# Markov chain Monte Carlo II

ESS 575 Models for Ecological Data

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# The MCMC algorithm

- Some intuition
- Accept-reject sampling with Metropolis algorithm
- Introduction to full-conditional distributions
- Gibbs sampling
- Metropolis-Hastings algorithm
- Implementing accept-reject sampling

# Implementing MCMC for multiple parameters and latent quantities

- Write an expression for the posterior and joint distribution using a DAG as a guide. Always.
- If you are using MCMC software (e.g. JAGS) use the expression for the posterior and joint distribution as template for writing code.
- If you are writing your own MCMC sampler:
  - Decompose the expression of the multivariate joint distribution into a series of univariate distributions called *full-conditional distributions*.
  - Choose a sampling method for each full-conditional distribution.
  - Cycle through each unobserved quantity, sampling from its full-conditional distribution, treating the others as if they were known and constant.
  - The accumulated samples approximate the marginal posterior distribution of each unobserved quantity.
  - Note that this takes a complex, multivariate problem and turns it into a series of simple, univariate problems that we solve, as in the example above, one at a time.

# Choosing a sampling method

1. Accept-reject:
    1.1 Metropolis
    1.2 Metropolis-Hastings
2. Gibbs: accepts all proposals because they are especially well chosen.

# When is accept-reject update mandatory?

We need to use Metropolis, Metropolis-Hastings or some other accept reject methods whenever

1. A conjugate relationship does not exist for the full-conditional distribution of a parameter, for example, for the shape parameter in the gamma distribution.
2. The deterministic model is non-linear, which almost always means a conjugate doesn't exist for its parameters.

# When is a model linear?

- A model is linear if it can be written as the sum of products of coefficients and predictor variables, i.e. $\mu_i = \beta_0 + \beta_1 x_{1,i} + .... + \beta_n x_{n,i}$ or in matrix form $\mu_i = \mathbf{x}_i \beta$. We can take powers and products of the $x$ and the model remains linear. We often transform models to linearize them using link functions (i.e., log, logit, probit).
- A model is non-linear if it cannot be written this way.

# Metropolis Updates

$$[\boldsymbol{\theta}^{*k+1}|y] = \frac{\overbrace{[y|\boldsymbol{\theta}^{*k+1}]}^{\text{likelihood}}\overbrace{[\boldsymbol{\theta}^{*k+1}]}^{\text{prior}}}{\cancel{\int [y|\boldsymbol{\theta}][\boldsymbol{\theta}]d\boldsymbol{\theta}}}$$

$$[\boldsymbol{\theta}^{k}|y] = \frac{\overbrace{[y|\boldsymbol{\theta}^{k}]}^{\text{likelihood}}\overbrace{[\boldsymbol{\theta}^{k}]}^{\text{prior}}}{\cancel{\int [y|\boldsymbol{\theta}][\boldsymbol{\theta}]d\boldsymbol{\theta}}}$$

$$R = \frac{[\boldsymbol{\theta}^{*k+1}|y]}{[\boldsymbol{\theta}^{k}|y]}$$

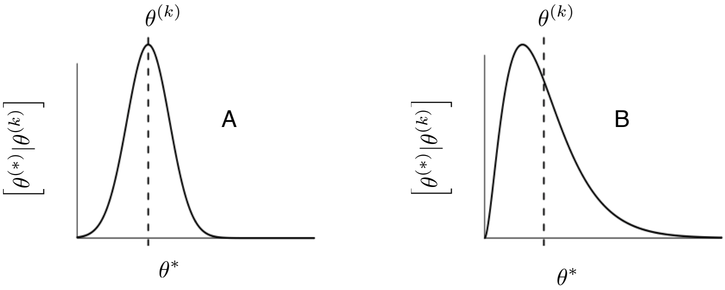
# Proposal distributions

- Independent chains have proposal distributions that do not depend on the current value $(\theta^k)$ in the chain. This is what we used in the fish disease example.
- Dependent chains, as you might expect, have proposal distributions that *do* depend on the current value of the chain $(\theta^k)$. In this case we draw from

$$[\theta^{*k+1}|\theta^k, \sigma] \tag{1}$$

  where $\sigma$ is a tuning parameter that we specify to obtain an acceptance rate of about 40%. Note that my notation and notation of others simplifies this distribution to $[\theta^{*k+1}|\theta^k]$ The $\sigma$ is implicit because it is a constant, not a random variable.
- Why are dependent chains usually more efficient that independent chains?

# Proposal distributions for dependent chains



# Metropolis-Hastings updates

- Metropolis updates require symmetric proposal distributions (e.g., uniform, normal).
- Metropolis-Hastings updates allow use of asymmetric (e.g., beta, gamma, lognormal).

# Definition of symmetry

A proposal distribution is symmetric if and only if

$$[\theta^{*k+1}|\theta^k] = [\theta^k|\theta^{*k+1}]. \tag{2}$$

Normal and uniform are symmetric. Gamma, beta, lognormal are not.

## Illustrating with code

```
#symmetric example
sigma=1
x = .8
z=rnorm(1,mean=x,sd=sigma);z
#[z|x]
dnorm(z,mean=x,sd=sigma)
#[x|z]
dnorm(x,mean=z,sd=sigma)
#asymmetric example
sigma=1
x = .8
a.x=x^2/sigma^2; b.x=x/sigma^2
z=rgamma(1,shape=a.x,rate=b.x);z
a.z=z^2/sigma^2; b.z=z/sigma^2
#[z|x]
dgamma(z,shape=a.x,rate=b.x)
#[x|z]
dgamma(x,shape=a.z,rate=b.z)
```

# Metropolis-Hastings updates

Metropolis R:

$$R = \frac{[\theta^{*k+1}|y]}{[\theta^k|y]} \tag{3}$$

Metropolis-Hastings R:

$$R = \frac{[\theta^{*k+1}|y]}{[\theta^k|y]} \frac{\overbrace{[\theta^k|\theta^{*k+1}]}^{\text{Proposal distribution}}}{\underbrace{[\theta^{*k+1}|\theta^k]}_{\text{Proposal distribution}}}, \tag{4}$$

which is the same as:

$$R = \frac{\overbrace{[y|\theta^{*k+1}]}^{\text{Likelihood}}\overbrace{[\theta^{*k+1}]}^{\text{Prior}}\quad\overbrace{[\theta^k|\theta^{*k+1}]}^{\text{Proposal distribution}}}{\underbrace{[y|\theta^k]}_{\text{Likelihood}}\ \underbrace{[\theta^k]}_{\text{Prior}}\quad\underbrace{[\theta^{*k+1}|\theta^k]}_{\text{Proposal distribution}}} \tag{5}$$

## Example using beta proposal distribution

1. Current value of parameter, $\theta^k = .42$, tuning parameter set at $\sigma = .10$
2. Make a draw from $\theta{*}^{k+1} \sim \text{beta}(m(.42, .10))$, where $m$ is moment matching function.
3. Calculate $R = \dfrac{\overbrace{[y \mid \theta^{*k+1}]}^{\text{Likelihood}}\overbrace{[\theta^{*k+1}]}^{\text{Prior}}\overbrace{[.42| \mid m(\theta^{*k+1}, .10)]}^{\text{beta proposal}}}{\underbrace{[y \mid \theta^k]}_{\text{Likelihood}}\underbrace{[\theta^k]}_{\text{Prior}}\underbrace{[\theta^{*k+1} \mid m(.42, .10)]}_{\text{beta proposal}}}$.
4. Choose proposed or current value based on $R$ as we did with Metropolis.

# MCMC

- Methods based on the Markov chain Monte Carlo algorithm allow us to approximate marginal posterior distributions of unobserved quantities without analytical integration.
- This makes it possible to estimate models that have many parameters, have multiple sources of uncertainty, and include latent quantities.
- We will learn a tool, JAGS, that simplifies the implementation of MCMC methods.
- Will will put this tool to use in building models that include nested levels in space, errors in the observations, differences among groups and processes that unfold over time.