More Mixed-Effects Models

Read in and Prepare Data for these Notes

```
# Load foreign package to be able to read in SPSS data
> library(foreign)

# Read in the level-1 (player-level) data
> nbaL1 = read.spss(file = "http://www.tc.umn.edu/~zief0002/data/nbaLevel1.sav",
    to.data.frame = TRUE)

# Read in the level-2 (team-level) data
> nbaL2 = read.spss(file = "http://www.tc.umn.edu/~zief0002/data/nbaLevel2.sav",
    to.data.frame = TRUE)

# Load ggplot2 library
> library(ggplot2)
```

Merge the Level-1 and Level-2 Data Sets

```
# Level-1 data
Team_ID Shots_on_five Life_Satisfaction
    01              3            18.804
    01              3            18.000
    01              4            21.000
    01              4            20.500
    01              3            19.000
     ⋮              ⋮                 ⋮
    02              4             24.70
    02              4             22.10
    02              5             23.90
    02              5             24.40
    02              4             23.30
     ⋮              ⋮                 ⋮
    03              2             16.00
    03              2             15.20
    03              3             17.70
    03              3             16.48
    03              2             15.77
     ⋮              ⋮                 ⋮
```

```
# Level-2 data
Team_ID Coach_Experience
     01                2
     02                3
     03                2
     04                2
     05                1
     06                2
      ⋮                ⋮
```

Merge (or join) combines records from two data frames (tables) by using variables common to each.

```
# Merge nbaL2 into nbaL1 using the Team_ID variable
> nba = merge(nbaL1, nbaL2, by = "Team_ID")

> head(nba)

  Team_ID Shots_on_five Life_Satisfaction Coach_Experience
1      01             3            18.804                2
2      01             3            18.000                2
3      01             4            21.000                2
4      01             4            20.500                2
5      01             3            19.000                2
6      01             2            12.100                2

> tail(nba)

    Team_ID Shots_on_five Life_Satisfaction Coach_Experience
295      30             3             19.90                3
296      30             1             13.90                3
297      30             2             14.01                3
298      30             2             12.99                3
299      30             3             13.01                3
300      30             3             14.78                3
```

If you have more than one variable that you want to match on, use the `c()` function in the by= argument.

Unconditional Random Intercepts Model

The Unconditional Random Intercepts Model

Partitioning Total Outcome Variation Between and Within Persons

Level-1 Model $$Y_{ij} = \beta^*_{0i} + \epsilon_{ij} \quad \text{where} \quad \epsilon_{ij} \sim N(0, \sigma^2_\epsilon)$$

Level-2 Model $$\beta^*_{0i} = \beta_{00} + b_{0j} \quad \text{where} \quad b_{0j} \sim N(0, \sigma^2_0)$$

Composite Model $$Y_{ij} = \boxed{\beta_{00}} + \boxed{b_{0j}} + \boxed{\epsilon_i}$$

Mean life satisfaction across players

Team-specific deviation

Player-specific deviation

j is the team subscript

i is the player subscript

```
# Unconditional random intercepts model
> lmer.a = lmer(Life_Satisfaction ~ 1 + (1 | Team_ID), data = nba)

> summary(lmer.a)

REML criterion at convergence: 1726.1

Random effects:
 Groups   Name        Variance Std.Dev.
 Team_ID  (Intercept) 14.96    3.868
 Residual             14.61    3.822
Number of obs: 300, groups: Team_ID, 30

Fixed effects:
            Estimate Std. Error      df t value            Pr(>|t|)
(Intercept)  14.8067     0.7398 28.9970   20.01 <0.0000000000000002 ***
```

Interpreting the Fixed-Effects

Predicted Level-1 Model

$$\widehat{\text{Life Satisfaction}} = 14.81$$

Interpretation of Intercept

The estimated mean life satisfaction score for all players is 14.81.

Interpretation of the variance component for the random-effect of intercept

The estimated variance for the random-effect of intercept provides a measure of the between-team (team-to-team) variation of life satisfaction scores.

In our example...

There seems to be between-team variation in life satisfaction.

$$\hat{\sigma}_0^2 = 14.96$$

Interpretation of the residual variance component

The estimated residual variance provides a measure of the within-team (player-to-player) variation of life satisfaction scores.

In our example...

There seems to be within-team variation in life satisfaction.

$$\hat{\sigma}_\epsilon^2 = 14.61$$

An estimated 50.6% of the total variation in life satisfaction is attributable to differences between teams.

$$\hat{\rho} = \frac{14.96}{14.96 + 14.61} = 0.506$$

...which means that an estimated 49.4% of the total variation in life satisfaction is attributable to differences between players.

Interpreting the Random-Effects

Composite Model

$$\hat{Y}_i = \hat{\beta}_0 + b_{0j}$$

Interpretation of RE of intercept

The b_{0j} estimate for each team is the difference in predicted life satisfaction between the team average and the sample average (grand mean).

To obtain the random-effects we will use the `ranef()` function.

```
# Get estimates of the random-effects
> ranef(lmer.a)

$Team_ID
   (Intercept)
01   1.6870159
02   5.5804726
03   4.3624008
04   1.8123761
05  -3.8844101
06   1.0334296
```

Team 1: $\hat{Y}_i = 14.81 + 1.69 = 16.5$

The estimated life satisfaction for a player on team 1 is 1.69 points higher than the grand mean.

The estimated life satisfaction for a player on team 1 is 16.5.

Team 2: $\hat{Y}_i = 14.81 + 5.58 = 20.39$

The estimated life satisfaction for a player on team 2 is 5.58 points higher than the grand mean.

The estimated life satisfaction for a player on team 1 is 20.39.

Team 5: $\hat{Y}_i = 14.81 - 3.88 = 10.93$

The estimated life satisfaction for a player on team 5 is 3.88 points lower than the grand mean.

The estimated life satisfaction for a player on team 1 is 10.93.

Unconditional Random Intercepts and Random Slopes Model

Next we will include any player-level (level-1) predictors to explain within-team variation.

Level-1 Model:

$$Y_{ij} = \beta^*_{0i} + \beta^*_{1i}(X_i) + \epsilon_{ij} \quad \text{where} \quad \epsilon_{ij} \sim N(0, \sigma^2_\epsilon)$$

Level-2 Model:

$$\begin{aligned} \beta^*_{0i} &= \beta_{00} + b_{0j} \\ \beta^*_{1i} &= \beta_{10} + b_{1j} \end{aligned} \quad \text{where} \quad \begin{bmatrix} b_{0j} \\ b_{1j} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_0 & \sigma_{01} \\ \sigma_{10} & \sigma^2_1 \end{bmatrix} \right)$$

Team-specific deviation in slope

Composite Model:

$$Y_{ij} = \beta_{00} + b_{0j} + (\beta_{10} + b_{1j})\, X_i + \epsilon_{ij}$$

Player-specific deviation

Mean life satisfaction for all players with $X_i = 0$

Team-specific deviation in intercept

Average difference in life satisfaction for a one-unit difference in X_i

Fitting the Unconditional Random Intercepts and Random Slopes Model

$$Y_{ij} = \beta_{00}(1) + \beta_{10}(X_i) + [b_{0j}(1) + b_{1j}(X_i) + \epsilon_{ij}]$$

```
# Unconditional random intercepts and random slopes model
> lmer.b = lmer(Life_Satisfaction ~ 1 + Shots_on_five +
    (1 + Shots_on_five | Team_ID), data = nba)

> summary(lmer.b)

REML criterion at convergence: 1379

Random effects:
 Groups   Name          Variance Std.Dev. Corr
 Team_ID  (Intercept)   0.09279  0.3046
          Shots_on_five 0.09913  0.3148   1.00
 Residual               5.10616  2.2597
Number of obs: 300, groups: Team_ID, 30

Fixed effects:
              Estimate Std. Error      df t value            Pr(>|t|)
(Intercept)     6.4296     0.3169 76.6400   20.29 <0.0000000000000002 ***
Shots_on_five   3.2887     0.1340 27.0600   24.55 <0.0000000000000002 ***
```

Interpreting the Fixed-Effects

Predicted Level-1 Model

$$\hat{\text{Life Satisfaction}} = 6.43 + 3.2(\text{SO5})$$

Interpretation of Intercept

The mean life satisfaction for players who have a shooting success (`Shots_on_five`) of 0 is 6.43.

Interpretation of the slope

Each one-unit difference in shooting success (`Shots_on_five`) is associated with a 3.29 unit change in life satisfaction, on average.

Interpretation of the residual variance component

The estimated residual variance provides a measure of the within-team (player-to-player) variation of life satisfaction scores after accounting for shooting success.

In our example...

There seems to be within-team variation in life satisfaction scores after accounting for shooting success.

$$\hat{\sigma}_{\epsilon}^{2} = 5.11$$

The residual variation (level-1) decreased from 14.61 to 5.11.

$$r^2 = \frac{14.61 - 5.11}{14.61} = 0.650$$

The change in level-1 residual variation should **always** be compared to the *unconditional random intercepts* model.

This is a **Pseudo R^2**. Similar to the R^2 in OLS models, it measures the reduction in the level-1 residual variance.

Interpreting the Random-Effects

Fitted Composite Model

$$\hat{Y}_{ij} = \hat{\beta}_{00} + b_{0j} + \hat{\beta}_{10}(X_i) + b_{1j}(X_i)$$

Interpretation of RE of intercept

The b_{0j} estimate for each team is the difference in predicted life satisfaction between the team average and the sample average, for a shooting success of 0.

Interpretation of RE of slope

The b_{1j} estimate for each team is the difference in the effect of shooting success on life satisfaction between the team and the overall sample.

Interpretation of the variance component for the REs of intercept

The variance in the b_{0j} estimates indicates the variation in average life satisfaction across teams for a shooting success of 0.

Interpretation of the variance component for the REs of slope

The variance in the b_{1j} estimates indicates the variation in the effect of shooting success on life satisfaction across teams.

```
# Get estimates of the random-effects
> ranef(lmer.b)

$Team_ID
   (Intercept) Shots_on_five
01  0.07792009    0.08053477
02  0.36112562    0.37324350
03  0.38440629    0.39730537
04 -0.07373235   -0.07620650
05 -0.23807533   -0.24606415
06  0.17342369    0.17924307
```

Team 1:

$$\begin{aligned} \hat{\text{Life Satisfaction}} &= 6.43 + 0.08 + (3.2 + 0.08)\text{SO5} \\ &= 6.51 + 3.37(\text{SO5}) \end{aligned}$$

The estimated life satisfaction for a player whose shooting success is 0 (`Shots_on_five` = 0) on team 1 is, on average, 6.51.

On team 1, each one-unit difference in shooting success (`Shots_on_five`) is associated with a 3.37 unit change in life satisfaction.

```
# Get estimates of the random-effects
> ranef(lmer.b)

$Team_ID
   (Intercept) Shots_on_five
01  0.07792009    0.08053477
02  0.36112562    0.37324350
03  0.38440629    0.39730537
04 -0.07373235   -0.07620650
05 -0.23807533   -0.24606415
06  0.17342369    0.17924307
```

There seems to be between-team variation in intercepts.

$$\hat{\sigma}_0^2 = 0.09$$

There seems to be between-team variation in slopes.

$$\hat{\sigma}_1^2 = 0.10$$

```
# Estimates of the variance-covariance matrix of the random effects
> varCorr(lmer.b)$Team_ID

              (Intercept) Shots_on_five
(Intercept)    0.09279463    0.09590843
Shots_on_five  0.09590843    0.09912672
attr(,"stddev")
  (Intercept) Shots_on_five
    0.3046221     0.3148440
attr(,"correlation")
              (Intercept) Shots_on_five
(Intercept)             1             1
Shots_on_five           1             1
```

Square roots of the variance estimates

$$\mathbf{G} = \begin{bmatrix} 0.093 & 0.096 \\ 0.096 & 0.099 \end{bmatrix}$$

The b_{0j} estimates and b_{1j} estimates are positively related. Teams that have a higher intercept also tend to have higher slopes.

Conditional Models: Adding Level-2 Predictors

Now we will include any team-level (level-2) predictors to explain between-team variation.

Level-1 Model:

$$Y_{ij} = \beta^*_{0i} + \beta^*_{1i}(X_i) + \epsilon_{ij} \qquad \text{where} \qquad \epsilon_{ij} \sim N(0, \sigma^2_\epsilon)$$

Level-2 Model:

$$\beta^*_{0i} = \beta_{00} + \beta_{01}(G_j) + b_{0j}$$

$$\beta^*_{1i} = \beta_{10} + \beta_{11}(G_j) + b_{1j}$$

$$\text{where} \quad \begin{bmatrix} b_{0j} \\ b_{1j} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2_0 & \sigma_{01} \\ \sigma_{10} & \sigma^2_1 \end{bmatrix} \right)$$

Composite Model

$$Y_{ij} = \beta_{00} + \beta_{01}(G_j) + b_{0j} + (\beta_{10} + \beta_{11}(G_j) + b_{1j})\, X_i + \epsilon_{ij}$$

In our example...

Level-1 Model:

$$\text{Life Satisfaction}_{ij} = \beta_0^* + \beta_1^*(\text{SO5}) + \epsilon_{ij}$$

Level-2 Model:

$$\beta_0^* = \beta_{00} + \beta_{01}(\text{CE}_j) + b_{0j}$$
$$\beta_1^* = \beta_{10} + \beta_{11}(\text{CE}_j) + b_{1j}$$

Composite Model

$$\text{Life Satisfaction} = \beta_{00} + \beta_{01}(\text{CE}) + b_{0j} + (\beta_{10} + \beta_{11}(\text{CE}) + b_{1j})\,\text{SO5} + \epsilon$$

$$\text{Life Satisfaction} = \beta_{00} + \beta_{01}(\text{CE}) + \beta_{10}(\text{SO5}) + \beta_{11}(\text{CE})(\text{SO5}) + b_{0j} + b_{1j}(\text{SO5}) + \epsilon$$

The fixed-effects include the intercept, the level-1 predictor, the level-2 predictor, and the cross-level interaction.

The random-effects include the intercept and level-1 predictor.

```
> lmer.c = lmer(Life_Satisfaction ~ 1 + Shots_on_five + Coach_Experience +
    Shots_on_five:Coach_Experience + (1 + Shots_on_five | Team_ID), data = nba)

> summary(lmer.c)

REML criterion at convergence: 1341.8

Random effects:
 Groups   Name          Variance Std.Dev. Corr
 Team_ID  (Intercept)   0.4202   0.6482
          Shots_on_five 0.1486   0.3855   -1.00
 Residual               4.8233   2.1962
Number of obs: 300, groups: Team_ID, 30

Fixed effects:
                               Estimate Std. Error      df t value Pr(>|t|)
(Intercept)                      4.5760     0.7951 32.2400   5.755 2.15e-06 ***
Shots_on_five                    2.6760     0.3788 26.3100   7.064 1.57e-07 ***
Coach_Experience                 1.2293     0.4391 56.9000   2.800  0.00698 **
Shots_on_five:Coach_Experience   0.2045     0.1745 24.1600   1.172  0.25268
```

Interpreting the Residual Variance Component

In our example...

The within-team variation in life satisfaction has changed a little.

$$\hat{\sigma}_\epsilon^2 = 4.82$$

The residual variation (level-1) has decreased from 67%.

$$r^2 = \frac{14.61 - 4.82}{14.61} = 0.670$$

Remember...the change in level-1 residual variation should **always** be compared to the *unconditional random intercepts* model.

Interpreting the Between-Teams Variance Components

```
# Estimates of the variance-covariance matrix of the random effects
> varCorr(lmer.c)$Team_ID

              (Intercept) Shots_on_five
(Intercept)     0.4201584    -0.2498698
Shots_on_five  -0.2498698     0.1485985
attr(,"stddev")
  (Intercept) Shots_on_five
    0.6481963     0.3854848
attr(,"correlation")
              (Intercept) Shots_on_five
(Intercept)             1            -1
Shots_on_five          -1             1
```

Square roots of the variance estimates

$$\mathbf{G} = \begin{bmatrix} 0.420 & -0.250 \\ -0.250 & 0.149 \end{bmatrix}$$

There seems to be between-team variation in intercepts.

$$\hat{\sigma}_0^2 = 0.420$$

There seems to be between-team variation in slopes.

$$\hat{\sigma}_1^2 = 0.149$$

The b_{0j} estimates and b_{1j} estimates are negatively related. Teams that have a higher intercept also tend to have lower slopes.

Pseudo-R^2 for the Random-Effects

We can compute two additional pseudo-R^2 values.

The change in level-2 residual variation should **always** be compared to the *unconditional random intercepts and random slopes* model.

The pseudo-R^2 for intercept represents the decrease in the residual variation in the level-2 intercepts.

$$r_0^2 = \frac{0.093 - 0.420}{0.093} = -3.51$$

The pseudo-R^2 for slope represents the decrease in the residual variation in the level-2 slopes.

$$r_1^2 = \frac{0.099 - 0.149}{0.093} = -0.505$$

Unlike OLS R^2 values, pseudo-R^2 values can be negative. They can also be greater than 1. This generally happens when all (or most) of the outcome's variation is exclusively at level-1 or level-2. Then, generally a predictor will reduce variation at one level, but increase the variation at the other level.

Interpreting the Fixed-Effects

Predicted Level-1 Model

$$\hat{\text{Life Satisfaction}} = 4.58 + 2.68(\text{SO5}) + 1.23(\text{CE}) + 0.20(\text{SO5})(\text{CE})$$

Interpretation of Intercept

The mean life satisfaction for players who have a shooting success of 0 and who are on teams where the coach has no experience is 4.58.

When an interaction is in the model, the constituent main-effects are **not** interpreted.

Interpretation of Interaction

To interpret the interaction, a plot is generally advisable (or at least computing fitted partial regression models for multiple prototypical values.

$$\widehat{\text{Life Satisfaction}} = 4.58 + 2.68(\text{SO5}) + 1.23(\text{CE}) + 0.20(\text{SO5})(\text{CE})$$

For CE = 1

$$\widehat{\text{Life Satisfaction}} = 4.58 + 2.68(\text{SO5}) + 1.23(1) + 0.20(\text{SO5})(1)$$

$$\widehat{\text{Life Satisfaction}} = 5.81 + 2.88(\text{SO5})$$

For CE = 3

$$\widehat{\text{Life Satisfaction}} = 4.58 + 2.68(\text{SO5}) + 1.23(2) + 0.20(\text{SO5})(2)$$

$$\widehat{\text{Life Satisfaction}} = 7.04 + 3.08(\text{SO5})$$

For CE = 3

$$\widehat{\text{Life Satisfaction}} = 4.58 + 2.68(\text{SO5}) + 1.23(3) + 0.20(\text{SO5})(3)$$

$$\widehat{\text{Life Satisfaction}} = 8.27 + 3.28(\text{SO5})$$

The effect of shooting success on life satisfaction **depends** on the coaches level of experience.

Table 1.

Taxonomy of Multi-Level Models Fitted Using REML to Explain Variation in Life Satisfaction for 300 NBA Players

		Model A	Model B	Model C
Fixed effects				
Intercept		14.81 (0.74)	6.43 (0.32)	4.58 (0.80)
Coaching Experience				1.23 (0.44)
Shooting success			3.29 (0.13)	2.68 (0.38)
Coaching Experience x Shooting success				0.20 (0.17)
Variance components				
Level-1	$\hat{\sigma}_\epsilon^2$	14.61	5.11	4.82
Level-2	$\hat{\sigma}_0^2$	14.96	0.09	0.42
	$\hat{\sigma}_1^2$		0.099	0.15
	$\hat{\sigma}_{01}$		0.1	–0.25
Pseudo R^2 statistics and Goodness-of-fit				
$R^2_{Y,\hat{Y}}$		0.547	0.837	0.842
Deviance		1726.1	1379	1341.8
AIC		1732.1	1391	1357.8
BIC		1743.3	1413.2	1387.4

General (Matrix) Form of the Mixed-Effects Regression Model

Consider the **unconditional random intercepts** model.

These equations represent the life satisfaction measurements for 10 players on a single team, *j*.

Thus $n_j = 10$.

$$
\begin{aligned}
y_1 &= \beta_0(1) + b_{0j}(1) + \epsilon_1 \\
y_2 &= \beta_0(1) + b_{0j}(1) + \epsilon_2 \\
y_3 &= \beta_0(1) + b_{0j}(1) + \epsilon_3 \\
&\vdots \\
y_{10} &= \beta_0(1) + b_{0j}(1) + \epsilon_{10}
\end{aligned}
$$

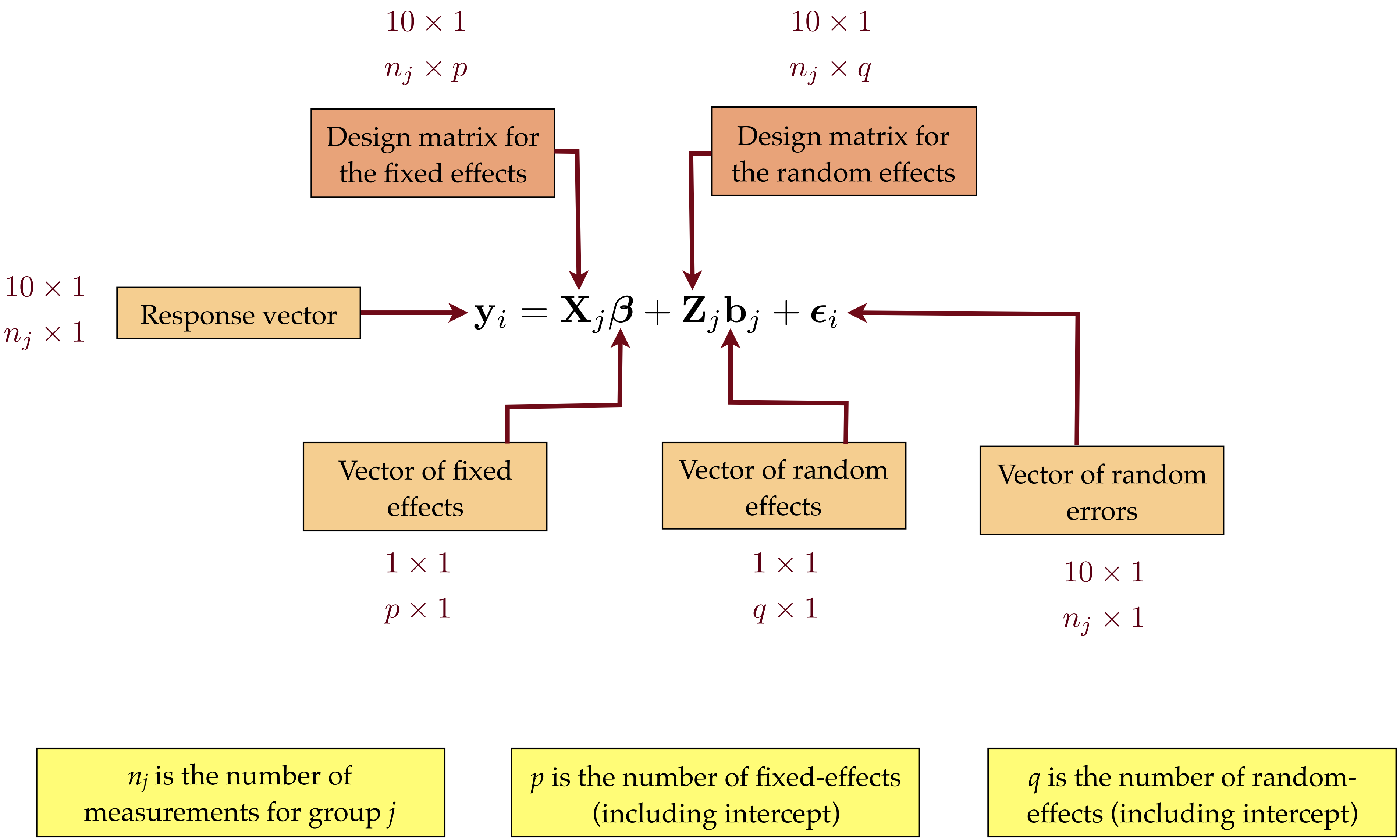
We can express these equations using matrices...

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_{10} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \beta_0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} b_{0j} \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \vdots \\ \epsilon_{10} \end{bmatrix}
$$

...which be denoted as

$$\mathbf{y}_i = \mathbf{X}_j \boldsymbol{\beta} + \mathbf{Z}_j \mathbf{b}_j + \boldsymbol{\epsilon}_i$$

This is the general form of the LMER model.


10×1
$n_j \times p$
10×1
$n_j \times q$
Design matrix for the fixed effects
Design matrix for the random effects
10×1
$n_j \times 1$
Response vector
$\mathbf{y}_i = \mathbf{X}_j\boldsymbol{\beta} + \mathbf{Z}_j\mathbf{b}_j + \boldsymbol{\epsilon}_i$
Vector of fixed effects
Vector of random effects
Vector of random errors
1×1
$p \times 1$
1×1
$q \times 1$
10×1
$n_j \times 1$
n_j is the number of measurements for group j
p is the number of fixed-effects (including intercept)
q is the number of random-effects (including intercept)

Any term with an j subscript can vary between teams

The lengths could differ because of missing data.

Z_j is typically a subset of X_j

Column dimension of X is enlarged (reduced) to accommodate models with different sets of predictors

$$\mathbf{y}_i = \mathbf{X}_j\boldsymbol{\beta} + \mathbf{Z}_j\mathbf{b}_j + \boldsymbol{\epsilon}_i$$

No j subscript
Shared by all teams

This equation has a direct connection to the `lmer()` syntax used in R, as X_j and Z_j are multipliers of the fixed and random effects, respectively.

- The columns of X_j are used in the fixed effects portion of the syntax.
- The columns of Z_j are used in the random effects portion of the syntax.

```
# Unconditional model: Random Intercepts
> lmer.a = lmer(Life_Satisfaction ~ 1 + (1|Team_ID), data = nba)
```

To obtain the design matrix for the fixed-effects we use the `model.matrix()` function.

```
> X = getME(lmer.a, “X”)

> head(X)

    (Intercept)
1             1
2             1
3             1
4             1
5             1
6             1
```

Variance-Covariance Matrix among the Level-1 Units

We now consider details regarding the random effects

The variance-covariance matrix for a single team, *j*, is written as $\mathbf{V}_j$

Symmetric matrix

$$10 \times 10$$

$$n_j \times n_j$$

$$\mathbf{V}_j = \begin{bmatrix} \boxed{\mathrm{Var}(y_{i1})} & \mathrm{Cov}(y_{i1}, y_{i2}) & \mathrm{Cov}(y_{i1}, y_{i3}) & \cdots & \mathrm{Cov}(y_{i1}, y_{i10}) \\ \mathrm{Cov}(y_{i2}, y_{i1}) & \boxed{\mathrm{Var}(y_{i2})} & \mathrm{Cov}(y_{i2}, y_{i3}) & \cdots & \mathrm{Cov}(y_{i2}, y_{i10}) \\ \mathrm{Cov}(y_{i3}, y_{i1}) & \mathrm{Cov}(y_{i3}, y_{i2}) & \boxed{\mathrm{Var}(y_{i3})} & \cdots & \mathrm{Cov}(y_{i3}, y_{i10}) \\ \vdots & \vdots & \vdots & \vdots & \\ \mathrm{Cov}(y_{i10}, y_{i1}) & \mathrm{Cov}(y_{i10}, y_{i2}) & \mathrm{Cov}(y_{i10}, y_{i3}) & \cdots & \boxed{\mathrm{Var}(y_{i10})} \end{bmatrix}$$

Covariances represent the dependency in the life satisfaction scores among players on the same team.

Variance of y_i for each player

This is decomposed into between-teams and within-teams components

$$\mathbf{V}_j = \mathbf{B}_j + \mathbf{W}_j$$

Between-teams matrix

Within-teams matrix

$$\mathbf{V}_j = \mathbf{B}_j + \mathbf{W}_j$$

Based on the variance-covariance matrix of the random effects, called *G*

$$\mathbf{G} = \left[\sigma_0^2\right]$$

No subscript...constant across teams

$$\mathbf{B}_j = \mathbf{Z}_j \mathbf{G} \mathbf{Z}_j^\intercal$$

Based on the variance of the random errors

$$\mathbf{W}_j = \sigma_\epsilon^2 \mathbf{I}_j$$

$$\mathbf{W}_j = \sigma_\epsilon^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{W}_j = \begin{bmatrix} \sigma_\epsilon^2 & 0 & 0 & 0 \\ 0 & \sigma_\epsilon^2 & 0 & 0 \\ 0 & 0 & \sigma_\epsilon^2 & 0 \\ 0 & 0 & 0 & \sigma_\epsilon^2 \end{bmatrix}$$

The 0 off-diagonals represent the independence assumption, and the equal diagonal values represent the homogeneity of variance assumption (within teams).

Finally we can rewrite *Vj* as

$$\mathbf{V}_j = \mathbf{Z}_j \mathbf{G} \mathbf{Z}_j^\intercal + \sigma_\epsilon^2 \mathbf{I}_j$$

Estimate the $\mathbf{Z}_j$ matrix

```
## Random-effects design matrix
> Z = getME(lmer.a, "Z")

> head(Z, 20)

 [1,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 [2,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 [3,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 [4,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 [5,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 [6,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 [7,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 [8,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 [9,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[10,] 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[11,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[12,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[13,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[14,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[15,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[16,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[17,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[18,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[19,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
[20,] . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . .
```

Estimate the **G** matrix

```
## Estimated variances, standard deviations, and correlations between the random-
effects terms
> est = as.data.frame(VarCorr(lmer.a))

> est

       grp        var1 var2     vcov    sdcor
1  Team_ID (Intercept) <NA> 14.95920 3.867713
2 Residual        <NA> <NA> 14.60563 3.821731

## Estimated variance of random-effect
> var.b0 = est$vcov[1]
> var.b0

[1] 14.9592

## Estimated G matrix
> I = diag(30)
> G = var.b0 * I
```

G matrix for the first team

```
> G[1:10, 1:10]

         [,1]    [,2]    [,3]    [,4]    [,5]    [,6]    [,7]    [,8]    [,9]   [,10]
 [1,] 14.9592  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
 [2,]  0.0000 14.9592  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
 [3,]  0.0000  0.0000 14.9592  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
 [4,]  0.0000  0.0000  0.0000 14.9592  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000
 [5,]  0.0000  0.0000  0.0000  0.0000 14.9592  0.0000  0.0000  0.0000  0.0000  0.0000
 [6,]  0.0000  0.0000  0.0000  0.0000  0.0000 14.9592  0.0000  0.0000  0.0000  0.0000
 [7,]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000 14.9592  0.0000  0.0000  0.0000
 [8,]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000 14.9592  0.0000  0.0000
 [9,]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000 14.9592  0.0000
[10,]  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000  0.0000 14.9592
```

Estimate the $\mathbf{B}_j$ matrix

```
## Estimated between-teams matrix
> B = Z %*% G %*% t(Z)

## B matrix for first team
> B[1:10, 1:10]

10 x 10 Matrix of class "dgeMatrix"
         [,1]    [,2]    [,3]    [,4]    [,5]    [,6]    [,7]    [,8]    [,9]   [,10]
 [1,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [2,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [3,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [4,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [5,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [6,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [7,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [8,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [9,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
[10,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
```

Remember, the `%*%` operator is how we carry out matrix multiplication in R. The `t()` function computes the transpose of a matrix.

The between-teams matrix suggests that, the diagonal elements and off-diagonal elements are all non-zero and constant.

$\mathbf{B_j}$ matrix for all teams

```
         [,1]    [,2] ...    [,9]   [,10]   [,11]    [,12] ...   [,19]   [,20] ...
 [1,] 14.9592 14.9592 ... 14.9592 14.9592
 [2,] 14.9592 14.9592 ... 14.9592 14.9592
   ⋮       ⋮       ⋮    ⋱      ⋮       ⋮
 [9,] 14.9592 14.9592 ... 14.9592 14.9592
[10,] 14.9592 14.9592 ... 14.9592 14.9592
[11,]                                   14.9592 14.9592 ... 14.9592 14.9592
[12,]                                   14.9592 14.9592 ... 14.9592 14.9592
   ⋮                                         ⋮       ⋮    ⋱      ⋮       ⋮
[19,]                                   14.9592 14.9592 ... 14.9592 14.9592
[20,]                                   14.9592 14.9592 ... 14.9592 14.9592
   ⋮
```

Estimate the $\mathbf{W_j}$ matrix

```
## Estimated value of the error variance
> var.err = est$vcov[2]
> var.err

[1] 14.60563

## Create a 10x10 identity matrix
> ident = diag(10)
> ident

      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
 [1,]    1    0    0    0    0    0    0    0    0     0
 [2,]    0    1    0    0    0    0    0    0    0     0
 [3,]    0    0    1    0    0    0    0    0    0     0
 [4,]    0    0    0    1    0    0    0    0    0     0
 [5,]    0    0    0    0    1    0    0    0    0     0
 [6,]    0    0    0    0    0    1    0    0    0     0
 [7,]    0    0    0    0    0    0    1    0    0     0
 [8,]    0    0    0    0    0    0    0    1    0     0
 [9,]    0    0    0    0    0    0    0    0    1     0
[10,]    0    0    0    0    0    0    0    0    0     1
```

Same value listed in the residual of the `summary()` output

```
## Estimated W matrix for team j
> W.j = var.err * ident
> W.j

          [,1]     [,2]     [,3]     [,4]     [,5]     [,6]     [,7]     [,8]     [,9]    [,10]
 [1,] 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000
 [2,]  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000
 [3,]  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000
 [4,]  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000
 [5,]  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000
 [6,]  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000
 [7,]  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000
 [8,]  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000
 [9,]  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000
[10,]  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563
```

An assumption of the model is

$$\epsilon_{ij} \sim N(0, \sigma_\epsilon^2)$$

The within-teams matrix ($\boldsymbol{W}_j$) has identical diagonal elements. In addition, the off-diagonal elements are all equal to 0.

$\mathbf{W_j}$ matrix for all teams

```
        [,1]     [,2]  ...     [,9]    [,10]     [,11]    [,12]  ...    [,19]    [,20]  ...
 [1,] 14.60563  0.00000  ...  0.00000  0.00000
 [2,]  0.00000 14.60563  ...  0.00000  0.00000
   ⋮      ⋮        ⋮      ⋱       ⋮        ⋮
 [9,]  0.00000  0.00000  ... 14.60563  0.00000
[10,]  0.00000  0.00000  ...  0.00000 14.60563
[11,]                                         14.60563  0.00000  ...  0.00000  0.00000
[12,]                                          0.00000 14.60563  ...  0.00000  0.00000
   ⋮                                               ⋮        ⋮      ⋱       ⋮        ⋮
[19,]                                          0.00000  0.00000  ... 14.60563  0.00000
[10,]                                          0.00000  0.00000  ...  0.00000 14.60563
  ⋮
```

$\mathbf{V}_\mathrm{j}$ matrix for team j

```
## Compute the estimated variance-covariance matrix for team j
> V = B[1:10, 1:10] + W.j

> V

          1        2        3        4        5        6        7        8        9       10
1  29.56483 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920
2  14.95920 29.56483 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920
3  14.95920 14.95920 29.56483 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920
4  14.95920 14.95920 14.95920 29.56483 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920
5  14.95920 14.95920 14.95920 14.95920 29.56483 14.95920 14.95920 14.95920 14.95920 14.95920
6  14.95920 14.95920 14.95920 14.95920 14.95920 29.56483 14.95920 14.95920 14.95920 14.95920
7  14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 29.56483 14.95920 14.95920 14.95920
8  14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 29.56483 14.95920 14.95920
9  14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 29.56483 14.95920
10 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 14.95920 29.56483
```

- The diagonal elements are **model-based estimates** of the variances for the life satisfaction scores.
  - ✓ Since there are no predictors in the model, these are all constant

- The off-diagonal elements are **model-based estimates** of the covariances between life satisfaction scores.
  - ✓ Since there are no predictors in the model, these are all constant.

```
          [,1]     [,2]     [,3]     [,4]     [,5]     [,6]     [,7]     [,8]     [,9]    [,10]
 [1,] 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000
 [2,]  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000
 [3,]  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000
 [4,]  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000
 [5,]  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000  0.00000
 [6,]  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000  0.00000
 [7,]  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000  0.00000
 [8,]  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000  0.00000
 [9,]  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563  0.00000
[10,]  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000  0.00000 14.60563
```

Comparing the within-teams matrix and the between-teams matrix, it becomes clear that the pattern shown in the variance-covariance matrix is accounted for by the between-teams matrix....

… meaning, the presence of random effects can account for possible heterogeneity and dependency among the measurements

```
         [,1]    [,2]    [,3]    [,4]    [,5]    [,6]    [,7]    [,8]    [,9]   [,10]
 [1,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [2,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [3,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [4,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [5,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [6,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [7,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [8,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
 [9,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
[10,] 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592 14.9592
```

Standardized $\mathbf{V}_\mathrm{j}$ matrix for team j

To help examine the correlational structure among the measurements, V_j can be standardized as

$$\mathbf{V}_j^* = \mathbf{D}_j \mathbf{V}_j \mathbf{D}_j$$

where $\boldsymbol{D}_j$ is a diagonal matrix with elements $\dfrac{1}{\sqrt{\mathrm{Var}(y_{ij})}}$

```
## Create the diagonal matrix D
> D = diag(1 / sqrt(diag(V)))

> D

         [,1]    [,2]    [,3]    [,4]    [,5]    [,6]    [,7]    [,8]    [,9]   [,10]
 [1,] 0.18391 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000
 [2,] 0.00000 0.18391 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000
 [3,] 0.00000 0.00000 0.18391 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000
 [4,] 0.00000 0.00000 0.00000 0.18391 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000
 [5,] 0.00000 0.00000 0.00000 0.00000 0.18391 0.00000 0.00000 0.00000 0.00000 0.00000
 [6,] 0.00000 0.00000 0.00000 0.00000 0.00000 0.18391 0.00000 0.00000 0.00000 0.00000
 [7,] 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.18391 0.00000 0.00000 0.00000
 [8,] 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.18391 0.00000 0.00000
 [9,] 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.18391 0.00000
[10,] 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.18391
```

Standardized $\mathbf{V}_j$ matrix for team j

```
## Compute the estimated standardized variance-covariance (correlation) matrix
> Vstar = D %*% V %*% D

> Vstar

         [,1]    [,2]    [,3]    [,4]    [,5]    [,6]    [,7]    [,8]    [,9]   [,10]
 [1,] 1.00000 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598
 [2,] 0.50598 1.00000 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598
 [3,] 0.50598 0.50598 1.00000 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598
 [4,] 0.50598 0.50598 0.50598 1.00000 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598
 [5,] 0.50598 0.50598 0.50598 0.50598 1.00000 0.50598 0.50598 0.50598 0.50598 0.50598
 [6,] 0.50598 0.50598 0.50598 0.50598 0.50598 1.00000 0.50598 0.50598 0.50598 0.50598
 [7,] 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 1.00000 0.50598 0.50598 0.50598
 [8,] 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 1.00000 0.50598 0.50598
 [9,] 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 1.00000 0.50598
[10,] 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 0.50598 1.00000
```

All the elements are **model-based estimates** of the correlations between life satisfaction scores.

The correlations show the same **pattern** shown by the covariances.