Additions on Pricing a Lucas Tree*

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1 Pricing a Lucas Tree

We assume there an tree in the economy that produces y_t (could be random) at time period t. A continuum of representative households can choose to trade the shares of the tree. The shares are denoted as z_t. Therefore, the maximization problem the consumer faces is

$$\max_{\{z_t\}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right]$$

subject to a budget constraint

$$\begin{aligned} c_t + p_z z_{t+1} &\leqslant z_t y_t + p_t z_t \\ z_t &\leqslant 1 \end{aligned}$$

Therefore, the first order necessary condition is

$$u'\left(c\right) p = \beta \mathbb{E} u'\left(c'\right)\left(p' + y'\right)$$

which yields

$$p = \beta \mathbb{E}\left[\frac{u'\left(c'\right)}{u'\left(c\right)}\left(p' + y'\right)\right] \tag{1}$$

We can denote m as the stochastic discount factor between the two periods, as $m = \beta \frac{u'(c')}{u'(c)}$, and denote R as the gross return of the asset $R = \frac{p'+y'}{p}$. Notice that both m and r are random variables – they could have various possible realizations. Then we have THE formula,

$$\mathbb{E}\left[mR\right] = 1$$

If we rewrite equation (1) in the original time series form,

$$p_t = \beta \mathbb{E}_t\left[\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\left(p_{t+1} + y_{t+1}\right)\right] \tag{2}$$

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then we can iterate p_t forward (and assume $\lim_{t\to\infty}\beta^t \frac{u'(c_t)}{u'(c_0)} p_t = 0$) . Denote $m_{t,t+n} = \beta^t \frac{u'(c_{t+n})}{u'(c_t)}$ then

$$p_t = \sum_{n=1}^{\infty} \mathbb{E}_t \left[m_{t,t+n} y_{t+n} \right]$$

which means the rational price of the asset today is the discounted sum of asset payments in the future, with the stochastic discount factor from the household optimization problem.

2 CCAPM

The idea of Consumption CAPM follows the logic above. With $\mathbb{E}[mR] = 1$, we have

$$1 = \mathbb{E}[mR] = \mathbb{E}[m]\,\mathbb{E}[R] + Cov(m, R)$$

which means "it's all about the covariance".

If we were to price a risk-free bond, $p^b = \mathbb{E}[m] = \frac{1}{R^f}$ where R^f is the gross risk-free rate. Therefore

$$\frac{\mathbb{E}[R]}{R^f} = 1 - Cov(m, R)$$
$$\Downarrow$$
$$\mathbb{E}[R] - R^f = -R^f Cov(m, R)$$

The left hand side is the expected excess return, and the right hand side characterizes it. We get a large risk premium if $Cov(m, R) < 0$.

We can compare different assets using this formula,

$$\frac{\mathbb{E}[R_1] - R^f}{\mathbb{E}[R_2] - R^f} = \frac{Cov(m, R_1)}{Cov(m, R_2)} = \frac{Cov(m, R_1)\,/Var(m)}{Cov(m, R_2)\,/Var(m)} = \frac{\beta_1}{\beta_2}$$