2.2 Linear Processes

Ignore this part, and go directly to here

The class of linear time series models, which includes the class of autoregressive moving-average (ARMA) models, provides a general framework for studying stationary processes. In fact, every second-order stationary process is either a linear process or can be transformed to a linear process by subtracting a *deterministic* component. This result is known as Wold's decomposition and is discussed in Section 2.6.

Definition 2.2.1

The time series $\{X_t\}$ is a **linear process** if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \tag{2.2.1}$$

for all t, where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constants with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$.

During lecture, I will read a fragment of the paper "The summation of random causes as the source of cyclic processes" (1936, E. Slutsky) to give an idea of where these type of models come from

First observation: MA(q) models are a special case of linear processes. (and also a special case of MA(∞).

In terms of the backward shift operator B, (2.2.1) can be written more compactly as

$$X_t = \psi(B) Z_t, \tag{2.2.2}$$

this is simply the operator defined as

where $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$. A linear process is called a **moving average or MA(∞)** if $\psi_j = 0$ for all $j < 0$, i.e., if

where $B^j Z_t = Z_{t-j}$

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

Remark 1. The condition $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ ensures that the infinite sum in (2.2.1) converges (with probability one), since $E|Z_t| \le \sigma$ and

$$E|X_t| \le \sum_{j=-\infty}^{\infty} (|\psi_j| E|Z_{t-j}|) \le \left(\sum_{j=-\infty}^{\infty} |\psi_j| \right) \sigma < \infty.$$

Pause for one second here and make sure you understand where the inequality $E[|X_t|] < \infty$ comes from!

It also ensures that $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ and hence (see Appendix C, Example C.1.1) that the series in (2.2.1) converges in mean square, i.e., that X_t is the mean square limit of the partial sums $\sum_{j=-n}^{n} \psi_j Z_{t-j}$. The condition $\sum_{j=-n}^{n} |\psi_j| < \infty$ also ensures convergence in both senses of the more general series (2.2.3) considered in Proposition 2.2.1 below. In Section 10.5 we consider a more general class of linear processes, the fractionally integrated ARMA processes, for which the coefficents are not absolutely summable but only square summable. □

Do you see why $E[|Z_t|] \le \sigma$?

Second observation: Any LINEAR PROCESS IS STATIONARY IN THE WEAK SENSE.

(I will clarify this in class).

Ignore (I mean ignore that not that)

The operator $\psi(B)$ can be thought of as a linear filter, which when applied to the white noise "input" series $\{Z_t\}$ produces the "output" $\{X_t\}$ (see Section 4.3). As established in the following proposition, a linear filter, when applied to any stationary input series, produces a stationary output series.

A linear filter is just a fancy word for a function (linear) that takes a time series $\{Z_t\}_{t \in T}$ and generates another one: $\{X_t\}_{t \in T}$

Proposition 2.2.1 *Let $\{Y_t\}$ be a stationary time series with mean 0 and covariance function γ_Y. If $\sum_{j=-\infty}^{\infty}|\psi_j| < \infty$, then the time series*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} = \psi(B)Y_t \tag{2.2.3}$$

is stationary with mean 0 and autocovariance function

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \psi_j\psi_k\gamma_Y(h+k-j). \tag{2.2.4}$$

In the special case where $\{X_t\}$ is a linear process,

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j\psi_{j+h}\sigma^2. \tag{2.2.5}$$

(or Y_t is white noise)

This proposition is just ALGEBRA, but it helps going over it just to get a sense of how "combining" a time series X_t linearly, alters the ACF.

Before going over the proof, let's figure out the acf of $X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$.

Just think of X_t as

$$X_t = \begin{array}{l} \vdots \\ \psi_{-2}\varepsilon_{t+2} \\ +\ \psi_{-1}\varepsilon_{t+1} \\ +\ \psi_0\varepsilon_t \\ +\ \psi_1\varepsilon_{t-1} \\ +\ \psi_2\varepsilon_{t-2} \\ \vdots \end{array}$$

These are called "leads" of ε_t

These are called lags of ε_t

Proof The argument used in Remark 1, with σ replaced by $\sqrt{\gamma_Y(0)}$, shows that the series in (2.2.3) is convergent. Since $EY_t = 0$, we have

$$E(X_t) = E\left(\sum_{j=-\infty}^{\infty}\psi_j Y_{t-j}\right) = \sum_{j=-\infty}^{\infty}\psi_j E(Y_{t-j}) = 0$$

and

$$\begin{aligned} E(X_{t+h}X_t) &= E\left[\left(\sum_{j=-\infty}^{\infty}\psi_j Y_{t+h-j}\right)\left(\sum_{k=-\infty}^{\infty}\psi_k Y_{t-k}\right)\right] \\ &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\psi_j\psi_k E(Y_{t+h-j}Y_{t-k}) \\ &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\psi_j\psi_k\gamma_Y(h-j+k), \end{aligned}$$

Make sure that you at least follow the case where $Y_t = \varepsilon_t$ (white noise)

which shows that $\{X_t\}$ is stationary with covariance function (2.2.4). (The interchange of summation and expectation operations in the above calculations can be justified by the absolute summability of ψ_j.) Finally, if $\{Y_t\}$ is the white noise sequence $\{Z_t\}$ in (2.2.1), then $\gamma_Y(h-j+k) = \sigma^2$ if $k = j-h$ and 0 otherwise, from which (2.2.5) follows. ∎

Remark 2. The absolute convergence of (2.2.3) implies (Problem 2.6) that filters of the form $\alpha(B) = \sum_{j=-\infty}^{\infty}\alpha_j B^j$ and $\beta(B) = \sum_{j=-\infty}^{\infty}\beta_j B^j$ with absolutely summable coefficients can be applied successively to a stationary series $\{Y_t\}$ to generate a new stationary series

$$W_t = \sum_{j=-\infty}^{\infty}\psi_j Y_{t-j},$$

O.K. Not a great remark !!

Imagine you want to get $E[X_t X_{t-1}]$

Think about what will happen with ε_t first.

ε_t appears in X_t with a coeff. of ψ_0. It also appears in X_{t-1} with a coeff. ψ_{-1} (a lead). So the movements in ε_t will cont. $\psi_{-1}\psi_0$ to the covariance.

How about ε_{t-1}? This contributes $\psi_0\psi_1$. Continue doing this and you will get $\gamma_X(1) = \sum_{j=-\infty}^{\infty}\psi_j\psi_{j+1}\sigma^2$ (σ^2 is the variance of epsilon) where

$$\psi_j = \sum_{k=-\infty}^{\infty} \alpha_k \beta_{j-k} = \sum_{k=-\infty}^{\infty} \beta_k \alpha_{j-k}. \tag{2.2.6}$$

These relations can be expressed in the equivalent form

$$W_t = \psi(B) Y_t,$$

where

$$\psi(B) = \alpha(B)\beta(B) = \beta(B)\alpha(B), \tag{2.2.7}$$

and the products are defined by (2.2.6) or equivalently by multiplying the series $\sum_{j=-\infty}^{\infty} \alpha_j B^j$ and $\sum_{j=-\infty}^{\infty} \beta_j B^j$ term by term and collecting powers of B. It is clear from (2.2.6) and (2.2.7) that the order of application of the filters $\alpha(B)$ and $\beta(B)$ is immaterial. □

We are finally ready for this example!

Example 2.2.1 An AR(1) process

In Example 1.4.5, an AR(1) process was defined as a stationary solution $\{X_t\}$ of the equations

$$X_t - \phi X_{t-1} = Z_t, \tag{2.2.8}$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, $|\phi| < 1$, and Z_t is uncorrelated with X_s for each $s < t$. To show that such a solution exists and is the unique stationary solution of (2.2.8), we consider the linear process defined by

Ignore this part

AR(1) process: for $|\phi| < 1$ (or theta if you want) define $X_t \equiv \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$ where $\{\varepsilon_t\}$ is white noise.

an start here

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}. \tag{2.2.9}$$

(The coefficients ϕ^j for $j \geq 0$ are absolutely summable, since $|\phi| < 1$.) It is easy to verify directly that the process (2.2.9) is a solution of (2.2.8), and by Proposition 2.2.1 it is also stationary with mean 0 and ACVF

$$\gamma_X(h) = \sum_{j=0}^{\infty} \phi^j \phi^{j+h} \sigma^2 = \frac{\sigma^2 \phi^h}{1 - \phi^2},$$

for $h \geq 0$.

Note that an AR(1) process is a SPECIAL case of an MA(∞) process in which the j-th MA coefficient is restricted to have the form ϕ^j.

To show that (2.2.9) is the *only* stationary solution of (2.2.8) let $\{Y_t\}$ be *any* stationary solution. Then, iterating (2.2.8), we obtain

$$\begin{aligned} Y_t &= \phi Y_{t-1} + Z_t \\ &= Z_t + \phi Z_{t-1} + \phi^2 Y_{t-2} \\ &= \cdots \\ &= Z_t + \phi Z_{t-1} + \cdots + \phi^k Z_{t-k} + \phi^{k+1} Y_{t-k-1}. \end{aligned}$$

We can ignore this part.

If we take (2.2.9) as the definition of an AR(1) we do not need to worry about this

If $\{Y_t\}$ is stationary, then EY_t^2 is finite and independent of t, so that

$$E(Y_t - \sum_{j=0}^{k} \phi^j Z_{t-j})^2 = \phi^{2k+2} E(Y_{t-k-1})^2$$

$$\to 0 \text{ as } k \to \infty.$$

This implies that Y_t is equal to the mean square limit $\sum_{j=0}^{\infty} \phi^j Z_{t-j}$ and hence that the process defined by (2.2.9) is the unique stationary solution of the equations (2.2.8).

It the case $|\phi| > 1$, the series in (2.2.9) does not converge. However, we can rewrite (2.2.8) in the form

$$X_t = -\phi^{-1} Z_{t+1} + \phi^{-1} X_{t+1}. \tag{2.2.10}$$

Iterating (2.2.10) gives

$$\begin{aligned} X_t &= -\phi^{-1} Z_{t+1} - \phi^{-2} Z_{t+2} + \phi^{-2} X_{t+2} \\ &= \cdots \\ &= -\phi^{-1} Z_{t+1} - \cdots - \phi^{-k-1} Z_{t+k+1} + \phi^{-k-1} X_{t+k+1}, \end{aligned}$$

which shows, by the same arguments used above, that

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j} \tag{2.2.11}$$

is the unique stationary solution of (2.2.8). This solution should not be confused with the nonstationary solution $\{X_t\}$ of (2.2.8) obtained when X_0 is any specified random variable that is uncorrelated with $\{Z_t\}$.

The solution (2.2.11) is frequently regarded as unnatural, since X_t as defined by (2.2.11) is correlated with *future* values of Z_s, contrasting with the solution (2.2.9), which has the property that X_t is uncorrelated with Z_s for all $s > t$. It is customary therefore in modeling stationary time series to restrict attention to AR(1) processes with $|\phi| < 1$. Then X_t has the representation (2.2.8) in terms of $\{Z_s, s \le t\}$, and we say that $\{X_t\}$ is a **causal or future-independent** function of $\{Z_t\}$, or more concisely that $\{X_t\}$ is a causal autoregressive process. It should be noted that every AR(1) process with $|\phi| > 1$ can be reexpressed as an AR(1) process with $|\phi| < 1$ and a new white noise sequence (Problem 3.8). From a second-order point of view, therefore, nothing is lost by eliminating AR(1) processes with $|\phi| > 1$ from consideration.

If $\phi = \pm 1$, there is no stationary solution of (2.2.8) (see Problem 2.8). □

This is another definition to keep in mind: A time series is causal w.r.t $\{Z_t\}$ (or it is a causal function of Z_t) if X_t can be written as a function of $Z_t, Z_{t-1}, \ldots$

Remark 3. It is worth remarking that when $|\phi| < 1$ the unique stationary solution (2.2.9) can be found immediately with the aid of (2.2.7). To do this let $\phi(B) = 1 - \phi B$ and $\pi(B) = \sum_{j=0}^{\infty} \phi^j B^j$. Then

$$\psi(B) := \phi(B)\pi(B) = 1.$$