Risk Aversion[*]

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Consider an utility function $u(\cdot)$ and a random variable x. In general, there will be a difference between $u(\mathbb{E}x)$ and $\mathbb{E}(u(x))$. A typical utility function that we use will be with positive marginal utility but decreasing, therefore $u'(x) > 0$, $u''(x) < 0$ for twice differentiable.

1 Absolute Risk Aversion

The definition of Absolute Risk Aversion (ARA) follows

$$ARA = -\frac{u''(x)}{u'(x)} \equiv R_A(x)$$

Intuitively, we consider the following question: Suppose instead of a deterministic payoff x, I have some random payoff structure that gives $x + h$ with probability π, and $x - h$ with probability $1 - \pi$. What should be the probability that makes the two options equivalent to me?

$$u(x) = \pi u(x + h) + (1 - \pi) u(x - h)$$

Taylor expansion on the right hand side will give

$$u(x) = u(x) + (2\pi - 1) h u'(x) + \frac{h^2}{2} u''(x) + O\left(h^2\right)$$

i.e.

$$\pi = \frac{1}{2} + \frac{h}{4} R_A(x)$$

2 Relative Risk Aversion

The definition of Relative Risk Aversion (RRA) is

$$RRA = -\frac{cu''(x)}{u'(x)} \equiv R_R(x)$$

Instead of the lottery example that we have in the last section, we assume that the lottery has possible payoffs $x + xh$ and $x - xh$, then

$$\pi = \frac{1}{2} + \frac{h}{4} R_R(x)$$

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3 Certainty Equivalence and Risk Premium

The maximal certain sum of money a person is willing to pay to acquire an uncertain opportunity is called **Certainty Equivalence.**

$$\mathbb{E}u\left(y+x\right) = u\left(y + CE\left(y,x\right)\right)$$

and the difference between $\mathbb{E}\left(x\right)$ and $CE(y,x)$ is called the **Risk Premium**.

$$\Pi\left(y,x\right) = \mathbb{E}x - CE\left(y,x\right)$$

and this amount can be characterized by Risk Aversion (by Taylor expansion on the right hand side),

$$\Pi\left(y,x\right) = \frac{1}{2}\sigma_x^2 R_A\left(y\right)$$

4 Application: Portfolio Choice

(I think) the most direct implication from risk aversion definitions are applied in the context of allocating wealth between 1) a risky asset and 2) a risk-free asset,

$$\max_a \mathbb{E}u\left(\left(y-a\right)r_f + ar\right)$$

where r_f is risk-free rate – a fixed number; and r is the return of the risky asset, a random variable. Under a concave utility function, we have

$$\mathbb{E}r > r_f \Leftrightarrow a > 0$$
$$\mathbb{E}r < r_f \Leftrightarrow a < 0$$

Moreover for ARA, we have

$$R_A'\left(y\right) \lesseqgtr 0 \Leftrightarrow a'\left(y\right) \gtreqless 0$$

and for RRA, instead of measuring the derivative $a'\left(y\right)$, we consider the elasticity $\eta = \frac{\mathrm{d}a}{\mathrm{d}y}\frac{y}{a}$,

$$R_R'\left(y\right) \lesseqgtr 0 \Leftrightarrow \eta \gtreqless 1$$