Basic Firm Dynamics Model

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1 The Simplest Model

We have a partial equilibrium investment model cast in discrete time with infinite horizon. We assume capital k is the only (fixed) factor of production.

The firm managers are risk-neutral, as they act on behalf of shareholders. Therefore, the value of the firm is the expected present value of the stream of future cash flows to the shareholders. Notice that the "cash flow" distributed to the shareholders can be negative.

The firm utilizes a constant returns to scale Cobb-Douglas production function on both capital k and labor l, scaled by a Hicks-neutral productivity shock.

$$y = sk^{\alpha}l^{1-\alpha}$$

and firms hold capital while hiring labor from the household at the wage rate w. Therefore, the firm's profit is $y - wl$ and the firm chooses l to maximize its profit.

$$\pi = \max_{l} sk^{\alpha}l^{1-\alpha} - wl$$

and this function boils down to

$$\pi = z\left(s,w\right)k^{\alpha}$$

where z is a function of s and w, but z is independent of k. If we assume w is fixed over time and s follows some exogenous Markov process, then we can treat z as an exogenous Markov process as well.

After production, the firm can choose its investment i and accumulate capital to bring into the next period,

$$k' = \left(1-\delta\right)k + i$$

where δ is the depreciation rate and i is the amount of investment.

The firm discounts its future cash flow with a discount rate β.

Therefore, we write down the Bellman equation for this problem

$$V\left(z,k\right) = \max_{k'}\left\{zk^{\alpha} - \left(k' - \left(1-\delta\right)k\right) + \beta\mathbb{E}V\left(z',k'\right)\right\}$$

Solution Characterization The Euler equation gives

$$\beta\mathbb{E}\left[(1-\delta)+\frac{\partial\pi'}{\partial k'}\right]=1$$
$$\Downarrow$$
$$\beta\mathbb{E}\left[(1-\delta)+\alpha z\left(k'\right)^{\alpha-1}\right]=1$$

2 Tobin's q

Tobin's q is the ratio between "a physical asset's market value and its replacement". Here's the original quote from James Tobin,

> *One, the numerator, is the market valuation: the going price in the market for exchanging existing assets. The other, the denominator, is the replacement or reproduction cost: the price in the market for newly produced commodities.*

Therefore, Tobin's q is the ratio of market value and book value, $\frac{V}{k}$.

While Tobin claims this "average" is an important indicator to evaluate the benefit of investment, we care about the "marginal" value of investment (or the shadow price of investment), $\frac{\partial V}{\partial k}$, i.e., *marginal Q* vs *average Q*.

3 A Discrete time version of Hayashi (1982)

We assume that firms incur a capital adjustment cost (in convex form) when firms invest or disinvest,

$$\Phi\left(\frac{i}{k}\right)=\frac{\phi}{2}\left(\frac{i}{k}-\delta\right)^2$$

More generally, we need this convex adjustment function $\Phi(\cdot)$ to follow $\Phi'(\cdot)>0, \Phi''(\cdot)\geqslant 0$ and $\Phi(\delta)=0, \Phi'(\delta)=0$. That means the firms must pay an increasing and convex cost of net investment or disinvestment.

Therefore, we can write down firm's problem

$$V(z,k)=\max_{k'}\left\{zk^{\alpha}-i-\Phi\left(\frac{i}{k}\right)k+\beta\mathbb{E}V\left(z',k'\right)\right\}$$

subject to firm's budget constraint

$$k'=i+(1-\delta)k$$

We write down the Lagrangian and denote the Lagrange multiplier as q (which resonates with *marginal Q* definitions) . Therefore,

$$q=1+\Phi'\left(\frac{i}{k}\right)$$
$$q=\beta\mathbb{E}\left(f\left(k'\right)-\Phi\left(\frac{i'}{k'}\right)+\Phi\left(\frac{i'}{k'}\right)\frac{i'}{k'}+(1-\delta)q'\right)$$

and we plug in the quadratic functional form,

$$\frac{i}{k} = \frac{1}{\phi}(q-1) + \delta$$

this says the net investment is positive if and only if $q \geqslant 1$. Put differently: a firm's investment as a fraction of its size (measured by capital stock k) should only be a function of q and it's parameters. That is to say q is a **sufficient statistic** for investment.

Hayashi (1982) shows a stronger result that under this functional form and under a constant returns to scale Cobb-Douglas production function, the marginal q and the average q are equal, i.e.

$$\frac{V}{k} = \frac{\partial V}{\partial k}$$

and V is linear in k.

But then this theory cannot explain the value premium.