# Models for Spatially Dependent Areal Data

ESS 575 Models for Ecological Data

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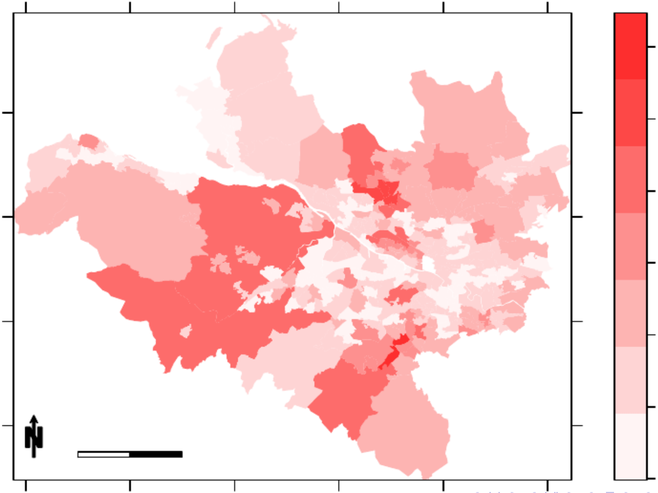
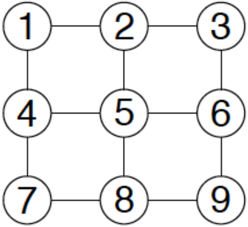
# Most ecological data are spatial

## Areal spatial processes



# Areal data and proximity




$$\mathbf{W} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Possibilities include, but are not limited to:

- $w_{ij} = 1$ if $i, j$ share a common boundary (possibly a common vertex)
- $w_{ij} = 1$ for $m$ nearest neighbors.

# Measures of regularity, clustering

Moran's I: similar to covariogram ( see `spdep` package ).

$$\hat{\mathbf{u}} = \mathbf{y} - \mathbf{X}\boldsymbol{\beta} \tag{1}$$

$$\mathbf{u} = \rho \mathbf{W}\mathbf{u} + \boldsymbol{\varepsilon},\ \boldsymbol{\varepsilon} \text{ multivariate normal}(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}) \tag{2}$$

Moran's I: (3)

$$I = \frac{\hat{u}'}{||\hat{\mathbf{u}}||^2} \tag{4}$$

- $E(I) = -\frac{1}{n-1}$
- $I > E(I)$ implies clustering.
- $I < E(I)$ implies regularity.

# Modeling areal data

Two general types of *spatial autoregressive models:*

- Simultaneous autoregressive models (SAR): less common in Bayesian analysis. Some references at end of lecture.
- Conditional autoregressive models (CAR): The probability of values estimated at any given location are conditional on neighboring values.

# Conditional autoregressive model

$$\begin{aligned}
\mathbf{y} &= g(\boldsymbol{\theta}, \mathbf{X}) + \boldsymbol{\eta} + \boldsymbol{\varepsilon} \\
\boldsymbol{\eta} &\sim \text{multivariate normal}(\mathbf{0}, \boldsymbol{\Sigma}) \\
\boldsymbol{\Sigma} &= \overbrace{\underbrace{\sigma^2 (\mathbf{I} - \rho \mathbf{W})}_{\text{precision matrix}}{}^{-1} \mathbf{I}}^{\text{covariance matrix}} \\
\boldsymbol{\varepsilon} &\sim \text{multivariate normal}(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I})
\end{aligned}$$

1. $\rho$ is an autocorrelation parameter.
2. Proximity matrix $\mathbf{W}$ must be symmetric.

# Alternative notation

$$
\begin{aligned}
\mathbf{y} &\sim \text{multivariate normal}(g(\boldsymbol{\theta}, \mathbf{X}) + \boldsymbol{\eta}, \sigma_\varepsilon^2 \mathbf{I}) \\
\boldsymbol{\eta} &\sim \text{multivariate normal}(0, \boldsymbol{\Sigma})
\end{aligned}
$$

# Conditional autoregressive model with row standardization

$$
\begin{aligned}
\mathbf{y} &\sim \text{multivariate normal}(g(\boldsymbol{\theta},\mathbf{X})+\boldsymbol{\eta},\sigma_\varepsilon^2\mathbf{I}) \\
\boldsymbol{\eta} &\sim \text{multivariate normal}(0,\boldsymbol{\Sigma}) \\
\boldsymbol{\Sigma} &= (\text{diag}(\mathbf{W1})-\rho\mathbf{W})^{-1}
\end{aligned}
$$

- Row standardization assures that $|\rho|<1$
- $\mathbf{1}$ is a column vector of $1's$.
- $\mathbf{W1}$ is the sums of the rows
- $\text{diag}(\mathbf{W1})$ is a matrix with the sums of the rows on the diagonal and zeros elsewhere.
- Equivalent to dividing each element of $\mathbf{W}$ by the sum of the rows to obtain $\mathbf{W}_+$ and using $\boldsymbol{\Sigma}=\boldsymbol{\sigma}^{\mathbf{2}}(\mathbf{I}-\rho\mathbf{W}_+)^{-\mathbf{1}}$
- $\rho\sim\text{Beta}(18,2)$ to favor values close to 1
- $\sigma^2\sim\text{IG}(r,q)$.

# CAR for non-negative observations

Let $\boldsymbol{\Sigma} = \sigma^2 \text{diag}(\mathbf{W}\mathbf{1} - \rho \mathbf{W})^{-1}$

$$\begin{aligned} g(\boldsymbol{\beta}, \mathbf{X}) &= \exp(\mathbf{X}\boldsymbol{\beta}) && (5) \\ \log(y) &\sim \text{multivariate normal}(\log(g(\boldsymbol{\beta}, \mathbf{X})) + \boldsymbol{\eta}, \boldsymbol{\sigma}_{\varepsilon}^2 \mathbf{I}) && (6) \end{aligned}$$

# CAR for counts

Let $\boldsymbol{\Sigma} = \sigma^2 \text{diag}(\mathbf{W}\mathbf{1} - \rho\mathbf{W})^{-1}$

$$
\begin{aligned}
g(\boldsymbol{\beta}, \mathbf{X}) &= \exp(\mathbf{X}\boldsymbol{\beta}) && (7)\\
y_i &\sim \text{Poisson}(\lambda_i) && (8)\\
\log(\boldsymbol{\lambda}) &\sim \text{multivariate normal}(\log(g(\boldsymbol{\beta}, \mathbf{X})) + \boldsymbol{\eta}, \boldsymbol{\sigma}_\varepsilon^2 \mathbf{I}) && (9)
\end{aligned}
$$

## CAR for binary observations

Let $\Sigma = \sigma^2 \text{diag}(\mathbf{W1} - \rho \mathbf{W})^{-1}$

$$
\begin{aligned}
g(\boldsymbol{\beta}, \mathbf{X}) &= \text{logit}^{-1}(\mathbf{X}\boldsymbol{\beta}) && (10) \\
y_i &\sim \text{Bernoulli}(p_i) && (11) \\
\text{logit}(\mathbf{p}) &\sim \text{multivariate normal}(\text{logit}(g(\boldsymbol{\beta}, \mathbf{X})) + \boldsymbol{\eta}, \sigma_\varepsilon^2 \mathbf{I}) && (12)
\end{aligned}
$$

# Take home

- Data taken over time or space are likely to be structured by physical and biological processes.
- Our deterministic model may account for this structure. However, if the *residuals* show correlation over time and/ or space, then we are obliged to model their covariance to assure that iid assumptions are met.
- Doing so requires estimating only a few more parameters, in most cases one or two, relative to the aspatial model.
- Fitting spatial models is computationally challenging.
- Deciding whether to use a spatial or aspatial model should probably be treated as a problem in model selection.

# Further study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- Peterson, E.E., E.M. Hanks, M.B. Hooten, J.M. Ver Hoef, and M.-J. Fortin. (2019). Spatially structured statistical network models for landscape genetics. Ecological Monographs, 89: e01355.
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