Risk Aversion*

October 29, 2020

Consider an utility function $u(\cdot)$ and a random variable x. In general, there will be a difference between $u(\mathbb{E}x)$ and $\mathbb{E}(u(x))$. A typical utility function that we use will be with positive marginal utility but decreasing, therefore $u'(x) > 0$, $u''(x) < 0$ for twice differentiable.

1 Absolute Risk Aversion

The definition of Absolute Risk Aversion (ARA) follows

$$ARA = -\frac{u''(x)}{u'(x)} \equiv R_A(x)$$

Intuitively, we consider the following question: Suppose instead of a deterministic payoff x, I have some random payoff structure that gives $x+h$ with probability π, and $x-h$ with probability $1-\pi$. What should be the probability that makes the two options equivalent to me?

$$u(x) = \pi u(x+h) + (1-\pi) u(x-h)$$

Taylor expansion on the right hand side will give

$$u(x) = u(x) + (2\pi - 1) hu'(x) + \frac{h^2}{2} u''(x) + O\left(h^2\right)$$

i.e.

$$\pi = \frac{1}{2} + \frac{h}{4} R_A(x)$$

2 Relative Risk Aversion

The definition of Relative Risk Aversion (RRA) is

$$RRA = -\frac{cu''(x)}{u'(x)} \equiv R_R(x)$$

Instead of the lottery example that we have in the last section, we assume that the lottery has possible payoffs $x + xh$ and $x - xh$, then

$$\pi = \frac{1}{2} + \frac{h}{4} R_R(x)$$

*Notes prepared for UBC ECON 407, 2020 fall

3 Certainty Equivalence and Risk Premium

The maximal certain sum of money a person is willing to pay to acquire an uncertain opportunity is called **Certainty Equivalence.**

$$\mathbb{E}u\left(y+x\right)=u\left(y+CE\left(y,x\right)\right)$$

and the difference between $\mathbb{E}\left(x\right)$ and $CE(y,x)$ is called the **Risk Premium**.

$$\Pi\left(y,x\right)=\mathbb{E}x-CE\left(y,x\right)$$

and this amount can be characterized by Risk Aversion (by Taylor expansion on the right hand side),

$$\Pi\left(y,x\right)=\frac{1}{2}\sigma_x^2R_A\left(y\right)$$

4 Application: Portfolio Choice

(I think) the most direct implication from risk aversion definitions are applied in the context of allocating wealth between 1) a risky asset and 2) a risk-free asset,

$$\max_a \mathbb{E}u\left(\left(y-a\right)r_f+ar\right)$$

where r_f is risk-free rate – a fixed number; and r is the return of the risky asset, a random variable. Under a concave utility function, we have

$$\mathbb{E}r>r_f\Leftrightarrow a>0$$
$$\mathbb{E}r<r_f\Leftrightarrow a<0$$

Moreover for ARA, we have

$$R_A'\left(y\right)\lesseqgtr 0\Leftrightarrow a'\left(y\right)\gtreqless 0$$

and for RRA, instead of measuring the derivative $a'\left(y\right)$, we consider the elasticity $\eta=\frac{\mathrm{d}a}{\mathrm{d}y}\frac{y}{a}$,

$$R_A'\left(y\right)\lesseqgtr 0\Leftrightarrow \eta\gtreqless 1$$