Notes on Growth Rates

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Growth rates

Growth rates are frequently used in this class, in economics, and in life in general. Below, we discuss calculating and interpreting growth rates. There are two types of growth rates that we encounter. The first growth rate is a simple *discretely compounded growth rate*, which is (if the time period is a year) analogous to an annually compounded interest rate you may know from your finance classes. The second growth rate is a *continuously compounded growth rate*. For bond-pricing aficionados, this is analogous to a continually compounded (as opposed to an annually compounded) interest rate.

Discrete compounded growth rates

The simplest growth rates are those that are compounded each period t at discrete time intervals. If the time period is a year (which will frequently be the case), then this corresponds with annual compounding. The annually compounded growth rate relates variable x across time periods as

$$x_{t+1} = (1+g)x_t,$$

where lower case g will denote the discrete compounded growth rate.

Notation note: We will always denote the discrete compounded growth rate as g.

To compute this growth rate from data on x, one can use the formula

$$g = \frac{x_{t+1}}{x_t} - 1.$$

If we want to express this growth rate as a percent, we multiply it by 100.

Example: The FRED database reports that annual US real Gross Domestic Product (GDP) (measured in 2005 dollars) in 2010 was 13088.0 billion. For 2011, annual US real GDP was 13315.1 billion. The annual (discrete compounded) growth rate of US real GDP between 2010 and 2011 was

$$g = \frac{13315.1}{13088.0} - 1 = .0174.$$

To express this growth rate as a percent, multiply 0.0174 by 100 to obtain 1.74 percent.

Multi-period growth. The formula above is for the growth rate from period t to $t+1$. The formula over many periods has a natural extension:

$$x_{t+n} = (1+g)^n x_t,$$

which follows from repeatedly multiplying x by $(1+g)$ and the first property of exponents discussed above. To calculate the growth rate based upon data on x, one can use the formula

$$g = \left(\frac{x_{t+n}}{x_t}\right)^{\frac{1}{n}} - 1.$$

If we want to express this growth rate as a percent, we multiply it by 100.

Example: The FRED database reports that annual US real GDP (measured in 2005 dollars) in 2011 was 13315.1 billion. Annual US real GDP in 1947 was 1774.6 billion. The average annual growth rate of US real GDP between 1947 and 2011 was

$$g = \left(\frac{13315.1}{1774.6}\right)^{\frac{1}{(2011-1947)}} - 1 = 0.0320.$$

To express this growth rate as a percent, multiply 0.0320 by 100 to obtain 3.20 percent.

Note the difference in the average growth rate of 3.20 percent for the US over the post-WWII time period versus the most recent annual growth rate of 1.74 percent in the previous example.

Continuously compounded growth rates

For many purposes in this course, it will be easier to use continuously compounded growth rates. Mathematically, this device is simply an extension of the discrete growth rate discussed above when the time interval becomes infinitesimal. While this growth rate is difficult to conceptualize, it has very useful features, which we discuss below.

The continuously compounded growth rate relates variable x across time periods as

$$x_{t+1} = \exp(\gamma)x_t.$$

Notation note: We will always denote the continuously compounded growth rate as γ.

To compute this growth rate from data on x, one can use the formula

$$\gamma \;=\; \ln x_{t+1} - \ln x_t,$$

which follows from the properties of logarithms listed above. If we wish to express this growth rate as a percent, we multiply it by 100.

Example: We can compute the continuously compounded growth rate using the same data described above. Recall that the FRED database reports that annual US real GDP (measured in 2005 dollars) in 2010 was 13088 billion. For 2011, annual US real GDP was 13315.1 billion. The continuously compounded growth rate is

$$\gamma = \ln 13315.1 - \ln 13088.0 = 0.0172 \tag{1}$$

To express this growth rate as a percent, multiply 0.0172 by 100 to obtain 1.72 percent. Note the similarity of the continuously compounded growth rate and the annually compounded growth rate (1.74 percent). This similarity is not a coincidence, as we discuss below.

Continuous compounding has three useful features for measuring growth rates:

1. **Continuous compounded growth rates approximate discretely compounded growth rates.** In the example above, the continuously compounded growth rate and the annually compounded growth rate are very similar. The similarity reflects the final property of logarithms listed above. Specifically,

$$\ln(1+a) \approx a \;\;\text{when}\;\; a \;\;\text{is a small fraction},$$

where $\approx$ means approximately and the value of a is small (less than 0.10 is a good rule of thumb). In words, the logarithm of one plus a is approximately equal to a, when a is small.

In the context of growth rates, take logarithms of both sides of the discrete compounded growth formula ($x_{t+1} = (1+g)x_t$) giving

$$\ln x_{t+1} = \ln(1+g) + \ln x_t,$$

which follows from the first property of logarithms. Rearranging and applying the approximation discussed above yields

$$\ln x_{t+1} - \ln x_t = \ln(1+g) \approx g.$$

Notice that $\ln x_{t+1} - \ln x_t$ *is* the continuously compounded growth rate, γ. Putting this information together shows that when the growth rate is small, the discrete compounded growth rate g will be approximately the same as the continuously compounded growth rate γ.

2. **Continuously compounded growth rates are additive.** Suppose that you're interested in the growth rate of a product xy. For example, x might be the price deflator and y real output, so that xy is nominal output. Using our definition:

$$\gamma_{xy} \;=\; \ln\left(\frac{x_{t+1}y_{t+1}}{x_t y_t}\right) \;=\; \ln\left(\frac{x_{t+1}}{x_t}\right) + \ln\left(\frac{y_{t+1}}{y_t}\right) \;=\; \gamma_x + \gamma_y.$$

They add up! Thus, the growth rate of a product is the sum of the growth rates. Mathematically, this result follows from the first two properties of logarithms discussed above. In the same way, the growth rate of x/y equals the growth rate of x minus the growth rate of y.

This additive feature of continuously compounded growth rates is the primary reason we use continuous compounding.

3. **Averages of continuously compounded growth rates are easy to compute**. Suppose that we want to know the *average* growth rate of x over n periods:

$$\gamma \;=\; \frac{(\ln x_t - \ln x_{t-1}) + (\ln x_{t-1} - \ln x_{t-2}) + \cdots + (\ln x_{t-n+1} - \ln x_{t-n})}{n}.$$

This expression is simply taking the average of the one-period growth rates $(\ln x_t - \ln x_{t-1})$. Now, if you look at this expression for a minute, you might notice that most of the terms cancel each other out. The term $\ln x_{t-1}$, for example, shows up twice, once with a positive sign, once with a negative sign. If we eliminate the redundant terms, we find that the average growth rate is

$$\gamma \;=\; \frac{\ln x_t - \ln x_{t-n}}{n}.$$

In other words, the average growth rate over the full period is simply the n-period growth rate divided by the number of time periods n.

Example: We can compute the average continuously compounded growth rate for post-WWII GDP data. The average annual growth rate of US real GDP between 1947 and 2011 was

$$\gamma = \frac{\ln 13315.1 - \ln 1774.6}{2011 - 1947} = 0.0315.$$

In percent terms, the average annual growth rate for the US is 3.15 percent. Note, again, that because the growth rate is relatively small, this value is similar to the annually compounded growth rate of 3.20 percent calculated in the previous example.