Real Business Cycles

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Household Problem

- Preferences:

$$\max E \sum_{t=0}^{\infty} \beta \left\{ \log c_t + \psi \log \left(1 - l_t\right) \right\}$$

- Budget constraint:

$$c_t + k_{t+1} = w_t l_t + r_t k_t + \left(1 - \delta\right) k_t, \; \forall \; t > 0$$

Problem of the Firm

- Neoclassical production function:

$$y_t = k_t^{\alpha} \left(e^{z_t} l_t\right)^{1-\alpha}$$

- By profit maximization:

$$\begin{aligned} \alpha k_t^{\alpha-1} \left(e^{z_t} l_t\right)^{1-\alpha} &= r_t \\ (1-\alpha) k_t^{\alpha} \left(e^{z_t} l_t\right)^{1-\alpha} l_t^{-1} &= w_t \end{aligned}$$

Evolution of the technology

- z_t changes over time.

- It follows the AR(1) process:

$$
\begin{aligned}
z_t &= \rho z_{t-1} + \varepsilon_t \\
\varepsilon_t &\sim \mathcal{N}(0, \sigma)
\end{aligned}
$$

- Interpretation of ρ.

A Competitive Equilibrium

- We can define a competitive equilibrium in the standard way.

- The competitive equilibrium is unique.

- This economy satisfies the conditions that assure that both welfare theorems hold.

- Why is this important? We could solve instead the Social Planner's Problem associated with it.

- Advantages and disadvantages of solving the social planner's problem.

The Social Planner's Problem

- It has the form:

$$\max E \sum_{t=0}^{\infty} \beta \left\{ \log c_t + \psi \log \left(1 - l_t\right) \right\}$$

$$\begin{aligned} c_t + k_{t+1} &= k_t^{\alpha} \left(e^{z_t} l_t\right)^{1-\alpha} + (1-\delta) k_t, \; \forall \; t > 0 \\ z_t &= \rho z_{t-1} + \varepsilon_t, \;\; \varepsilon_t \sim \mathcal{N}(0, \sigma) \end{aligned}$$

- This is a dynamic optimization problem.

Computing the RBC

- The previous problem does not have a known "paper and pencil" solution.

- We will work with an approximation: Perturbation Theory.

- We will undertake a first order perturbation of the model.

- How well will the approximation work?

Equilibrium Conditions

From the household problem+firms's problem+aggregate conditions:

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left(1 + \alpha k_t^{\alpha-1} \left(e^{z_t} l_t \right)^{1-\alpha} - \delta \right) \right\}$$

$$\psi \frac{c_t}{1 - l_t} = (1-\alpha) k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} l_t^{-1}$$

$$c_t + k_{t+1} = k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

Finding a Deterministic Solution

- We search for the firts component of the solution.

- If $\sigma = 0$, the equilibrium conditions are:

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} \left(1 + \alpha k_t^{\alpha-1} l_t^{1-\alpha} - \delta\right)$$
$$\psi \frac{c_t}{1 - l_t} = (1 - \alpha) k_t^{\alpha} l_t^{-\alpha}$$
$$c_t + k_{t+1} = k_t^{\alpha} l_t^{1-\alpha} + (1 - \delta) k_t$$

Deterministic Steady State

- The equilibrium conditions imply a steady state:

$$\frac{1}{c} = \beta \frac{1}{c}\left(1 + \alpha k^{\alpha-1} l^{1-\alpha} - \delta\right)$$
$$\psi \frac{c}{1-l} = (1-\alpha) k^{\alpha} l^{-\alpha}$$
$$c + \delta k = k^{\alpha} l^{1-\alpha}$$

- Or symplifying:

$$\frac{1}{\beta} = 1 + \alpha k^{\alpha-1} l^{1-\alpha} - \delta$$
$$\psi \frac{c}{1-l} = (1-\alpha) k^{\alpha} l^{-\alpha}$$
$$c + \delta k = k^{\alpha} l^{1-\alpha}$$

Solving the Steady State

Solution:

$$
\begin{aligned}
k &= \frac{\mu}{\Omega + \varphi\mu} \\
l &= \varphi k \\
c &= \Omega k \\
y &= k^{\alpha} l^{1-\alpha}
\end{aligned}
$$

where $\varphi = \left(\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right)^{\frac{1}{1-\alpha}}$, $\Omega = \varphi^{1-\alpha} - \delta$ and $\mu = \frac{1}{\psi}(1-\alpha)\varphi^{-\alpha}$.

Linearization I

- Loglinearization or linearization?
- Advantages and disadvantages
- We can linearize and perform later a change of variables.

Linearization II

We linearize:

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left(1 + \alpha k_t^{\alpha-1} \left(e^{z_t} l_t \right)^{1-\alpha} - \delta \right) \right\}$$
$$\psi \frac{c_t}{1 - l_t} = (1 - \alpha) \, k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} l_t^{-1}$$
$$c_t + k_{t+1} = k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} + (1 - \delta) \, k_t$$
$$z_t = \rho z_{t-1} + \varepsilon_t$$

around l, k, and c with a First-order Taylor Expansion.

Linearization III

We get:

$$-\frac{1}{c}\left(c_t - c\right) = E_t \left\{ \begin{array}{c} -\frac{1}{c}\left(c_{t+1} - c\right) + \alpha\left(1-\alpha\right)\beta\frac{y}{k}z_{t+1} + \\ \alpha\left(\alpha-1\right)\beta\frac{y}{k^2}\left(k_{t+1} - k\right) + \alpha\left(1-\alpha\right)\beta\frac{y}{kl}\left(l_{t+1} - l\right) \end{array} \right\}$$

$$\frac{1}{c}\left(c_t - c\right) + \frac{1}{\left(1-l\right)}\left(l_t - l\right) = \left(1-\alpha\right)z_t + \frac{\alpha}{k}\left(k_t - k\right) - \frac{\alpha}{l}\left(l_t - l\right)$$

$$\left(c_t - c\right) + \left(k_{t+1} - k\right) = \left\{ \begin{array}{c} y\left(\left(1-\alpha\right)z_t + \frac{\alpha}{k}\left(k_t - k\right) + \frac{\left(1-\alpha\right)}{l}\left(l_t - l\right)\right) \\ +\left(1-\delta\right)\left(k_t - k\right) \end{array} \right\}$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

Rewriting the System I

Or:

$$\alpha_1 (c_t - c) = E_t \{\alpha_1 (c_{t+1} - c) + \alpha_2 z_{t+1} + \alpha_3 (k_{t+1} - k) + \alpha_4 (l_{t+1} - l)\}$$

$$(c_t - c) = \alpha_5 z_t + \frac{\alpha}{k} c (k_t - k) + \alpha_6 (l_t - l)$$

$$(c_t - c) + (k_{t+1} - k) = \alpha_7 z_t + \alpha_8 (k_t - k) + \alpha_9 (l_t - l)$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

Rewriting the System II

where

$$
\begin{array}{ll}
\alpha_1 = -\frac{1}{c} & \alpha_2 = \alpha\left(1-\alpha\right)\beta\frac{y}{k} \\
\alpha_3 = \alpha\left(\alpha-1\right)\beta\frac{y}{k^2} & \alpha_4 = \alpha\left(1-\alpha\right)\beta\frac{y}{kl} \\
\alpha_5 = \left(1-\alpha\right)c & \alpha_6 = -\left(\frac{\alpha}{l}+\frac{1}{(1-l)}\right)c \\
\alpha_7 = \left(1-\alpha\right)y & \alpha_8 = y\frac{\alpha}{k}+\left(1-\delta\right) \\
\alpha_9 = y\frac{(1-\alpha)}{l} & y = k^{\alpha}l^{1-\alpha}
\end{array}
$$

Rewriting the System III

After some algebra the system is reduced to:

$$A\left(k_{t+1}-k\right)+B\left(k_{t}-k\right)+C\left(l_{t}-l\right)+Dz_{t}=0$$

$$E_{t}\left(G\left(k_{t+1}-k\right)+H\left(k_{t}-k\right)+J\left(l_{t+1}-l\right)+K\left(l_{t}-l\right)+Lz_{t+1}+Mz_{t}\right)=0$$

$$E_{t}z_{t+1}=\rho z_{t}$$

Guess Policy Functions

We guess policy functions of the form $(k_{t+1} - k) = P(k_t - k) + Qz_t$ and $(l_t - l) = R(k_t - k) + Sz_t$, plug them in and get:

$$A(P(k_t - k) + Qz_t) + B(k_t - k)$$
$$+C(R(k_t - k) + Sz_t) + Dz_t = 0$$

$$G(P(k_t - k) + Qz_t) + H(k_t - k) + J(R(P(k_t - k) + Qz_t) + SNz_t)$$
$$+K(R(k_t - k) + Sz_t) + (LN + M)z_t = 0$$

Solving the System I

Since these equations need to hold for any value $(k_{t+1} - k)$ or z_t we need to equate each coefficient to zero, on $(k_t - k)$:

$$\begin{aligned} AP + B + CR &= 0 \\ GP + H + JRP + KR &= 0 \end{aligned}$$

and on z_t:

$$\begin{aligned} AQ + CS + D &= 0 \\ (G + JR)\,Q + JSN + KS + LN + M &= 0 \end{aligned}$$

Solving the System II

- We have a system of four equations on four unknowns.

- To solve it note that $R = -\frac{1}{C}(AP + B) = -\frac{1}{C}AP - \frac{1}{C}B$

- Then:

$$P^2 + \left(\frac{B}{A} + \frac{K}{J} - \frac{GC}{JA}\right)P + \frac{KB - HC}{JA} = 0$$

a quadratic equation on P.

Solving the System III

- We have two solutions:

$$P = -\frac{1}{2}\left(-\frac{B}{A} - \frac{K}{J} + \frac{GC}{JA} \pm \left(\left(\frac{B}{A} + \frac{K}{J} - \frac{GC}{JA}\right)^2 - 4\frac{KB - HC}{JA}\right)^{0.5}\right)$$

one stable and another unstable.

- If we pick the stable root and find $R = -\frac{1}{C}(AP + B)$ we have to a system of two linear equations on two unknowns with solution:

$$Q = \frac{-D(JN + K) + CLN + CM}{AJN + AK - CG - CJR}$$

$$S = \frac{-ALN - AM + DG + DJR}{AJN + AK - CG - CJR}$$

Calibration

- What does it mean to calibrate a model?

- Our choices

Calibrated Parameters

Parameter	β	ψ	α	δ	ρ	σ
Value	0.99	1.75	0.33	0.023	0.95	0.01