Lecture 11

1 Estimation in an MA(1) model: θ_1 and σ^2

Consider the MA(1) model:

$$X_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1}, \quad E[\epsilon_t] = 0 \quad E[\epsilon_t^2] = \sigma^2.$$

We are trying to figure out how to use time series data $(X_1, X_2, \ldots X_T)$ to estimate the parameters μ, θ_1, σ^2.

For μ (which is the population mean of X_t) we have already discussed the properties of the estimator:

$$\widehat{\mu} = \frac{1}{T} \sum_{t=1}^{T} X_t.$$

We also know that:

$$\gamma_0 = \sigma^2 + \theta_1^2 \sigma^2, \quad \gamma_1 = \theta_1 \sigma^2.$$

This gives θ_1 and σ^2 as a function of γ_0 and γ_1 which have direct counterparts in the data (denote them $\widehat{\gamma}_0, \widehat{\gamma}_1$). Thus, one way to try to figure out what are the values of θ_1 and σ^2 is to solve the system of equations:

$$\gamma_0 = \sigma^2 + \theta_1^2 \sigma^2 \tag{1}$$

$$\gamma_1 = \theta_1 \sigma^2 \tag{2}$$

From (2) we get that $\theta_1 = \gamma_1 / \sigma^2$ and from (1), we get that:

$$\gamma_0 = \sigma^2 + \frac{\gamma_1^2}{\sigma^2}$$

Multiplying the expression by σ^2 we have the expression:

$$(\sigma^2)^2 - \gamma_0\sigma^2 + \gamma_1^2$$

which is a quadratic equation in σ^2. Solving for σ^2 we get

$$\sigma^2 = \frac{\gamma_0 \pm \sqrt{\gamma_0^2 - 4\gamma_1^2}}{2},$$

which we can further re-write as:

$$\sigma^2 = \gamma_0 \left(\frac{1 \pm \sqrt{1 - 4(\gamma_1/\gamma_0)^2}}{2} \right)$$

and

$$\theta_1 = \frac{\gamma_1/\gamma_0}{\left(\frac{1 \pm \sqrt{1 - 4(\gamma_1/\gamma_0)^2}}{2} \right)}$$

A natural estimator for σ^2 and θ_1 replaces γ_0 and γ_1 by $\widehat{\gamma}_0$ and $\widehat{\gamma}_1$. Note that we are finding two solutions, but only one them satisfies $|\theta_1| < 1$.