mu & \gamma_0 + \mu\mu' & \gamma_1 + \mu\mu' & \dots & \gamma_{p-1} + \mu\mu' \\ \mu & \gamma_1 + \mu\mu' & \gamma_0 + \mu\mu' & \dots & \gamma_{p-2} + \mu\mu' \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mu & \gamma_{p-1} + \mu\mu' & \gamma_{p-2} + \mu\mu' & \dots & \gamma_0 + \mu\mu' \end{bmatrix}. \quad (9.48)$$

*Proof.* See Appendix 10.5. □

The following proposition extends the previous proposition and includes covariances between different  $\beta_i$ 's as well as the asymptotic distribution of the ML estimates of  $\Omega$ .

**Proposition 9.14** (Asymptotic distribution of the OLS estimates). *If  $y_t$  follows a VAR model, as defined in Definition 9.21, we have:*

$$\sqrt{T} \begin{bmatrix} \text{vec}(\hat{\Pi} - \Pi) \\ \text{vec}(\hat{\Omega} - \Omega) \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} \Omega \otimes \mathbf{Q}^{-1} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} \right), \quad (9.49)$$

where the component of  $\Sigma_{22}$  corresponding to the covariance between  $\hat{\sigma}_{i,j}$  and  $\hat{\sigma}_{k,l}$  (for  $i, j, l, m \in \{1, \dots, n\}^4$ ) is equal to  $\sigma_{i,l}\sigma_{j,m} + \sigma_{i,m}\sigma_{j,l}$ .

*Proof.* See Hamilton (1994), Appendix of Chapter 11.  $\square$

Naturally, in practice,  $\Omega$  is replaced with  $\hat{\Omega}$ ,  $\mathbf{Q}$  is replaced with  $\hat{\mathbf{Q}} = \frac{1}{T} \sum_{t=p}^T x_t x_t'$  and  $\Sigma$  with the matrix whose components are of the form  $\hat{\sigma}_{i,l}\hat{\sigma}_{j,m} + \hat{\sigma}_{i,m}\hat{\sigma}_{j,l}$ , where the  $\hat{\sigma}_{i,l}$ 's are the components of  $\hat{\Omega}$ .

The simplicity of the VAR framework and the tractability of its MLE open the way to convenient econometric testing. Let's illustrate this with the likelihood ratio test (see Def. 7.8). The maximum value achieved by the MLE is

$$\log \mathcal{L}(Y_T; \hat{\Pi}, \hat{\Omega}) = -\frac{Tn}{2} \log(2\pi) + \frac{T}{2} \log |\hat{\Omega}^{-1}| - \frac{1}{2} \sum_{t=1}^T [\hat{\varepsilon}'_t \hat{\Omega}^{-1} \hat{\varepsilon}_t].$$

The last term is:

$$\begin{aligned} \sum_{t=1}^T \hat{\varepsilon}'_t \hat{\Omega}^{-1} \hat{\varepsilon}_t &= \text{Tr} \left[ \sum_{t=1}^T \hat{\varepsilon}'_t \hat{\Omega}^{-1} \hat{\varepsilon}_t \right] = \text{Tr} \left[ \sum_{t=1}^T \hat{\Omega}^{-1} \hat{\varepsilon}_t \hat{\varepsilon}'_t \right] \\ &= \text{Tr} \left[ \hat{\Omega}^{-1} \sum_{t=1}^T \hat{\varepsilon}_t \hat{\varepsilon}'_t \right] = \text{Tr} [\hat{\Omega}^{-1} (T\hat{\Omega})] = Tn. \end{aligned}$$

Therefore, the optimized log-likelihood is simply obtained by:

$$\log \mathcal{L}(Y_T; \hat{\Pi}, \hat{\Omega}) = -(Tn/2) \log(2\pi) + (T/2) \log |\hat{\Omega}^{-1}| - Tn/2. \quad (9.50)$$

Assume that we want to test the null hypothesis that a set of variables follows a  $\text{VAR}(p_0)$  against the alternative specification of  $p_1 (> p_0)$ .

Let us denote by  $\hat{L}_0$  and  $\hat{L}_1$  the maximum log-likelihoods obtained with  $p_0$  and  $p_1$  lags, respectively.

Under the null hypothesis ( $H_0: p = p_0$ ), we have:

$$2(\hat{L}_1 - \hat{L}_0) = T(\log |\hat{\Omega}_1^{-1}| - \log |\hat{\Omega}_0^{-1}|) \sim \chi^2(n^2(p_1 - p_0)).$$

What precedes can be used to help determine the appropriate number of lags to use in the specification. In a VAR, using too many lags consumes numerous degrees of freedom: with  $p$  lags, each of the  $n$  equations in the VAR contains  $n \times p$  coefficients plus the intercept term. Adding lags improve in-sample fit, but is likely to result in over-parameterization and affect the **out-of-sample** prediction performance.

To select appropriate lag length, **selection criteria** can be used (see Definition 9.20). In the context of VAR models, using Eq. (9.50), we have:

$$\begin{aligned} AIC &= cst + \log |\hat{\Omega}| + \frac{2}{T} N \\ BIC &= cst + \log |\hat{\Omega}| + \frac{\log T}{T} N, \end{aligned}$$

where  $N = p \times n^2$ .

### 9.3.5 Block exogeneity and Granger causality

#### Block exogeneity

Let's decompose  $y_t$  into two subvectors  $y_t^{(1)}$  ( $n_1 \times 1$ ) and  $y_t^{(2)}$  ( $n_2 \times 1$ ), with  $y_t' = [y_t^{(1)'}', y_t^{(2)'}']$  (and therefore  $n = n_1 + n_2$ ), such that:

$$\begin{bmatrix} y_t^{(1)} \\ y_t^{(2)} \end{bmatrix} = \begin{bmatrix} \Phi^{(1,1)} & \Phi^{(1,2)} \\ \Phi^{(2,1)} & \Phi^{(2,2)} \end{bmatrix} \begin{bmatrix} y_{t-1}^{(1)} \\ y_{t-1}^{(2)} \end{bmatrix} + \varepsilon_t.$$

Using, e.g., a likelihood ratio test (see Def. 7.8), one can easily test for block exogeneity of  $y_t^{(2)}$  (say). The null assumption can be expressed as  $\Phi^{(2,1)} = 0$ .

#### Granger Causality

Granger (1969) developed a method to explore **causal relationships** among variables. The approach consists in determining whether the past values of  $y_{1,t}$  can help explain the current  $y_{2,t}$  (beyond the information already included in the past values of  $y_{2,t}$ ).

Formally, let us denote three information sets:

$$\begin{aligned} \mathcal{I}_{1,t} &= \{y_{1,t}, y_{1,t-1}, \dots\} \\ \mathcal{I}_{2,t} &= \{y_{2,t}, y_{2,t-1}, \dots\} \\ \mathcal{I}_t &= \{y_{1,t}, y_{1,t-1}, \dots, y_{2,t}, y_{2,t-1}, \dots\}. \end{aligned}$$

We say that  $y_{1,t}$  Granger-causes  $y_{2,t}$  if

$$\mathbb{E}[y_{2,t} | \mathcal{I}_{2,t-1}] \neq \mathbb{E}[y_{2,t} | \mathcal{I}_{t-1}].$$

To get the intuition behind the testing procedure, consider the following bivariate VAR( $p$ ) process:

$$\begin{aligned} y_{1,t} &= c_1 + \sum_{i=1}^p \Phi_i^{(11)} y_{1,t-i} + \sum_{i=1}^p \Phi_i^{(12)} y_{2,t-i} + \varepsilon_{1,t} \\ y_{2,t} &= c_2 + \sum_{i=1}^p \Phi_i^{(21)} y_{1,t-i} + \sum_{i=1}^p \Phi_i^{(22)} y_{2,t-i} + \varepsilon_{2,t}, \end{aligned}$$

where  $\Phi_k^{(ij)}$  denotes the element  $(i, j)$  of  $\Phi_k$ .

Then,  $y_{1,t}$  is said not to Granger-cause  $y_{2,t}$  if

$$\Phi_1^{(21)} = \Phi_2^{(21)} = \dots = \Phi_p^{(21)} = 0.$$

Therefore the hypothesis testing is

$$\begin{cases} H_0 : \Phi_1^{(21)} = \Phi_2^{(21)} = \dots = \Phi_p^{(21)} = 0 \\ H_1 : \Phi_1^{(21)} \neq 0 \text{ or } \Phi_2^{(21)} \neq 0 \text{ or } \dots \Phi_p^{(21)} \neq 0. \end{cases}$$

Loosely speaking, we reject  $H_0$  if some of the coefficients on the lagged  $y_{1,t}$ 's are statistically significant. Formally, this can be tested using the  $F$ -test or asymptotic chi-square test. The  $F$ -statistic is

$$F = \frac{(RSS - USS)/p}{USS/(T - 2p - 1)},$$

where RSS is the Restricted sum of squared residuals and USS is the Unrestricted sum of squared residuals. Under  $H_0$ , the  $F$ -statistic is distributed as  $\mathcal{F}(p, T - 2p - 1)$ . (We have  $pF \xrightarrow[T \rightarrow \infty]{} \chi^2(p)$ .)

### 9.3.6 Identification problem and standard identification techniques

In Section 9.3.4, we have seen how to estimate  $\text{Var}(\varepsilon_t) = \Omega$  and the  $\Phi_k$  matrices in the context of a VAR model. But the IRFs are functions of  $B$  and the  $\Phi_k$ 's, not of  $\Omega$  the  $\Phi_k$ 's (see Section 9.3.2). We have  $\Omega = BB'$ , but this is not sufficient to recover  $B$ .

Indeed, seen a system of equations whose unknowns are the  $b_{i,j}$ 's (components of  $B$ ), the system  $\Omega = BB'$  contains only  $n(n + 1)/2$  linearly independent equations. For instance, for  $n = 2$ :

$$\begin{aligned} \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \\ \Leftrightarrow \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} &= \begin{bmatrix} b_{11}^2 + b_{12}^2 & b_{11}b_{21} + b_{12}b_{22} \\ b_{11}b_{21} + b_{12}b_{22} & b_{22}^2 + b_{21}^2 \end{bmatrix}. \end{aligned}$$

We then have 3 linearly independent equations but 4 unknowns. Therefore,  $B$  is not identified based on second-order moments. Additional restrictions are required to identify  $B$ . This section covers two standard identification schemes: **short-run** and **long-run** restrictions:

1. A **short-run restriction (SRR)** prevents a structural shock from affecting an endogenous variable contemporaneously.
  - Easy to implement: the appropriate entries of  $B$  are set to 0.
  - Particular case: **Cholesky, or recursive approach.**
  - Examples: Bernanke (1986), Sims (1986), Galí (1992), Rubio-Ramírez et al. (2010).
2. A **long-run restriction (LRR)** prevents a structural shock from having a cumulative impact on one of the endogenous variables.
  - Additional computations are required to implement this. One needs to compute the cumulative effect of one of the structural shocks  $u_t$  on one of the endogenous variable.
  - Examples: Blanchard and Quah (1989), Faust and Leeper (1997), Galí (1999), Erceg et al. (2005), Christiano et al. (2007).

The two approaches can be combined (see, e.g., Gerlach and Smets (1995)).

Let us consider a simple example that could motivate short-run restrictions. Consider the following stylized macro model:

$$\begin{aligned} g_t &= \bar{g} - \lambda(i_{t-1} - \mathbb{E}_{t-1}\pi_t) + \underbrace{\sigma_d \eta_{d,t}}_{\text{demand shock}} \quad (\text{IS curve}) \\ \Delta\pi_t &= \beta(g_t - \bar{g}) + \underbrace{\sigma_\pi \eta_{\pi,t}}_{\text{cost push shock}} \quad (\text{Phillips curve}) \\ i_t &= \rho i_{t-1} + [\gamma_\pi \mathbb{E}_t \pi_{t+1} + \gamma_g(g_t - \bar{g})] \\ &\quad + \underbrace{\sigma_{mp} \eta_{mp,t}}_{\text{Mon. Pol. shock}} \quad (\text{Taylor rule}), \end{aligned} \tag{9.51}$$

where:

$$\eta_t = \begin{bmatrix} \eta_{\pi,t} \\ \eta_{d,t} \\ \eta_{mp,t} \end{bmatrix} \sim i.i.d. \mathcal{N}(0, I). \tag{9.52}$$

Vector  $\eta_t$  is assumed to be a vector of structural shocks, mutually and serially independent. On date  $t$ :

- $g_t$  is contemporaneously affected by  $\eta_{d,t}$  only;
- $\pi_t$  is contemporaneously affected by  $\eta_{\pi,t}$  and  $\eta_{d,t}$ ;
- $i_t$  is contemporaneously affected by  $\eta_{mp,t}$ ,  $\eta_{\pi,t}$  and  $\eta_{d,t}$ .

System (9.51) could be rewritten in the form:

$$\begin{bmatrix} d_t \\ \pi_t \\ i_t \end{bmatrix} = \Phi(L) \begin{bmatrix} d_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \bullet & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}}_{\substack{=B \\ =\varepsilon_t}} \eta_t \tag{9.53}$$

This is the **reduced-form** of the model. This representation suggests three additional restrictions on the entries of  $B$ ; the latter matrix is therefore identified (up to the signs of its columns) as soon as  $\Omega = BB'$  is known.

There are particular cases in which some well-known matrix decomposition of  $\Omega = \text{Var}(\varepsilon_t)$  can be used to easily estimate some specific SVAR.

Consider the following context:

- A first shock (say,  $\eta_{n_1,t}$ ) can affect instantaneously (i.e., on date  $t$ ) only one of the endogenous variable (say,  $y_{n_1,t}$ );
- A second shock (say,  $\eta_{n_2,t}$ ) can affect instantaneously (i.e., on date  $t$ ) two endogenous variables,  $y_{n_1,t}$  (the same as before) and  $y_{n_2,t}$ ;
- ...

This implies (1) that column  $n_1$  of  $B$  has only 1 non-zero entry (this is the  $n_1^{th}$  entry), (2) that column  $n_2$  of  $B$  has 2 non-zero entries (the  $n_1^{th}$  and the  $n_2^{th}$  ones), etc. Without loss of generality, we can set  $n_1 = n$ ,  $n_2 = n - 1$ , etc. In this context, matrix  $B$  is lower triangular.

The Cholesky decomposition of  $\Omega_\varepsilon$  then provides an appropriate estimate of  $B$ , since this matrix decomposition yields to a lower triangular matrix satisfying:

$$\Omega_\varepsilon = BB'.$$

For instance, Dedola and Lippi (2005) estimate 5 structural VAR models for the US, the UK, Germany, France and Italy to analyse the monetary-policy transmission mechanisms. They estimate SVAR(5) models over the period 1975-1997. The shock-identification scheme is based on Cholesky decompositions, the ordering of the endogenous variables being: the industrial production, the consumer price index, a commodity price index, the short-term rate, monetary aggregate and the effective exchange rate (except for the US). This ordering implies that monetary policy reacts to the shocks affecting the first three variables but that the latter react to monetary policy shocks with a one-period lag only.

Importantly, the Cholesky approach can be useful when one is interested in one specific structural shock. This was the case, e.g., of Christiano et al. (1996). Their identification is based on the following relationship between  $\varepsilon_t$  and  $\eta_t$ :

$$\begin{bmatrix} \varepsilon_{S,t} \\ \varepsilon_{r,t} \\ \varepsilon_{F,t} \end{bmatrix} = \begin{bmatrix} B_{SS} & 0 & 0 \\ B_{rS} & B_{rr} & 0 \\ B_{FS} & B_{Fr} & B_{FF} \end{bmatrix} \begin{bmatrix} \eta_{S,t} \\ \eta_{r,t} \\ \eta_{F,t} \end{bmatrix},$$

where  $S$ ,  $r$  and  $F$  respectively correspond to *slow-moving variables*, the policy variable (short-term rate) and *fast-moving variables*. While  $\eta_{r,t}$  is scalar,  $\eta_{S,t}$  and  $\eta_{F,t}$  may be vectors. The space spanned by  $\varepsilon_{S,t}$  is the same as that spanned by  $\eta_{S,t}$ . As a result, because  $\varepsilon_{r,t}$  is a linear combination of  $\eta_{r,t}$  and  $\eta_{S,t}$  (which are  $\perp$ ), it comes that the  $B_{rr}\eta_{r,t}$ 's are the (population) residuals in the regression of  $\varepsilon_{r,t}$  on  $\varepsilon_{S,t}$ . Because  $\text{Var}(\eta_{r,t}) = 1$ ,  $B_{rr}$  is given by the square root of the variance of  $B_{rr}\eta_{r,t}$ .  $B_{F,r}$  is finally obtained by regressing the components of  $\varepsilon_{F,t}$  on the estimates of  $\eta_{r,t}$ .

An equivalent approach consists in computing the Cholesky decomposition of  $BB'$  and the contemporaneous impacts of the monetary policy shock (on the  $n$  endogenous variables) are the components of the column of  $B$  corresponding to the policy variable.

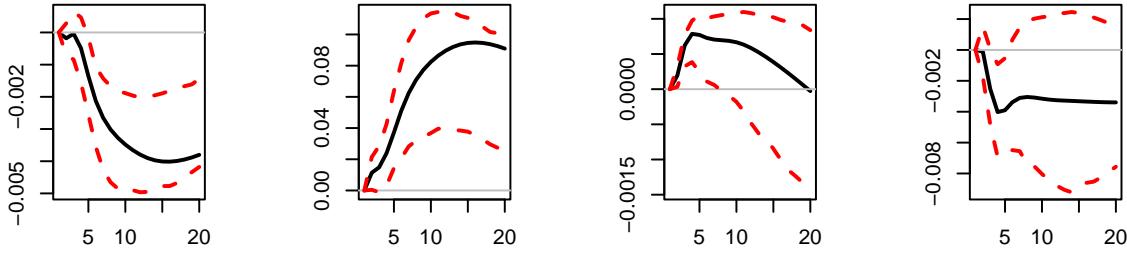
```
library(AEC)
library(vars)
data("USmonthly")
# Select sample period:
First.date <- "1965-01-01"; Last.date <- "1995-06-01"
indic.first <- which(USmonthly$DATES==First.date)
indic.last <- which(USmonthly$DATES==Last.date)
USmonthly <- USmonthly[indic.first:indic.last,]
considered.variables <- c("LIP", "UNEMP", "LCPI", "LPCOM", "FFR", "NBR", "TTR", "M1")
```

```

y <- as.matrix(USmonthly[considered.variables])
res.svar.ordering <- svar.ordering(y,p=3,
                                     posit.of.shock = 5,
                                     nb.periods.IRF = 20,
                                     nb.bootstrap.replications = 100,
                                     confidence.interval = 0.90, # expressed in pp.
                                     indic.plot = 1 # Plots are displayed if = 1.
)

```

**Effect of shock on LIF Effect of shock on UNEI Effect of shock on LCF Effect of shock on LPC**



**Effect of shock on FFI Effect of shock on NB Effect of shock on TTI Effect of shock on M1**

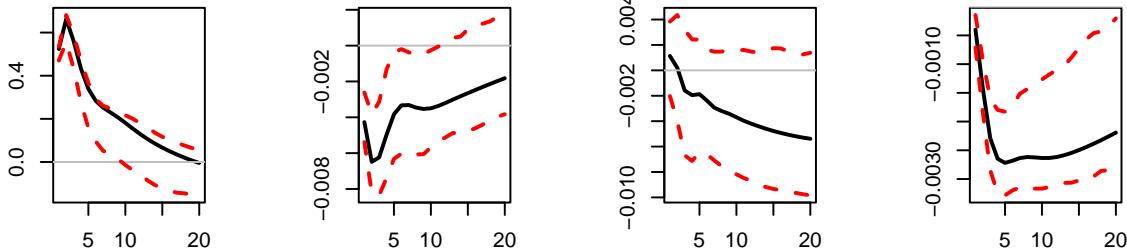


Figure 9.15: Response to a monetary-policy shock. Identification approach of Christiano, Eichenbaum and Evans (1996). Confidence intervals are obtained by bootstrapping the estimated VAR model (see inference section).

Let us now turn to **Long-run restrictions**. Such a restriction concerns the long-run influence of a shock on an endogenous variable. Let us consider for instance a structural shock that is assumed to have no “long-run influence” on GDP. How to express this? The long-run change in GDP can be expressed as  $GDP_{t+h} - GDP_t$ , with  $h$  large. Note further that:

$$GDP_{t+h} - GDP_t = \Delta GDP_{t+h} + \Delta GDP_{t+h-1} + \cdots + \Delta GDP_{t+1}.$$

Hence, the fact that a given structural shock ( $\eta_{i,t}$ , say) has no long-run influence on GDP means that

$$\lim_{h \rightarrow \infty} \frac{\partial GDP_{t+h}}{\partial \eta_{i,t}} = \lim_{h \rightarrow \infty} \frac{\partial}{\partial \eta_{i,t}} \left( \sum_{k=1}^h \Delta GDP_{t+k} \right) = 0.$$

This can be easily formulated as a function of  $B$  and of the matrices  $\Phi_i$  when  $y_t$  (including  $\Delta GDP_t$ ) follows a VAR process.

Without loss of generality, we will only consider the VAR(1) case. Indeed, one can always write a VAR( $p$ ) as a VAR(1). To see that, stack the last  $p$  values of vector  $y_t$  in vector  $y_t^* = [y_t', \dots, y_{t-p+1}']'$ ; Eq. (9.36) can then be rewritten in its **companion form**:

$$y_t^* = \underbrace{\begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=c^*} + \underbrace{\begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_p \\ I & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}}_{=\Phi} y_{t-1}^* + \underbrace{\begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=\varepsilon_t^*}, \quad (9.54)$$

where matrices  $\Phi$  and  $\Omega^* = \text{Var}(\varepsilon_t^*)$  are of dimension  $np \times np$ ;  $\Omega^*$  is filled with zeros, except the  $n \times n$  upper-left block that is equal to  $\Omega = \text{Var}(\varepsilon_t)$ . (Matrix  $\Phi$  had been introduced in Eq. (9.42).)

Focusing on the VAR(1) case:

$$\begin{aligned} y_t &= c + \Phi y_{t-1} + \varepsilon_t \\ &= c + \varepsilon_t + \Phi(c + \varepsilon_{t-1}) + \dots + \Phi^k(c + \varepsilon_{t-k}) + \dots \\ &= \mu + \varepsilon_t + \Phi \varepsilon_{t-1} + \dots + \Phi^k \varepsilon_{t-k} + \dots \\ &= \mu + B\eta_t + \Phi B\eta_{t-1} + \dots + \Phi^k B\eta_{t-k} + \dots, \end{aligned}$$

The sequence of shocks  $\{\eta_t\}$  determines the sequence  $\{y_t\}$ . What if  $\{\eta_t\}$  is replaced with  $\{\tilde{\eta}_t\}$ , where  $\tilde{\eta}_t = \eta_t$  if  $t \neq s$  and  $\tilde{\eta}_s = \eta_s + \gamma$ ? Assume  $\{\tilde{y}_t\}$  is the associated “perturbed” sequence. We have  $\tilde{y}_t = y_t$  if  $t < s$ . For  $t \geq s$ , the Wold decomposition of  $\{\tilde{y}_t\}$  implies:

$$\tilde{y}_t = y_t + \Phi^{t-s} B\gamma.$$

Therefore, the cumulative impact of  $\gamma$  on  $\tilde{y}_t$  will be (for  $t \geq s$ ):

$$\begin{aligned} (\tilde{y}_t - y_t) + (\tilde{y}_{t-1} - y_{t-1}) + \dots + (\tilde{y}_s - y_s) &= \\ (Id + \Phi + \Phi^2 + \dots + \Phi^{t-s})B\gamma. \end{aligned} \tag{9.55}$$

Consider a shock on  $\eta_{1,t}$ , with a magnitude of 1. This shock corresponds to  $\gamma = [1, 0, \dots, 0]'$ . Given Eq. (9.55), the long-run cumulative effect of this shock on the endogenous variables is given by:

$$\underbrace{(Id + \Phi + \dots + \Phi^k + \dots)}_{=(Id - \Phi)^{-1}} B \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

that is the first column of  $\Theta \equiv (Id - \Phi)^{-1}B$ .

In this context, consider the following long-run restriction: “ $j^{th}$  structural shock has no cumulative impact on the  $i^{th}$  endogenous variable”. It is equivalent to

$$\Theta_{ij} = 0,$$

where  $\Theta_{ij}$  is the element  $(i, j)$  of  $\Theta$ .

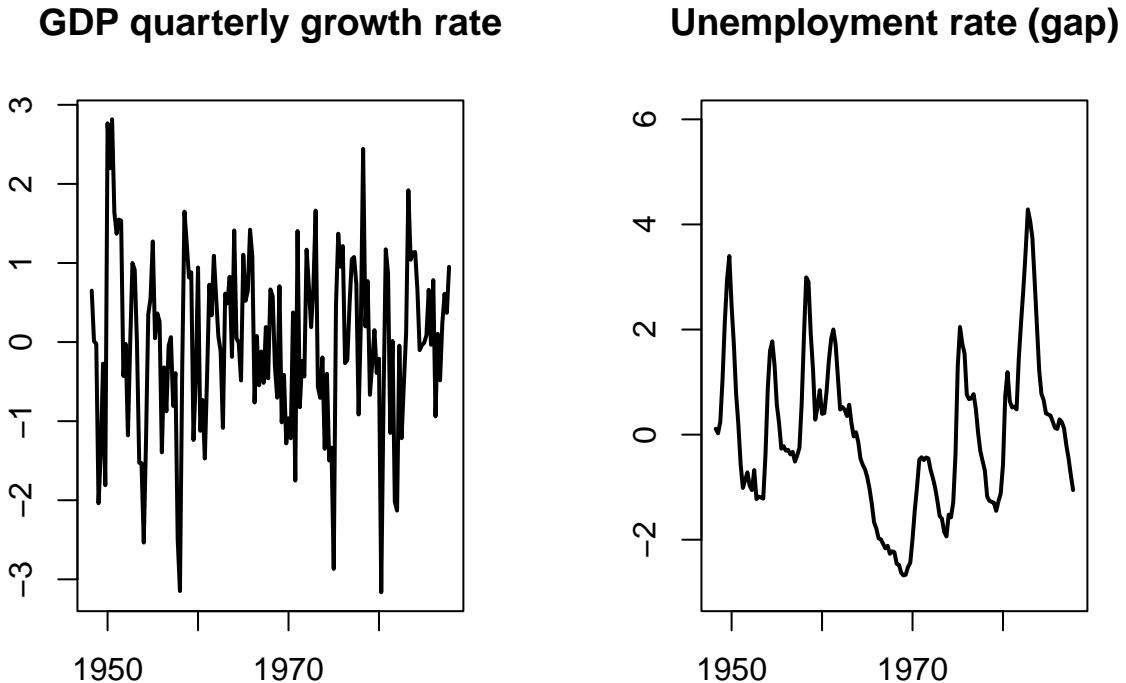
Blanchard and Quah (1989) have implemented such long-run restrictions in a small-scale VAR. Two variables are considered: GDP and unemployment. Consequently, the VAR is affected by two types of shocks. Specifically, authors want to identify **supply shocks** (that can have a permanent effect on output) and **demand shocks** (that cannot have a permanent effect on output).<sup>2</sup>

Blanchard and Quah (1989)'s dataset is quarterly, spanning the period from 1950:2 to 1987:4. Their VAR features 8 lags. Here are the data they use:

```
library(AEC)
data(BQ)
par(mfrow=c(1,2))
plot(BQ$Date, BQ$Dgdp, type="l", main="GDP quarterly growth rate",
      xlab="", ylab="", lwd=2)
plot(BQ$Date, BQ$unemp, type="l", ylim=c(-3,6), main="Unemployment rate (gap)",
      xlab="", ylab="", lwd=2)
```

---

<sup>2</sup>The motivation of the authors regarding their long-run restrictions can be obtained from a traditional Keynesian view of fluctuations. The authors propose a variant of a model from Fischer (1977).



Estimate a reduced-form VAR(8) model:

```
library(vars)
y <- BQ[,2:3]
est.VAR <- VAR(y,p=8)
Omega <- var(residuals(est.VAR))
```

Now, let us define a loss function (`loss`) that is equal to zero if (a)  $BB' = \Omega$  and (b) the element (1,1) of  $\Theta B$  is equal to zero:

```
# Compute (Id - Phi)^{-1}:
Phi <- Acoef(est.VAR)
PHI <- make.PHI(Phi)
sum.PHI.k <- solve(diag(dim(PHI)[1]) - PHI)[1:2,1:2]
loss <- function(param){
  B <- matrix(param,2,2)
  X <- Omega - B %*% t(B)
  Theta <- sum.PHI.k[1:2,1:2] %*% B
  loss <- 10000 * ( X[1,1]^2 + X[2,1]^2 + X[2,2]^2 + Theta[1,1]^2 )
  return(loss)
}
res.opt <- optim(c(1,0,0,1),loss,method="BFGS",hessian=FALSE)
print(res.opt$par)
```

```
## [1] 0.8570358 -0.2396345 0.1541395 0.1921221
```

(Note: one can use that type of approach, based on a loss function, to mix short- and long-run restrictions.)

Figure 9.16 displays the resulting IRFs. Note that, for GDP, we cumulate the GDP growth IRF, so as to have the response of the GDP in level.

```
B.hat <- matrix(res.opt$par,2,2)
print(cbind(Omega,B.hat %*% t(B.hat)))

##          Dgdp      unemp
## Dgdp  0.7582704 -0.17576173  0.7582694 -0.17576173
## unemp -0.1757617  0.09433658 -0.1757617  0.09433558
```

```

nb.sim <- 40
par(mfrow=c(2,2)); par(plt=c(.15,.95,.15,.8))
Y <- simul.VAR(c=matrix(0,2,1),Phi,B.hat,nb.sim,y0.star=rep(0,2*8),
                 indic.IRF = 1,u.shock = c(1,0))
plot(cumsum(Y[,1]),type="l",lwd=2,xlab="",ylab="",main="Demand shock on GDP")
plot(Y[,2],type="l",lwd=2,xlab="",ylab="",main="Demand shock on UNEMP")
Y <- simul.VAR(c=matrix(0,2,1),Phi,B.hat,nb.sim,y0.star=rep(0,2*8),
                 indic.IRF = 1,u.shock = c(0,1))
plot(cumsum(Y[,1]),type="l",lwd=2,xlab="",ylab="",main="Supply shock on GDP")
plot(Y[,2],type="l",lwd=2,xlab="",ylab="",main="Supply shock on UNEMP")

```

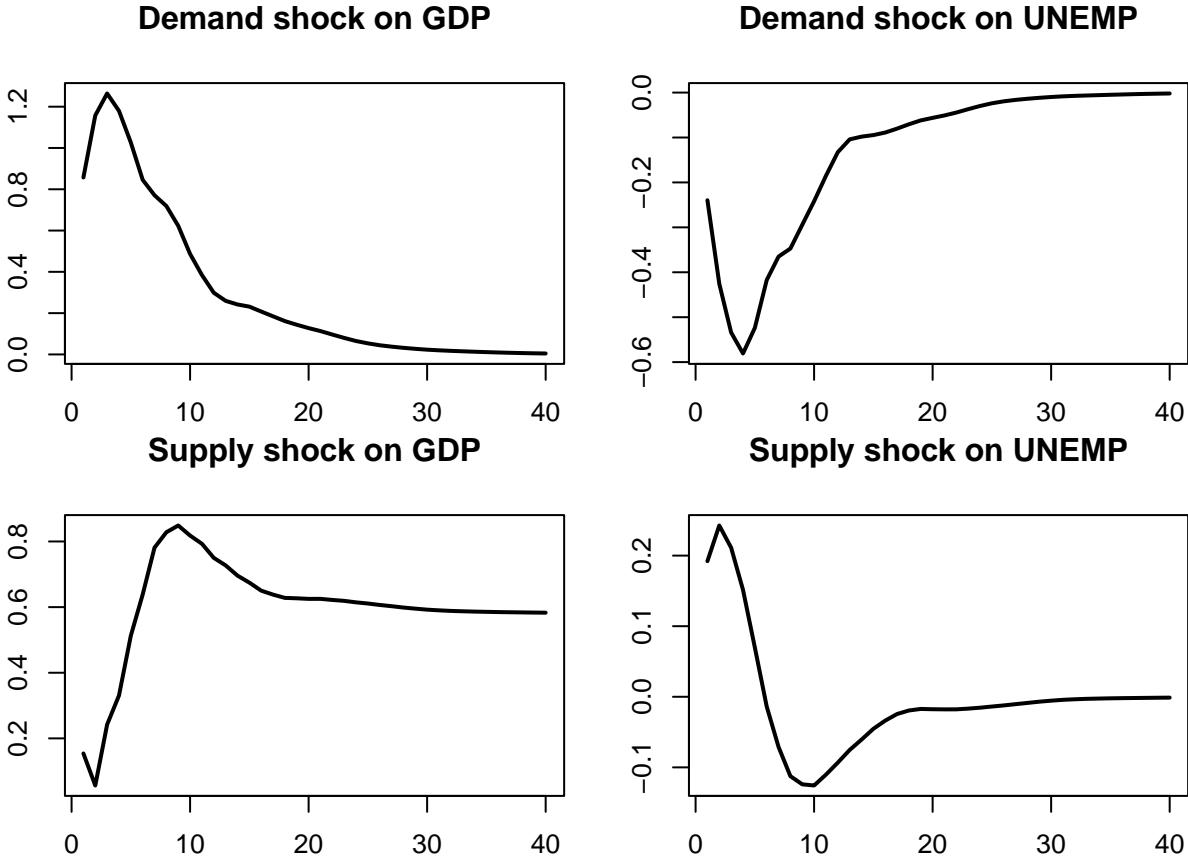


Figure 9.16: IRF of GDP and unemployment to demand and supply shocks.

### 9.3.7 Sign restrictions

To identify the structural shocks, we need to find a matrix  $B$  that satisfies  $\Omega = BB'$  (with  $\Omega = \text{Var}(\varepsilon_t)$ ) and other restrictions. Indeed, as explained above,  $\Omega = BB'$  is not sufficient to identify  $B$  since, if we take any orthogonal matrix  $Q$  (see Def. 9.23), then  $\mathcal{P} = BQ$  also satisfies  $\Omega = \mathcal{P}\mathcal{P}'$ .

**Definition 9.23** (Orthogonal matrix). An orthogonal matrix  $Q$  is a matrix such that  $QQ' = I$ , i.e., all columns (rows) of  $Q$  are orthogonal and unit vectors:

$$q_i' q_j = 0 \text{ if } i \neq j \text{ and } q_i' q_j = 1 \text{ if } i = j,$$

where  $q_i$  is the  $i^{th}$  column of  $Q$ .

The idea behind the sign-restriction approach is to “draw” random matrices  $\mathcal{P}$  that satisfy  $\Omega = \mathcal{P}\mathcal{P}'$ , and then to constitute a set of admissible matrices, keeping in this set only the simulated  $\mathcal{P}$  matrices that satisfy some predefined sign-based restriction. An example of restriction is “*after one year, a contractionary monetary-policy shocks has a negative impact on inflation*”.

As suggested above, if  $B$  is any matrix that satisfies  $\Omega = BB'$  (for instance,  $B$  can be based on the Cholesky decomposition of  $\Omega$ ), then we also have  $\Omega = \mathcal{P}\mathcal{P}'$  as soon as  $\mathcal{P} = BQ$ ,

where  $Q$  is an orthogonal matrix. Therefore, to draw  $\mathcal{P}$  matrices, it suffices to draw in the set of orthogonal matrices.

To fix ideas, consider dimension 2. In that case, the orthogonal matrices are rotation matrices, and the set of orthogonal matrices can be parameterized by the angle  $x$ , with:

$$Q_x = \begin{pmatrix} \cos(x) & \cos(x + \frac{\pi}{2}) \\ \sin(x) & \sin(x + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}.$$

(This is an angle- $x$  counter-clockwise rotation.) Hence, in that case, by drawing  $x$  randomly from  $[0, 2\pi]$ , we draw randomly from the set of  $2 \times 2$  rotation matrices. For high-dimensional VAR, we lose this simple geometrical representation, though. It is not always possible to parametrize a rotation matrix (high-dimentional VARs).

How to proceed, then? Arias et al. (2018) provide a procedure. Their approach is based on the so-called  $QR$  decomposition: any square matrix  $X$  may be decomposed as  $X = QR$  where  $Q$  is an orthogonal matrix and  $R$  is an upper diagonal matrix. With this in mind, they propose a two-step approach:

- i. Draw a random matrix  $X$  by drawing each element from independent standard normal distribution.
- ii. Let  $X = QR$  be the  $QR$  decomposition of  $X$  with the diagonal of  $R$  normalized to be positive. The random matrix  $Q$  is orthogonal and is a draw from the uniform distribution over the set of orthogonal matrices.

Equipped with this procedure, the sign-restriction is based on the following algorithm:

1. Draw a random orthogonal matrix  $Q$  (using step i. and ii. described above).
2. Compute  $B = PQ$  where  $P$  is the Cholesky decomposition of the reduced form residuals  $\Omega_\varepsilon$ .
3. Compute the impulse response associated with  $B$   $y_{t,t+k} = \Phi^k B$  or the cumulated response  $\bar{y}_{t,t+k} = \sum_{j=0}^k \Phi^j B$ .
4. Are the sign restrictions satisfied?
  - a. **Yes.** Store the impulse response in the set of admissible response.
  - b. **No.** Discard the impulse response.
5. Perform  $N$  replications and report the median impulse response (and its “confidence” intervals).

Note: to take into account the uncertainty in  $B$  and  $\Phi$ , you can draw  $B$  and  $\Phi$  in Steps 2 and 3 using an inference method (see Section 9.3.12).

The sign-restriction approach method has the advantage of being relatively agnostic. Moreover, it is fairly flexible, as one can impose sign restrictions on any variable, at any horizon. A prominent example is Uhlig (2005). Using US monthly data from 1965.I to 2003.XII, he employs sign restrictions to estimate the effect of monetary policy shocks.

According to conventional wisdom, monetary contractions should:<sup>3</sup>

- Raise the federal funds rate,
- Lower prices,
- Decrease non-borrowed reserves,
- Reduce real output.

The restrictitons considered by Uhlig (2005) are as follows: an expansionary monetary policy shock leads to:

---

<sup>3</sup>Standard identification schemes often fail to achieve these 4 points Two puzzles regularly arise: *liquidity puzzle*: when identifying monetary policy shocks as surprise increases in the stock of money, interest rates tend to go up, not down; *price puzzle*: after a contractionary monetary policy shock, even with interest rates going up and money supply going down, inflation goes up rather than down.

- Increases in prices
- Increase in nonborrowed reserves
- Decreases in the federal funds rate

What about output? Since is the response of interest, we leave it un-restricted.

```
library(AEC);library(vars);library(Matrix)
data("USmonthly")
First.date <- "1965-01-01"
Last.date <- "1995-06-01"
indic.first <- which(USmonthly$DATES==First.date)
indic.last <- which(USmonthly$DATES==Last.date)
USmonthly <- USmonthly[indic.first:indic.last,]
considered.variables<-c("LIP","UNEMP","LCPI","LPCOM","FFR","NBR","TTR","M1")
n <- length(considered.variables)
y <- as.matrix(USmonthly[considered.variables])
sign.restrictions <- list()
horizon <- list()
#Define sign restrictions and horizon for restrictions
for(i in 1:n){
  sign.restrictions[[i]] <- matrix(0,n,n)
  horizon[[i]] <- 1
}
sign.restrictions[[1]][1,3] <- 1
sign.restrictions[[1]][2,5] <- -1
sign.restrictions[[1]][3,6] <- 1
horizon[[1]] <- 1:5
res.svar.signs <-
  svar.signs(y,p=3,
             nb.shocks = 1, #number of identified shocks
             nb.periods.IRF = 20,
             bootstrap.replications = 1, # = 0 if no bootstrap
             confidence.interval = 0.80, # expressed in pp.
             indic.plot = 1, # Plots are displayed if = 1.
             nb.draws = 10000, # number of draws
             sign.restrictions,
             horizon,
             recursive =1 #  =0 <- draw Q directly, =1 <- draw q recursively
  )

```

It has to be stressed that the sign restriction approach does not lead to a unique IRF, but to a set of admissible IRFs. Also, we say that this approach is set-identified, not point-identified.

An alternative approach is the so-called **penalty-function approach** (PFA, Uhlig (2005), present in Danne (2015)'s package). This approach relies on a *penalty function*:

$$\begin{aligned} f(x) &= x && \text{if } x \leq 0 \\ &100.x && \text{if } x > 0 \end{aligned}$$

which penalizes positive responses and rewards negative responses.

Let  $\psi_k^j(q)$  be the impulse response of variable  $j$ . The  $\psi_k^j(q)$ 's are the elements of  $\psi_k(q) = \Psi_k q$ .

Let  $\sigma_j$  be the standard deviation of variable  $j$ . Let  $\iota_{j,k} = 1$  if we restrict the response of variable  $j$  at the  $k^{th}$  horizon to be negative,  $\iota_{j,k} = -1$  if we restrict it to be positive, and  $\iota_{j,k} = 0$  if there is no restriction. The total penalty is given by

$$\mathbf{P}(q) = \sum_{j=1}^m \sum_{k=0}^K f\left(\iota_{j,k} \frac{\psi_k^j(q)}{\sigma_j}\right).$$

We are looking for a solution to

$$\min_q \mathbf{P}(q)$$

$$\text{s.t. } q'q = 1.$$

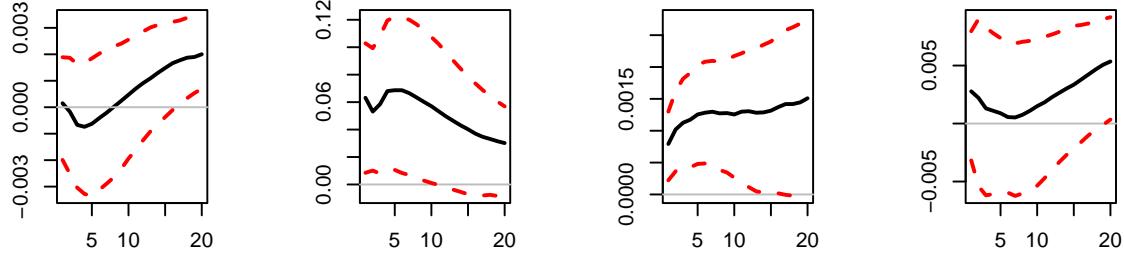
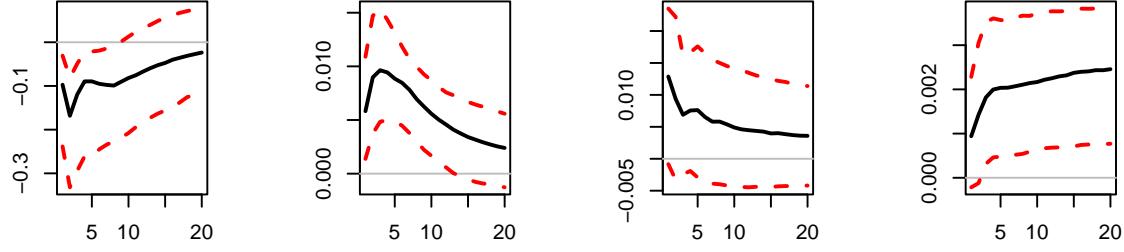
**Effect of shock 1 on LIEffect of shock 1 on UNE Effect of shock 1 on LCEffect of shock 1 on LPC**

**Effect of shock 1 on FF Effect of shock 1 on NE Effect of shock 1 on T1 Effect of shock 1 on M**


Figure 9.17: IRF associated with a monetary policy shock; sign-restriction approach.

The problem is solved numerically.

### 9.3.8 Forecast error variance maximization

The approach presented in this section exploits the derivations of Uhlig (2004). Barsky and Sims (2011) exploit this approach to identify a TFP news shock, that they define as the shock (a) that is orthogonal to the innovation in current utilization-adjusted TFP and (b) that best explains variation in future TFP.

Consider a process  $\{y_t\}$  that admits the infinite MA representation of Eq. (9.38). Let  $Q$  be an orthogonal matrix, an alternative decomposition is:

$$y_t = \sum_{h=0}^{+\infty} \Psi_h \underbrace{\eta_{t-h}}_{Q\tilde{\eta}_{t-h}} = \sum_{h=0}^{+\infty} \underbrace{\Psi_h Q}_{\tilde{\Psi}_h} \tilde{\eta}_{t-h} = \sum_{h=0}^{+\infty} \tilde{\Psi}_h \tilde{\eta}_{t-h}, \quad (9.56)$$

where  $\tilde{\eta}_{t-h} = Q'\eta_{t-h}$  are the white-noise shocks associated with the new MA representation. (They also satisfy  $\text{Var}(\tilde{\eta}_t) = Id$ .)

The  $h$ -step ahead prediction error of  $y_{t+h}$ , given all the data up to and including  $t-1$  is given by

$$e_{t+h}(h) = y_{t+h} - \mathbb{E}_{t-1}(y_{t+h}) = \sum_{j=0}^h \tilde{\Psi}_h \tilde{\eta}_{t+h-j}.$$

The variance-covariance matrix of  $e_{t+h}(h)$  is

$$\Omega(h) = \sum_{j=0}^h \tilde{\Psi}_j \tilde{\Psi}'_j = \sum_{j=0}^h \Psi_j \Psi'_j.$$

We can decompose  $\Omega(h)$  into the contribution of each shock  $l$  ( $l^{th}$  component of  $\tilde{\eta}_t$ ):

$$\Omega^{(h)} = \sum_{l=1}^n \Omega_l^{(h)}(Q)$$

with

$$\Omega_l^{(h)}(Q) = \sum_{j=0}^h (\Psi_j q_l)(\Psi_j q_l)',$$

where  $q_l$  is the  $l^{th}$  column of  $Q$ .

This decomposition can be used with the objective of finding the **impulse vector**  $b$  that is s.t. that it explains as much as possible of the sum of the  $h$ -step ahead prediction error variance of some variable  $i$ , say, for prediction horizons  $h \in [\underline{h}, \bar{h}]$ .

Formally, the task is to explain as much as possible of the variance

$$\sigma^2(\underline{h}, \bar{h}, q_1) = \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h [(\Psi_j q_1)(\Psi_j q_1)']_{i,i}$$

with a single impulse vector  $q_1$ .

Denote by  $E_{ii}$  the matrix that is filled with zeros, except for its  $(i, i)$  entry, set to 1. We have:

$$\begin{aligned} \sigma^2(\underline{h}, \bar{h}, q_1) &= \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h [(\Psi_j q_1)(\Psi_j q_1)']_{i,i} = \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h \text{Tr}[E_{ii}(\Psi_j q_1)(\Psi_j q_1)'] \\ &= \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h \text{Tr}[q_1' \Psi_j' E_{ii} \Psi_j q_1] \\ &= q_1' S q_1, \end{aligned}$$

where

$$\begin{aligned} S &= \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h \Psi_j' E_{ii} \Psi_j \\ &= \sum_{j=0}^{\bar{h}} (\bar{h} + 1 - \max(\underline{h}, j)) \Psi_j' E_{ii} \Psi_j \\ &= \sum_{j=0}^{\bar{h}} (\bar{h} + 1 - \max(\underline{h}, j)) \Psi_{j,i}' \Psi_{j,i} \end{aligned}$$

where  $\Psi_{j,i}$  denotes row  $i$  of  $\Psi_j$ , i.e., the response of variable  $i$  at horizon  $j$  (when  $Q = Id$ ).

The maximization problem subject to the side constraint  $q_1' q_1 = 1$  can be written as a Lagrangian:

$$L = q_1' S q_1 - \lambda(q_1' q_1 - 1),$$

with the first-order condition  $S q_1 = \lambda q_1$  (the side constraint is  $q_1' q_1 = 1$ ). From this equation, we see that the solution  $q_1$  is an eigenvector of  $S$ , the one associated with eigenvalue  $\lambda$ . We also see that  $\sigma^2(\underline{h}, \bar{h}, q_1) = \lambda$ . Thus, to maximize this variance, we need to find the eigenvector of  $S$  that is associated with the maximal eigenvalue  $\lambda$ . That defines the first principal component (see Section 10.1). That is, if  $S$  admits the following spectral decomposition:

$$S = \mathcal{P} D \mathcal{P}',$$

where  $D$  is diagonal matrix whose entries are the (ordered) eigenvalues:  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , then  $\sigma^2(\underline{h}, \bar{h}, q_1)$  is maximized for  $q_1 = p_1$ , where  $p_1$  is the first column of  $\mathcal{P}$ .

### 9.3.9 Identification based on non-normality of the shocks

In this section, we show that the non-identification of the structural shocks ( $\eta_t$ ) is specific to the Gaussian case. We propose consistent estimation approaches for SVAR in the context of non-Gaussian shocks.

We have seen in what precedes that we cannot identify  $B$  based on first and second moments only. Since a Gaussian distribution is perfectly determined by the first two moments, it comes that one cannot achieve identification when the structural shocks are Gaussian. That is, even if we observe an infinite number of i.i.d.  $B\eta_t$ , we cannot recover  $B$  if the  $\eta_t$ 's are Gaussian.

Indeed, if  $\eta_t \sim \mathcal{N}(0, Id)$ , then the distribution of  $\varepsilon_t \equiv B\eta_t$  is  $\mathcal{N}(0, BB')$ . Hence  $\Omega = BB'$  is observed (in the population), but for any orthogonal matrix  $Q$  (i.e.  $QQ' = Id$ ), we also have  $BQ\eta_t \sim \mathcal{N}(0, \Omega)$ .

To illustrate, consider the following bivariate Gaussian situations, with  $\Theta_1 = 0$ :

$$\begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix} \sim \mathcal{N}(0, Id), \text{ with } B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} \text{ (rotation).}$$

Figure 9.18 shows that the distributions of  $B\eta_t$  and of  $BQ\eta_t$  are identical. However, the impulse response functions associated with one of the other impulse matrix ( $B$  or  $BQ$ ) are different. This is illustrated by Figure 9.19, that shows the IRFs associated with two identical models (defined by Eq. (9.39)), the only difference being the impulse matrix ( $B$  or  $BQ$ ).

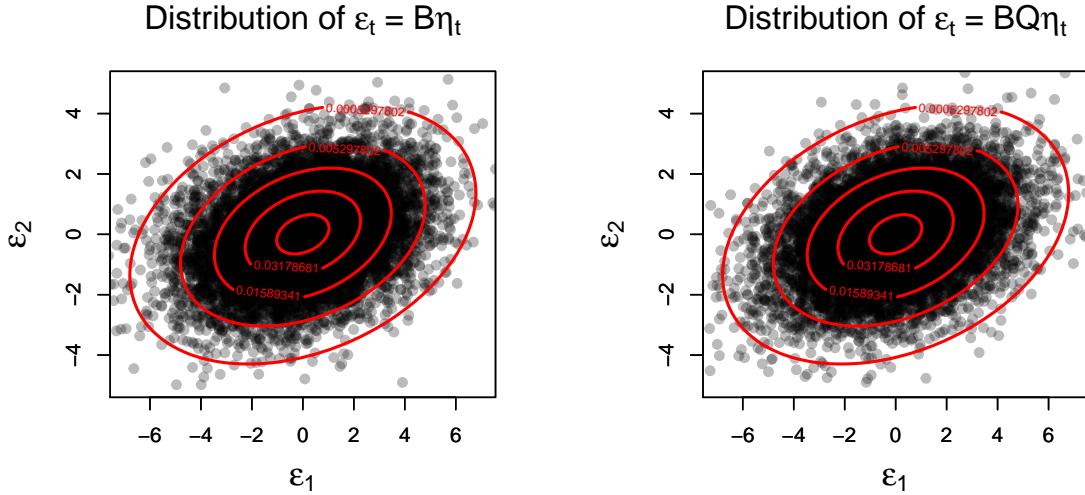


Figure 9.18: This figure compares the distributions of two Gaussian bivariate vectors,  $B\eta_t$  and  $BQ\eta_t$ , where  $\eta_t \sim \mathcal{N}(0, Id)$  (therefore  $\eta_{1,t}$  and  $\eta_{2,t}$  are independent), and  $Q$  is an orthogonal matrix.

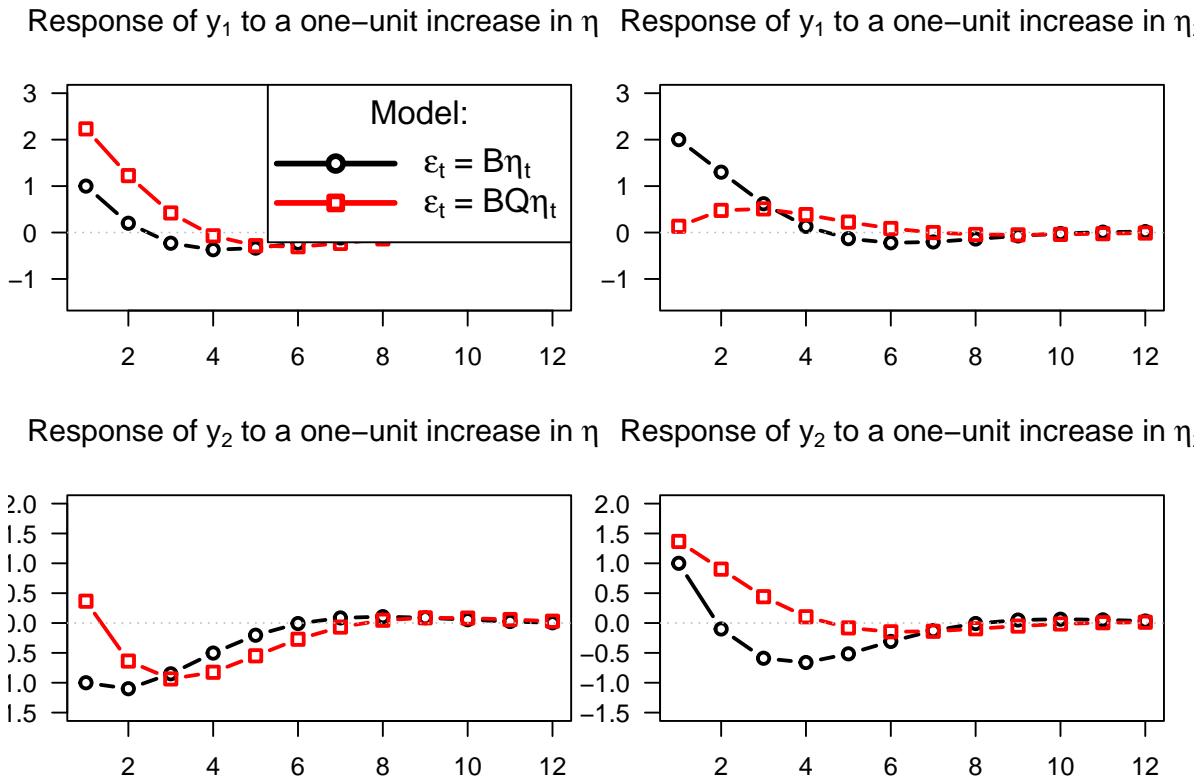


Figure 9.19: This figure shows that the impulse response functions associated with an impulse matrix equal to  $B$  (black line) or  $BQ$  (red line) are different (even if  $BB' = BQ(BQ)'$ ).

Hence, in the Gaussian case, external restrictions (economic hypotheses) are needed to identify  $B$  (see previous sections). But such restrictions may not be necessary if the structural shocks are not Gaussian. That is, the identification problem is very specific to normally-distributed  $\eta_t$ 's (Rigobon (2003), Normandin and Phaneuf (2004), Lanne and Lütkepohl (2008)).

To better see why this can be the case, consider again a bivariate vector of independent structural shocks ( $\eta_{1,t}$  and  $\eta_{2,t}$ ) but, now, assume that one of them is not Gaussian any more. Specifically, assume that  $\eta_{2,t}$  is drawn from a Student distribution with 5 degrees of freedom:  $\eta_{1,t} \sim \mathcal{N}(0, 1)$ ,  $\eta_{2,t} \sim t(5)$ ,  $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$  and  $Q = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix}$ .

Figure 9.20 shows that, in this case,  $B\eta_t$  and  $BQ\eta_t$  do not have the same distribution any more (in spite of the fact that, in both cases, we have  $\text{Var}(\varepsilon_t) = BB'$ ). This opens the door to the identification of the impulse matrix ( $BQ$ ) in the non-Gaussian case.

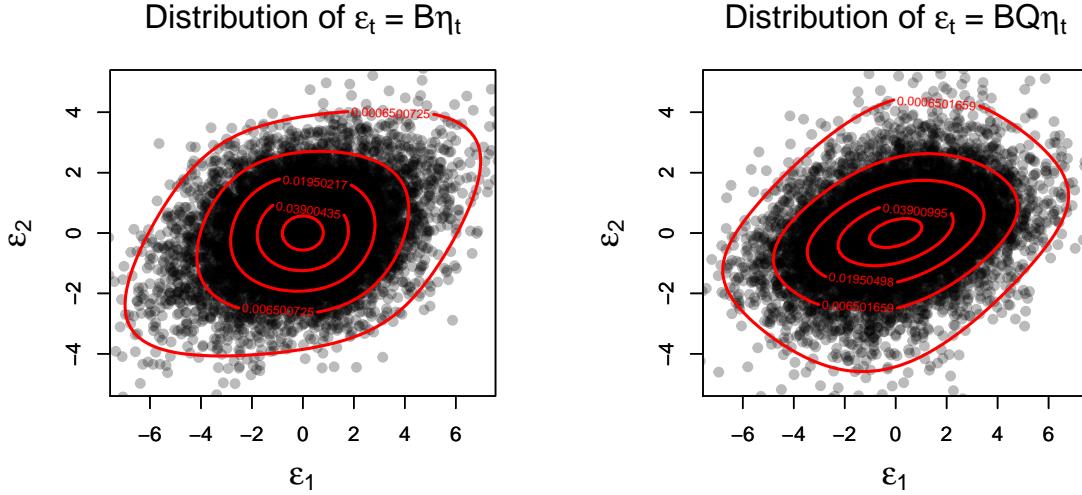


Figure 9.20: This figure compares the distributions of two Gaussian bivariate vectors,  $B\eta_t$  and  $BQ\eta_t$ , where  $\eta_t 1, t \sim \mathcal{N}(0, 1)$ ,  $\eta_t 2, t \sim t(5)$ , and  $Q$  is an orthogonal matrix.

The exercise that consists in identifying non-Gaussian independent shocks out of linear combinations of these shocks is a well-known problem of the signal-processing literature, called **independent component analysis (ICA)**. Without loss of generality, we can assume that  $BB' = Id$  (i.e.  $B$  is orthogonal). (If this is not the case, i.e. if  $\text{Var}(\varepsilon_t) = \Omega \neq Id$ , then one can pre-multiply the data by  $\Omega^{-1/2}$ .) The classical ICA problem is as follows: Find  $B$  such that  $\varepsilon_t = B\eta_t$  (or  $\varepsilon_t = B\eta_t$ ) given that

- i. We observe the  $\varepsilon_t$ 's,
- ii. The components of  $\eta_t$  are independent,
- iii.  $BB' = Id$  (i.e.,  $B$  is orthogonal).

Figure 9.21 represents again some bivariate distributions. The black (red) lines correspond to the distributions of  $\eta_t$  ( $B\eta_t$ ). It is important to note that the two components of vector  $B\eta_t$  are not independent (contrary to those of  $\eta_t$ ).

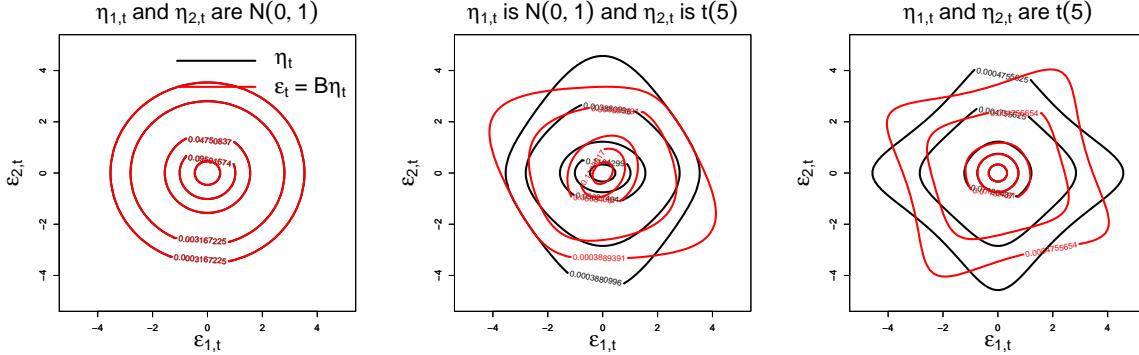


Figure 9.21: The three plots represent the bivariate distributions of  $\eta_t$  (black) and of  $B\eta_t$  (red), where the two components of  $\eta_t$  are independent, of unit variance, and  $B$  is orthogonal. Hence, for each of the three plots,  $\text{Var}(B\eta_t) = Id$ .

In all cases, we have  $\text{Var}(\varepsilon_t) = \text{Var}(\eta_t) = Id$ . But the two components of  $\varepsilon_t$  are not independent. For instance: We have  $\mathbb{E}(\varepsilon_{2,t} | \varepsilon_{1,t} > 4) < 0$  (whereas  $\mathbb{E}(\eta_{2,t} | \eta_{1,t} > 4) = 0$ ). The objective of ICA is to rotate  $\varepsilon_t$  to retrieve independent components ( $\eta_t$ ).

**Hypothesis 9.1.** Process  $\eta_t$  satisfies:

- i. The  $\eta_t$ 's are i.i.d. (across time) with  $\mathbb{E}(\eta_t) = 0$  and  $\text{Var}(\eta_t) = Id$ .

ii. The components  $\eta_{1,t}, \dots, \eta_{n,t}$  are mutually independent. iii We have

$$\varepsilon_t = B_0 \eta_t,$$

with  $\text{Var}(\varepsilon_t) = Id$  (i.e.  $B_0$  is orthogonal).

**Theorem 9.4** (Eriksson, Koivunen (2004)). *If Hypothesis 9.1 is satisfied and if at most one of the components of  $\eta$  is Gaussian, then matrix  $B_0$  is identifiable up to the post multiplication by  $DP$ , where  $P$  is a permutation matrix and  $D$  is a diagonal matrix whose diagonal entries are 1 or  $-1$ .*

Hence, the structural shocks are identifiable. But how to estimate them based on observations of the  $\varepsilon_t$ 's? Gouriéroux et al. (2017) have proposed a **Pseudo-Maximum Likelihood (PML)** approach. This approach consists in maximizing a so-called **pseudo log-likelihood function**, based on a set of p.d.f.  $g_i(\eta_i), i = 1, \dots, n$  (that may be different from the true p.d.f. of the  $\eta_{i,t}$ 's):

$$\log \mathcal{L}_T(B) = \sum_{t=1}^T \sum_{i=1}^n \log g_i(b'_i Y_t), \quad (9.57)$$

where  $b_i$  is the  $i^{th}$  column of matrix  $B$  (or  $b'_i$  is the  $i^{th}$  row of  $B^{-1}$  since  $B^{-1} = B'$ ).

The log-likelihood function (9.57) is computed as if the errors  $\eta_{i,t}$  had the p.d.f.  $g_i(\eta_i)$ . The PML estimator of matrix  $B$  maximizes the pseudo log-likelihood function:

$$\widehat{B}_T = \arg \max_B \sum_{t=1}^T \sum_{i=1}^n \log g_i(b'_i \varepsilon_t), \quad (9.58)$$

$$s.t. B'B = Id.$$

The restrictions  $B'B = Id$  can be eliminated by parameterizing  $B$  in such a way that, whatever the consider parameters,  $B$  is orthogonal. Gouriéroux et al. (2017) propose to use, for that, the Cayley's representation: any orthogonal matrix with no eigenvalue equal to  $-1$  can be written as

$$B(A) = (Id + A)(Id - A)^{-1}, \quad (9.59)$$

where  $A$  is a skew symmetric (or antisymmetric) matrix, such that  $A' = -A$ . There is a one-to-one relationship with  $A$ , since:

$$A = (B(A) + Id)^{-1}(B(A) - Id). \quad (9.60)$$

Hence, the PML estimator of matrix  $B$  is obtained as  $\widehat{B}_T = B(\widehat{A}_T)$ , where:

$$\widehat{A}_T = \arg \max_{a_{i,j}, i > j} \sum_{t=1}^T \sum_{i=1}^n \log g_i[b_i(A)' \varepsilon_t]. \quad (9.61)$$

Table 9.1: This table reports usual p.d.f. and their derivatives.

$\log g(x)$	$\frac{d \log g(x)}{dx}$	$\frac{d^2 \log g(x)}{dx^2}$
Gaussian $cst - x^2/2$	$-x$	$-1$
Student $t(\nu > 4)$ $-\frac{1-\nu}{2} \log \left(1 + \frac{x^2}{\nu-2}\right)$	$-\frac{x(1+\nu)}{\nu-2+x^2}$	$-(1+\nu)\frac{\nu-2-x^2}{\nu-2+x^2}$
Hyperbolic secant $-\log \left(\cosh \left\{\frac{\pi}{2}x\right\}\right)$	$-\frac{\pi}{2} \operatorname{anh} \left(\frac{\pi}{2}x\right)$	$-\left(\frac{\pi}{2} \frac{1}{\cosh \left(\frac{\pi}{2}x\right)}\right)^2$

$\log g(x)$	$\frac{d \log g(x)}{dx}$	$\frac{d^2 \log g(x)}{dx^2}$
Subgaussian: $\pi x^2 + \log \left( \cosh \left\{ \frac{\pi}{2} x \right\} \right)$	$2\pi x + \frac{\pi}{2} \tanh \left( x \frac{\pi}{2} \right)$	$2\pi + \left( \frac{\pi}{2} \frac{1}{\cosh \left( \frac{\pi}{2} x \right)} \right)^2$

Under assumptions on the  $g_i$  functions (excluding the Gaussian distributions), Gouriéroux et al. (2017) derive the asymptotic properties of the PML estimator. Specifically, the PML estimator  $\widehat{B}_T$  of  $B_0$  is consistent (in  $\mathcal{P}_0$ , the set of matrices obtained by permutation and sign change of the columns of  $B_0$ ) and asymptotically normal, with speed of convergence  $1/\sqrt{T}$ .

The asymptotic variance-covariance matrix of  $\text{vec}\sqrt{T}(\widehat{B}_T - B_0)$  is  $A^{-1} \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} (A')^{-1}$ , where matrices  $A$  and  $\Gamma$  are detailed in Gouriéroux et al. (2017).

Note that the potential misspecification of pseudo-distributions  $g_i$  has no effect on the consistency of these specific PML estimators.

**Example 9.8** (Non-Gaussian monetary-policy shocks). We apply the PML-ICA approach on U.S. data covering the period 1959:IV to 2015:I at the quarterly frequency ( $T = 224$ ). We consider three dependent variables: inflation ( $\pi_t$ ), economic activity ( $z_t$ , the output gap) and the nominal short-term interest rate ( $r_t$ ). Changes in the log of oil prices added as an exogenous variable ( $x_t$ ).

```
library(AEC)
First.date <- "1959-04-01"
Last.date <- "2015-01-01"
data <- US3var
data <- data[(data$date>=First.date)&(data$date<=Last.date),]
Y <- as.matrix(data[c("infl","y.gdp.gap","r")])
names.var <- c("inflation","real activity","short-term rate")
T <- dim(Y)[1]
n <- dim(Y)[2]
```

Let us denote by  $W_t$  the set of information made of the past values of  $y_t = [\pi_t, z_t, r_t]$ , that is  $\{y_{t-1}, y_{t-2}, \dots\}$ , and of exogenous variables  $\{x_t, x_{t-1}, \dots\}$ . The reduced-form VAR model reads:

$$y_t = \mu + \underbrace{\sum_{i=1}^p \Phi_i y_{t-i} + \Theta x_t}_{a(W_t; \theta)} + u_t$$

where the  $u_t$ 's are assumed to be serially independent, with zero mean and variance-covariance matrix  $\Sigma$ .

Matrices  $\mu$ ,  $\Phi_i$ ,  $\Theta$  and  $\Sigma$  are consistently estimated by OLS. Jarque-Bera tests support the hypothesis of non-normality for all residuals.

```
nb.lags <- 6 # number of lags used in the VAR model
X <- NULL
for(i in 1:nb.lags){
  lagged.Y <- rbind(matrix(NaN,i,n),Y[1:(T-i),])
  X <- cbind(X,lagged.Y)}
X <- cbind(X,data$commo) # add exogenous variables
Phi <- matrix(0,n,n*nb.lags);mu <- rep(0,n)
effect.commo <- rep(0,n)
U <- NULL # Eta is the matrix of OLS residuals
for(i in 1:n){
  eq <- lm(Y[,i] ~ X)
  Phi[i,] <- eq$coef[2:(dim(Phi)[2]+1)]
```

```

mu[i] <- eq$coef[1]
U <- cbind(U,eq$residuals)
effect.commo[i] <- eq$coef[length(eq$coef)]
}
Omega <- var(U) # Covariance matrix of the OLS residuals.
B <- t(chol(Omega)) # Cholesky matrix associated with Omega (lower triang.)
Eps <- U %*% t(solve(B)) # Recover associated structural shocks

```

We want to estimate the orthogonal matrix  $B$  such that  $u_t = SB\eta_t$ , where

- $S$  results from the Cholesky decomposition of  $\Sigma$  and
- the components of  $\eta_t$  are independent, zero-mean with unit variance.

The PML approach is applied on standardized VAR residuals given by:

$$\hat{\varepsilon}_t = \hat{S}_T^{-1} \underbrace{[y_t - a(W_t; \hat{\theta}_T)]}_{\text{VAR residuals}}$$

By construction of  $\hat{S}_T^{-1}$ , it comes that the covariance matrix of these residuals is  $Id$ .

The pseudo density functions are distinct and asymmetric mixtures of Gaussian distributions.

```

distri <- list(
  type=c("mixt.gaussian","mixt.gaussian","mixt.gaussian"),
  df=c(NaN,NaN,NaN),
  p=c(0.5,.5,.5),mu=c(.1,.1,.1),sigma=c(.5,.7,1.3))
AA.0 <- c(0,0,0)
res.optim <- optim(AA.0,func.2.minimize,
  Y = Eps, distri = distri,
  gr = d.func.2.minimize,
  method="Nelder-Mead",
  control=list(trace=FALSE,maxit=1000))
AA.0 <- res.optim$par
res.optim <- optim(AA.0,func.2.minimize,d.func.2.minimize,
  Y = Eps, distri = distri,
  method="BFGS",
  control=list(trace=FALSE))
AA.est <- res.optim$par
n <- ncol(Y)
M <- make.M(n)
A.est <- matrix(M %*% AA.est,n,n)
C.PML <- (diag(n) + A.est) %*% solve(diag(n) - A.est)

eta.PML <- Eps %*% C.PML # eta.PML are the ICA-estimated structural shocks

A <- make.A.matrix(eta.PML,distri,C.PML)
Omega <- make.Omega(eta.PML,distri)
# Compute asymptotic covariance matrix of C.PML:
V <- make.Asympt.Cov.delta(eta.PML,distri,C.PML)
param <- c(C.PML)
st.dev <- sqrt(diag(V))
t.stat <- c(C.PML)/sqrt(diag(V))
cbind(param,st.dev,t.stat) # print results of PML estimation

```

	param	st.dev	t.stat
## [1,]	0.94417705	0.040848382	23.1141845
## [2,]	-0.32711569	0.118802653	-2.7534376
## [3,]	0.03905164	0.074172945	0.5264944
## [4,]	0.32070293	0.119270893	2.6888616

```
## [5,] 0.93977707 0.041629110 22.5749976
## [6,] 0.11818924 0.060821400 1.9432179
## [7,] -0.07536139 0.071980455 -1.0469702
## [8,] -0.09906759 0.062185577 -1.5930959
## [9,] 0.99222290 0.007785691 127.4418551
```

(Note: it is always useful to combine two optimization algorithms, such as Nelder–Mead and BFGS.)

We would obtain close results by neglecting commodity prices. In that case, one can simply use the function `estim.SVAR.ICA` of the `AEC` package. Let us compare the  $C$  matrix obtained in the two cases (with or without commodity prices):

```
ICA.res.no.commo <- estim.SVAR.ICA(Y,distri = distri,p=6)
round(cbind(ICA.res.no.commo$C.PML,NaN,C.PML),3)
```

```
##      [,1]  [,2]  [,3]  [,4]  [,5]  [,6]  [,7]
## [1,] 0.956 0.287 -0.059  NaN  0.944 0.321 -0.075
## [2,] -0.292 0.950 -0.108  NaN -0.327 0.940 -0.099
## [3,]  0.025 0.121  0.992  NaN  0.039 0.118  0.992
```

Once  $B$  has been estimated, it remains to label the resulting structural shocks (components of  $\eta_t$ ). Postulated shocks are monetary-policy, supply, and demand shocks. This labelling can be based on the following considerations:

- Contractionary **monetary-policy shocks** have a negative impact on real activity and on inflation.
- **Supply shock** have influences of opposite signs on economic activity and on inflation.
- **Demand shock** have influences of same signs on economic activity and on inflation.

Let us compute the IRFs associated with the three structural shocks. (For the sake of comparison, the first line of plots shows the IRFs to a monetary-policy shock obtained from a Cholesky-based approach where the short-term rate is ordered last.)

```
IRF.Chol <- array(NaN,c(n,41,n))
IRF.ICA <- array(NaN,c(n,41,n))
PHI <- list();for(i in 1:nb.lags){PHI[[i]]<-array(Phi,c(3,3,nb.lags))[, , i]}
for(jjjj in 1:n){
  u.shock <- rep(0,n)
  u.shock[jjjj] <- 1
  IRF.Cho1[,,jjjj] <-
    t(simul.VAR(c=rep(0,3),Phi=PHI,B=B,nb.sim=41,
                y0.star=rep(0,3*nb.lags),indic.IRF = 1,u.shock = u.shock))
  IRF.ICA[,,jjjj] <-
    t(simul.VAR(c=rep(0,3),Phi=PHI,B=B%*%C.PML,nb.sim=41,
                y0.star=rep(0,3*nb.lags),indic.IRF = 1,u.shock = u.shock))
}
```

According to Figure 9.22, Shock 1 is a supply shock, Shock 2 is a demand shock, and Shock 3 is a monetary-policy shock. Note that Shock 3 is close to the one resulting from the Cholesky approach.

### Relation with the Heteroskedasticity Identification

In some cases, where the  $\varepsilon_t$ 's are heteroskedastic, the  $B$  matrix can be identified (Rigobon (2003), Lanne et al. (2010)).

Consider the case where we still have  $\varepsilon_t = B\eta_t$  but where  $\eta_t$ 's variance conditionally depends on a regime  $s_t \in \{1, \dots, M\}$ . That is:

$$\text{Var}(\eta_{k,t}|s_t) = \lambda_{s_t,k} \quad \text{for } k \in \{1, \dots, n\}$$

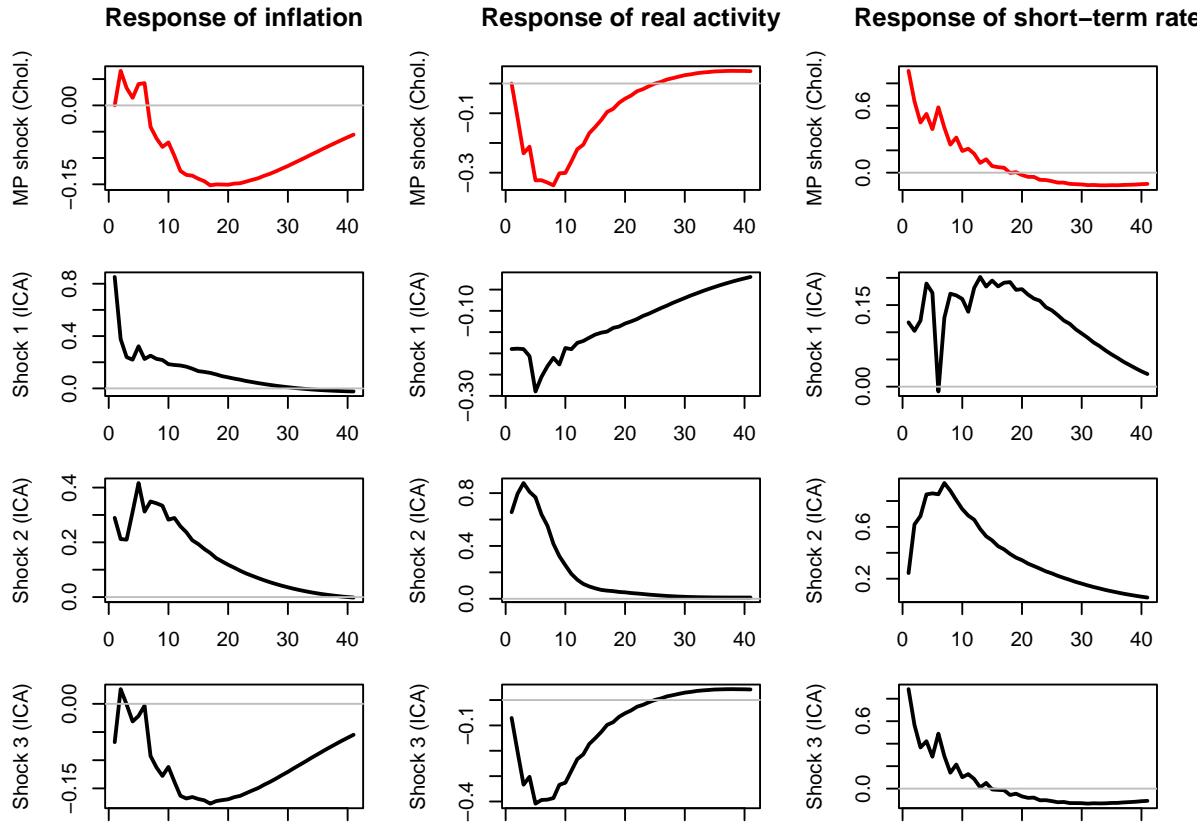


Figure 9.22: The first row of plots shows the responses of the three endogenous variables to the monetary policy shock in the context of a Cholesky-identified SVAR (ordering: inflation, output gap, interest rate). The next three rows of plots show the responses of the endogenous variables to the three structural shocks identified by ICA. The last one (Shock 3) is close to the Cholesky-identified monetary policy shock.

Denoting by  $\Lambda_i$  the diagonal matrix whose diagonal entries are the  $\lambda_{i,k}$ 's, it comes that:

$$\text{Var}(\eta_t | s_t) = \Lambda_{s_t}, \quad \text{and} \quad \text{Var}(\varepsilon_t | s_t) = B\Lambda_{s_t}B'.$$

Without loss of generality, it can be assumed that  $\Lambda_1 = Id$ .

In this context,  $B$  is identified, apart from sign reversal of its columns if for all  $k \neq j \in \{1, \dots, n\}$ , there is a regime  $i$  s.t.  $\lambda_{i,k} \neq \lambda_{i,j}$ . (Prop.1 in @Lanne et al. (2010)).

Bivariate regime case ( $M = 2$ ):  $B$  identified if the  $\lambda_{2,k}$ 's are all different. That is, identification is ensured if “there is sufficient heterogeneity in the volatility changes” (Lütkepohl and Netšunajev (2017)).

If the regimes  $s_t$  are exogenous and serially independent, then this situation is consistent with the “non-Gaussian” situation described above.

### 9.3.10 Factor-Augmented VAR (FAVAR)

VAR models are subject to the curse of dimensionality: If  $n$ , is large, then the number of parameters (in  $n^2$ ) explodes.

In the case where one suspects that the  $y_{i,t}$ 's are mainly driven by a small number of random sources, a factor structure may be imposed, and **principal component analysis** (PCA, see Appendix 10.1) can be employed to estimate the relevant factors (Bernanke et al. (2005)).

Let us denote by  $F_t$  a  $k$ -dimensional vector of latent factors accounting for important shares of the variances of the  $y_{i,t}$ 's (with  $K \ll n$ ) and by  $x_t$  is a small  $M$ -dimensional subset of  $y_t$  (with  $M \ll n$ ). The following factor structure is posited:

$$y_t = \Lambda^f F_t + \Lambda^x x_t + e_t,$$

where the  $e_t$  are “small” serially and mutually i.i.d. error terms. That is  $F_t$  and  $x_t$  are supposed to drive most of the fluctuations of  $y_t$ 's components.

The model is complemented by positing a VAR dynamics for  $[F'_t, x'_t]'$ :

$$\begin{bmatrix} F_t \\ x_t \end{bmatrix} = \Phi(L) \begin{bmatrix} F_{t-1} \\ x_{t-1} \end{bmatrix} + v_t. \quad (9.62)$$

Standard identification techniques of structural shocks can be employed in Eq. (9.62): Cholesky approach can be used for instance if the last component of  $x_t$  is the short-term interest rate and if it is assumed that a MP shock has no contemporaneous impact on other macro-variables (in  $x_t$ ).

In their identification procedure, Bernanke et al. (2005) exploit the fact that macro-finance variables can be decomposed in two sets—fast-moving and slow-moving variables—and that only the former reacts contemporaneously to monetary-policy shocks. Now, how to estimate the (unobserved) factors  $F_t$ ? Bernanke et al. (2005) note that the first  $K + M$  PCA of the whole dataset ( $y_t$ ), that they denote by  $\hat{C}(F_t, x_t)$  should span the same space as  $F_t$  and  $x_t$ . To get an estimate of  $F_t$ , the dependence of  $\hat{C}(F_t, x_t)$  in  $x_t$  has to be removed. This is done by regressing, by OLS,  $\hat{C}(F_t, x_t)$  on  $x_t$  and on  $\hat{C}^*(F_t)$ , where the latter is an estimate of the common components other than  $x_t$ . To proxy for  $\hat{C}^*(F_t)$ , Bernanke et al. (2005) take principal components from the set of slow-moving variables, that are not contemporaneously correlated to  $x_t$ . Vector  $\hat{F}_t$  is then computed as  $\hat{C}(F_t, x_t) - b_x x_t$ , where  $b_x$  are the coefficients coming from the previous OLS regressions.

Note that this approach implies that the vectorial space spanned by  $(\hat{F}_t, x_t)$  is the same as that spanned by  $\hat{C}(F_t, x_t)$ .

Below, we employ this method on the dataset built by McCracken and Ng (2016)—the FRED:MD database—that includes 119 time series.

```
library(BVAR) # contains the fred_md dataset
library(vars)
data <- fred_transform(fred_md, na.rm = FALSE, type = "fred_md")
First.date <- "1959-02-01"
Last.date <- "2020-01-01"
data <- data[(rownames(data)>First.date)&(rownames(data)<Last.date),]
variables.with.na <- which(is.na(apply(data, 2, sum)))
data <- data[,-variables.with.na]
data.values <- scale(data, center = TRUE, scale = TRUE)
data_scaled <- data
data_scaled[1:dim(data)[1], 1:dim(data)[2]] <- data.values
K <- 3
M <- 1
PCA <- prcomp(data_scaled) # implies that PCA$x %*% t(PCA$rotation) = data
C.hat <- PCA$x[, 1:(K+M)]
fast_moving <- c("HOUST", "HOUSTNE", "HOUSTMW", "HOUSTS", "HOUSTW", "HOUSTS", "AMDMNOx",
                 "FEDFUNDS", "CP3Mx", "TB3MS", "TB6MS", "GS1", "GS5", "GS10",
                 "COMPAPFFx", "TB3SMFFM", "TB6SMFFM", "T1YFFM", "T5YFFM", "T10YFFM",
                 "AAAFFM", "EXSZUSx", "EXJPUSx", "EXUSUKx", "EXCAUSx")
data.slow <- data_scaled[,-which(fast_moving %in% names(data))]
PCA.star <- prcomp(data.slow) # implies that PCA$x %*% t(PCA$rotation) = data
C.hat.star <- PCA.star$x[, 1:K]
D <- cbind(data$FEDFUNDS, C.hat.star)
b.x <- solve(t(D) %*% D) %*% t(D) %*% C.hat
F.hat <- C.hat - data$FEDFUNDS %*% matrix(b.x[1,], nrow=1)
data_var <- data.frame(F.hat, FEDFUNDS = data$FEDFUNDS)
p <- 10
var <- VAR(data_var, p)
Omega <- var(residuals(var))
B <- t(chol(Omega))
D <- cbind(F.hat, data$FEDFUNDS)
loadings <- solve(t(D) %*% D) %*% t(D) %*% as.matrix(data_scaled)
irf <- simul.VAR(c=rep(0, (K+M)*p), Phi=Acoef(var), B, nb.sim=120,
```

```

y0.star=rep(0,(K+M)*p),indic.IRF = 1,
u.shock = c(rep(0,K+1),1)
irf.all <- irf %*% loadings
par(mfrow=c(2,2))
variables.2.plot <- c("FEDFUNDS","INDPRO","UNRATE","CPIAUCSL")
par(plt=c(.2,.95,.3,.95))
for(i in 1:length(variables.2.plot)){
  plot(cumsum(irf.all[,which(variables.2.plot[i]==names(data))]),lwd=2,
    type="l",xlab="months after shock",ylab=variables.2.plot[i])
}

```

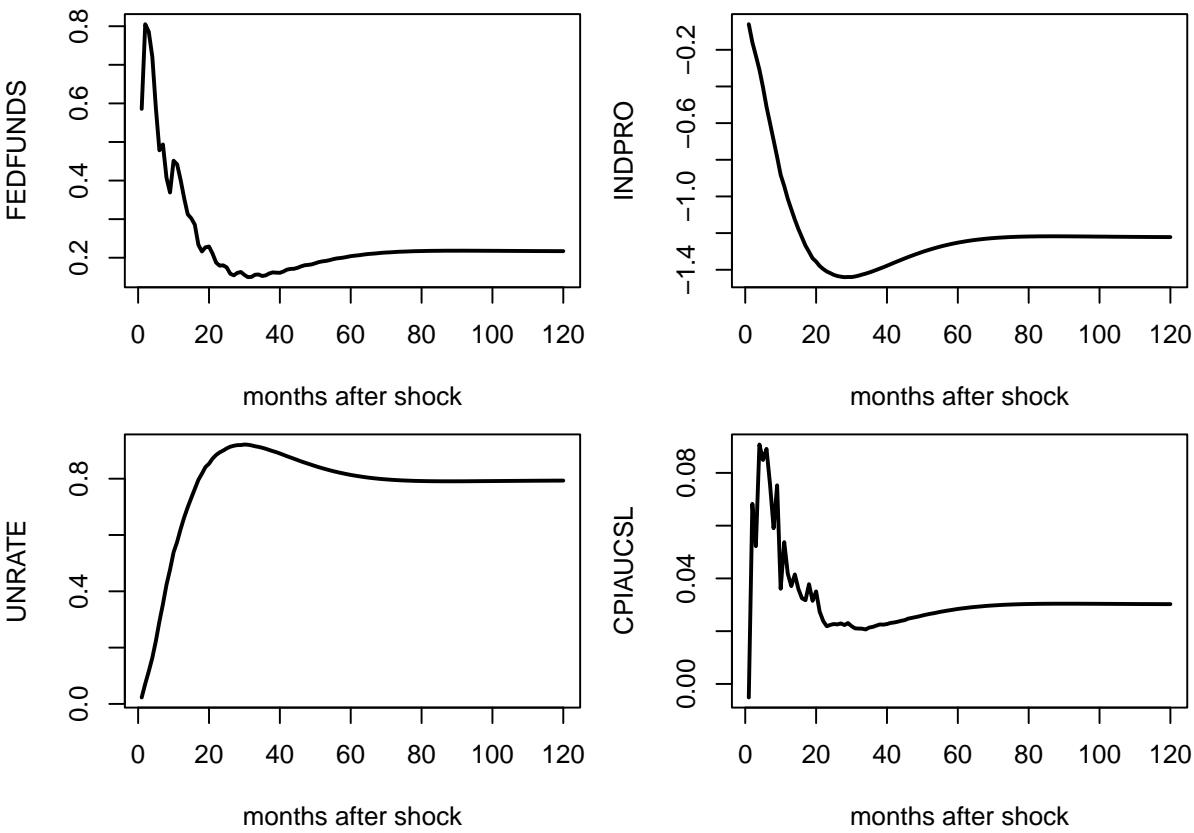


Figure 9.23: Responses of a monetary-policy shock. FAVAR approach of Bernanke, Boivin, and Eliasz (2005). FRED-MD dataset.

### 9.3.11 Projection Methods

Consider the infinite MA representation of  $y_t$  (Eq. (9.38)):

$$y_t = \mu + \sum_{h=0}^{\infty} \Psi_h \eta_{t-h}.$$

As seen in Section 9.3.2, the entries  $(i, j)$  of the sequence of the  $\Psi_h$  matrices define the IRF of  $\eta_{j,t}$  on  $y_{i,t}$ .

Assume that you observe  $\eta_{j,t}$ , then a consistent estimate of  $\Psi_{i,j,h}$  is simply obtained by the OLS regression of  $y_{i,t+h}$  on  $\eta_{j,t}$ :

$$y_{i,t+h} = \mu_i + \Psi_{i,j,h} \eta_{j,t} + u_{i,j,t+h}. \quad (9.63)$$

Because the residuals  $u_{i,j,t+h}$  are autocorrelated (for  $h > 0$ ), estimates of the covariance of the OLS estimators of the  $\Psi_{i,j,h}$  then have to be based on robust estimators (e.g. Newey-West, see Eq. (5.39)). This is the core idea of the **local projection approach** proposed by Jordà (2005).

Now, how to proceed in the (usual) case where  $\eta_{j,t}$  is not observed? We consider two situations.

#### Situation A: Without IV

This corresponds to the original Jordà (2005)'s approach.

Assume that the structural shock of interest ( $\eta_{1,t}$ , say) can be consistently obtained as the residual of a regression of a variable  $x_t$  on a set of control variables  $w_t$  independent from  $\eta_{1,t}$ :

$$\eta_{1,t} = x_t - \mathbb{E}(x_t|w_t), \quad (9.64)$$

where  $\mathbb{E}(x_t|w_t)$  is affine in  $w_t$  and where  $w_t$  is an affine transformation of  $\eta_{2:n,t}$  and of past shocks  $\eta_{t-1}, \eta_{t-2}, \dots$ .

Eq. (9.64) implies that, conditional on  $w_t$ , the additional knowledge of  $x_t$  is useful only when it comes to forecast something that depends on  $\eta_{1,t}$ . Hence, given that  $u_{i,1,t+h}$  (see Eq. (9.63)) is independent from  $\eta_{1,t}$  (it depends on  $\eta_{t+h}, \dots, \eta_{t+1}, \eta_{2:n,t}, \eta_{t-1}, \eta_{t-2}, \dots$ ), it comes that

$$\mathbb{E}(u_{i,1,t+h}|x_t, w_t) = \mathbb{E}(u_{i,1,t+h}|w_t).$$

This is the *conditional mean independence* case.

Let's rewrite Eq. (9.63) as follows:

$$\begin{aligned} y_{i,t+h} &= \mu_i + \Psi_{i,1,h}\eta_{1,t} + u_{i,1,t+h} \\ &= \mu_i + \Psi_{i,1,h}x_t - \Psi_{i,1,h}\mathbb{E}(x_t|w_t) + u_{i,1,t+h}, \end{aligned}$$

What precedes implies that the expectation of the blue term, conditional on  $x_t$  and  $w_t$ , is linear in  $w_t$ . Standard results in the conditional mean independence case imply that the regression of  $y_{i,t+h}$  on  $x_t$ , controlling for  $w_t$ , provides a consistent estimate of  $\Psi_{i,1,h}$ :

$$y_{i,t+h} = \alpha_i + \Psi_{i,1,h}x_t + \beta'w_t + v_{i,t+h}. \quad (9.65)$$

This is for instance consistent with the case where  $[\Delta GDP_t, \pi_t, i_t]'$  follows a VAR(1) and the monetary-policy shock do not contemporaneously affect  $\Delta GDP_t$  and  $\pi_t$ .

The IRFs can be estimated by LP, taking  $x_t = i_t$  and  $w_t = [\Delta GDP_t, \pi_t, \Delta GDP_{t-1}, \pi_{t-1}, i_{t-1}]'$ .

This approach closely relates to the SVAR Cholesky-based identification approach. Specifically, if  $w_t = [y_{1,t}, \dots, y_{k-1,t}, y'_{t-1}, \dots, y'_{t-p}]'$ , with  $k \leq n$ , and  $x_t = y_{k,t}$ , then this approach corresponds, for  $h = 0$ , to the SVAR( $p$ ) Cholesky-based IRF (focusing on the responses to the  $k^{th}$  structural shock). However, the two approaches differ for  $h > 0$ , because the LP methodology does not assumes a VAR dynamics for  $y_t$ .<sup>4</sup>

### Situation B: IV approach

Consider now that we have a valid instrument  $z_t$  for  $\eta_{1,t}$  (with  $\mathbb{E}(z_t) = 0$ ). That is:

$$\begin{cases} (IV.i) \quad \mathbb{E}(z_t\eta_{1,t}) \neq 0 & \text{(relevance condition)} \\ (IV.ii) \quad \mathbb{E}(z_t\eta_{j,t}) = 0 \quad \text{for } j > 1 & \text{(exogeneity condition)} \end{cases} \quad (9.66)$$

The instrument  $z_t$  can be used to identify the structural shock. Eq. (9.66) implies that there exist  $\rho \neq 0$  and a mean-zero variable  $\xi_t$  such that:

$$\eta_{1,t} = \rho z_t + \xi_t,$$

where  $\xi_t$  is correlated neither to  $z_t$ , nor to  $\eta_{j,t}$ ,  $j \geq 2$ .

*Proof.* Define  $\rho = \frac{\mathbb{E}(\eta_{1,t}z_t)}{\text{Var}(z_t)}$  and  $\xi_t = \eta_{1,t} - \rho z_t$ . It is easily seen that  $\xi_t$  satisfies the moment restrictions given above.  $\square$

Ramey (2016) reviews the different approaches employed to construct monetary policy-shocks (the two main approaches are presented in 9.9 and 9.10 below). She has also collected time series of such shocks, see her website.

---

<sup>4</sup>This is reminiscent of the distinction between direct forecasting –based on regressions of  $y_{t+h}$  on  $\{y_t, y_{t-1}, \dots\}$ – and iterated forecasting –based on a recursive model where  $y_{t+1} = g(y_t, y_{t-1}, \dots) + \varepsilon_{t+1}$ , see Marcellino et al. (2006).

**Example 9.9** (Identification of Monetary-Policy Shocks Based on High-Frequency Data). Instruments for monetary-policy shocks can be extracted from high-frequency market data associated with interest-rate products.

The quotes of all interest-rate-related financial products are sensitive to monetary-policy announcements. That is because these quotes mainly depends on investors' expectations regarding future short-term rates:  $\mathbb{E}_t(i_{t+s})$ . Typically, if agents were risk-neutral, the maturity- $h$  interest rate would approximatively be given by:

$$i_{t,h} \approx \mathbb{E}_t \left( \frac{1}{h} \int_0^h i_{t+s} ds \right) = \frac{1}{h} \int_0^h \mathbb{E}_t (i_{t+s}) ds.$$

In general, changes in  $\mathbb{E}_t(i_{t+s})$ , for  $s > 0$ , can be affected by all types of shocks that may trigger a reaction by the central bank.

However, if a MP announcement takes place between  $t$  and  $t + \epsilon$ , then most of  $\mathbb{E}_{t+\epsilon}(i_{t+s}) - \mathbb{E}_t(i_{t+s})$  is to be attributed to the MP shock (see Figure 9.24, from Gürkaynak et al. (2005)). Hence, a monthly time series of MP shocks can be obtained by summing, over each month, the changes  $i_{t+\epsilon,h} - i_{t,h}$  associated with a given interest rate (T-bills, futures, swaps) and a given maturity  $h$ .

See among others: Kuttner (2001), Cochrane and Piazzesi (2002), Gürkaynak et al. (2005), Piazzesi and Swanson (2008), Gertler and Karadi (2015).

**Figure 1. Intraday Trading in Federal Funds Futures Contracts**

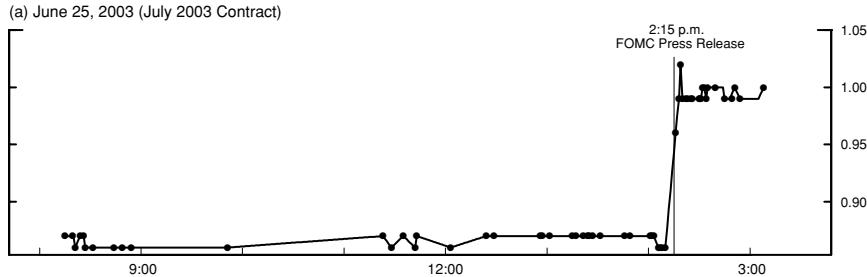


Figure 9.24: Source: Gürkaynak, Sack and Swanson (2005). Transaction rates of Federal funds futures on June 25, 2003, day on which a regularly scheduled FOMC meeting was scheduled. At 2:15 p.m., the FOMC announced that it was lowering its target for the federal funds rate from 1.25% to 1%, while many market participants were expecting a 50 bp cut. This shows that (i) financial markets seem to fully adjust to the policy action within just a few minutes and (ii) the federal funds rate surprise is not necessarily in the same direction as the federal funds rate action itself.

**Example 9.10** (Identification of Monetary-Policy Shocks Based on the Narrative Approach). Romer and Romer (2004) propose a two-step approach:

- derive a series for Federal Reserve intentions for the federal funds rate (the explicit target of the Fed) around FOMC meetings,
- control for Federal Reserve forecasts.

This gives a measure of intended monetary policy actions not driven by information about future economic developments. a. "intentions" are measured as a combination of narrative and quantitative evidence. Sources: (among others) Minutes of FOMC and "Blue Books". b. Controls = variables spanning the information the Federal Reserve has about future developments. Data: Federal Reserve's internal forecasts (inflation, real output and unemployment), "Greenbook's forecasts" – usually issued 6 days before the FOMC meeting.

The shock measure is the residual series in the linear regression of (a) on (b).

There are two main IV approaches to estimate IRFs see Stock and Watson (2018):

- The LP-IV approach, where  $y_t$ 's DGP is left unspecified,

b. The SVAR-IV approach.

The LP-IV approach is based on a set of IV regressions (for each variable of interest, one for each forecast horizon). The SVAR-IV approach is based on IV regressions of VAR innovations only (one for each series of VAR innovations).

If the VAR adequately captures the DGP, then the IV-SVAR is optimal for all horizons. However, if the VAR is misspecified, then specification errors are compounded at each horizon and a local projection method would lead to better results.

### Situation B.1: SVAR-IV approach

Assume you have consistent estimates of  $\varepsilon_t = B\eta_t$ , these estimates ( $\hat{\varepsilon}_t$ ) coming from the estimation of a VAR model. You have, for  $i \in \{1, \dots, n\}$ :

$$\begin{aligned}\varepsilon_{i,t} &= b_{i,1}\eta_{1,t} + u_{i,t} \quad (\#eq:eps_rho) \\ &= b_{i,1}\rho z_t + \underbrace{b_{i,1}\xi_t + u_{i,t}}_{\perp z_t}.\end{aligned}\tag{9.67}$$

( $u_{i,t}$  is a linear combination of the  $\eta_{j,t}$ 's,  $j \geq 2$ ).

Hence, up to a multiplicative factor ( $\rho$ ), the (OLS) regressions of the  $\hat{\varepsilon}_{i,t}$ 's on  $z_t$  provide consistent estimates of the  $b_{i,1}$ 's.

Combined with the estimated VAR (the  $\Phi_k$  matrices), this provides consistent estimates of the IRFs of  $\eta_{1,t}$  on  $y_t$ , though up to a multiplicative factor. This scale ambiguity can be solved by rescaling the structural shock (“unit-effect normalisation”, see Stock and Watson (2018)). Let us consider  $\tilde{\eta}_{1,t} = b_{1,1}\eta_{1,t}$ ; by construction,  $\tilde{\eta}_{1,t}$  has a one-unit contemporaneous effect on  $y_{1,t}$ . Denoting by  $\tilde{B}_{i,1}$  the contemporaneous impact of  $\tilde{\eta}_{1,t}$ , we get:

$$\tilde{B}_1 = \frac{1}{b_{1,1}}B_1,$$

where  $B_1$  denotes the 1<sup>st</sup> column of  $B$  and  $\tilde{B}_1 = [1, \tilde{B}_{2,1}, \dots, \tilde{B}_{n,1}]'$ .

Eq. @ref(eq:eps\_rho) gives:

$$\begin{aligned}\varepsilon_{1,t} &= \tilde{\eta}_{1,t} + u_{1,t} \\ \varepsilon_{i,t} &= \tilde{B}_{i,1}\tilde{\eta}_{1,t} + u_{i,t}.\end{aligned}$$

This suggests that  $\tilde{B}_{i,1}$  can be estimated by regressing  $\varepsilon_{i,t}$  on  $\varepsilon_{1,t}$ , using  $z_t$  as an instrument.

What about inference? Once cannot use the usual TSLS standard deviations because the  $\varepsilon_{i,t}$ 's are not directly observed. Bootstrap procedures can be resorted to. Stock and Watson (2018) propose, in particular, a Gaussian parametric bootstrap:

Assume you have estimated  $\{\widehat{\Phi}_1, \dots, \widehat{\Phi}_p, \widehat{B}_1\}$  using the SVAR-IV approach based on a size- $T$  sample. Generate  $N$  (where  $N$  is large) size- $T$  samples from the following VAR:

$$\begin{bmatrix} \widehat{\Phi}(L) & 0 \\ 0 & \widehat{\rho}(L) \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \varepsilon_t \\ e_t \end{bmatrix},$$

$$\text{where } \begin{bmatrix} \varepsilon_t \\ e_t \end{bmatrix} \sim i.i.d. \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega & S'_{\varepsilon,e} \\ S_{\varepsilon,e} & \sigma_e^2 \end{bmatrix} \right),$$

where  $\widehat{\rho}(L)$  and  $\sigma_e^2$  result from the estimation of an AR process for  $z_t$ , and where  $\Omega$  and  $S_{\varepsilon,e}$  are sample covariances for the VAR/AR residuals.

For each simulated sample (of  $\tilde{y}_t$  and  $\tilde{z}_t$ , say), estimate  $\{\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_p, \widetilde{B}_1\}$  and associated  $\widetilde{\Psi}_{i,1,h}$ . This provides e.g. a sequence of  $N$  estimates of  $\Psi_{i,1,h}$ , from which quantiles and conf. intervals can be deduced.

```
# Load vars package:
library(vars)
library(AEC)
data("USmonthly")
First.date <- "1990-05-01"
Last.date <- "2012-6-01"
indic.first <- which(USmonthly$DATES==First.date)
indic.last <- which(USmonthly$DATES==Last.date)
USmonthly <- USmonthly[indic.first:indic.last,]
shock.name <- "FF4_TC" # "FF4_TC", "ED2_TC", "ff1_vr", "rrshock83b"
indic.shock.name <- which(names(USmonthly)==shock.name)
Z <- matrix(USmonthly[,indic.shock.name],ncol=1)
par(plt=c(.1,.95,.1,.95))
plot(USmonthly$DATES,Z,type="l",xlab="",ylab="",lwd=2)
```

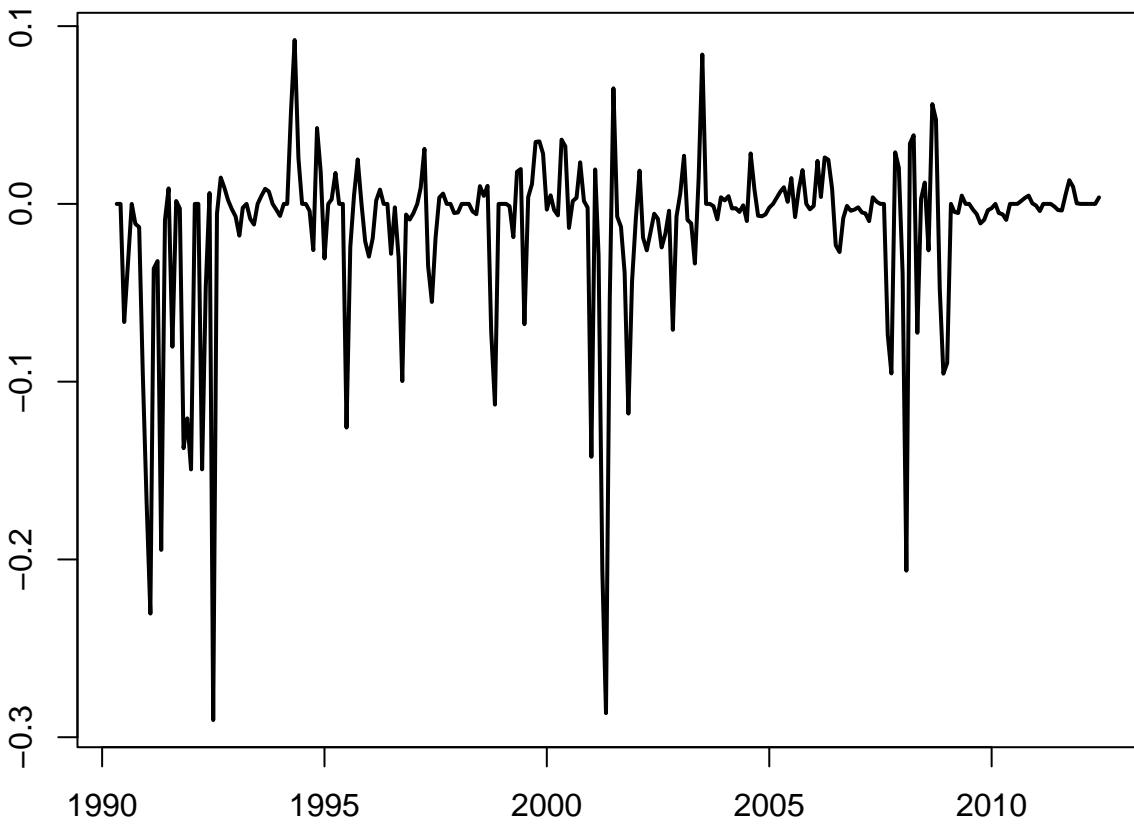


Figure 9.25: Gertler-Karadi monthly shocks, fed funds futures 3 months.

```
considered.variables <- c("GS1","LIP","LCPI","EBP")
y <- as.matrix(USmonthly[,considered.variables])
n <- length(considered.variables)
colnames(y) <- considered.variables
par(plt=c(.15,.95,.15,.8))
res.svar.iv <-
  svar.iv(y,Z,p = 4,names.of.variables=considered.variables,
          nb.periods.IRF = 20,
          z.AR.order=1,
          nb.bootstrap.replications = 100,
          confidence.interval = 0.90,
          indic.plot=1)
```

### Situation B.2: LP-IV

If you do not want to posit a VAR-type dynamics for  $y_t$  –e.g. because you suspect that the true generating model may be a non-invertible VARMA model– you can directly proceed by IV-projection methods to obtain the  $\tilde{\Psi}_{i,1,h} \equiv \Psi_{i,1,h}/b_{1,1}$  (that are the IRFs of  $\tilde{\eta}_{1,t}$  on  $y_{i,t}$ ).

However, Assumptions (IV.i) and (IV.ii) (Eq. (9.66)) have to be complemented with (IV.iii):

$$(IV.iii) \quad \mathbb{E}(z_t \eta_{j,t+h}) = 0 \text{ for } h \neq 0 \quad (\text{lead-lag exogeneity})$$

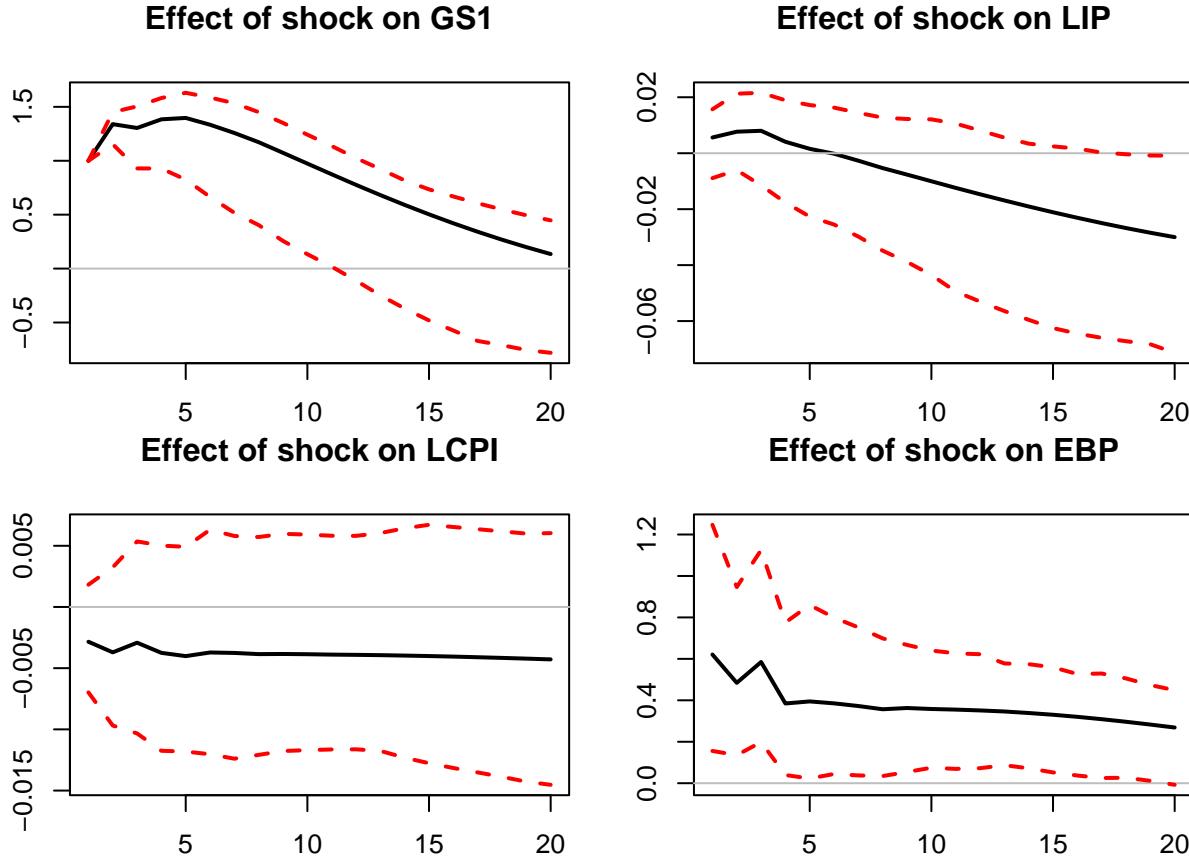


Figure 9.26: Reponses to a monetary-policy shock, SVAR-IV approach.

When (IV.i), (IV.ii) and (IV.iii) are satisfied,  $\tilde{\Psi}_{i,1,h}$  can be estimated by regressing  $y_{i,t+h}$  on  $y_{1,t}$ , using  $z_t$  as an instrument, i.e. by considering the TSLS estimation of:

$$y_{i,t+h} = \alpha_i + \tilde{\Psi}_{i,1,h} y_{1,t} + \nu_{i,t+h}, \quad (9.68)$$

where  $\nu_{i,t+h}$  is correlated to  $y_{1,t}$ , but not to  $z_t$ .

We have indeed:

$$\begin{aligned} y_{1,t} &= \alpha_1 + \tilde{\eta}_{1,t} + v_{1,t} \\ y_{i,t+h} &= \alpha_i + \tilde{\Psi}_{i,1,h} \tilde{\eta}_{1,t} + v_{i,t+h}, \end{aligned}$$

where the  $v_{i,t+h}$ 's are uncorrelated to  $z_t$  under (IV.i), (IV.ii) and (IV.iii).

Note again that, for  $h > 0$ , the  $v_{i,t+h}$  (and  $\nu_{i,t+h}$ ) are auto-correlated. Newey-West corrections therefore have to be used to compute std errors of the  $\tilde{\Psi}_{i,1,h}$ 's estimates.

Consider the linear regression:

$$\mathbf{Y} = \mathbf{X}\beta + \varepsilon,$$

where  $\mathbb{E}(\varepsilon) = 0$ , but where the explicative variables  $\mathbf{X}$  are supposed to be correlated to the residuals  $\varepsilon$ .

Moreover, the  $\varepsilon$  are supposed to be possibly heteroskedastic and auto-correlated.

We consider the instruments  $\mathbf{Z}$ , with  $\mathbb{E}(\mathbf{X}'\mathbf{Z}) \neq 0$  but  $\mathbb{E}(\varepsilon'\mathbf{Z}) = 0$ .

The IV estimator of  $\beta$  is obtained by regressing  $\hat{\mathbf{Y}}$  on  $\hat{\mathbf{X}}$ , where  $\hat{\mathbf{Y}}$  and  $\hat{\mathbf{X}}$  are the respective residuals of the regressions of  $\mathbf{Y}$  and  $\mathbf{X}$  on  $\mathbf{Z}$ .

$$\begin{aligned} \mathbf{b}_{iv} &= [\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y} \\ \mathbf{b}_{iv} &= \beta + \frac{1}{\sqrt{T}} \underbrace{T[\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}}_{=Q(\mathbf{X}, \mathbf{Z}) \xrightarrow{p} \mathbf{Q}_{xz}} \underbrace{\sqrt{T} \left( \frac{1}{T} \mathbf{Z}' \varepsilon \right)}_{\xrightarrow{d} \mathcal{N}(0, S)}, \end{aligned}$$

where  $\mathbf{S}$  is the long-run variance of  $\mathbf{z}_t \varepsilon_t$  (see next slide).

The asymptotic covariance matrix of  $\sqrt{T}\mathbf{b}_{iv}$  is  $\mathbf{Q}_{xz}\mathbf{S}\mathbf{Q}_{xz}'$ .

The covariance matrix of  $\mathbf{b}_{iv}$  can be approximated by  $\frac{1}{T}Q(\mathbf{X}, \mathbf{Z})\hat{\mathbf{S}}Q(\mathbf{X}, \mathbf{Z})'$  where  $\hat{\mathbf{S}}$  is the Newey-West estimator of  $\mathbf{S}$  (see Eq. (5.39))

(IV.iii) is usually not restrictive for  $h > 0$  ( $z_t$  is usually not affected by future shocks). By contrast, it may be restrictive for  $h < 0$ . This can be solved by adding controls in Regression (9.68). These controls should span the space of  $\{\eta_{t-1}, \eta_{t-2}, \dots\}$ .

If  $z_t$  is suspected to be correlated to past values of  $\eta_{1,t}$  but not to the  $\eta_{j,t}$ 's,  $j > 1$ , then one can add lags of  $z_t$  as controls (method e.g. advocated by Ramey, 2016, p.108, considering the instrument by Gertler and Karadi (2015)).

In the general case, one can use lags of  $y_t$  as controls. Note that, even if (IV.iii) holds, adding controls may reduce the variance of the regression error.

As noted by Stock and Watson (2018), the relevant variance is the long-run variance of the instrument-times-error term. They also recommend (p.926) using leads and lags of  $z_t$  to improve efficiency.

```
res.LP.IV <- make.LPIV.irf(y,Z,
                           nb.periods.IRF = 20,
                           nb.lags.Y.4.control=4,
                           nb.lags.Z.4.control=4,
                           indic.plot = 1, # Plots are displayed if = 1.
                           confidence.interval = 0.90)
```

### 9.3.12 Inference

Consider the following SVAR model:

$$y_t = \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p} + \varepsilon_t$$

with  $\varepsilon_t = B\eta_t$ ,  $\Omega_\varepsilon = BB'$ .

The corresponding infinite MA representation (Eq. (9.38), or Wold theorem, Theorem 9.3) is:

$$y_t = \sum_{h=0}^{\infty} \Psi_h \eta_{t-h},$$

where  $\Psi_0 = B$  and for  $h = 1, 2, \dots$ :

$$\Psi_h = \sum_{j=1}^h \Psi_{h-j} \Phi_j,$$

with  $\Phi_j = 0$  for  $j > p$  (see Prop. 9.8 for this recursive computation of the  $\Psi_j$ 's).

Inference on the VAR coefficients  $\{\Phi_j\}_{j=1,\dots,p}$  is straightforward (standard OLS inference). But inference is more complicated regarding IRF. Indeed, as shown by the previous equation, the (infinite) MA coefficients  $\{\Psi_j\}_{j=1,\dots}$  are non-linear functions of the  $\{\Phi_j\}_{j=1,\dots,p}$  and of  $\Omega_\varepsilon$ . An other issue pertain to small sample bias: typically, for persistent process, auto-regressive parameters are known to be downward biased.

The main inference methods are the following:

- Monte Carlo method (Hamilton (1994))
- Asymptotic normal approximation (Lütkepohl (1990)), or Delta method
- Bootstrap method (Kilian\_1998)

#### Monte Carlo method

We use Monte Carlo when we need to approximate the distribution of a variable whose distribution is unknown (here: the  $\Psi_j$ 's) but which is a function of another variable whose distribution is known (here, the  $\Phi_j$ 's).

For instance, suppose we know the distribution of a random variable  $X$ , which takes values in  $\mathbb{R}$ , with density function  $p$ . Assume we want to compute the mean of  $\varphi(X)$ . We have:

$$\mathbb{E}(\varphi(X)) = \int_{-\infty}^{+\infty} \varphi(x)p(x)dx$$

Suppose that the above integral does not have a simple expression. We cannot compute  $\mathbb{E}(\varphi(X))$  but, by virtue of the law of large numbers (Theorem ??), we can approximate it as follows:

$$\mathbb{E}(\varphi(X)) \approx \frac{1}{N} \sum_{i=1}^N \varphi(X^{(i)}),$$

where  $\{X^{(i)}\}_{i=1,\dots,N}$  are  $N$  independent draws of  $X$ . More generally, the distribution of  $\varphi(X)$  can be approximated by the empirical distribution of the  $\varphi(X^{(i)})$ 's. Typically, if 10'000 values of  $\varphi(X^{(i)})$  are drawn, the 5<sup>th</sup> percentile of the p.d.f. of  $\varphi(X)$  can be approximated by the 500<sup>th</sup> value of the 10'000 draws of  $\varphi(X^{(i)})$  (after arranging these values in ascending order).

As regards the computation of confidence intervals around IRFs, one has to think of  $\{\widehat{\Phi}_j\}_{j=1,\dots,p}$ , and of  $\widehat{\Omega}$  as  $X$  and  $\{\widehat{\Psi}_j\}_{j=1,\dots}$  as  $\varphi(X)$ . (Proposition 9.14 provides us with the asymptotic distribution of the "X.")

To summarize, here are the steps one can implement to derive confidence intervals for the IRFs using the Monte-Carlo approach:

For each iteration  $k$ :

1. Draw  $\{\widehat{\Phi}_j^{(k)}\}_{j=1,\dots,p}$  and  $\widehat{\Omega}^{(k)}$  from their asymptotic distribution (using Proposition 9.14).
2. Compute the matrix  $B^{(k)}$  so that  $\widehat{\Omega}^{(k)} = B^{(k)}B^{(k)'}_{'}$ , according to your identification strategy.
3. Compute the associated IRFs  $\{\widehat{\Psi}_j\}^{(k)}$ .

Perform  $N$  replications and report the median impulse response (and its confidence intervals).

### Delta method

Suppose  $\beta$  is a vector of parameters and  $\hat{\beta}$  is an estimator such that

$$\sqrt{T}(\hat{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, \Sigma_\beta),$$

where  $d$  denotes convergence in distribution,  $\mathcal{N}(0, \Sigma_\beta)$  denotes the multivariate normal distribution with mean vector 0 and covariance matrix  $\Sigma_\beta$  and  $T$  is the sample size used for estimation.

Let  $g(\beta) = (g_1(\beta), \dots, g_m(\beta))'$  be a continuously differentiable function with values in  $\mathbb{R}^m$ , and assume that  $\partial g_i / \partial \beta' = (\partial g_i / \partial \beta_j)$  is nonzero at  $\beta$  for  $i = 1, \dots, m$ . Then

$$\sqrt{T}(g(\hat{\beta}) - g(\beta)) \xrightarrow{d} \mathcal{N}\left(0, \frac{\partial g}{\partial \beta'} \Sigma_\beta \frac{\partial g'}{\partial \beta}\right).$$

(This formula underlies the Delta method, see Eq. (7.15).)

Using this property, Lütkepohl (1990) provides the asymptotic distributions of the  $\Psi_j$ 's.

A limit of the last two approaches (Monte Carlo and the Delta method) is that they critically rely on asymptotic results. Bootstrapping approaches are more robust in small-sample situations.

### Bootstrap

IRFs' confidence intervals are intervals where 90% (or 95%, 75%, ...) of the IRFs would lie, if we were to repeat the estimation a large number of times in similar conditions ( $T$  observations). We obviously cannot do this, because we have only one sample:  $\{y_t\}_{t=1,\dots,T}$ . But we can try to *construct* such samples.

Bootstrapping consists in:

- re-sampling  $N$  times, i.e., constructing  $N$  samples of  $T$  observations, using the estimated VAR coefficients and
  - a. a sample of residuals from the distribution  $N(0, BB')$  (**parametric approach**), or

- b. a sample of residuals drawn randomly from the set of the actual estimated residuals  $\{\hat{\varepsilon}_t\}_{t=1,\dots,T}$ . (**non-parametric approach**).
- re-estimating the SVAR  $N$  times.

Here is the algorithm:

1. Construct a sample

$$y_t^{(k)} = \widehat{\Phi}_1 y_{t-1}^{(k)} + \cdots + \widehat{\Phi}_p y_{t-p}^{(k)} + \hat{\varepsilon}_t^{(k)},$$

with  $\hat{\varepsilon}_t^{(k)} = \hat{\varepsilon}_{s_t^{(k)}}$ , where  $\{s_1^{(k)}, \dots, s_T^{(k)}\}$  is a random set from  $\{1, \dots, T\}^T$ .

2. Re-estimate the SVAR and compute the IRFs  $\{\widehat{\Psi}_j\}^{(k)}$ .

Perform  $N$  replications and report the median impulse response (and its confidence intervals).

#### Bootstrap-after-bootstrap (Kilian (1998))

The previous simple bootstrapping procedure deals with non-normality and small sample distribution, since we use the actual residuals. However, it does not deal with the *small sample bias*, stemming, in particular, from small-sample bias associated with OLS coefficient estimates  $\{\widehat{\Phi}_j\}_{j=1,\dots,p}$ . The main idea of the bootstrap-after-bootstrap of Kilian (1998) is to run two consecutive bootstraps: the objective of the first is to compute the bias, which can further be used to correct the initial estimates of the  $\Phi_i$ 's. Further, these corrected estimates are used—in the second bootstrap—to compute a set of IRFs (as in the standard bootstrap).

More formally, the algorithm is as follows:

1. Estimate the SVAR coefficients  $\{\widehat{\Phi}_j\}_{j=1,\dots,p}$  and  $\widehat{\Omega}$
2. **First bootstrap.** For each iteration  $k$ :

- a. Construct a sample

$$y_t^{(k)} = \widehat{\Phi}_1 y_{t-1}^{(k)} + \cdots + \widehat{\Phi}_p y_{t-p}^{(k)} + \hat{\varepsilon}_t^{(k)},$$

with  $\hat{\varepsilon}_t^{(k)} = \hat{\varepsilon}_{s_t^{(k)}}$ , where  $\{s_1^{(k)}, \dots, s_T^{(k)}\}$  is a random set from  $\{1, \dots, T\}^T$ .

- b. Re-estimate the VAR and compute the coefficients  $\{\widehat{\Phi}_j\}_{j=1,\dots,p}^{(k)}$ .

3. Perform  $N$  replications and compute the median coefficients  $\{\widehat{\Phi}_j\}_{j=1,\dots,p}^*$ .

4. Approximate the bias terms by  $\widehat{\Theta}_j = \widehat{\Phi}_j^* - \widehat{\Phi}_j$ .

5. Construct the bias-corrected terms  $\widetilde{\Phi}_j = \widehat{\Phi}_j - \widehat{\Theta}_j$ .

6. **Second bootstrap.** For each iteration  $k$ :

- a. Construct a sample now from

$$y_t^{(k)} = \widetilde{\Phi}_1 y_{t-1}^{(k)} + \cdots + \widetilde{\Phi}_p y_{t-p}^{(k)} + \hat{\varepsilon}_t^{(k)}.$$

- b. Re-estimate the VAR and compute the coefficients  $\{\widehat{\Phi}_j^*\}_{j=1,\dots,p}^{(k)}$ .

- c. Construct the bias-corrected estimates  $\widetilde{\Phi}_j^{*(k)} = \widehat{\Phi}_j^{*(k)} - \widehat{\Theta}_j$ .

- d. Compute the associated IRFs  $\{\widetilde{\Psi}_j^{*(k)}\}_{j \geq 1}$ .

7. Perform  $N$  replications and compute the median and the confidence interval of the set of IRFs.

It should be noted that correcting for the bias can generate non-stationary results ( $\tilde{\Phi}$  with eigenvalue with modulus  $> 1$ ). Solution (Kilian (1998)):

In step 5, check if the largest eigenvalue of  $\tilde{\Phi}$  is of modulus  $< 1$ . If not, shrink the bias: for all  $j$ s, set  $\widehat{\Theta}_j^{(i+1)} = \delta_{i+1} \widehat{\Theta}_j^{(i)}$ , with  $\delta_{i+1} = \delta_i - 0.01$ , starting with  $\delta_1 = 1$  and  $\widehat{\Theta}_j^{(1)} = \widehat{\Theta}_j$ , and compute  $\widetilde{\Phi}_j^{(i+1)} = \widehat{\Phi}_j - \widehat{\Theta}_j^{(i+1)}$  until the largest eigenvalue of  $\widetilde{\Phi}^{(i+1)}$  has modulus  $< 1$ .

Function `VAR.Boot` of package `VAR.etp` (Kim (2022)) can be used to operate the bias-correction approach of Kilian (1998):

```

library(VAR.etc)
library(vars) #standard VAR models
data(dat) # part of VAR.etc package
corrected <- VAR.Boot(dat,p=2,nb=200,type="const")
noncorrec <- VAR(dat,p=2)
rbind(corrected$coef[1,],
      (corrected$coef+corrected$Bias)[1,],
      noncorrec$varresult$inv$coefficients)

##           inv(-1)   inc(-1)   con(-1)   inv(-2)   inc(-2)   con(-2)       const
## [1,] -0.3228276 0.0698514 1.066291 -0.1555651 0.1350925 1.0044744 -0.01895355
## [2,] -0.3196310 0.1459888 0.961219 -0.1605511 0.1146050 0.9343938 -0.01672199
## [3,] -0.3196310 0.1459888 0.961219 -0.1605511 0.1146050 0.9343938 -0.01672199

```

## 9.4 Forecasting

Forecasting has always been an important part of the time series field (De Gooijer and Hyndman (2006)). Macroeconomic forecasts are done in many places: Public Administration (notably Treasuries), Central Banks, International Institutions (e.g. IMF, OECD), banks, big firms. These institutions are interested in the **point estimates** ( $\sim$  most likely value) of the variable of interest. They also sometimes need to measure the **uncertainty** ( $\sim$  dispersion of likely outcomes) associated to the point estimates.<sup>5</sup>

Forecasts produced by professional forecasters are available on these web pages:

- Philly Fed Survey of Professional Forecasters.
- ECB Survey of Professional Forecasters.
- IMF World Economic Outlook.
- OECD Global Economic Outlook.
- European Commission Economic Forecasts.

How to formalize the forecasting problem? Assume the current date is  $t$ . We want to forecast the value that variable  $y_t$  will take on date  $t + 1$  (i.e.,  $y_{t+1}$ ) based on the observation of a set of variables gathered in vector  $x_t$  ( $x_t$  may contain lagged values of  $y_t$ ).

The forecaster aims at minimizing (a function of) the forecast error. It is usual to consider the following (quadratic) loss function:

$$\underbrace{\mathbb{E}([y_{t+1} - y_{t+1}^*]^2)}_{\text{Mean square error (MSE)}}$$

where  $y_{t+1}^*$  is the forecast of  $y_{t+1}$  (function of  $x_t$ ).

**Proposition 9.15** (Smallest MSE). *The smallest MSE is obtained with MSE the expectation of  $y_{t+1}$  conditional on  $x_t$ .*

*Proof.* See Appendix 10.5. □

**Proposition 9.16.** *Among the class of linear forecasts, the smallest MSE is obtained with the linear projection of  $y_{t+1}$  on  $x_t$ . This projection, denoted by  $\hat{P}(y_{t+1}|x_t) := \alpha' x_t$ , satisfies:*

$$\mathbb{E}([y_{t+1} - \alpha' x_t] x_t) = \mathbf{0}. \quad (9.69)$$

*Proof.* Consider the function  $f : \alpha \rightarrow \mathbb{E}([y_{t+1} - \alpha' x_t]^2)$ . We have:

$$f(\alpha) = \mathbb{E}(y_{t+1}^2 - 2y_{t+1} \alpha' x_t + \alpha' x_t x_t' \alpha).$$

We have  $\partial f(\alpha)/\partial \alpha = \mathbb{E}(-2y_{t+1} x_t + 2x_t x_t' \alpha)$ . The function is minimised for  $\partial f(\alpha)/\partial \alpha = 0$ . □

---

<sup>5</sup>In its inflation report, the Bank of England displays charts showing the conditional distribution of future inflation, called fan charts. This fan charts show the uncertainty associated with future inflation. See this page.

Eq. (9.69) implies that  $\mathbb{E}(y_{t+1}x_t) = \mathbb{E}(x_tx'_t)\alpha$ . (Note that  $x_tx'_t\alpha = x_t(x'_t\alpha) = (\alpha'x_t)x'_t$ .) Hence, if  $\mathbb{E}(x_tx'_t)$  is nonsingular,

$$\alpha = [\mathbb{E}(x_tx'_t)]^{-1}\mathbb{E}(y_{t+1}x_t). \quad (9.70)$$

The MSE then is:

$$\mathbb{E}([y_{t+1} - \alpha'x_t]^2) = \mathbb{E}(y_{t+1}^2) - \mathbb{E}(y_{t+1}x_t)[\mathbb{E}(x_tx'_t)]^{-1}\mathbb{E}(x_ty_{t+1}).$$

Consider the regression  $y_{t+1} = \beta'x_t + \varepsilon_{t+1}$ . The OLS estimate is:

$$\mathbf{b} = \left[ \underbrace{\frac{1}{T} \sum_{i=1}^T \mathbf{x}_t \mathbf{x}'_t}_{\mathbf{m}_1} \right]^{-1} \left[ \underbrace{\frac{1}{T} \sum_{i=1}^T \mathbf{x}'_t y_{t+1}}_{\mathbf{m}_2} \right].$$

If  $\{x_t, y_t\}$  is covariance-stationary and ergodic for the second moments then the sample moments ( $\mathbf{m}_1$  and  $\mathbf{m}_2$ ) converges in probability to the associated population moments and  $\mathbf{b} \xrightarrow{p} \alpha$  (where  $\alpha$  is defined in Eq. (9.70)).

**Example 9.11** (Forecasting an MA(q) process). Consider the MA(q) process:

$$y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q},$$

where  $\{\varepsilon_t\}$  is a white noise sequence (Def. 9.1).

We have:

$$\begin{aligned} \mathbb{E}(y_{t+h}|\varepsilon_t, \varepsilon_{t-1}, \dots) &= \\ \begin{cases} \mu + \theta_h\varepsilon_t + \cdots + \theta_q\varepsilon_{t-q+h} & \text{for } h \in [1, q] \\ \mu & \text{for } h > q \end{cases} \end{aligned}$$

and

$$\begin{aligned} \text{Var}(y_{t+h}|\varepsilon_t, \varepsilon_{t-1}, \dots) &= \mathbb{E}([y_{t+h} - \mathbb{E}(y_{t+h}|\varepsilon_t, \varepsilon_{t-1}, \dots)]^2) = \\ \begin{cases} \sigma^2(1 + \theta_1^2 + \cdots + \theta_{h-1}^2) & \text{for } h \in [1, q] \\ \sigma^2(1 + \theta_1^2 + \cdots + \theta_q^2) & \text{for } h > q. \end{cases} \end{aligned}$$

Remark: The previous reasoning relies on the assumption that the  $\varepsilon_t$ s are observed. But this is generally not the case in practice. Note that consistent estimates are available if the MA process is invertible (see Eq. (9.32)).

**Example 9.12** (Forecasting an AR(p) process). (See this web interface.) Consider the AR(p) process:

$$y_t = c + \phi_1y_{t-1} + \phi_2y_{t-2} + \cdots + \phi_py_{t-p} + \varepsilon_t,$$

where  $\{\varepsilon_t\}$  is a white noise sequence (Def. 9.1).

Using the notation of Eq. (9.9), we have:

$$\mathbf{y}_t - \boldsymbol{\mu} = F(\mathbf{y}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\xi}_t,$$

with  $\boldsymbol{\mu} = [\mu, \dots, \mu]'$  ( $\boldsymbol{\mu}$  is defined in Eq. (9.14)). Hence:

$$\mathbf{y}_{t+h} - \boldsymbol{\mu} = \boldsymbol{\xi}_{t+h} + F\boldsymbol{\xi}_{t+h-1} + \cdots + F^{h-1}\boldsymbol{\xi}_{t+1} + F^h(\mathbf{y}_t - \boldsymbol{\mu}).$$

Therefore:

$$\begin{aligned} \mathbb{E}(\mathbf{y}_{t+h}|\mathbf{y}_t, \mathbf{y}_{t-1}, \dots) &= \boldsymbol{\mu} + F^h(\mathbf{y}_t - \boldsymbol{\mu}) \\ \text{Var}([\mathbf{y}_{t+h} - \mathbb{E}(\mathbf{y}_{t+h}|\mathbf{y}_t, \mathbf{y}_{t-1}, \dots)]) &= \boldsymbol{\Sigma} + F\Sigma F' + \cdots + F^{h-1}\boldsymbol{\Sigma}(F^{h-1})', \end{aligned}$$

where:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2 & 0 & \cdots \\ 0 & 0 & \ddots \\ \vdots & & \ddots \end{bmatrix}.$$

Alternative approach: Taking the (conditional) expectations of both sides of

$$y_{t+h} - \mu = \phi_1(y_{t+h-1} - \mu) + \phi_2(y_{t+h-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \varepsilon_{t+h},$$

we obtain:

$$\begin{aligned}\mathbb{E}(y_{t+h}|y_t, y_{t-1}, \dots) &= \mu + \phi_1(\mathbb{E}[y_{t+h-1}|y_t, y_{t-1}, \dots] - \mu) + \\ &\quad \phi_2(\mathbb{E}[y_{t+h-2}|y_t, y_{t-1}, \dots] - \mu) + \cdots + \\ &\quad \phi_p(\mathbb{E}[y_{t+h-p}|y_t, y_{t-1}, \dots] - \mu),\end{aligned}$$

which can be exploited recursively.

The recursion begins with  $\mathbb{E}(y_{t-k}|y_t, y_{t-1}, \dots) = y_{t-k}$  (for any  $k \geq 0$ ).

**Example 9.13** (Forecasting an ARMA(p,q) process). Consider the process:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \quad (9.71)$$

where  $\{\varepsilon_t\}$  is a white noise sequence (Def. 9.1). We assume that the MA part of the process is invertible (see Eq. (9.32)), which implies that the information contained in  $\{y_t, y_{t-1}, y_{t-2}, \dots\}$  is identical to that in  $\{\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \dots\}$ .

While one could use a recursive algorithm to compute the conditional mean (as in Example 9.12), it is convenient to employ the Wold decomposition of this process (see Theorem 9.3 and Prop. 9.8 for the computation of the  $\psi_i$ 's in the context of ARMA processes):

$$y_t = \mu + \sum_{i=0}^{+\infty} \psi_i \varepsilon_{t-i}.$$

This implies:

$$\begin{aligned}y_{t+h} &= \mu + \sum_{i=0}^{h-1} \psi_i \varepsilon_{t+h-i} + \sum_{i=h}^{+\infty} \psi_i \varepsilon_{t+h-i} \\ &= \mu + \sum_{i=0}^{h-1} \psi_i \varepsilon_{t+h-i} + \sum_{i=0}^{+\infty} \psi_{i+h} \varepsilon_{t-i}.\end{aligned}$$

Since  $\mathbb{E}(y_{t+h}|y_t, y_{t-1}, \dots) = \mu + \sum_{i=0}^{+\infty} \psi_{i+h} \varepsilon_{t-i}$ , we get:

$$\text{Var}(y_{t+h}|y_t, y_{t-1}, \dots) = \text{Var}\left(\sum_{i=0}^{h-1} \psi_i \varepsilon_{t+h-i}\right) = \sigma^2 \sum_{i=0}^{h-1} \psi_i^2.$$

How to use the previous formulas in practice?

One has first to select a specification and to estimate the model. Two methods to determine relevant specifications:

- a. Information criteria (see Definition 9.20).
- b. Box-Jenkins approach.

Box and Jenkins (1976) have proposed an approach that is now widely used.

1. Data transformation. The data should be transformed to “make them stationary”. To do so, one can e.g. take logarithms, take changes in the considered series, remove (deterministic) trends.
2. Select  $p$  and  $q$ . This can be based on the PACF approach (see Section 9.2.4), or on selection criteria (see Definition 9.20).
3. Estimate the model parameters. See Section 9.2.8.
4. Check that the estimated model is consistent with the data. See below.

### Assessing the performances of a forecasting model

Once one has fitted a model on a given dataset (of length  $T$ , say), one compute MSE (mean square errors) to evaluate the performance of the model. But this MSE is the **in-sample** one. It is easy to reduce in-sample MSE. Typically, if the model is estimated by OLS, adding covariates mechanically reduces the MSE (see Props. 5.4 and 5.5). That is, even if additional data are irrelevant, the  $R^2$  of the regression increases. Adding irrelevant variables increases the (in-sample)  $R^2$  but is bound to increase the **out-of-sample** MSE.

Therefore, it is important to analyse **out-of-sample** performances of the forecasting model:

- Estimate a model on a sample of reduced size  $(1, \dots, T^*$ , with  $T^* < T)$
- Use the remaining available periods  $(T^* + 1, \dots, T)$  to compute **out-of-sample** forecasting errors (and compute their MSE). In an out-of-sample exercise, it is important to make sure that the data used to produce a forecasts (as of date  $T^*$ ) where indeed available on date  $T^*$ .

### Diebold-Mariano test

How to compare different forecasting approaches? Diebold and Mariano (1995) have proposed a simple test to address this question.

Assume that you want to compare approaches A and B. You have historical data sets and you have implemented both approaches in the past, providing you with two sets of forecasting errors:  $\{e_t^A\}_{t=1,\dots,T}$  and  $\{e_t^B\}_{t=1,\dots,T}$ .

It may be the case that your forecasts serve a specific purpose and that, for instance, you dislike positive forecasting errors and you care less about negative errors. We assume you are able to formalise this by means of a **loss function**  $L(e)$ . For instance:

- If you dislike large positive errors, you may set  $L(e) = \exp(e)$ .
- If you are concerned about both positive and negative errors (indifferently), you may set  $L(e) = e^2$  (standard approach).

Let us define the sequence  $\{d_t\}_{t=1,\dots,T} \equiv \{L(e_t^A) - L(e_t^B)\}_{t=1,\dots,T}$  and assume that this sequence is covariance stationary. We consider the following null hypothesis:  $H_0 : \bar{d} = 0$ , where  $\bar{d}$  denotes the population mean of the  $d_t$ s. Under  $H_0$  and under the assumption of covariance-stationarity of  $d_t$ , we have (Theorem @ref{(hm:CLTcovstat)}):

$$\sqrt{T}\bar{d}_T \xrightarrow{d} \mathcal{N}\left(0, \sum_{j=-\infty}^{+\infty} \gamma_j\right),$$

where the  $\gamma_j$ s are the autocovariances of  $d_t$ .

Hence, assuming that  $\hat{\sigma}^2$  is a consistent estimate of  $\sum_{j=-\infty}^{+\infty} \gamma_j$  (for instance the one given by the Newey-West formula, see Def. ??), we have, under  $H_0$ :

$$DM_T := \sqrt{T} \frac{\bar{d}_T}{\sqrt{\hat{\sigma}^2}} \xrightarrow{d} \mathcal{N}(0, 1).$$

$DM_T$  is the test statistics. For a test of size  $\alpha$ , the critical region is:<sup>6</sup>

$$] -\infty, -\Phi^{-1}(1 - \alpha/2)] \cup [\Phi^{-1}(1 - \alpha/2), +\infty[,$$

where  $\Phi$  is the c.d.f. of the standard normal distribution.

**Example 9.14** (Forecasting Swiss GDP growth). We use a long historical time series of the Swiss GDP growth taken from the Jordà et al. (2017) dataset (see Figure 9.3, and Example 9.6).

We want to forecast this GDP growth. We envision two specifications : an AR(1) specification (the one advocated by the AIC criteria, see Example 9.6), and an ARMA(2,2) specification. We are interested in 2-year-ahead forecasts (i.e.,  $h = 2$  since the data are yearly).

---

<sup>6</sup>This ShinyApp application illustrates the notion of statistical test (illustrating the p-value and the critical region, in particular).

```

library(AEC)
library(forecast)

## Registered S3 method overwritten by 'quantmod':
##   method           from
##   as.zoo.data.frame zoo

data <- subset(JST, iso=="CHE")
T <- dim(data)[1]
y <- c(NaN, log(data$gdp[2:T]/data$gdp[1:(T-1)]))
first.date <- T-50
e1 <- NULL; e2 <- NULL; h<-2
for(T.star in first.date:(T-h)){
  estim.model.1 <- arima(y[1:T.star], order=c(1,0,0))
  estim.model.2 <- arima(y[1:T.star], order=c(2,0,2))
  e1 <- c(e1, y[T.star+h] - predict(estim.model.1, n.ahead=h)$pred[h])
  e2 <- c(e2, y[T.star+h] - predict(estim.model.2, n.ahead=h)$pred[h])
}
res.DM <- dm.test(e1, e2, h = h, alternative = "greater")
res.DM

##
## Diebold-Mariano Test
##
## data: e1e2
## DM = -0.82989, Forecast horizon = 2, Loss function power = 2, p-value =
## 0.7946
## alternative hypothesis: greater

```

With `alternative = "greater"` The alternative hypothesis is that method 2 is more accurate than method 1. Since we do not reject the null (the p-value being of 0.795), we are not led to use the more sophisticated model (ARMA(2,2)) and we keep the simple AR(1) model.

Assume now that we want to compare the AR(1) process to a VAR model (see Def. 9.21). We consider a bivariate VAR, where GDP growth is complemented with CPI-based inflation rate.

```

library(vars)
infl <- c(NaN, log(data$cpi[2:T]/data$cpi[1:(T-1)]))
y_var <- cbind(y, infl)
e3 <- NULL
for(T.star in first.date:(T-h)){
  estim.model.3 <- VAR(y_var[2:T.star,], p=1)
  e3 <- c(e3, y[T.star+h] - predict(estim.model.3, n.ahead=h)$fcst$y[h,1])
}
res.DM <- dm.test(e1, e2, h = h, alternative = "greater")
res.DM

##
## Diebold-Mariano Test
##
## data: e1e2
## DM = -0.82989, Forecast horizon = 2, Loss function power = 2, p-value =
## 0.7946
## alternative hypothesis: greater

```

Again, we do not find that the alternative model (here the VAR(1) model) is better than the AR(1) model to forecast GDP growth.



# Chapter 10

## Appendix

### 10.1 Principal component analysis (PCA)

**Principal component analysis (PCA)** is a classical and easy-to-use statistical method to reduce the dimension of large datasets containing variables that are linearly driven by a relatively small number of factors. This approach is widely used in data analysis and image compression.

Suppose that we have  $T$  observations of a  $n$ -dimensional random vector  $x$ , denoted by  $x_1, x_2, \dots, x_T$ . We suppose that each component of  $x$  is of mean zero.

Let denote with  $X$  the matrix given by  $[x_1 \ x_2 \ \dots \ x_T]'$ . Denote the  $j^{th}$  column of  $X$  by  $X_j$ .

We want to find the linear combination of the  $x_i$ 's ( $x.u$ ), with  $\|u\| = 1$ , with “maximum variance.” That is, we want to solve:

$$\begin{array}{ll} \arg \max_u & u' X' X u. \\ \text{s.t.} & |u| = 1 \end{array} \quad (10.1)$$

Since  $X'X$  is a positive definite matrix, it admits the following decomposition:

$$\begin{aligned} X'X &= PDP' \\ &= P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P', \end{aligned}$$

where  $P$  is an orthogonal matrix whose columns are the eigenvectors of  $X'X$ .

We can order the eigenvalues such that  $\lambda_1 \geq \dots \geq \lambda_n$ . (Since  $X'X$  is positive definite, all these eigenvalues are positive.)

Since  $P$  is orthogonal, we have  $u' X' X u = u' P D P' u = y' D y$  where  $\|y\| = 1$ . Therefore, we have  $y_i^2 \leq 1$  for any  $i \leq n$ .

As a consequence:

$$y' D y = \sum_{i=1}^n y_i^2 \lambda_i \leq \lambda_1 \sum_{i=1}^n y_i^2 = \lambda_1.$$

It is easily seen that the maximum is reached for  $y = [1, 0, \dots, 0]'$ . Therefore, the maximum of the optimization program (Eq. (10.1)) is obtained for  $u = P[1, 0, \dots, 0]'$ . That is,  $u$  is the eigenvector of  $X'X$  that is associated with its larger eigenvalue (first column of  $P$ ).

Let us denote with  $F$  the vector that is given by the matrix product  $XP$  (note that its last column is equal to  $Xu$ ). The columns of  $F$ , denoted by  $F_j$ , are called **factors**. We have:

$$F' F = P' X' X P = D.$$

Therefore, in particular, the  $F_j$ 's are orthogonal.

Since  $X = FP'$ , the  $X_j$ 's are linear combinations of the factors. Let us then denote with  $\hat{X}_{i,j}$  the part of  $X_i$  that is explained by factor  $F_j$ , we have:

$$\begin{aligned}\hat{X}_{i,j} &= p_{ij}F_j \\ X_i &= \sum_j \hat{X}_{i,j} = \sum_j p_{ij}F_j.\end{aligned}$$

Consider the share of variance that is explained –through the  $n$  variables ( $X_1, \dots, X_n$ )– by the first factor  $F_1$ :

$$\frac{\sum_i \hat{X}_{i,1} \hat{X}'_{i,1}}{\sum_i X_i X'_i} = \frac{\sum_i p_{i1} F_1 F'_1 p_{i1}}{\text{tr}(X'X)} = \frac{\sum_i p_{i1}^2 \lambda_1}{\text{tr}(X'X)} = \frac{\lambda_1}{\sum_i \lambda_i}.$$

Intuitively, if the first eigenvalue is large, it means that the first factor embed a large share of the fluctuations of the  $n$   $X_i$ 's.

Let us illustrate PCA on the term structure of yields. The term structure of yields (or yield curve) is known to be driven by only a small number of factors (e.g., Litterman and Scheinkman (1991)). One can typically employ PCA to recover such factors. The data used in the example below are taken from the Fred database (tickers: “DGS6MO”, “DGS1”, ...). The second plot shows the factor loadings, that indicate that the first factor is a level factor (loadings = black line), the second factor is a slope factor (loadings = blue line), the third factor is a curvature factor (loadings = red line).

To run a PCA, one simply has to apply function `prcomp` to a matrix of data:

```
library(AEC)
USyields <- USyields[complete.cases(USyields),]
yds <- USyields[c("Y1", "Y2", "Y3", "Y5", "Y7", "Y10", "Y20", "Y30")]
PCA.yds <- prcomp(yds, center=TRUE, scale. = TRUE)
```

Let us now visualize some results. The first plot of Figure 10.1 shows the share of total variance explained by the different principal components (PCs). The second plot shows the factor loadings. The two bottom plots show how yields (in black) are fitted by linear combinations of the first two PCs only.

```
par(mfrow=c(2,2))
par(plt=c(.1,.95,.2,.8))
barplot(PCA.yds$sdev^2/sum(PCA.yds$sdev^2),
        main="Share of variance expl. by PC's")
axis(1, at=1:dim(yds)[2], labels=colnames(PCA.yds$x))
nb.PC <- 2
plot(-PCA.yds$rotation[,1], type="l", lwd=2, ylim=c(-1,1),
      main="Factor loadings (1st 3 PCs)", xaxt="n", xlab="")
axis(1, at=1:dim(yds)[2], labels=colnames(yds))
lines(PCA.yds$rotation[,2], type="l", lwd=2, col="blue")
lines(PCA.yds$rotation[,3], type="l", lwd=2, col="red")
Y1.hat <- PCA.yds$x[,1:nb.PC] %*% PCA.yds$rotation["Y1",1:2]
Y1.hat <- mean(USyields$Y1) + sd(USyields$Y1) * Y1.hat
plot(USyields$date, USyields$Y1, type="l", lwd=2,
      main="Fit of 1-year yields (2 PCs)",
      ylab="Obs (black) / Fitted by 2PCs (dashed blue)")
lines(USyields$date, Y1.hat, col="blue", lty=2, lwd=2)
Y10.hat <- PCA.yds$x[,1:nb.PC] %*% PCA.yds$rotation["Y10",1:2]
Y10.hat <- mean(USyields$Y10) + sd(USyields$Y10) * Y10.hat
plot(USyields$date, USyields$Y10, type="l", lwd=2,
      main="Fit of 10-year yields (2 PCs)",
      ylab="Obs (black) / Fitted by 2PCs (dashed blue)")
lines(USyields$date, Y10.hat, col="blue", lty=2, lwd=2)
```

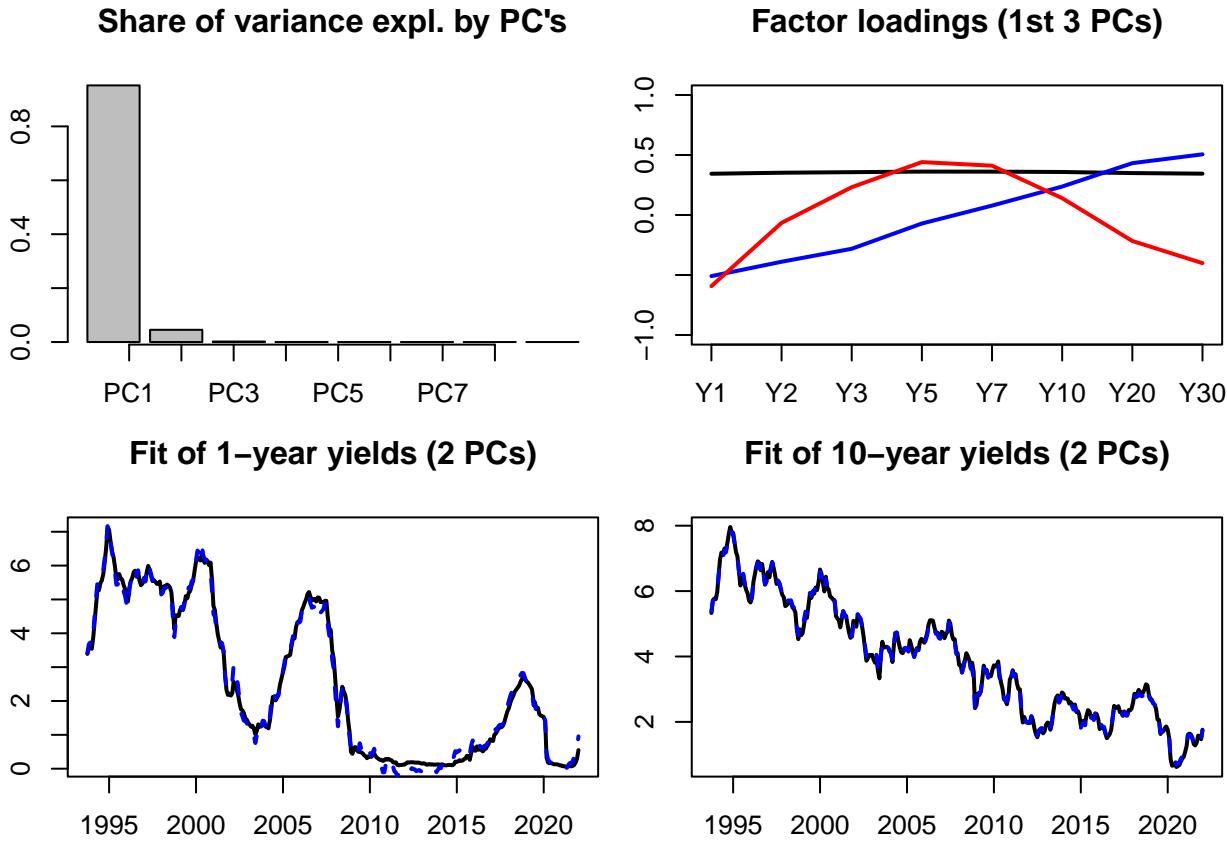


Figure 10.1: Some PCA results. The dataset contains 8 time series of U.S. interest rates of different maturities.

## 10.2 Linear algebra: definitions and results

**Definition 10.1** (Eigenvalues). The eigenvalues of a matrix  $M$  are the numbers  $\lambda$  for which:

$$|M - \lambda I| = 0,$$

where  $|\bullet|$  is the determinant operator.

**Proposition 10.1** (Properties of the determinant). *We have:*

- $|MN| = |M| \times |N|$ .
- $|M^{-1}| = |M|^{-1}$ .
- If  $M$  admits the diagonal representation  $M = TDT^{-1}$ , where  $D$  is a diagonal matrix whose diagonal entries are  $\{\lambda_i\}_{i=1,\dots,n}$ , then:

$$|M - \lambda I| = \prod_{i=1}^n (\lambda_i - \lambda).$$

**Definition 10.2** (Moore-Penrose inverse). If  $M \in \mathbb{R}^{m \times n}$ , then its Moore-Penrose pseudo inverse (exists and) is the unique matrix  $M^* \in \mathbb{R}^{n \times m}$  that satisfies:

- i.  $MM^*M = M$
- ii.  $M^*MM^* = M^*$
- iii.  $(MM^*)' = MM^*$
- .iv  $(M^*M)' = M^*M$ .

**Proposition 10.2** (Properties of the Moore-Penrose inverse). • If  $M$  is invertible then

$$M^* = M^{-1}.$$

- The pseudo-inverse of a zero matrix is its transpose. \*

\*

The pseudo-inverse of the pseudo-inverse is the original matrix.

**Definition 10.3** (Idempotent matrix). Matrix  $M$  is idempotent if  $M^2 = M$ .

If  $M$  is a symmetric idempotent matrix, then  $M'M = M$ .

**Proposition 10.3** (Roots of an idempotent matrix). The eigenvalues of an idempotent matrix are either 1 or 0.

*Proof.* If  $\lambda$  is an eigenvalue of an idempotent matrix  $M$  then  $\exists x \neq 0$  s.t.  $Mx = \lambda x$ . Hence  $M^2x = \lambda Mx \Rightarrow (1 - \lambda)Mx = 0$ . Either all element of  $Mx$  are zero, in which case  $\lambda = 0$  or at least one element of  $Mx$  is nonzero, in which case  $\lambda = 1$ .  $\square$

**Proposition 10.4** (Idempotent matrix and chi-square distribution). *The rank of a symmetric idempotent matrix is equal to its trace.*

*Proof.* The result follows from Prop. 10.3, combined with the fact that the rank of a symmetric matrix is equal to the number of its nonzero eigenvalues.  $\square$

**Proposition 10.5** (Constrained least squares). *The solution of the following optimisation problem:*

$$\begin{aligned} \min_{\beta} \quad & \|y - X\beta\|^2 \\ \text{subject to } & R\beta = q \end{aligned}$$

is given by:

$$\beta^r = \beta_0 - (X'X)^{-1}R'\{R(X'X)^{-1}R'\}^{-1}(R\beta_0 - q),$$

where  $\beta_0 = (X'X)^{-1}X'y$ .

*Proof.* See for instance Jackman, 2007.  $\square$

**Proposition 10.6** (Inverse of a partitioned matrix). *We have:*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}.$$

**Definition 10.4** (Matrix derivatives). Consider a fonction  $f : \mathbb{R}^K \rightarrow \mathbb{R}$ . Its first-order derivative is:

$$\frac{\partial f}{\partial \mathbf{b}}(\mathbf{b}) = \begin{bmatrix} \frac{\partial f}{\partial b_1}(\mathbf{b}) \\ \vdots \\ \frac{\partial f}{\partial b_K}(\mathbf{b}) \end{bmatrix}.$$

We use the notation:

$$\frac{\partial f}{\partial \mathbf{b}'}(\mathbf{b}) = \left( \frac{\partial f}{\partial \mathbf{b}}(\mathbf{b}) \right)'.$$

**Proposition 10.7.** *We have:*

- If  $f(\mathbf{b}) = A'\mathbf{b}$  where  $A$  is a  $K \times 1$  vector then  $\frac{\partial f}{\partial \mathbf{b}}(\mathbf{b}) = A$ .
- If  $f(\mathbf{b}) = \mathbf{b}'A\mathbf{b}$  where  $A$  is a  $K \times K$  matrix, then  $\frac{\partial f}{\partial \mathbf{b}}(\mathbf{b}) = 2A\mathbf{b}$ .

**Proposition 10.8** (Square and absolute summability). *We have:*

$$\underbrace{\sum_{i=0}^{\infty} |\theta_i| < +\infty}_{\text{Absolute summability}} \Rightarrow \underbrace{\sum_{i=0}^{\infty} \theta_i^2 < +\infty}_{\text{Square summability}}.$$

*Proof.* See Appendix 3.A in Hamilton. Idea: Absolute summability implies that there exist  $N$  such that, for  $j > N$ ,  $|\theta_j| < 1$  (deduced from Cauchy criterion, Theorem 10.2 and therefore  $\theta_j^2 < |\theta_j|$ ).  $\square$

## 10.3 Statistical analysis: definitions and results

### 10.3.1 Moments and statistics

**Definition 10.5** (Partial correlation). The **partial correlation** between  $y$  and  $z$ , controlling for some variables  $\mathbf{X}$  is the sample correlation between  $y^*$  and  $z^*$ , where the latter two variables are the residuals in regressions of  $y$  on  $\mathbf{X}$  and of  $z$  on  $\mathbf{X}$ , respectively.

This correlation is denoted by  $r_{yz}^{\mathbf{X}}$ . By definition, we have:

$$r_{yz}^{\mathbf{X}} = \frac{\mathbf{z}^{*\prime} \mathbf{y}^*}{\sqrt{(\mathbf{z}^{*\prime} \mathbf{z}^*)(\mathbf{y}^{*\prime} \mathbf{y}^*)}}. \quad (10.2)$$

**Definition 10.6** (Skewness and kurtosis). Let  $Y$  be a random variable whose fourth moment exists. The expectation of  $Y$  is denoted by  $\mu$ .

- The skewness of  $Y$  is given by:

$$\frac{\mathbb{E}[(Y - \mu)^3]}{\{\mathbb{E}[(Y - \mu)^2]\}^{3/2}}.$$

- The kurtosis of  $Y$  is given by:

$$\frac{\mathbb{E}[(Y - \mu)^4]}{\{\mathbb{E}[(Y - \mu)^2]\}^2}.$$

**Theorem 10.1** (Cauchy-Schwarz inequality). *We have:*

$$|\mathbb{C}\text{ov}(X, Y)| \leq \sqrt{\mathbb{V}\text{ar}(X)\mathbb{V}\text{ar}(Y)}$$

and, if  $X \neq 0$  and  $Y \neq 0$ , the equality holds iff  $X$  and  $Y$  are the same up to an affine transformation.

*Proof.* If  $\mathbb{V}\text{ar}(X) = 0$ , this is trivial. If this is not the case, then let's define  $Z$  as  $Z = Y - \frac{\mathbb{C}\text{ov}(X, Y)}{\mathbb{V}\text{ar}(X)}X$ . It is easily seen that  $\mathbb{C}\text{ov}(X, Z) = 0$ . Then, the variance of  $Y = Z + \frac{\mathbb{C}\text{ov}(X, Y)}{\mathbb{V}\text{ar}(X)}X$  is equal to the sum of the variance of  $Z$  and of the variance of  $\frac{\mathbb{C}\text{ov}(X, Y)}{\mathbb{V}\text{ar}(X)}X$ , that is:

$$\mathbb{V}\text{ar}(Y) = \mathbb{V}\text{ar}(Z) + \left( \frac{\mathbb{C}\text{ov}(X, Y)}{\mathbb{V}\text{ar}(X)} \right)^2 \mathbb{V}\text{ar}(X) \geq \left( \frac{\mathbb{C}\text{ov}(X, Y)}{\mathbb{V}\text{ar}(X)} \right)^2 \mathbb{V}\text{ar}(X).$$

The equality holds iff  $\mathbb{V}\text{ar}(Z) = 0$ , i.e. iff  $Y = \frac{\mathbb{C}\text{ov}(X, Y)}{\mathbb{V}\text{ar}(X)}X + cst$ .  $\square$

**Definition 10.7** (Asymptotic level). An asymptotic test with critical region  $\Omega_n$  has an asymptotic level equal to  $\alpha$  if:

$$\sup_{\theta \in \Theta} \lim_{n \rightarrow \infty} \mathbb{P}_{\theta}(S_n \in \Omega_n) = \alpha,$$

where  $S_n$  is the test statistic and  $\Theta$  is such that the null hypothesis  $H_0$  is equivalent to  $\theta \in \Theta$ .

**Definition 10.8** (Asymptotically consistent test). An asymptotic test with critical region  $\Omega_n$  is consistent if:

$$\forall \theta \in \Theta^c, \quad \mathbb{P}_{\theta}(S_n \in \Omega_n) \rightarrow 1,$$

where  $S_n$  is the test statistic and  $\Theta^c$  is such that the null hypothesis  $H_0$  is equivalent to  $\theta \notin \Theta^c$ .

**Definition 10.9** (Kullback discrepancy). Given two p.d.f.  $f$  and  $f^*$ , the Kullback discrepancy is defined by:

$$I(f, f^*) = \mathbb{E}^* \left( \log \frac{f^*(y)}{f(y)} \right) = \int \log \frac{f^*(y)}{f(y)} f^*(y) dy.$$

**Proposition 10.9** (Properties of the Kullback discrepancy). *We have:*

- i.  $I(f, f^*) \geq 0$
- ii.  $I(f, f^*) = 0$  iff  $f \equiv f^*$ .

*Proof.*  $x \rightarrow -\log(x)$  is a convex function. Therefore  $\mathbb{E}^*(-\log f(Y)/f^*(Y)) \geq -\log \mathbb{E}^*(f(Y)/f^*(Y)) = 0$  (proves (i)). Since  $x \rightarrow -\log(x)$  is strictly convex, equality in (i) holds if and only if  $f(Y)/f^*(Y)$  is constant (proves (ii)).  $\square$

**Definition 10.10** (Characteristic function). For any real-valued random variable  $X$ , the characteristic function is defined by:

$$\phi_X : u \rightarrow \mathbb{E}[\exp(iuX)].$$

### 10.3.2 Standard distributions

**Definition 10.11** (F distribution). Consider  $n = n_1 + n_2$  i.i.d.  $\mathcal{N}(0, 1)$  r.v.  $X_i$ . If the r.v.  $F$  is defined by:

$$F = \frac{\sum_{i=1}^{n_1} X_i^2}{\sum_{j=n_1+1}^{n_1+n_2} X_j^2} \frac{n_2}{n_1}$$

then  $F \sim \mathcal{F}(n_1, n_2)$ . (See Table 10.4 for quantiles.)

**Definition 10.12** (Student-t distribution).  $Z$  follows a Student-t (or  $t$ ) distribution with  $\nu$  degrees of freedom (d.f.) if:

$$Z = X_0 \sqrt{\frac{\sum_{i=1}^{\nu} X_i^2}{\nu}}, \quad X_i \sim i.i.d. \mathcal{N}(0, 1).$$

We have  $\mathbb{E}(Z) = 0$ , and  $\text{Var}(Z) = \frac{\nu}{\nu-2}$  if  $\nu > 2$ . (See Table 10.2 for quantiles.)

**Definition 10.13** (Chi-square distribution).  $Z$  follows a  $\chi^2$  distribution with  $\nu$  d.f. if  $Z = \sum_{i=1}^{\nu} X_i^2$  where  $X_i \sim i.i.d. \mathcal{N}(0, 1)$ . We have  $\mathbb{E}(Z) = \nu$ . (See Table 10.3 for quantiles.)

**Definition 10.14** (Cauchy distribution). The probability distribution function of the Cauchy distribution defined by a location parameter  $\mu$  and a scale parameter  $\gamma$  is:

$$f(x) = \frac{1}{\pi\gamma \left(1 + \left[\frac{x-\mu}{\gamma}\right]^2\right)}.$$

The mean and variance of this distribution are undefined.

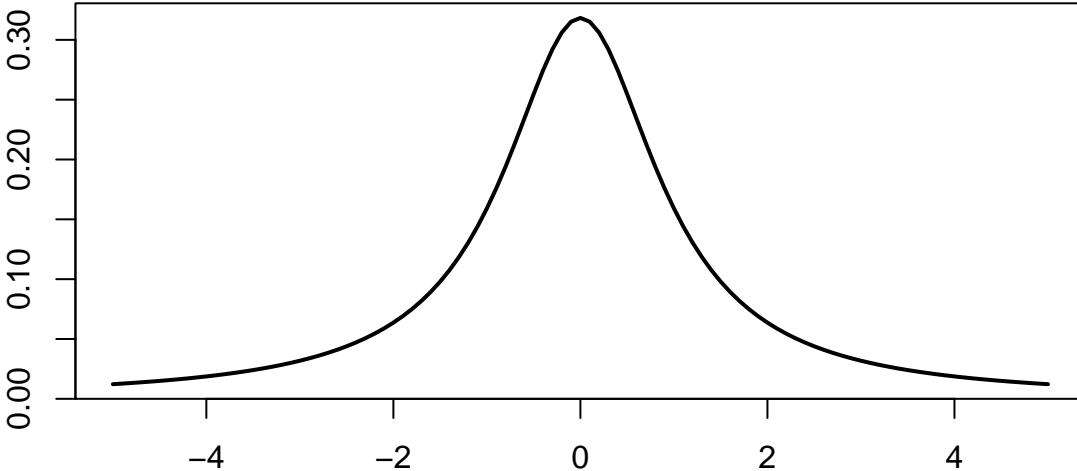


Figure 10.2: Pdf of the Cauchy distribution ( $\mu = 0, \gamma = 1$ ).

**Proposition 10.10** (Inner product of a multivariate Gaussian variable). Let  $X$  be a  $n$ -dimensional multivariate Gaussian variable:  $X \sim \mathcal{N}(0, \Sigma)$ . We have:

$$X' \Sigma^{-1} X \sim \chi^2(n).$$

*Proof.* Because  $\Sigma$  is a symmetrical definite positive matrix, it admits the spectral decomposition  $PDP'$  where  $P$  is an orthogonal matrix (i.e.  $PP' = Id$ ) and  $D$  is a diagonal matrix with non-negative entries. Denoting by  $\sqrt{D^{-1}}$  the diagonal matrix whose diagonal entries are the inverse of those of  $D$ , it is easily checked that the covariance matrix of  $Y := \sqrt{D^{-1}}P'X$  is  $Id$ . Therefore  $Y$  is a vector of uncorrelated Gaussian variables. The properties of Gaussian variables imply that the components of  $Y$  are then also independent. Hence  $Y'Y = \sum_i Y_i^2 \sim \chi^2(n)$ .

It remains to note that  $Y'Y = X'PD^{-1}P'X = X'\text{Var}(X)^{-1}X$  to conclude.  $\square$

**Definition 10.15** (Generalized Extreme Value (GEV) distribution). The vector of disturbances  $\varepsilon = [\varepsilon_{1,1}, \dots, \varepsilon_{1,K_1}, \dots, \varepsilon_{J,1}, \dots, \varepsilon_{J,K_J}]'$  follows the Generalized Extreme Value (GEV) distribution if its c.d.f. is:

$$F(\varepsilon, \rho) = \exp(-G(e^{-\varepsilon_{1,1}}, \dots, e^{-\varepsilon_{J,K_J}}; \rho))$$

with

$$\begin{aligned} G(\mathbf{Y}; \rho) &\equiv G(Y_{1,1}, \dots, Y_{1,K_1}, \dots, Y_{J,1}, \dots, Y_{J,K_J}; \rho) \\ &= \sum_{j=1}^J \left( \sum_{k=1}^{K_j} Y_{jk}^{1/\rho_j} \right)^{\rho_j} \end{aligned}$$

### 10.3.3 Stochastic convergences

**Proposition 10.11** (Chebychev's inequality). *If  $\mathbb{E}(|X|^r)$  is finite for some  $r > 0$  then:*

$$\forall \varepsilon > 0, \quad \mathbb{P}(|X - c| > \varepsilon) \leq \frac{\mathbb{E}[|X - c|^r]}{\varepsilon^r}.$$

In particular, for  $r = 2$ :

$$\forall \varepsilon > 0, \quad \mathbb{P}(|X - c| > \varepsilon) \leq \frac{\mathbb{E}[(X - c)^2]}{\varepsilon^2}.$$

*Proof.* Remark that  $\varepsilon^r \mathbb{I}_{\{|X| \geq \varepsilon\}} \leq |X|^r$  and take the expectation of both sides.  $\square$

**Definition 10.16** (Convergence in probability). The random variable sequence  $x_n$  converges in probability to a constant  $c$  if  $\forall \varepsilon, \lim_{n \rightarrow \infty} \mathbb{P}(|x_n - c| > \varepsilon) = 0$ .

It is denoted as:  $\text{plim } x_n = c$ .

**Definition 10.17** (Convergence in the L<sup>r</sup> norm).  $x_n$  converges in the  $r$ -th mean (or in the  $L^r$ -norm) towards  $x$ , if  $\mathbb{E}(|x_n|^r)$  and  $\mathbb{E}(|x|^r)$  exist and if

$$\lim_{n \rightarrow \infty} \mathbb{E}(|x_n - x|^r) = 0.$$

It is denoted as:  $x_n \xrightarrow{L^r} c$ .

For  $r = 2$ , this convergence is called **mean square convergence**.

**Definition 10.18** (Almost sure convergence). The random variable sequence  $x_n$  converges almost surely to  $c$  if  $\mathbb{P}(\lim_{n \rightarrow \infty} x_n = c) = 1$ .

It is denoted as:  $x_n \xrightarrow{a.s.} c$ .

**Definition 10.19** (Convergence in distribution).  $x_n$  is said to converge in distribution (or in law) to  $x$  if

$$\lim_{n \rightarrow \infty} F_{x_n}(s) = F_x(s)$$

for all  $s$  at which  $F_X$  –the cumulative distribution of  $X$ – is continuous.

It is denoted as:  $x_n \xrightarrow{d} x$ .

**Proposition 10.12** (Rules for limiting distributions (Slutsky)). *We have:*

i. **Slutsky's theorem:** If  $x_n \xrightarrow{d} x$  and  $y_n \xrightarrow{p} c$  then

$$\begin{aligned} x_n y_n &\xrightarrow{d} xc \\ x_n + y_n &\xrightarrow{d} x + c \\ x_n / y_n &\xrightarrow{d} x/c \quad (\text{if } c \neq 0) \end{aligned}$$

ii. **Continuous mapping theorem:** If  $x_n \xrightarrow{d} x$  and  $g$  is a continuous function then  $g(x_n) \xrightarrow{d} g(x)$ .

**Proposition 10.13** (Implications of stochastic convergences). *We have:*

$$\begin{array}{ccccc} \boxed{\xrightarrow{L^s}} & \xrightarrow[1 \leq r \leq s]{\Rightarrow} & \boxed{\xrightarrow{L^r}} & & \\ & & \Downarrow & & \\ \boxed{\xrightarrow{\text{a.s.}}} & \Rightarrow & \boxed{\xrightarrow{p}} & \Rightarrow & \boxed{\xrightarrow{d}}. \end{array}$$

*Proof.* (of the fact that  $\left(\xrightarrow{p}\right) \Rightarrow \left(\xrightarrow{d}\right)$ ). Assume that  $X_n \xrightarrow{p} X$ . Denoting by  $F$  and  $F_n$  the c.d.f. of  $X$  and  $X_n$ , respectively:

$$F_n(x) = \mathbb{P}(X_n \leq x, X \leq x+\varepsilon) + \mathbb{P}(X_n \leq x, X > x+\varepsilon) \leq F(x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon). \quad (10.3)$$

Besides,

$$F(x-\varepsilon) = \mathbb{P}(X \leq x-\varepsilon, X_n \leq x) + \mathbb{P}(X \leq x-\varepsilon, X_n > x) \leq F_n(x) + \mathbb{P}(|X_n - X| > \varepsilon),$$

which implies:

$$F(x-\varepsilon) - \mathbb{P}(|X_n - X| > \varepsilon) \leq F_n(x). \quad (10.4)$$

Eqs. (10.3) and (10.4) imply:

$$F(x-\varepsilon) - \mathbb{P}(|X_n - X| > \varepsilon) \leq F_n(x) \leq F(x+\varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon).$$

Taking limits as  $n \rightarrow \infty$  yields

$$F(x-\varepsilon) \leq \liminf_{n \rightarrow \infty} F_n(x) \leq \limsup_{n \rightarrow \infty} F_n(x) \leq F(x+\varepsilon).$$

The result is then obtained by taking limits as  $\varepsilon \rightarrow 0$  (if  $F$  is continuous at  $x$ ).  $\square$

**Proposition 10.14** (Convergence in distribution to a constant). *If  $X_n$  converges in distribution to a constant  $c$ , then  $X_n$  converges in probability to  $c$ .*

*Proof.* If  $\varepsilon > 0$ , we have  $\mathbb{P}(X_n < c - \varepsilon) \xrightarrow{n \rightarrow \infty} 0$  i.e.  $\mathbb{P}(X_n \geq c - \varepsilon) \xrightarrow{n \rightarrow \infty} 1$  and  $\mathbb{P}(X_n < c + \varepsilon) \xrightarrow{n \rightarrow \infty} 1$ . Therefore  $\mathbb{P}(c - \varepsilon \leq X_n < c + \varepsilon) \xrightarrow{n \rightarrow \infty} 1$ , which gives the result.  $\square$

**Example 10.1** (Convergence in probability but not  $L^r$ ). Let  $\{x_n\}_{n \in \mathbb{N}}$  be a series of random variables defined by:

$$x_n = n u_n,$$

where  $u_n$  are independent random variables s.t.  $u_n \sim \mathcal{B}(1/n)$ .

We have  $x_n \xrightarrow{p} 0$  but  $x_n \not\xrightarrow{L^r} 0$  because  $\mathbb{E}(|X_n - 0|) = \mathbb{E}(X_n) = 1$ .

**Theorem 10.2** (Cauchy criterion (non-stochastic case)). *We have that  $\sum_{i=0}^T a_i$  converges ( $T \rightarrow \infty$ ) iff, for any  $\eta > 0$ , there exists an integer  $N$  such that, for all  $M \geq N$ ,*

$$\left| \sum_{i=N+1}^M a_i \right| < \eta.$$

**Theorem 10.3** (Cauchy criterion (stochastic case)). *We have that  $\sum_{i=0}^T \theta_i \varepsilon_{t-i}$  converges in mean square ( $T \rightarrow \infty$ ) to a random variable iff, for any  $\eta > 0$ , there exists an integer  $N$  such that, for all  $M \geq N$ ,*

$$\mathbb{E} \left[ \left( \sum_{i=N+1}^M \theta_i \varepsilon_{t-i} \right)^2 \right] < \eta.$$

### 10.3.4 Central limit theorem

**Theorem 10.4** (Law of large numbers). *The sample mean is a consistent estimator of the population mean.*

*Proof.* Let's denote by  $\phi_{X_i}$  the characteristic function of a r.v.  $X_i$ . If the mean of  $X_i$  is  $\mu$  then the Taylor expansion of the characteristic function is:

$$\phi_{X_i}(u) = \mathbb{E}(\exp(iuX)) = 1 + iu\mu + o(u).$$

The properties of the characteristic function (see Def. 10.10) imply that:

$$\phi_{\frac{1}{n}(X_1+\dots+X_n)}(u) = \prod_{i=1}^n \left(1 + i\frac{u}{n}\mu + o\left(\frac{u}{n}\right)\right) \rightarrow e^{iu\mu}.$$

The facts that (a)  $e^{iu\mu}$  is the characteristic function of the constant  $\mu$  and (b) that a characteristic function uniquely characterises a distribution imply that the sample mean converges in distribution to the constant  $\mu$ , which further implies that it converges in probability to  $\mu$ .  $\square$

**Theorem 10.5** (Lindberg-Levy Central limit theorem, CLT). *If  $x_n$  is an i.i.d. sequence of random variables with mean  $\mu$  and variance  $\sigma^2$  ( $\in ]0, +\infty[$ ), then:*

$$\sqrt{n}(\bar{x}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \quad \text{where } \bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i.$$

*Proof.* Let us introduce the r.v.  $Y_n := \sqrt{n}(\bar{X}_n - \mu)$ . We have  $\phi_{Y_n}(u) = [\mathbb{E}(\exp(i\frac{1}{\sqrt{n}}u(X_1 - \mu)))]^n$ . We have:

$$\begin{aligned} \left[\mathbb{E}\left(\exp\left(i\frac{1}{\sqrt{n}}u(X_1 - \mu)\right)\right)\right]^n &= \left[\mathbb{E}\left(1 + i\frac{1}{\sqrt{n}}u(X_1 - \mu) - \frac{1}{2n}u^2(X_1 - \mu)^2 + o(u^2)\right)\right]^n \\ &= \left(1 - \frac{1}{2n}u^2\sigma^2 + o(u^2)\right)^n. \end{aligned}$$

Therefore  $\phi_{Y_n}(u) \xrightarrow[n \rightarrow \infty]{} \exp(-\frac{1}{2}u^2\sigma^2)$ , which is the characteristic function of  $\mathcal{N}(0, \sigma^2)$ .  $\square$

## 10.4 Some properties of Gaussian variables

**Proposition 10.15.** *If  $\mathbf{A}$  is idempotent and if  $\mathbf{x}$  is Gaussian,  $\mathbf{Lx}$  and  $\mathbf{x}'\mathbf{Ax}$  are independent if  $\mathbf{LA} = \mathbf{0}$ .*

*Proof.* If  $\mathbf{LA} = \mathbf{0}$ , then the two Gaussian vectors  $\mathbf{Lx}$  and  $\mathbf{Ax}$  are independent. This implies the independence of any function of  $\mathbf{Lx}$  and any function of  $\mathbf{Ax}$ . The results then follows from the observation that  $\mathbf{x}'\mathbf{Ax} = (\mathbf{Ax})'(\mathbf{Ax})$ , which is a function of  $\mathbf{Ax}$ .  $\square$

**Proposition 10.16** (Bayesian update in a vector of Gaussian variables). *If*

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}\right),$$

then

$$Y_2|Y_1 \sim \mathcal{N}(\Omega_{21}\Omega_{11}^{-1}Y_1, \Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12}).$$

$$Y_1|Y_2 \sim \mathcal{N}(\Omega_{12}\Omega_{22}^{-1}Y_2, \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}).$$

**Proposition 10.17** (Truncated distributions). *If  $X$  is a random variable distributed according to some p.d.f.  $f$ , with c.d.f.  $F$ , with infinite support. Then the p.d.f. of  $X|a \leq X < b$  is*

$$g(x) = \frac{f(x)}{F(b) - F(a)} \mathbb{I}_{\{a \leq x < b\}},$$

for any  $a < b$ .

In particular, for a Gaussian variable  $X \sim \mathcal{N}(\mu, \sigma^2)$ , we have

$$f(X = x | a \leq X < b) = \frac{\frac{1}{\sigma} \phi\left(\frac{x - \mu}{\sigma}\right)}{Z}.$$

with  $Z = \Phi(\beta) - \Phi(\alpha)$ , where  $\alpha = \frac{a - \mu}{\sigma}$  and  $\beta = \frac{b - \mu}{\sigma}$ .

Moreover:

$$\mathbb{E}(X | a \leq X < b) = \mu - \frac{\phi(\beta) - \phi(\alpha)}{Z} \sigma. \quad (10.5)$$

We also have:

$$\begin{aligned} & \text{Var}(X | a \leq X < b) \\ &= \sigma^2 \left[ 1 - \frac{\beta \phi(\beta) - \alpha \phi(\alpha)}{Z} - \left( \frac{\phi(\beta) - \phi(\alpha)}{Z} \right)^2 \right] \end{aligned} \quad (10.6)$$

In particular, for  $b \rightarrow \infty$ , we get:

$$\text{Var}(X | a < X) = \sigma^2 [1 + \alpha \lambda(-\alpha) - \lambda(-\alpha)^2], \quad (10.7)$$

with  $\lambda(x) = \frac{\phi(x)}{\Phi(x)}$  is called the **inverse Mills ratio**.

Consider the case where  $a \rightarrow -\infty$  (i.e. the conditioning set is  $X < b$ ) and  $\mu = 0, \sigma = 1$ . Then Eq. (10.5) gives  $\mathbb{E}(X | X < b) = -\lambda(b) = -\frac{\phi(b)}{\Phi(b)}$ , where  $\lambda$  is the function computing the inverse Mills ratio.

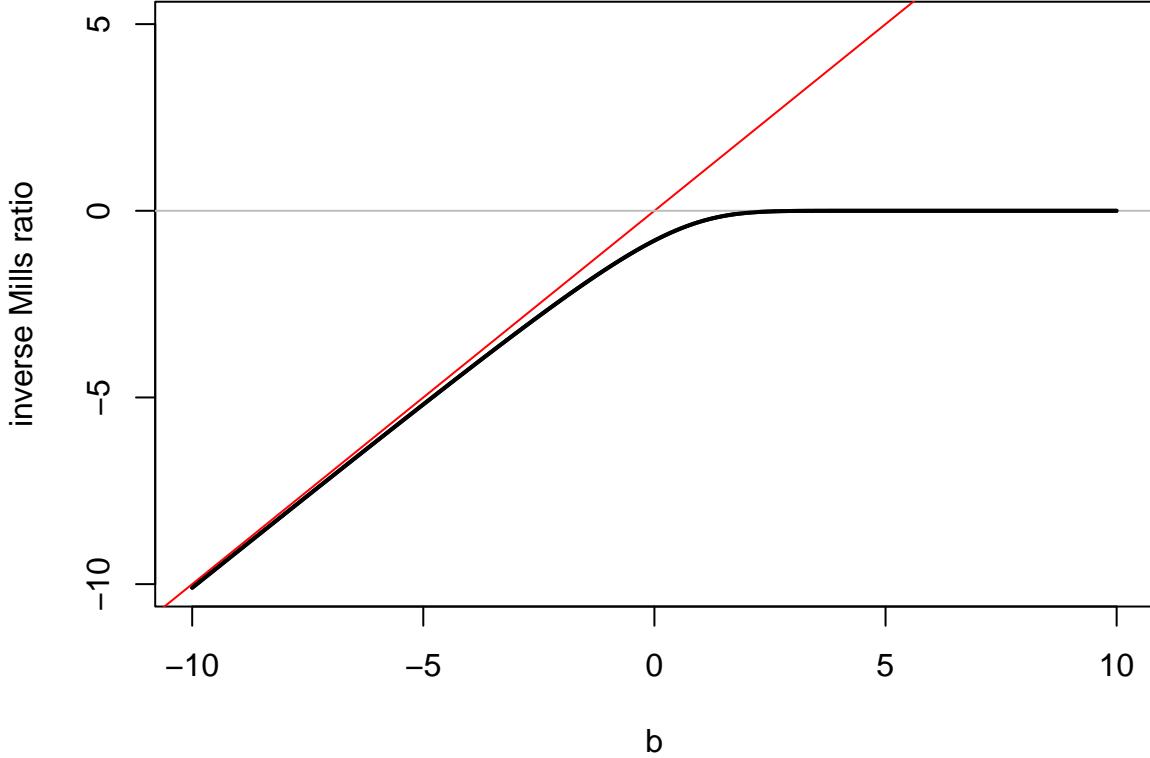


Figure 10.3:  $\mathbb{E}(X | X < b)$  as a function of  $b$  when  $X \sim \mathcal{N}(0, 1)$  (in black).

**Proposition 10.18** (p.d.f. of a multivariate Gaussian variable). *If  $Y \sim \mathcal{N}(\mu, \Omega)$  and if  $Y$  is a  $n$ -dimensional vector, then the density function of  $Y$  is:*

$$\frac{1}{(2\pi)^{n/2} |\Omega|^{1/2}} \exp \left[ -\frac{1}{2} (Y - \mu)' \Omega^{-1} (Y - \mu) \right].$$

## 10.5 Proofs

### Proof of Proposition 7.4

*Proof.* Assumptions (i) and (ii) (in the set of Assumptions 7.1) imply that  $\theta_{MLE}$  exists ( $= \operatorname{argmax}_{\theta} (1/n) \log \mathcal{L}(\theta; \mathbf{y})$ ).

$(1/n) \log \mathcal{L}(\theta; \mathbf{y})$  can be interpreted as the sample mean of the r.v.  $\log f(Y_i; \theta)$  that are i.i.d. Therefore  $(1/n) \log \mathcal{L}(\theta; \mathbf{y})$  converges to  $\mathbb{E}_{\theta_0}(\log f(Y; \theta))$  – which exists (Assumption iv).

Because the latter convergence is uniform (Assumption v), the solution  $\theta_{MLE}$  almost surely converges to the solution to the limit problem:

$$\operatorname{argmax}_{\theta} \mathbb{E}_{\theta_0}(\log f(Y; \theta)) = \operatorname{argmax}_{\theta} \int_{\mathcal{Y}} \log f(y; \theta) f(y; \theta_0) dy.$$

Properties of the Kullback information measure (see Prop. 10.9), together with the identifiability assumption (ii) implies that the solution to the limit problem is unique and equal to  $\theta_0$ .

Consider a r.v. sequence  $\theta$  that converges to  $\theta_0$ . The Taylor expansion of the score in a neighborhood of  $\theta_0$  yields to:

$$\frac{\partial \log \mathcal{L}(\theta; \mathbf{y})}{\partial \theta} = \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} + \frac{\partial^2 \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta \partial \theta'} (\theta - \theta_0) + o_p(\theta - \theta_0)$$

$\theta_{MLE}$  converges to  $\theta_0$  and satisfies the likelihood equation  $\frac{\partial \log \mathcal{L}(\theta; \mathbf{y})}{\partial \theta} = \mathbf{0}$ . Therefore:

$$\frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} \approx -\frac{\partial^2 \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta \partial \theta'} (\theta_{MLE} - \theta_0),$$

or equivalently:

$$\frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} \approx \left( -\frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \log f(y_i; \theta_0)}{\partial \theta \partial \theta'} \right) \sqrt{n} (\theta_{MLE} - \theta_0),$$

By the law of large numbers, we have:  $\left( -\frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \log f(y_i; \theta_0)}{\partial \theta \partial \theta'} \right) \rightarrow \frac{1}{n} \mathbf{I}(\theta_0) = \mathcal{J}_Y(\theta_0)$ .

Besides, we have:

$$\begin{aligned} \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} &= \sqrt{n} \left( \frac{1}{n} \sum_i \frac{\partial \log f(y_i; \theta_0)}{\partial \theta} \right) \\ &= \sqrt{n} \left( \frac{1}{n} \sum_i \left\{ \frac{\partial \log f(y_i; \theta_0)}{\partial \theta} - \mathbb{E}_{\theta_0} \frac{\partial \log f(Y_i; \theta_0)}{\partial \theta} \right\} \right) \end{aligned}$$

which converges to  $\mathcal{N}(0, \mathcal{J}_Y(\theta_0))$  by the CLT.

Collecting the preceding results leads to (b). The fact that  $\theta_{MLE}$  achieves the FDCR bound proves (c).  $\square$

### Proof of Proposition 7.5

*Proof.* We have  $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0, \mathcal{J}(\theta_0)^{-1})$  (Eq. ??eq:normMLE)). A Taylor expansion around  $\theta_0$  yields to:

$$\sqrt{n}(h(\hat{\theta}_n) - h(\theta_0)) \xrightarrow{d} \mathcal{N}\left(0, \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h(\theta_0)'}{\partial \theta}\right). \quad (10.8)$$

Under  $H_0$ ,  $h(\theta_0) = 0$  therefore:

$$\sqrt{n}h(\hat{\theta}_n) \xrightarrow{d} \mathcal{N}\left(0, \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h(\theta_0)'}{\partial \theta}\right). \quad (10.9)$$

Hence

$$\sqrt{n} \left( \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{I}(\theta_0)^{-1} \frac{\partial h(\theta_0)'}{\partial \theta} \right)^{-1/2} h(\hat{\theta}_n) \xrightarrow{d} \mathcal{N}(0, Id).$$

Taking the quadratic form, we obtain:

$$nh(\hat{\theta}_n)' \left( \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{I}(\theta_0)^{-1} \frac{\partial h(\theta_0)'}{\partial \theta} \right)^{-1} h(\hat{\theta}_n) \xrightarrow{d} \chi^2(r).$$

The fact that the test has asymptotic level  $\alpha$  directly stems from what precedes. **Consistency of the test:** Consider  $\theta_0 \in \Theta$ . Because the MLE is consistent,  $h(\hat{\theta}_n)$  converges to  $h(\theta_0) \neq 0$ . Eq. (10.8) is still valid. It implies that  $\xi_n^W$  converges to  $+\infty$  and therefore that  $\mathbb{P}_\theta(\xi_n^W \geq \chi_{1-\alpha}^2(r)) \rightarrow 1$ .  $\square$

### Proof of Proposition 7.6

*Proof.* Notations: “ $\approx$ ” means “equal up to a term that converges to 0 in probability”. We are under  $H_0$ .  $\hat{\theta}^0$  is the constrained ML estimator;  $\hat{\theta}$  denotes the unconstrained one.

We combine the two Taylor expansion:  $h(\hat{\theta}_n) \approx \frac{\partial h(\theta_0)}{\partial \theta'} (\hat{\theta}_n - \theta_0)$  and  $h(\hat{\theta}_n^0) \approx \frac{\partial h(\theta_0)}{\partial \theta'} (\hat{\theta}_n^0 - \theta_0)$  and we use  $h(\hat{\theta}_n^0) = 0$  (by definition) to get:

$$\sqrt{n}h(\hat{\theta}_n) \approx \frac{\partial h(\theta_0)}{\partial \theta'} \sqrt{n}(\hat{\theta}_n - \hat{\theta}_n^0). \quad (10.10)$$

Besides, we have (using the definition of the information matrix):

$$\frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \approx \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} - \mathcal{I}(\theta_0) \sqrt{n}(\hat{\theta}_n^0 - \theta_0) \quad (10.11)$$

and:

$$0 = \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n; \mathbf{y})}{\partial \theta} \approx \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} - \mathcal{I}(\theta_0) \sqrt{n}(\hat{\theta}_n - \theta_0). \quad (10.12)$$

Taking the difference and multiplying by  $\mathcal{I}(\theta_0)^{-1}$ :

$$\sqrt{n}(\hat{\theta}_n - \hat{\theta}_n^0) \approx \mathcal{I}(\theta_0)^{-1} \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \mathcal{I}(\theta_0). \quad (10.13)$$

Eqs. (10.10) and (10.13) yield to:

$$\sqrt{n}h(\hat{\theta}_n) \approx \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{I}(\theta_0)^{-1} \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta}. \quad (10.14)$$

Recall that  $\hat{\theta}_n^0$  is the MLE of  $\theta_0$  under the constraint  $h(\theta) = 0$ . The vector of Lagrange multipliers  $\hat{\lambda}_n$  associated to this program satisfies:

$$\frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} + \frac{\partial h'(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \hat{\lambda}_n = 0. \quad (10.15)$$

Substituting the latter equation in Eq. (10.14) gives:

$$\sqrt{n}h(\hat{\theta}_n) \approx - \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{I}(\theta_0)^{-1} \frac{\partial h'(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \frac{\hat{\lambda}_n}{\sqrt{n}} \approx - \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{I}(\theta_0)^{-1} \frac{\partial h'(\theta_0; \mathbf{y})}{\partial \theta} \frac{\hat{\lambda}_n}{\sqrt{n}}$$

which yields:

$$\frac{\hat{\lambda}_n}{\sqrt{n}} \approx - \left( \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{I}(\theta_0)^{-1} \frac{\partial h'(\theta_0; \mathbf{y})}{\partial \theta} \right)^{-1} \sqrt{n}h(\hat{\theta}_n). \quad (10.16)$$

It follows, from Eq. (10.9), that:

$$\frac{\hat{\lambda}_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N} \left( 0, \left( \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h'(\theta_0; \mathbf{y})}{\partial \theta} \right)^{-1} \right).$$

Taking the quadratic form of the last equation gives:

$$\frac{1}{n} \hat{\lambda}'_n \frac{\partial h(\hat{\theta}_n^0)}{\partial \theta'} \mathcal{J}(\hat{\theta}_n^0)^{-1} \frac{\partial h'(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \hat{\lambda}_n \xrightarrow{d} \chi^2(r).$$

Using Eq. (10.15), it appears that the left-hand side term of the last equation is  $\xi^{LM}$  as defined in Eq. (7.18). Consistency: see Remark 17.3 in Gouriéroux and Monfort (1995).  $\square$

### Proof of Proposition 7.7

*Proof.* Let us first demonstrate the asymptotic equivalence of  $\xi^{LM}$  and  $\xi^{LR}$ .

The second-order taylor expansions of  $\log \mathcal{L}(\hat{\theta}_n^0, \mathbf{y})$  and  $\log \mathcal{L}(\hat{\theta}_n, \mathbf{y})$  are:

$$\begin{aligned} \log \mathcal{L}(\hat{\theta}_n, \mathbf{y}) &\approx \log \mathcal{L}(\theta_0, \mathbf{y}) + \frac{\partial \log \mathcal{L}(\theta_0, \mathbf{y})}{\partial \theta'} (\hat{\theta}_n - \theta_0) - \frac{n}{2} (\hat{\theta}_n - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n - \theta_0) \\ \log \mathcal{L}(\hat{\theta}_n^0, \mathbf{y}) &\approx \log \mathcal{L}(\theta_0, \mathbf{y}) + \frac{\partial \log \mathcal{L}(\theta_0, \mathbf{y})}{\partial \theta'} (\hat{\theta}_n^0 - \theta_0) - \frac{n}{2} (\hat{\theta}_n^0 - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n^0 - \theta_0). \end{aligned}$$

Taking the difference, we obtain:

$$\xi_n^{LR} \approx 2 \frac{\partial \log \mathcal{L}(\theta_0, \mathbf{y})}{\partial \theta'} (\hat{\theta}_n - \hat{\theta}_n^0) + n (\hat{\theta}_n^0 - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n^0 - \theta_0) - n (\hat{\theta}_n - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n - \theta_0).$$

Using  $\frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0, \mathbf{y})}{\partial \theta} \approx \mathcal{J}(\theta_0) \sqrt{n} (\hat{\theta}_n - \theta_0)$  (Eq. (10.12)), we have:

$$\xi_n^{LR} \approx 2n (\hat{\theta}_n - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n - \hat{\theta}_n^0) + n (\hat{\theta}_n^0 - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n^0 - \theta_0) - n (\hat{\theta}_n - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n - \theta_0).$$

In the second of the three terms in the sum, we replace  $(\hat{\theta}_n^0 - \theta_0)$  by  $(\hat{\theta}_n^0 - \hat{\theta}_n + \hat{\theta}_n - \theta_0)$  and we develop the associated product. This leads to:

$$\xi_n^{LR} \approx n (\hat{\theta}_n^0 - \hat{\theta}_n)' \mathcal{J}(\theta_0)^{-1} (\hat{\theta}_n^0 - \hat{\theta}_n). \quad (10.17)$$

The difference between Eqs. (10.11) and (10.12) implies:

$$\frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \approx \mathcal{J}(\theta_0) \sqrt{n} (\hat{\theta}_n - \hat{\theta}_n^0),$$

which, associated to Eq. @??eq:lr10), gives:

$$\xi_n^{LR} \approx \frac{1}{n} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \approx \xi_n^{LM}.$$

Hence  $\xi_n^{LR}$  has the same asymptotic distribution as  $\xi_n^{LM}$ .

Let's show that the LR test is consistent. For this, note that:

$$\frac{\log \mathcal{L}(\hat{\theta}, \mathbf{y}) - \log \mathcal{L}(\hat{\theta}^0, \mathbf{y})}{n} = \frac{1}{n} \sum_{i=1}^n [\log f(y_i; \hat{\theta}_n) - \log f(y_i; \hat{\theta}_n^0)] \rightarrow \mathbb{E}_0 [\log f(Y; \theta_0) - \log f(Y; \theta_\infty)],$$

where  $\theta_\infty$ , the pseudo true value, is such that  $h(\theta_\infty) \neq 0$  (by definition of  $H_1$ ). From the Kullback inequality and the asymptotic identifiability of  $\theta_0$ , it follows that  $\mathbb{E}_0 [\log f(Y; \theta_0) - \log f(Y; \theta_\infty)] > 0$ . Therefore  $\xi_n^{LR} \rightarrow +\infty$  under  $H_1$ .

Let us now demonstrate the equivalence of  $\xi^{LM}$  and  $\xi^W$ .

We have (using Eq. ??eq:multiplier)):

$$\xi_n^{LM} = \frac{1}{n} \hat{\lambda}'_n \frac{\partial h(\hat{\theta}_n^0)}{\partial \theta'} \mathcal{J}(\hat{\theta}_n^0)^{-1} \frac{\partial h'(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \hat{\lambda}_n.$$

Since, under  $H_0$ ,  $\hat{\theta}_n^0 \approx \hat{\theta}_n \approx \theta_0$ , Eq. (10.16) therefore implies that:

$$\xi_n^{LM} \approx nh(\hat{\theta}_n)' \left( \frac{\partial h(\hat{\theta}_n)}{\partial \theta'} \mathcal{J}(\hat{\theta}_n)^{-1} \frac{\partial h'(\hat{\theta}_n; \mathbf{y})}{\partial \theta} \right)^{-1} h(\hat{\theta}_n) = \xi_n^W,$$

which gives the result.  $\square$

### Proof of Eq. (9.3)

*Proof.* We have:

$$\begin{aligned} & T\mathbb{E}[(\bar{y}_T - \mu)^2] \\ &= T\mathbb{E}\left[\left(\frac{1}{T} \sum_{t=1}^T (y_t - \mu)\right)^2\right] = \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^T (y_t - \mu)^2 + 2 \sum_{s < t \leq T} (y_t - \mu)(y_s - \mu)\right] \\ &= \gamma_0 + \frac{2}{T} \left( \sum_{t=2}^T \mathbb{E}[(y_t - \mu)(y_{t-1} - \mu)] \right) + \frac{2}{T} \left( \sum_{t=3}^T \mathbb{E}[(y_t - \mu)(y_{t-2} - \mu)] \right) + \dots \\ &\quad + \frac{2}{T} \left( \sum_{t=T-1}^T \mathbb{E}[(y_t - \mu)(y_{t-(T-2)} - \mu)] \right) + \frac{2}{T}\mathbb{E}[(y_T - \mu)(y_{T-(T-1)} - \mu)] \\ &= \gamma_0 + 2\frac{T-1}{T}\gamma_1 + \dots + 2\frac{1}{T}\gamma_{T-1}. \end{aligned}$$

Therefore:

$$T\mathbb{E}[(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j = -2\frac{1}{T}\gamma_1 - 2\frac{2}{T}\gamma_2 - \dots - 2\frac{T-1}{T}\gamma_{T-1} - 2\gamma_T - 2\gamma_{T+1} + \dots$$

And then:

$$\left| T\mathbb{E}[(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j \right| \leq 2\frac{1}{T}|\gamma_1| + 2\frac{2}{T}|\gamma_2| + \dots + 2\frac{T-1}{T}|\gamma_{T-1}| + 2|\gamma_T| + 2|\gamma_{T+1}| + \dots$$

For any  $q \leq T$ , we have:

$$\begin{aligned} \left| T\mathbb{E}[(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j \right| &\leq 2\frac{1}{T}|\gamma_1| + 2\frac{2}{T}|\gamma_2| + \dots + 2\frac{q-1}{T}|\gamma_{q-1}| + 2\frac{q}{T}|\gamma_q| + \\ &\quad 2\frac{q+1}{T}|\gamma_{q+1}| + \dots + 2\frac{T-1}{T}|\gamma_{T-1}| + 2|\gamma_T| + 2|\gamma_{T+1}| + \dots \\ &\leq \frac{2}{T}(|\gamma_1| + 2|\gamma_2| + \dots + (q-1)|\gamma_{q-1}| + q|\gamma_q|) + \\ &\quad 2|\gamma_{q+1}| + \dots + 2|\gamma_{T-1}| + 2|\gamma_T| + 2|\gamma_{T+1}| + \dots \end{aligned}$$

Consider  $\varepsilon > 0$ . The fact that the autocovariances are absolutely summable implies that there exists  $q_0$  such that (Cauchy criterion, Theorem 10.2):

$$2|\gamma_{q_0+1}| + 2|\gamma_{q_0+2}| + 2|\gamma_{q_0+3}| + \dots < \varepsilon/2.$$

Then, if  $T > q_0$ , it comes that:

$$\left| T\mathbb{E}[(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j \right| \leq \frac{2}{T}(|\gamma_1| + 2|\gamma_2| + \dots + (q_0-1)|\gamma_{q_0-1}| + q_0|\gamma_{q_0}|) + \varepsilon/2.$$

If  $T \geq 2(|\gamma_1| + 2|\gamma_2| + \dots + (q_0 - 1)|\gamma_{q_0-1}| + q_0|\gamma_{q_0}|) / (\varepsilon/2)$  ( $= f(q_0)$ , say) then

$$\frac{2}{T} (|\gamma_1| + 2|\gamma_2| + \dots + (q_0 - 1)|\gamma_{q_0-1}| + q_0|\gamma_{q_0}|) \leq \varepsilon/2.$$

Then, if  $T > f(q_0)$  and  $T > q_0$ , i.e. if  $T > \max(f(q_0), q_0)$ , we have:

$$\left| T \mathbb{E}[(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j \right| \leq \varepsilon.$$

□

### Proof of Proposition 9.15

*Proof.* We have:

$$\begin{aligned} \mathbb{E}([y_{t+1} - y_{t+1}^*]^2) &= \mathbb{E}(\{\textcolor{blue}{y_{t+1}} - \mathbb{E}(y_{t+1}|x_t)\} + \{\textcolor{red}{\mathbb{E}(y_{t+1}|x_t)} - y_{t+1}^*\})^2 \\ &= \mathbb{E}([\textcolor{blue}{y_{t+1}} - \mathbb{E}(y_{t+1}|x_t)]^2) + \mathbb{E}([\textcolor{red}{\mathbb{E}(y_{t+1}|x_t)} - y_{t+1}^*]^2) \\ &\quad + 2\mathbb{E}([\textcolor{blue}{y_{t+1}} - \mathbb{E}(y_{t+1}|x_t)][\textcolor{red}{\mathbb{E}(y_{t+1}|x_t)} - y_{t+1}^*]). \end{aligned} \quad (10.18)$$

Let us focus on the last term. We have:

$$\begin{aligned} &\mathbb{E}([\textcolor{blue}{y_{t+1}} - \mathbb{E}(y_{t+1}|x_t)][\textcolor{red}{\mathbb{E}(y_{t+1}|x_t)} - y_{t+1}^*]) \\ &= \mathbb{E}(\mathbb{E}([\textcolor{blue}{y_{t+1}} - \mathbb{E}(y_{t+1}|x_t)][\textcolor{red}{\mathbb{E}(y_{t+1}|x_t)} - y_{t+1}^*]|x_t)) \\ &\quad \text{function of } x_t \\ &= \mathbb{E}([\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*]\mathbb{E}([\textcolor{blue}{y_{t+1}} - \mathbb{E}(y_{t+1}|x_t)]|x_t)) \\ &= \mathbb{E}([\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*]\underbrace{[\mathbb{E}(y_{t+1}|x_t) - \mathbb{E}(y_{t+1}|x_t)]}_{=0}) = 0. \end{aligned}$$

Therefore, Eq. (10.18) becomes:

$$\begin{aligned} &\mathbb{E}([y_{t+1} - y_{t+1}^*]^2) \\ &= \underbrace{\mathbb{E}([\textcolor{blue}{y_{t+1}} - \mathbb{E}(y_{t+1}|x_t)]^2)}_{\geq 0 \text{ and does not depend on } y_{t+1}^*} + \underbrace{\mathbb{E}([\textcolor{red}{\mathbb{E}(y_{t+1}|x_t)} - y_{t+1}^*]^2)}_{\geq 0 \text{ and depends on } y_{t+1}^*}. \end{aligned}$$

This implies that  $\mathbb{E}([y_{t+1} - y_{t+1}^*]^2)$  is always larger than  $\mathbb{E}([\textcolor{blue}{y_{t+1}} - \mathbb{E}(y_{t+1}|x_t)]^2)$ , and is therefore minimized if the second term is equal to zero, that is if  $\mathbb{E}(y_{t+1}|x_t) = y_{t+1}^*$ . □

### Proof of Proposition 9.12

*Proof.* Using Proposition ?? (in Appendix ??), we obtain that, conditionally on  $x_1$ , the log-likelihood is given by

$$\begin{aligned} \log \mathcal{L}(Y_T; \theta) &= -(Tn/2) \log(2\pi) + (T/2) \log |\Omega^{-1}| \\ &\quad - \frac{1}{2} \sum_{t=1}^T [(y_t - \Pi' x_t)' \Omega^{-1} (y_t - \Pi' x_t)]. \end{aligned}$$

Let's rewrite the last term of the log-likelihood:

$$\begin{aligned} &\sum_{t=1}^T [(y_t - \Pi' x_t)' \Omega^{-1} (y_t - \Pi' x_t)] = \\ &\sum_{t=1}^T [(y_t - \hat{\Pi}' x_t + \hat{\Pi}' x_t - \Pi' x_t)' \Omega^{-1} (y_t - \hat{\Pi}' x_t + \hat{\Pi}' x_t - \Pi' x_t)] = \\ &\sum_{t=1}^T [(\hat{\varepsilon}_t + (\hat{\Pi} - \Pi)' x_t)' \Omega^{-1} (\hat{\varepsilon}_t + (\hat{\Pi} - \Pi)' x_t)], \end{aligned}$$

where the  $j^{th}$  element of the  $(n \times 1)$  vector  $\hat{\varepsilon}_t$  is the sample residual, for observation  $t$ , from an OLS regression of  $y_{j,t}$  on  $x_t$ . Expanding the previous equation, we get:

$$\begin{aligned} \sum_{t=1}^T [(y_t - \Pi' x_t)' \Omega^{-1} (y_t - \Pi' x_t)] &= \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} \hat{\varepsilon}_t \\ &+ 2 \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} (\hat{\Pi} - \Pi)' x_t + \sum_{t=1}^T x_t' (\hat{\Pi} - \Pi) \Omega^{-1} (\hat{\Pi} - \Pi)' x_t. \end{aligned}$$

Let's apply the trace operator on the second term (that is a scalar):

$$\begin{aligned} \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} (\hat{\Pi} - \Pi)' x_t &= Tr \left( \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} (\hat{\Pi} - \Pi)' x_t \right) \\ = Tr \left( \sum_{t=1}^T \Omega^{-1} (\hat{\Pi} - \Pi)' x_t \hat{\varepsilon}_t' \right) &= Tr \left( \Omega^{-1} (\hat{\Pi} - \Pi)' \sum_{t=1}^T x_t \hat{\varepsilon}_t' \right). \end{aligned}$$

Given that, by construction (property of OLS estimates), the sample residuals are orthogonal to the explanatory variables, this term is zero. Introducing  $\tilde{x}_t = (\hat{\Pi} - \Pi)' x_t$ , we have

$$\sum_{t=1}^T [(y_t - \Pi' x_t)' \Omega^{-1} (y_t - \Pi' x_t)] = \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} \hat{\varepsilon}_t + \sum_{t=1}^T \tilde{x}_t' \Omega^{-1} \tilde{x}_t.$$

Since  $\Omega$  is a positive definite matrix,  $\Omega^{-1}$  is as well. Consequently, the smallest value that the last term can take is obtained for  $\tilde{x}_t = 0$ , i.e. when  $\Pi = \hat{\Pi}$ .

The MLE of  $\Omega$  is the matrix  $\hat{\Omega}$  that maximizes  $\Omega \xrightarrow{\ell} L(Y_T; \hat{\Pi}, \Omega)$ . We have:

$$\log \mathcal{L}(Y_T; \hat{\Pi}, \Omega) = -(Tn/2) \log(2\pi) + (T/2) \log |\Omega^{-1}| - \frac{1}{2} \sum_{t=1}^T [\hat{\varepsilon}_t' \Omega^{-1} \hat{\varepsilon}_t].$$

Matrix  $\hat{\Omega}$  is a symmetric positive definite. It is easily checked that the (unrestricted) matrix that maximizes the latter expression is symmetric positive definite matrix. Indeed:

$$\frac{\partial \log \mathcal{L}(Y_T; \hat{\Pi}, \Omega)}{\partial \Omega} = \frac{T}{2} \Omega' - \frac{1}{2} \sum_{t=1}^T \hat{\varepsilon}_t \hat{\varepsilon}_t' \Rightarrow \hat{\Omega}' = \frac{1}{T} \sum_{t=1}^T \hat{\varepsilon}_t \hat{\varepsilon}_t',$$

which leads to the result.  $\square$

### Proof of Proposition 9.13

*Proof.* Let us drop the  $i$  subscript. Rearranging Eq. (9.47), we have:

$$\sqrt{T}(\mathbf{b} - \beta) = (X' X/T)^{-1} \sqrt{T}(X' \varepsilon/T).$$

Let us consider the autocovariances of  $\mathbf{v}_t = x_t \varepsilon_t$ , denoted by  $\gamma_j^v$ . Using the fact that  $x_t$  is a linear combination of past  $\varepsilon_t$ s and that  $\varepsilon_t$  is a white noise, we get that  $\mathbb{E}(\varepsilon_t x_t) = 0$ . Therefore

$$\gamma_j^v = \mathbb{E}(\varepsilon_t \varepsilon_{t-j} x_t x_{t-j}').$$

If  $j > 0$ , we have  $\mathbb{E}(\varepsilon_t \varepsilon_{t-j} x_t x_{t-j}') = \mathbb{E}(\mathbb{E}[\varepsilon_t \varepsilon_{t-j} x_t x_{t-j}' | \varepsilon_{t-j}, x_t, x_{t-j}]) = \mathbb{E}(\varepsilon_{t-j} x_t x_{t-j}' \mathbb{E}[\varepsilon_t | \varepsilon_{t-j}, x_t, x_{t-j}]) = 0$ . Note that we have  $\mathbb{E}[\varepsilon_t | \varepsilon_{t-j}, x_t, x_{t-j}] = 0$  because  $\{\varepsilon_t\}$  is an i.i.d. white noise sequence.

If  $j = 0$ , we have:

$$\gamma_0^v = \mathbb{E}(\varepsilon_t^2 x_t x_t') = \mathbb{E}(\varepsilon_t^2) \mathbb{E}(x_t x_t') = \sigma^2 \mathbf{Q}.$$

The convergence in distribution of  $\sqrt{T}(X' \varepsilon/T) = \sqrt{T} \frac{1}{T} \sum_{t=1}^T v_t$  results from the Central Limit Theorem for covariance-stationary processes, using the  $\gamma_j^v$  computed above.  $\square$

## 10.6 Additional codes

### 10.6.1 Simulating GEV distributions

The following lines of code have been used to generate Figure 8.7.

```

n.sim <- 4000
par(mfrow=c(1,3),
     plt=c(.2,.95,.2,.85))
all.rhos <- c(.3,.6,.95)
for(j in 1:length(all.rhos)){
  theta <- 1/all.rhos[j]
  v1 <- runif(n.sim)
  v2 <- runif(n.sim)
  w <- rep(.000001,n.sim)
  # solve for f(w) = w*(1 - log(w)/theta) - v2 = 0
  for(i in 1:20){
    f.i <- w * (1 - log(w)/theta) - v2
    f.prime <- 1 - log(w)/theta - 1/theta
    w <- w - f.i/f.prime
  }
  u1 <- exp(v1^(1/theta) * log(w))
  u2 <- exp((1-v1)^(1/theta) * log(w))

  # Get eps1 and eps2 using the inverse of
  # the Gumbel distribution's cdf:
  eps1 <- -log(-log(u1))
  eps2 <- -log(-log(u2))
  cbind(cor(eps1,eps2),1-all.rhos[j]^2)
  plot(eps1,eps2,pch=19,col="#FF000044",
       main=paste("rho = ",toString(all.rhos[j]),sep=""),
       xlab=expression(epsilon[1]),
       ylab=expression(epsilon[2]),
       cex.lab=2,cex.main=1.5)
}

```

## 10.6.2 Computing the covariance matrix of IRF using the delta method

```

irf.function <- function(THETA){
  c <- THETA[1]
  phi <- THETA[2:(p+1)]
  if(q>0){
    theta <- c(1,THETA[(1+p+1):(1+p+q)])
  }else{
    theta <- 1
  }
  sigma <- THETA[1+p+q+1]
  r <- dim(Matrix.of.Exog)[2] - 1
  beta <- THETA[(1+p+q+1+1):(1+p+q+1+(r+1))]

  irf <- sim.arma(0,phi,beta,sigma=sd(Ramey$ED3_TC,na.rm=TRUE),T=60,
                 y.0=rep(0,length(x$phi)),nb.sim=1,make.IRF=1,
                 X=NaN,beta=NaN)
  return(irf)
}

IRF.0 <- 100*irf.function(x$THETA)
eps <- .00000001
d.IRF <- NULL
for(i in 1:length(x$THETA)){
  THETA.i <- x$THETA
  THETA.i[i] <- THETA.i[i] + eps
}

```

Table 10.1: Quantiles of the  $\mathcal{N}(0, 1)$  distribution. If  $a$  and  $b$  are respectively the row and column number; then the corresponding cell gives  $\mathbb{P}(0 < X \leq a + b)$ , where  $X \sim \mathcal{N}(0, 1)$ .

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.6179	0.7257	0.8159	0.8849	0.9332	0.9641	0.9821	0.9918	0.9965
0.1	0.5040	0.6217	0.7291	0.8186	0.8869	0.9345	0.9649	0.9826	0.9920	0.9966
0.2	0.5080	0.6255	0.7324	0.8212	0.8888	0.9357	0.9656	0.9830	0.9922	0.9967
0.3	0.5120	0.6293	0.7357	0.8238	0.8907	0.9370	0.9664	0.9834	0.9925	0.9968
0.4	0.5160	0.6331	0.7389	0.8264	0.8925	0.9382	0.9671	0.9838	0.9927	0.9969
0.5	0.5199	0.6368	0.7422	0.8289	0.8944	0.9394	0.9678	0.9842	0.9929	0.9970
0.6	0.5239	0.6406	0.7454	0.8315	0.8962	0.9406	0.9686	0.9846	0.9931	0.9971
0.7	0.5279	0.6443	0.7486	0.8340	0.8980	0.9418	0.9693	0.9850	0.9932	0.9972
0.8	0.5319	0.6480	0.7517	0.8365	0.8997	0.9429	0.9699	0.9854	0.9934	0.9973
0.9	0.5359	0.6517	0.7549	0.8389	0.9015	0.9441	0.9706	0.9857	0.9936	0.9974
1	0.5398	0.6554	0.7580	0.8413	0.9032	0.9452	0.9713	0.9861	0.9938	0.9974
1.1	0.5438	0.6591	0.7611	0.8438	0.9049	0.9463	0.9719	0.9864	0.9940	0.9975
1.2	0.5478	0.6628	0.7642	0.8461	0.9066	0.9474	0.9726	0.9868	0.9941	0.9976
1.3	0.5517	0.6664	0.7673	0.8485	0.9082	0.9484	0.9732	0.9871	0.9943	0.9977
1.4	0.5557	0.6700	0.7704	0.8508	0.9099	0.9495	0.9738	0.9875	0.9945	0.9977
1.5	0.5596	0.6736	0.7734	0.8531	0.9115	0.9505	0.9744	0.9878	0.9946	0.9978
1.6	0.5636	0.6772	0.7764	0.8554	0.9131	0.9515	0.9750	0.9881	0.9948	0.9979
1.7	0.5675	0.6808	0.7794	0.8577	0.9147	0.9525	0.9756	0.9884	0.9949	0.9979
1.8	0.5714	0.6844	0.7823	0.8599	0.9162	0.9535	0.9761	0.9887	0.9951	0.9980
1.9	0.5753	0.6879	0.7852	0.8621	0.9177	0.9545	0.9767	0.9890	0.9952	0.9981
2	0.5793	0.6915	0.7881	0.8643	0.9192	0.9554	0.9772	0.9893	0.9953	0.9981
2.1	0.5832	0.6950	0.7910	0.8665	0.9207	0.9564	0.9778	0.9896	0.9955	0.9982
2.2	0.5871	0.6985	0.7939	0.8686	0.9222	0.9573	0.9783	0.9898	0.9956	0.9982
2.3	0.5910	0.7019	0.7967	0.8708	0.9236	0.9582	0.9788	0.9901	0.9957	0.9983
2.4	0.5948	0.7054	0.7995	0.8729	0.9251	0.9591	0.9793	0.9904	0.9959	0.9984
2.5	0.5987	0.7088	0.8023	0.8749	0.9265	0.9599	0.9798	0.9906	0.9960	0.9984
2.6	0.6026	0.7123	0.8051	0.8770	0.9279	0.9608	0.9803	0.9909	0.9961	0.9985
2.7	0.6064	0.7157	0.8078	0.8790	0.9292	0.9616	0.9808	0.9911	0.9962	0.9985
2.8	0.6103	0.7190	0.8106	0.8810	0.9306	0.9625	0.9812	0.9913	0.9963	0.9986
2.9	0.6141	0.7224	0.8133	0.8830	0.9319	0.9633	0.9817	0.9916	0.9964	0.9986

```

IRF.i <- 100*irf.function(THETA.i)
d.IRF <- cbind(d.IRF,
                 (IRF.i - IRF.0)/eps
                 )
}
mat.var.cov.IRF <- d.IRF %*% x$I %*% t(d.IRF)

```

## 10.7 Statistical Tables

Table 10.2: Quantiles of the Student- $t$  distribution. The rows correspond to different degrees of freedom ( $\nu$ , say); the columns correspond to different probabilities ( $z$ , say). The cell gives  $q$  that is s.t.  $\mathbb{P}(-q < X < q) = z$ , with  $X \sim t(\nu)$ .

	0.05	0.1	0.75	0.9	0.95	0.975	0.99	0.999
1	0.079	0.158	2.414	6.314	12.706	25.452	63.657	636.619
2	0.071	0.142	1.604	2.920	4.303	6.205	9.925	31.599
3	0.068	0.137	1.423	2.353	3.182	4.177	5.841	12.924
4	0.067	0.134	1.344	2.132	2.776	3.495	4.604	8.610
5	0.066	0.132	1.301	2.015	2.571	3.163	4.032	6.869
6	0.065	0.131	1.273	1.943	2.447	2.969	3.707	5.959
7	0.065	0.130	1.254	1.895	2.365	2.841	3.499	5.408
8	0.065	0.130	1.240	1.860	2.306	2.752	3.355	5.041
9	0.064	0.129	1.230	1.833	2.262	2.685	3.250	4.781
10	0.064	0.129	1.221	1.812	2.228	2.634	3.169	4.587
20	0.063	0.127	1.185	1.725	2.086	2.423	2.845	3.850
30	0.063	0.127	1.173	1.697	2.042	2.360	2.750	3.646
40	0.063	0.126	1.167	1.684	2.021	2.329	2.704	3.551
50	0.063	0.126	1.164	1.676	2.009	2.311	2.678	3.496
60	0.063	0.126	1.162	1.671	2.000	2.299	2.660	3.460
70	0.063	0.126	1.160	1.667	1.994	2.291	2.648	3.435
80	0.063	0.126	1.159	1.664	1.990	2.284	2.639	3.416
90	0.063	0.126	1.158	1.662	1.987	2.280	2.632	3.402
100	0.063	0.126	1.157	1.660	1.984	2.276	2.626	3.390
200	0.063	0.126	1.154	1.653	1.972	2.258	2.601	3.340
500	0.063	0.126	1.152	1.648	1.965	2.248	2.586	3.310

Table 10.3: Quantiles of the  $\chi^2$  distribution. The rows correspond to different degrees of freedom; the columns correspond to different probabilities.

	0.05	0.1	0.75	0.9	0.95	0.975	0.99	0.999
1	0.004	0.016	1.323	2.706	3.841	5.024	6.635	10.828
2	0.103	0.211	2.773	4.605	5.991	7.378	9.210	13.816
3	0.352	0.584	4.108	6.251	7.815	9.348	11.345	16.266
4	0.711	1.064	5.385	7.779	9.488	11.143	13.277	18.467
5	1.145	1.610	6.626	9.236	11.070	12.833	15.086	20.515
6	1.635	2.204	7.841	10.645	12.592	14.449	16.812	22.458
7	2.167	2.833	9.037	12.017	14.067	16.013	18.475	24.322
8	2.733	3.490	10.219	13.362	15.507	17.535	20.090	26.124
9	3.325	4.168	11.389	14.684	16.919	19.023	21.666	27.877
10	3.940	4.865	12.549	15.987	18.307	20.483	23.209	29.588
20	10.851	12.443	23.828	28.412	31.410	34.170	37.566	45.315
30	18.493	20.599	34.800	40.256	43.773	46.979	50.892	59.703
40	26.509	29.051	45.616	51.805	55.758	59.342	63.691	73.402
50	34.764	37.689	56.334	63.167	67.505	71.420	76.154	86.661
60	43.188	46.459	66.981	74.397	79.082	83.298	88.379	99.607
70	51.739	55.329	77.577	85.527	90.531	95.023	100.425	112.317
80	60.391	64.278	88.130	96.578	101.879	106.629	112.329	124.839
90	69.126	73.291	98.650	107.565	113.145	118.136	124.116	137.208
100	77.929	82.358	109.141	118.498	124.342	129.561	135.807	149.449
200	168.279	174.835	213.102	226.021	233.994	241.058	249.445	267.541
500	449.147	459.926	520.950	540.930	553.127	563.852	576.493	603.446

Table 10.4: Quantiles of the  $\mathcal{F}$  distribution. The columns and rows correspond to different degrees of freedom (resp.  $n_1$  and  $n_2$ ). The different panels correspond to different probabilities ( $\alpha$ ) The corresponding cell gives  $z$  that is s.t.  $\mathbb{P}(X \leq z) = \alpha$ , with  $X \sim \mathcal{F}(n_1, n_2)$ .

	1	2	3	4	5	6	7	8	9	10
alpha = 0.9										
5	4.060	3.780	3.619	3.520	3.453	3.405	3.368	3.339	3.316	3.297
10	3.285	2.924	2.728	2.605	2.522	2.461	2.414	2.377	2.347	2.323
15	3.073	2.695	2.490	2.361	2.273	2.208	2.158	2.119	2.086	2.059
20	2.975	2.589	2.380	2.249	2.158	2.091	2.040	1.999	1.965	1.937
50	2.809	2.412	2.197	2.061	1.966	1.895	1.840	1.796	1.760	1.729
100	2.756	2.356	2.139	2.002	1.906	1.834	1.778	1.732	1.695	1.663
500	2.716	2.313	2.095	1.956	1.859	1.786	1.729	1.683	1.644	1.612
alpha = 0.95										
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348
50	4.034	3.183	2.790	2.557	2.400	2.286	2.199	2.130	2.073	2.026
100	3.936	3.087	2.696	2.463	2.305	2.191	2.103	2.032	1.975	1.927
500	3.860	3.014	2.623	2.390	2.232	2.117	2.028	1.957	1.899	1.850
alpha = 0.99										
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158	10.051
10	10.044	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.942	4.849
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368
50	7.171	5.057	4.199	3.720	3.408	3.186	3.020	2.890	2.785	2.698
100	6.895	4.824	3.984	3.513	3.206	2.988	2.823	2.694	2.590	2.503
500	6.686	4.648	3.821	3.357	3.054	2.838	2.675	2.547	2.443	2.356

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