Midterm

March 14, 2019

This sample Midterm has 100 points. Please read the questions carefully and provide clean and concise answers (I cannot grant partial credit for unreadable answers). I suggest you to answer the questions in the order that they appear.

The grades for the midterm will be based on relative performance. Bottom 20% gets a C, Top 30% gets an A, the rest gets a B. Pluses or minuses, would be determined by the relative quality of the answers.

Good luck!

Question 1 (10 points):

a) (10 points) Suppose $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ is i.i.d. over time. Is the process:

$$X_t \equiv \frac{1}{\sqrt{t}} \sum_{j=1}^{t} \epsilon_j .$$

weakly stationary? Choose one (and only one) of the following answers

- ☐ Yes, it is weakly stationary.
- ☐ Yes, it is in an MA(q) process.
- ☐ No! It is not weakly stationary because the variance depends on t!
- ✓ No! Even though it is true that both the mean and the variance are constant over time, the autocovariance function depends on t!

Answer: To make this point consider $cov(X_t, X_{t-1})$. This equals

$$\frac{1}{\sqrt{t}} \frac{1}{\sqrt{t-1}} \text{cov}(\epsilon_1 + \ldots + \epsilon_{t-1} + \epsilon_t, \epsilon_1 + \ldots + \epsilon_{t-1}).$$

Because ϵ_t are i.i.d. over time, the covariance above equals

$$(t-1)\sigma^2.$$

Consequently,

$$cov(X_t, X_{t-1}) = \sigma^2\sqrt{t-1}/\sqrt{t}.$$

Question 2 (35 points): Consider the time series model—with parameters ϕ $(|\phi| < 1), \theta, \sigma^2$—given by:

$$X_t = \sum_{j=0}^{\infty} \phi^j \eta_{t-j}, \quad \eta_t = \epsilon_t + \theta\epsilon_{t-1}, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2), \quad \text{i.i.d.} \tag{1}$$

Note that η_t is an MA(1) model with parameter θ. This means that:

$$V(\eta_t) = \sigma^2 + \theta^2\sigma^2 \quad \text{and} \quad \text{Cov}(\eta_{t+1}, \eta_t) = \theta\sigma^2.$$

The model in (1) can be written as:

$$X_t = \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1}, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2), \quad \text{i.i.d.} \tag{2}$$

This is called an ARMA(1,1) model (AR for autoregressive, MA for moving average).

a) (12.5 points) What is the variance of X_t? (HINT: $\text{Cov}(X_t, \epsilon_t) = \sigma^2$, X_t is weakly stationary).

Answer: We compute the variance as we did in class.

$$\begin{aligned} V(X_t) &= V(\phi X_{t-1} + \eta_t) \\ &= \phi^2 V(X_{t-1}) + V(\eta_t) + 2\text{Cov}(X_{t-1}, \eta_t). \end{aligned}$$

Weak stationarity implies $V(X_t) = V(X_{t-1})$. Consequently:

$$V(X_t) = \frac{1}{1-\phi^2}\left[V(\eta_t) + 2\text{Cov}(X_{t-1}, \eta_t)\right].$$

We know $V(\eta_t) = \epsilon_t + \theta\epsilon_{t-1}$, because η_t is an MA(1) model. We only need to compute

$$\begin{aligned}
\mathrm{Cov}(X_{t-1}, \eta_t) &= \mathrm{Cov}(X_{t-1}, \epsilon_t + \theta\epsilon_{t-1}), \\
&= \mathrm{Cov}(X_{t-1}, \epsilon_t) + \theta \mathrm{Cov}(X_{t-1}, \epsilon_{t-1}), \\
&= \theta \mathrm{Cov}(X_{t-1}, \epsilon_{t-1}), \\
& \quad \text{(by equation (1))} \\
&= \theta\sigma^2. \\
& \quad \text{(by the Hint in the previous page).}
\end{aligned}$$

We conclude

$$V(X_t) = \frac{1}{1-\phi^2}\left[\sigma^2 + \theta^2\sigma^2 + 2\theta\sigma^2\right] = \frac{\sigma^2}{1-\phi^2}(1+\theta)^2.$$

b) (12.5 points) Suppose that you try to estimate the parameter ϕ by OLS. That is, you regress X_t on X_{t-1} and estimate ϕ as

$$\widehat{\phi} = \sum_{t=2}^{T} x_t x_{t-1} \Big/ \sum_{t=2}^{T} x_{t-1}^2.$$

Is $\widehat{\phi}$ a consistent estimator? If not, what is the probability limit of ϕ as a function of (ϕ, θ, σ^2)? (HINT: Use the LLN for averages of X_t and/or products of the form $X_t X_{t-h}$. Also, if you have two sequences X_n, Y_n such that $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y \neq 0$ then $X_n/Y_n \to X/Y$)

Answer: The regressor X_{t-1} is correlated with the residual $\eta_t = \epsilon_t + \theta\epsilon_{t-1}$. This means that we do not expect OLS to be consistent. To formalize this observation we first apply the LLN

$$\begin{aligned}
\widehat{\phi} &= \sum_{t=2}^{T} x_t x_{t-1} \Big/ \sum_{t=2}^{T} x_{t-1}^2, \\
&\xrightarrow{p} \mathbb{E}[X_t X_{t-1}]/E[X_{t-1}^2], \\
&= \mathbb{E}[X_t X_{t-1}]/V[X_{t-1}].
\end{aligned}$$

And note that for the ARMA(1,1) model

$$\begin{aligned}
\mathbb{E}[X_t X_{t-1}] &= \mathbb{E}[(\phi X_{t-1} + \eta_t)\, X_{t-1}], \\
&= \phi \mathbb{E}[X_{t-1}^2] + \mathbb{E}[\eta_t X_{t-1}], \\
&= \phi \mathbb{E}[X_{t-1}^2] + \mathbb{E}[(\epsilon_t + \theta \epsilon_{t-1}) X_{t-1}], \\
&= \phi \mathbb{E}[X_{t-1}^2] + \theta \sigma^2.
\end{aligned}$$

Therefore

$$\widehat{\phi} \xrightarrow{p} \phi + \frac{\theta \sigma^2}{V(X_t)}.$$

This shows that $\widehat{\phi}$ is not consistent for ϕ.

c) (10 points) Suppose that you know σ^2 and θ. How could you construct a consistent estimator for ϕ? (HINT: Could you remove the bias of the OLS estimator by estimating it? Again, if you have two sequences X_n, Y_n such that $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y \neq 0$ then $X_n / Y_n \to X/Y$ If this hint does not help you, construct your own estimator).

Answer: The bias of the OLS estimator depends on θ, σ^2, and $V(X_t)$. Consider the estimator

$$\widehat{\phi} - \frac{\theta \sigma^2}{\widehat{V}(X_t)},$$

where

$$\widehat{V}(X_t) \equiv \frac{1}{T} \sum_{t=1}^{T} X_t^2.$$

By part b) and the hint to this question

$$\widehat{\phi} - \frac{\theta \sigma^2}{\widehat{V}(X_t)} \xrightarrow{p} \phi.$$

Question 3 (25 points): Consider the time series model—with parameters $\mu \neq 1, \mu > 0, \sigma^2$—given by:

$$X_t = \mu^t \exp(\epsilon_t) \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2), \text{ i.i.d.} \tag{3}$$

a) (25 points) Is the estimator

$$\frac{1}{T}\sum_{t=2}^{T} \ln(X_t) - \ln(X_{t-1}),$$

a consistent estimator for $\ln(\mu)$?

Answer: According to (3)

$$\ln(X_t) = t\ln(\mu) + \epsilon_t$$

and

$$\ln(X_{t-1}) = (t-1)\ln(\mu) + \epsilon_{t-1}$$

Therefore,

$$\ln(X_t) - \ln(X_{t-1}) = \ln(\mu) + \epsilon_t - \epsilon_{t-1},$$

and

$$\begin{aligned}
\frac{1}{T}\sum_{t=2}^{T}(\ln(X_t) - \ln(X_{t-1})) - \ln(\mu) &= -\frac{1}{T}\ln(\mu) + \frac{1}{T}\sum_{t=1}^{T}\epsilon_t + \frac{1}{T}\sum_{t=2}^{T}\epsilon_{t-1}, \\
&= -\frac{1}{T}\ln(\mu) + \left(\frac{T-1}{T}\right)\frac{1}{T-1}\sum_{t=1}^{T}\epsilon_t \\
&- \left(\frac{T-1}{T}\right)\frac{1}{T-1}\sum_{t=2}^{T}\epsilon_{t-1}.
\end{aligned}$$

Since $\ln(\mu)/T \to 0$, the Law of Large Numbers imply

$$\frac{1}{T}\sum_{t=2}^{T}(\ln(X_t) - \ln(X_{t-1})) - \ln(\mu) \xrightarrow{p} 0.$$

Question 4 (30 points): Consider the nonstationary model for $(X_1, \ldots, X_T)$ given by:

$$X_t = (\mu)(t) + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2) \quad \text{i.i.d.} \tag{4}$$

with parameter σ^2. Assume that μ is known. The model above is a simple model of random variable with a trend (the mean μ grows linearly over time).

We would like to estimate scale of the residuals.

a) (5 points) What is the log-likelihood function of the data $(X_1, X_2, \ldots, X_T)$?
HINT: If $Z \sim \mathcal{N}(m, s^2)$, then the p.d.f of Z is given by:

$$f(Z; m, s) = \frac{1}{\sqrt{2\pi}s} \exp\left(-\frac{1}{2s^2}(Z-m)^2\right)$$

Answer: The p.d.f of X_t is given by

$$f(X_t|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2\sigma^2}(X_t - \mu t)^2\right).$$

By independence, the Likelihood function is

$$\Pi_{t=1}^T f(X_t|\sigma^2) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^T \exp\left(-\frac{1}{2\sigma^2}\sum_{t=1}^T (X_t - \mu t)^2\right).$$

Thus, the log-likelihood function is

$$T\left(\ln\frac{1}{\sqrt{2\pi}}\right) - \frac{T}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^T (X_t - \mu t)^2$$

b) (10 points) What is the Maximum Likelihood Estimator of σ^2 in the model given by (4)?

Answer: The necessary first order conditions are

$$-\frac{T}{2}\frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{t=1}^T (X_t - t\mu)^2.$$

Because μ is known (by assumption)

$$\widehat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^T (X_t - t\mu)^2$$

c) (10 points) Is the Maximum Likelihood Estimator for σ^2 consistent?

Answer: Yes, it is.

$$X_t - t\mu = \epsilon_t.$$

Consequently

$$\widehat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}(\epsilon_t)^2.$$

The Law of Large Numbers implies $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2$.