Midterm

March 14, 2019

This sample Midterm has 100 points. Please read the questions carefully and provide clean and concise answers (I cannot grant partial credit for unreadable answers). I suggest you to answer the questions in the order that they appear.

The grades for the midterm will be based on relative performance. Bottom 20% gets a C, Top 30% gets an A, the rest gets a B. Pluses or minuses, would be determined by the relative quality of the answers.

Good luck!

Question 1 (10 points):

a) (10 points) Suppose $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ is i.i.d. over time. Is the process:

$$X_t \equiv \frac{1}{\sqrt{t}} \sum_{j=1}^{t} \epsilon_j.$$

weakly stationary? Choose one (and only one) of the following answers

- ☐ Yes, it is weakly stationary.
- ☐ Yes, it is in an MA(q) process.
- ☐ No! It is not weakly stationary because the variance depends on t!
- ☐ No! Even though it is true that both the mean and the variance are constant over time, the autocovariance function depends on t!

Question 2 (35 points): Consider the time series model—with parameters

ϕ ($|\phi| < 1$), θ, σ^2—given by:

$$X_t = \sum_{j=0}^{\infty} \phi^j \eta_{t-j}, \quad \eta_t = \epsilon_t + \theta\epsilon_{t-1}, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2), \quad \text{i.i.d.} \tag{1}$$

Note that η_t is an MA(1) model with parameter θ. This means that:

$$V(\eta_t) = \sigma^2 + \theta^2\sigma^2 \quad \text{and} \quad \text{Cov}(\eta_{t+1}, \eta_t) = \theta\sigma^2.$$

The model in (1) can be written as:

$$X_t = \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1}, \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2), \quad \text{i.i.d.} \tag{2}$$

This is called an ARMA(1,1) model (AR for autoregressive, MA for moving average).

a) (12.5 points) What is the variance of X_t? (HINT: $\text{Cov}(X_t, \epsilon_t) = \sigma^2$, X_t is weakly stationary).

b) (12.5 points) Suppose that you try to estimate the parameter ϕ by OLS. That is, you regress X_t on X_{t-1} and estimate ϕ as

$$\widehat{\phi} = \sum_{t=2}^{T} x_t x_{t-1} \Big/ \sum_{t=2}^{T} x_{t-1}^2.$$

Is $\widehat{\phi}$ a consistent estimator? If not, what is the probability limit of ϕ as a function of (ϕ, θ, σ^2)? (HINT: Use the LLN for averages of X_t and/or products of the form $X_t X_{t-h}$. Also, if you have two sequences X_n, Y_n such that $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y \neq 0$ then $X_n/Y_n \xrightarrow{p} X/Y$)

c) (10 points) Suppose that you know σ^2 and θ. How could you construct a consistent estimator for ϕ? (HINT: Could you remove the bias of the OLS estimator by estimating it? Again, if you have two sequences X_n, Y_n such that $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y \neq 0$ then $X_n/Y_n \xrightarrow{p} X/Y$ If this hint does not help you, construct your own estimator).

Question 3 (25 points): Consider the time series model—with parameters $\mu \neq 1, \mu > 0, \sigma^2$—given by:

$$X_t = \mu^t \exp(\epsilon_t) \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2), \text{ i.i.d.} \tag{3}$$

a) (25 points) Is the estimator

$$\frac{1}{T} \sum_{t=2}^{T} \ln(X_t) - \ln(X_{t-1}),$$

a consistent estimator for $\ln(\mu)$?

Question 4 (30 points): Consider the nonstationary model for $(X_1, \ldots, X_T)$ given by:

$$X_t = (\mu)(t) + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2) \quad \text{i.i.d.} \tag{4}$$

with parameter σ^2. Assume that μ is known. The model above is a simple model of random variable with a trend (the mean μ grows linearly over time). We would like to estimate scale of the residuals.

a) (10 points) What is the log-likelihood function of the data $(X_1, X_2, \ldots, X_T)$?
Hint: If $Z \sim \mathcal{N}(m, s^2)$, then the p.d.f of Z is given by:

$$f(Z; m, s) = \frac{1}{\sqrt{2\pi} s} \exp\left(-\frac{1}{2s^2}(Z - m)^2\right)$$

b) (10 points) What is the Maximum Likelihood Estimator of σ^2 in the model given by (4)?

c) (10 points) Is the Maximum Likelihood Estimator for σ^2 consistent?