Lecture 9-10

Definition (Linear Process) A time series X_t is a linear process if it has the representation:

$$X_t = \sum_{j=-\infty}^{\infty} \theta_j \varepsilon_{t-j} \quad \forall\, t,$$

where $\varepsilon_t \sim \mathrm{WN}(0, \sigma^2)$ and $\{\theta_j\}$ is a sequence of constants satisfying the *summability* condition:

$$\sum_{j=-\infty}^{\infty} |\theta_j| < \infty.$$

The sequence $\{\theta_j\}$ is usually referred to as the impulse-response function of the linear process X_t. The j-th element of the sequence is called the j-th impulse-response coefficient.

COMMENT 1: A linear process allows us to model the time series X_t as the sum of *uncorrelated random fluctuations* (the historical reference for this type of device in macroeconomics is Slutsky's random waves model).

COMMENT 2: The term impulse-response comes from the fact that:

$$\frac{\partial X_{t+j}}{\partial \varepsilon_t} = \theta_j.$$

Consequently, θ_j captures the response of X_{t+j} to an 'impulse' in ε_t.

Mean function: The mean function of the linear process is zero; that is:

$$E[X_t] = 0 \quad \forall\, t.$$

This result does not require the summability condition.

Variance: The variance of X_t equals:

$$\text{Var}(X_t) = \left(\sum_{j=-\infty}^{\infty} \theta_j^2 \right) \sigma^2$$

The summability condition implies that the Variance is well defined; i.e., $\text{Var}(X_t) < \infty$.[1] Note that the Variance does not depend on the particular index t.

AutoCovariance Function: Note that for any elements X_t and X_{t+h} we can show that (we did this in class that):

$$Cov(X_t, X_{t+h}) = \sum_{j=-\infty}^{\infty} \theta_j \theta_{j+h} \sigma^2$$

(which only depends on h).[2]

Claim: Let P be any distribution for the white noise ε_t. The time series model based on P and any linear process X_t is stationary in the weak sense.

Examples of Linear Processes:

Definition (MA(∞)) A linear process

$$X_t = \sum_{j=-\infty}^{\infty} \theta_j \varepsilon_{t-j}$$

is called a moving average of infinite order if $\theta_j = 0$ for all $j < 0$. That is, if

[1] To see this, just note that

$$\sum_{j=-\infty}^{\infty} |\theta_j| < \infty$$

implies that there is J sufficiently large such that $|\theta_j| < 1$ for all $|j| > J$. This implies that:

$$\sum_{|j|>J} \theta_j^2 < \sum_{|j|>J} |\theta_j| < \infty.$$

Consequently,

$$\sum_{j=-\infty}^{\infty} \theta_j^2 < \infty$$

[2] The Cauchy-Schwarz inequality and the summability condition implies that $|Cov(X_t, X_{t+h})| < \infty$

$$X_t = \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}.$$

MA(∞) are THE most popular time series models. Another popular way of referring to MA(∞) models is is using the definition of causality.

Definition: A time series X_t is a causal function of $\{\epsilon_t\}$, if X_t depends only on $\epsilon_t, \epsilon_{t-1}, \epsilon_{t-2}, \ldots$.

We can define MA(∞) models as <u>causal linear processes.</u>

Consider the causal linear process that satisfies the equation:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \epsilon_t. \tag{1}$$

We will refer to (1) as the AR(p) model (autoregressive model of order p).

1 The IRF coefficients of an AR(p) model

We would like to find the IRF coefficients $\psi_0, \psi_1, \psi_2, \ldots$ such that:

$$X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}.$$

Note that whatever these coefficients are, we can write:

$$\begin{aligned}
X_t &= \psi_0\epsilon_t + \psi_1\epsilon_{t-1} + \psi_2\epsilon_{t-2} + \psi_3\epsilon_{t-3} + \psi_4\epsilon_{t-4} + \ldots \psi_p \ \epsilon_{t-p} \ + \ldots . \\
\phi_1 X_{t-1} &= \phi_1 [\psi_0\epsilon_{t-1} + \psi_1\epsilon_{t-2} + \psi_2\epsilon_{t-3} + \psi_3\epsilon_{t-4} + \ldots \psi_{p-1}\epsilon_{t-p} + \ldots \\
\phi_2 X_{t-2} &= \phi_2 [\psi_0\epsilon_{t-2} + \psi_1\epsilon_{t-3} + \psi_2\epsilon_{t-4} + \ldots \psi_{p-2}\epsilon_{t-p} + \ldots \\
\vdots \quad & \vdots \\
\phi_p X_{t-p} &= \phi_p [\psi_0 \ \epsilon_{t-p} \ + \ldots
\end{aligned}$$

Moreover, we know the AR(p) model satisfies:

$$\begin{gathered}
X_t \\
-\phi_1 X_{t-1} \\
\vdots \\
-\phi X_{t-p} \\
= \\
\epsilon_t
\end{gathered}$$

Consequently, the IRF coefficients of an AR(p) model can be written recursively as $\psi_0 = 1$ and:

$$\psi_j \equiv \sum_{k=1}^{j} \phi_k \psi_{j-k}, \quad \text{where } \phi_j = 0 \text{ if } j > p.$$

Getting analytical expressions for these coefficients even in simple models (like the AR(2) discussed in the Homework) can easily become complicated. The main takeaway from this section is that you can always write a simple Matlab function that takes the AR coefficients as inputs and returns the IRFs.

2 Summability condition for the AR(p) model

How do we know that the IRF coefficients corresponding to some AR coefficients satisfy the summability condition for linear processes? The answer requires some algebra tricks (nothing very deep).

2.1 A necessary condition for summability

Note that if the IRF coefficients are summable, then for any number z such that $|z| \leq 1$:[3]

$$-\infty < \psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots < \infty.$$

Moreover, note that the definition of ψ_j implies that for any $|z|$ such that $|z| \leq 1$:

$$(\psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots)(1 - \phi_1 z - \phi_2 z^2 - \ldots \phi_p z^p) = 1,$$

(we did the algebra for this equation in class). Consequently, the summability of the IRF coefficients implies that for any number such that $|z| \leq 1$, the equation:

$$(1 - \phi_1 z - \phi_2 z^2 - \ldots \phi_p z^p)$$

cannot equal zero! Thus, a <u>necessary</u> condition on $(\phi_1, \ldots, \phi_p)$ for summability of the IRF coefficients is that the equation:

[3] z can a <u>real</u> number or a <u>complex</u> number. If you do not know what the latter is, do not worry: just follow the argument assuming that z is a real number.

$$(1 - \phi_1 z - \phi_2 z^2 - \dots \phi_p z^p) = 0 \tag{2}$$

cannot have a solution z^* such that $|z^*| \leq 1$.

Using Matlab to check the necessary condition: Verifying the condition above with pen and paper can be quite complicated (even in simple models like the AR(2)!). However, it is pretty straightforward to use Matlab to figure out whether or not some values of AR coefficients satisfy the necessary condition for summability. Here is an example.

Suppose that I give you the AR(2) model:

$$X_t = .5X_{t-1} + .8X_{t-2} + \epsilon_{t-1}, \tag{3}$$

and I ask you if you can represent this equation as a causal linear process. Based on the result above, you only need to solve the equation:

$$.8z^2 + .5z - 1 = 0,$$

and get the absolute value of the roots. You can do this using the Matlab command:

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roots[.8 .5 -1].
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In this case, there are two real roots:

$$z_1^* = -1.4734, \quad z_2^* = .8484.$$

and, clearly, $|z_2^*| < 1$. This means that we have found a number with 'module'smaller than 1 such that:

$$.8z^2 + .5z - 1 = 0.$$

This means that (3) cannot be represented as a causal linear process!

2.2 A sufficient condition for summability

Now we want to show that if:

$$1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p \neq 0$$

for all $|z| \leq 1$, then:

$$\sum_{j=0}^{\infty} |\psi_j| < \infty. \tag{4}$$

This is actually easier than the necessary condition. We have shown that the definition of the IRF coefficients implies that:

$$(\psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots)(1 - \phi_1 z - \phi_2 z^2 - \ldots \phi_p z^p) = 1.$$

This means that:

$$(\psi_0 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \ldots) = \frac{1}{(1 - \phi_1 z - \phi_2 z^2 - \ldots \phi_p z^p)}.$$

for all $|z| \leq 1$. Note that function:

$$\frac{1}{(1 - \phi_1 z - \phi_2 z^2 - \ldots \phi_p z^p)}$$

is continuous around 1, therefore there is some $\epsilon > 0$ such that:

$$-\infty < \psi_0 + \psi_1(1+\epsilon) + \psi_2(1+\epsilon)^2 + \psi_3(1+\epsilon)^3 + \ldots \infty.$$

This implies that:

$$\psi_j(1+\epsilon)^j \to 0,$$

and therefore, there is some $K > 0$ such that for j large enough:

$$|\psi_j| \leq \frac{K}{(1+\epsilon)^j}.$$

This gives the desired result.