Some Matrix Algebra

To enter a matrix in R, use the `matrix()` function.

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & -6 \\ 7 & 8 & 9 \\ 0 & 0 & 10 \end{pmatrix}$$

```
> X = matrix(
    x = c(1, 4, 7, 0, -2, -5, 8, 0, 3, -6, 9, 10),
    nrow = 4,
    ncol = 3
    )

> X
     [,1] [,2] [,3]
[1,]    1   -2    3
[2,]    4   -5   -6
[3,]    7    8    9
[4,]    0    0   10
```

By default, elements are filled in columns.

The `byrow=TRUE` argument will fill the elements by rows rather than columns.

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & -6 \\ 7 & 8 & 9 \\ 0 & 0 & 10 \end{pmatrix}$$

```
> X = matrix(c(1, -2,  3,
               4, -5, -6,
               7,  8,  9,
               0,  0, 10),
    byrow = TRUE,
    nrow = 4
    )

> X
     [,1] [,2] [,3]
[1,]    1   -2    3
[2,]    4   -5   -6
[3,]    7    8    9
[4,]    0    0   10
```

Enter the matrix **B** (Fox, p.3) into R.

$$\mathbf{B} = \begin{pmatrix} -5 & 1 & 3 \\ 2 & 2 & 6 \\ 7 & 3 & -4 \end{pmatrix}$$

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```
> B = matrix(c(-5, 1,  3,
                2, 2,  6,
                7, 3, -4),
    byrow = TRUE,
    nrow = 3
    )

> B
     [,1] [,2] [,3]
[1,]   -5    1    3
[2,]    2    2    6
[3,]    7    3   -4
```

The `dim()` function will return the dimensions of a matrix.

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & -6 \\ 7 & 8 & 9 \\ 0 & 0 & 10 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -5 & 1 & 3 \\ 2 & 2 & 6 \\ 7 & 3 & -4 \end{pmatrix}$$

```
> dim(X)

[1] 4 3

> dim(B)

[1] 3 3
```

The `diag()` function will return the diagonal of a square matrix.

$$\mathbf{X} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & -6 \\ 7 & 8 & 9 \\ 0 & 0 & 10 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -5 & 1 & 3 \\ 2 & 2 & 6 \\ 7 & 3 & -4 \end{pmatrix}$$

```
> diag(B)

[1] -5  2 -4

# The trace of a matrix is the sum of its diagonal elements
> sum(diag(B))

[1] -7
```

```
> library(psych)
> tr(B)

[1] -7
```

Find the diagonal and trace of **X**.

The `diag()` function can also be used to create an **identity matrix**.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
# The argument is the number of rows and columns
> diag(3)

     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
```

Why is **I** a scalar matrix?

Enter the matrices **A** and **B** (Fox, p.5) into R.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -5 & 1 & 2 \\ 3 & 0 & 4 \end{pmatrix}$$

Add them together.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -5 & 1 & 2 \\ 3 & 0 & 4 \end{pmatrix}$$

```
> A = matrix(c(1, 2,  3,
               4, 5,  6),
    byrow = TRUE,
    nrow = 2
    )

> B = matrix(c(-5, 1, 2,
                3, 0, 4),
    byrow = TRUE,
    nrow = 2
    )

> A + B
     [,1] [,2] [,3]
[1,]   -4    3    5
[2,]    7    5   10
```

Compute 3 x **B**.

$$\mathbf{B} = \begin{pmatrix} -5 & 1 & 2 \\ 3 & 0 & 4 \end{pmatrix}$$

```
> 3 * B
     [,1] [,2] [,3]
[1,]  -15    3    6
[2,]    9    0   12
```

Compute **A** x **B.**

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -5 & 1 & 2 \\ 3 & 0 & 4 \end{pmatrix}$$

```
> A * B

     [,1] [,2] [,3]
[1,]   -5    2    6
[2,]   12    0   24
```

How did we get these elements?

Multiply **A** by **I.**

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Check your answer with p. 8 of Fox.

To carry out **matrix multiplication** we use the `%*%` operator.

```
> A %*% diag(3)

     [,1] [,2] [,3]
[1,]    1    2    3
[2,]    4    5    6
```

What happens when you **postmultiply** a matrix by the identity matrix?

Is matrix multiplication commutative?

A vector can be input into R using the `c()` function or the `matrix()` function.

$$\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 6 \\ 0 \\ 9 \end{pmatrix}$$

```
> a = c(2, 0, 1, 3)
> a

[1] 2 0 1 3

> b = matrix(c(-1, 6, 0, 9), ncol = 1)
> b

     [,1]
[1,]   -1
[2,]    6
[3,]    0
[4,]    9
```

What are the dimensions of **a** and **b**?

$$\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 6 \\ 0 \\ 9 \end{pmatrix}$$

```
> dim(a)

NULL

> dim(b)

[1] 4 1
```

Technically, the `c()` function produces a column vector and the `matrix()` function produces a one-column matrix. The difference is in their classes and we can, for all intents and purposes, just use the `c()` function..

The **dot product** (or inner product) of two vectors can be computed using the `%*%` operator..

$$\mathbf{a}^{\mathsf{T}} \bullet \mathbf{b} = \begin{pmatrix} 2 & 0 & 1 & 3 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 6 \\ 0 \\ 9 \end{pmatrix}$$

```
> t(a) %*% b


     [,1]
[1,]   25
```

What are the dimensions of $\mathbf{a}^{\mathsf{T}}$ and $\mathbf{b}$?

What is the dot product of $\mathbf{a}^{\mathsf{T}}$ and $\mathbf{a}$?

Matrix multiplication is essentially computing a dot product for each element. To find element *ij*, compute the dot product between row *i* of matrix **A** and column *j* of matrix **I**.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The inverse of a matrix is the matrix that you postmultiply by to get the identity matrix.

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
> D = matrix(c(2, 5,
               1, 3),
    byrow = TRUE,
    nrow = 2
    )

> D

     [,1] [,2]
[1,]    2    5
[2,]    1    3

> solve(D)

     [,1] [,2]
[1,]    3   -5
[2,]   -1    2
```

```
> D %*% solve(D)

     [,1] [,2]
[1,]    1    0
[2,]    0    1
```

What are the dimensions of **D**?

What are the dimensions of $\mathbf{D}^{-1}$?

What are the dimensions of $\mathbf{D}\mathbf{D}^{-1}$?

Is matrix multiplication commutative when a matrix is multiplied by its inverse?

There are many computational methods for computing the inverse of a matrix. For example, Gaussian elimination, Gauss–Jordan elimination, Choleski decomposition, QR decomposition, LU decomposition, and many others. R implements many of these methods in different functions.

```
> qr.solve(D)

     [,1] [,2]
[1,]    3   -5
[2,]   -1    2
```

If a matrix has an inverse it is referred to as **nonsingular**.

If the **determinant** of a matrix is nonzero, it is nonsingular.

```
> det(D)

[1] 1
```

Since the determinant of $\mathbf{D} \neq 0$, $\mathbf{D}$ is nonsingular; it has an inverse.