The Real Business Cycle Model*

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1 Model

Consider a representative agent that lives forever. Time is discrete. The agent wishes to maximize her expected discounted sum of utility,

$$\max_{\{c_t\}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right] \tag{1}$$

This agent owns capital k_t at the beginning of time period t. With capital she can produce goods following a production function $y = f(z, k)$ where z is the productivity shock this period. Therefore, production is stochastic.

The output can either be consumed or invested (we allow free-disposal, but it is not hard to show that the maximization will exploit the resource). If invested, the good transforms one-for-one into capital. We assume current consumption is c_t, and then investment is i_t. Therefore,

$$\begin{aligned} c_t + i_t &\leqslant y_t \\ y_t &= f\left(z_t, k_t\right) \end{aligned}$$

Moreover, we assume that the capital will depreciate every period with a depreciation rate δ, therefore

$$k_{t+1} = (1-\delta) k_t + i_t$$

and we can write the budget constraint as

$$c_t + \left(k_{t+1} - (1-\delta) k_t\right) = f\left(z_t, k_t\right) \tag{2}$$

2 Model Solution

The trade-off here is clear: the agent can choose a higher consumption today for a higher utility, or invest for a higher output tomorrow.

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2.1 Sequential Solution

Denote the Lagrange multiplier for (2) as λ_t. Then we write down the Lagrangian,

$$\mathcal{L} = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right] - \lambda_t \left(c_t + \left(k_{t+1} - \left(1-\delta\right) k_t\right) - f\left(z_t, k_t\right)\right)$$

Differentiate w.r.t. c_t and k_t,

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'\left(c_t\right) - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial k_t} = \lambda_t \left[\left(1-\delta\right) + f_k\left(z_t, k_t\right)\right] - \lambda_{t-1} = 0$$

and we combine both of the equations,

$$\mathbb{E}\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\left[\left(1-\delta\right) + f_k\left(z_{t+1}, k_{t+1}\right)\right] = 1$$

2.2 Solve a Value Function

We can write the original problem in a Bellman equation form. How?