The Real Business Cycle Model*

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1 Model

Consider a representative agent that lives forever. Time is discrete. The agent wishes to maximize her expected discounted sum of utility,

$$\max_{\{c_t\}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right] \tag{1}$$

This agent owns capital k_t at the beginning of time period t. With capital she can produce goods following a production function $y = f\left(z, k\right)$ where z is the productivity shock this period. Therefore, production is stochastic.

The output can either be consumed or invested (we allow free-disposal, but it is not hard to show that the maximization will exploit the resource). If invested, the good transforms one-for-one into capital. We assume current consumption is c_t, and then investment is i_t. Therefore,

$$\begin{aligned} c_t + i_t &\leqslant y_t \\ y_t &= f\left(z_t, k_t\right) \end{aligned}$$

Moreover, we assume that the capital will depreciate every period with a depreciation rate δ, therefore

$$k_{t+1} = \left(1 - \delta\right) k_t + i_t$$

and we can write the budget constraint as

$$c_t + \left(k_{t+1} - \left(1 - \delta\right) k_t\right) = f\left(z_t, k_t\right) \tag{2}$$

2 Model Solution

The trade-off here is clear: the agent can choose a higher consumption today for a higher utility, or invest for a higher output tomorrow.

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2.1 Sequential Solution

Denote the Lagrange multiplier for (2) as λ_t. Then we write down the Lagrangian,

$$\mathcal{L} = \mathbb{E}\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right] - \sum_{t=0}^{\infty}\lambda_t\left(c_t + \left(k_{t+1} - (1-\delta)\,k_t\right) - f\left(z_t, k_t\right)\right)$$

Differentiate w.r.t. c_t and k_{t+1},

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'\left(c_t\right) - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \lambda_{t+1}\left[(1-\delta) + f_k\left(z_{t+1}, k_{t+1}\right)\right] - \lambda_t = 0$$

and we combine both of the equations,

$$\mathbb{E}\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\left[(1-\delta) + f_k\left(z_{t+1}, k_{t+1}\right)\right] = 1$$

This equation above is called Euler equation.

2.2 Solve a Value Function

We can write the original problem in a Bellman equation form. As we Markovize this problem, we denote variables at time t as c, k, y, and variables at time $t+1$ as c', k', y'.

$$V\left(z, k\right) = \max_{c}\left\{u\left(c\right) + \beta\mathbb{E}V\left(z', k'\right)\right\} \tag{3}$$

subject to the budget constraint

$$c + \left(k' - (1-\delta)\,k\right) = y$$
$$y = f\left(z, k\right)$$

To solve this optimization problem, we still write down the Lagrangian (denote the Lagrange multiplier as λ),

$$\mathcal{L} = u\left(c\right) + \beta\mathbb{E}V\left(z', k'\right) - \lambda\left(c + \left(k' - (1-\delta)\,k\right) - f\left(z, k\right)\right)$$

and differentiate w.r.t. c and k',

$$\frac{\partial \mathcal{L}}{\partial c} = \beta u'\left(c\right) - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial k'} = \beta\mathbb{E}\frac{\partial V\left(z', k'\right)}{\partial k'} - \lambda k' = 0$$

To compute $\frac{\partial V(z',k')}{\partial k'}$, we can try $\frac{\partial V(z,k)}{\partial k}$ and then just replace all the variables with next period ones. We need to introduce **envelope theorem** at this point,

Theorem 1. *(Envelope Theorem, no constraint) Let x^*be the extreme point of differentiable function $f\left(x, \alpha\right)$ and let*

$V(\alpha) = f(x^*, \alpha)$, then

$$\frac{\mathrm{d}V}{\mathrm{d}\alpha} = \frac{\partial f}{\partial \alpha}|_{x=x^*}$$

Corollary 1. *(Envelope Theorem, with constraint) Let $\mathcal{L}$ be the associated Lagrangian, then*

$$\frac{\mathrm{d}V}{\mathrm{d}\alpha} = \frac{\partial \mathcal{L}}{\partial \alpha}|_{x=x^*}$$

We rewrite Equation (3) by plugging in the budget constraint,

$$V(z,k) = \max_{k'} \left\{ u\left(f(z,k) + (1-\delta)k - k'\right) + \beta \mathbb{E} V\left(z', k'\right) \right\}$$

therefore we apply envelope theorem,

$$\frac{\partial V(z,k)}{\partial k} = u'(c)\left[f_k(z,k) + (1-\delta)\right]$$

and finally we put them together, we yield the same Euler equation as we did in the last class,

$$\mathbb{E}\frac{\beta u'(c')}{u'(c)}\left[(1-\delta) + f_k\left(z', k'\right)\right] = 1$$

3 Parameterization

There are several parameters and functions that govern the whole model.

Timing and Frequency The typical and the most common choice of one period is a quarter. If there's only annual data to match with, then annual frequency is somewhat fine.

Function Choices We assume consumption utility function as Constant Relative Risk Aversion (CRRA),

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

where γ is the risk-aversion.

We assume the production function as Cobb-Douglas, Decreasing Returns to Scale (DRS) on capital with z being a Hicks-neutral productivity,

$$f(z,k) = \exp(z) k^{\alpha}$$

where $\alpha < 1$ is the "span of control".

The z process can be any Markov process. In the original RBC paper, the authors assumed z following an AR(1) process,

$$z' = \rho z + \epsilon$$

where $\rho < 1$ is the persistence, and $\epsilon \sim \mathcal{N}\left(0, \sigma^2\right)$.

Parameters For a quarterly calibration,

Table 1: Parameter Choices

Name	Value	Notes
α	1/3	Capital share is about to be 1/3
β	0.99	Quarterly interest rate 1%
γ	1.0	For simplicity
δ	0.025	Annual depreciation rate 10%

4 The Simplest General Equilibrium: RBC model with Labor

The method applies in contexts in which a "representative firm" or a "representative household" is a price taker operating in a competitive equilibrium. Some people (and some literature) call it "Big K, Little k" trick. I don't like that name in particular though. Other people use "a continuum (of measure 1) of atomic representative firms/households" to indicate that the households and firms are price takers.

The rules we impose are

- The representative agent (either firm or household) takes aggregate quantities (or prices that would clear the market) as exogenous and fixed
- The quantities that the representative agents choose will clear the market

We consider a variation of the RBC model, with labor choice added.

4.1 Setting

The representative household chooses both consumption and labor every period to maximize their expected discounted sum of utility,

$$\max_{\{c_t, l_t\}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right)\right]$$

The representative firm rents the capital[1] and hires labor from the household every period. Then the firm applies a production function $y = f\left(z, k, l\right)$ for output. After that, the representative firm pays back capital rent and wages to the household. Assume the capital rent per one unit is r and the wage per one unit of labor is w. Given these prices, the representative firm chooses to hire labor and rent capital to maximize its profit,

$$\max_{l,k} f\left(z, k, l\right) - wl - rk$$

We rewrite the household's problem into Bellman equation form.

$$V\left(z, k\right) = \max_{l.c} \left\{u\left(c, l\right) + \beta \mathbb{E} V\left(z', k'\right)\right\}$$

Therefore, the household's budget constraint is

$$c + i \leqslant wl + rk$$

where

$$k' = \left(1 - \delta\right) k + i$$

[1]There are several ways to model the representative firm, all of which are equivalent.

Both the labor market and the asset market should clear.

4.2 Optimization

For the representative firm,

$$f_l(z,k,l) = w$$
$$f_k(z,k,l) = r$$

For the representative household, the Euler equation (or the intertemporal condition) is similar to what we had before, but

$$\mathbb{E}\frac{\beta u_c(c',l')}{u_c(c,l')}\left[(1-\delta)+r'\right] = 1$$

and the labor supply choice by the household is intratemporal,

$$-\frac{u_l(c,l)}{u_c(c,l)} = w$$

4.3 Additional Function Choices and Parameterization

We assume a Cobb-Douglas production function on labor and capital,

$$f(z,k,l) = zk^{\alpha}l^{1-\alpha}$$

and we assume

$$u(c,l) = \frac{c^{1-\gamma}}{1-\gamma} - \psi\frac{l^{1+\eta}}{1+\eta}$$

where ψ is the disutility of labor, and η is the inverse of Frisch elasticity of labor.