Chapter 7

The Ramsey Model

7.1 Explicit dynamics: government spending

Households have an intertemporal logarithmic utility function:

$$U = \sum_t \beta^t Log C_t$$

Firms have a Cobb-Douglas production function:

$$Y_t = A K_t^\alpha L_t^{1-\alpha}$$

$\delta = 1$

Capital depreciates fully in one period, so that:

$$K_{t+1} = I_t \quad + (1 - \delta) K_t$$

We further assume that the government wants to carry public spending G_t in the amount:

$$G_t = \zeta Y_t$$

These government expenses are financed by a lump sum tax $T_t = G_t$.

7.1.1 Questions

1. Write the first order conditions of the household.

2. Compute consumption and investment.

7.1.2 Solution

1. The first order conditions

The household maximizes utility:

$$U = \sum_t \beta^t LogC_t \tag{7.1}$$

subject to the budget constraints:

lump sum

$$C_t + K_{t+1} = R_t K_t + \omega_t L_t - T_t \qquad \forall t = 1 \dots T \tag{7.2}$$

The Lagrangian is:

$$+\lambda_{t+1}[R_{t+1}K_{t+1} \dots]$$

$$\sum_t \beta^t \{LogC_t + \lambda_t [R_t K_t + \omega_t L_t - T_t - C_t - K_{t+1}]\} \tag{7.3}$$

at t: The first order conditions in C_t and K_t yield: K_t is state

$$\frac{1}{C_t} = \lambda_t \tag{7.4}$$

w.r.t. K_{t+1} $\quad -\lambda_t$

$$\lambda_t = \beta R_{t+1} \lambda_{t+1} \tag{7.5}$$

capital share

Now in view of the Cobb-Douglas utility function:

Firm maximize profit subjected to budget

$$K_{t+1} \cdot R_{t+1} = \alpha Y_{t+1}$$

$$R_{t+1} = \frac{\alpha Y_{t+1}}{K_{t+1}} = \frac{\alpha Y_{t+1}}{I_t} \tag{7.6}$$

$1-\alpha$

Combining (4), (5) and (6) we obtain:

$$\boxed{\frac{I_t}{C_t} = \alpha\beta \frac{Y_{t+1}}{C_{t+1}}} \tag{7.7}$$

$$\frac{I_t}{C_t}$$

2. Equilibrium consumption and investment

Now we have the accounting identity:

$$(1-\zeta)Y_t = C_t + I_t$$

$$Y_t = C_t + I_t + G_t \tag{7.8}$$

$$Y_t = \frac{C_t + I_t}{1-\zeta}$$

with:

$$G_t = \zeta Y_t \tag{7.9}$$

Inserting (8) and (9) into (7), we obtain the dynamic equation:

$$\frac{I_t}{C_t} = \frac{\alpha\beta}{1-\zeta} + \frac{\alpha\beta}{1-\zeta}\frac{I_{t+1}}{C_{t+1}}$$

1st order difference equation

S.S.

$t \to \infty$

$$\frac{I_t}{C_t} \to \left(\frac{I_t}{C_t}\right) = A$$

$$\bar{A} = \frac{\alpha\beta}{1-\zeta} + \frac{\alpha\beta}{1-\zeta} \cdot \bar{A}$$

$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$

$\frac{I_t + C_t}{1-\zeta} = Y_t$

A

steady state ?

$Y_t = I + C_t \downarrow\ G \uparrow$

Integrating forward we find:

$$A = \frac{I_t}{C_t} = \frac{\alpha\beta}{1-\zeta-\alpha\beta} \tag{7.11}$$

B how

$G = \zeta Y_t$

So we have:

$$\frac{I_t}{\alpha\beta} = \frac{C_t}{1-\zeta-\alpha\beta} = \frac{I_t + C_t}{1-\zeta} = Y_t \tag{7.12}$$

So the values of consumption and investment are:

G lumpsum

crowd out

$$C_t = (1-\zeta-\alpha\beta)\, Y_t \tag{7.13}$$

$$I_t = \alpha\beta Y_t \tag{7.14}$$

$\zeta \uparrow\ C \downarrow$

7.2 Explicit dynamics: distortionary taxation

We take the same model as in problem 7.1, but we now assume that G_t is financed by a proportional tax on all components of income Y_t:

$$G_t = \tau Y_t$$

The budget is balanced, so that $\tau = \zeta$.

7.2.1 Questions

1. Compute the dynamics of the economy.

2. What is the effect of distortionary taxation on investment?

7.2.2 Solution

1. The dynamics of consumption and investment

The problem is similar to that of problem 7.1, but the budget constraint of the consumer is now:

proportional tax

B.C.

$$C_t + K_{t+1} = (1-\tau)\,(R_t K_t + \omega_t L_t) \tag{7.15}$$

with $\tau = \zeta$. Maximizing the utility function (1) subject to the budget constraints (15) yields the following equation, which replaces (7):

replace

$$\frac{I_t}{C_t} = \alpha\beta\,(1-\tau)\,\frac{Y_{t+1}}{C_{t+1}} = \alpha\beta\,(1-\zeta)\,\frac{Y_{t+1}}{C_{t+1}} \tag{7.16}$$

Combining with (8) and (9), this yields the dynamic equation in I_t/C_t:

first order diff linear

$$\boxed{\frac{I_t}{C_t} = \alpha\beta + \alpha\beta\frac{I_{t+1}}{C_{t+1}}} \tag{7.17}$$

S.S.

$A = \alpha\beta + \alpha\beta A$

$$\frac{a}{b}=\frac{c}{d}=\frac{a+c}{b+d}$$

A

2. Distortionary taxation and investment

$Y_t = I + C + G$

Integrating (17) forward we obtain:

$$\frac{I_t}{C_t} = \frac{\alpha\beta}{1-\alpha\beta} \tag{7.18}$$

B ?

how

and so:

$$\frac{I_t}{\alpha\beta} = \frac{C_t}{1-\alpha\beta} = I_t + C_t = (1-\zeta)\,Y_t \tag{7.19}$$

$= Y_t + G$

from which we deduce consumption and investment:

$$C_t = (1-\alpha\beta)\,(1-\zeta)\,Y_t \tag{7.20}$$

$$I_t = \alpha\beta\,(1-\zeta)\,Y_t \tag{7.21}$$

Comparing formulas (14) and (21), we see that distortionary taxation reduces investment.

7.3 Labor versus capital taxation

Distortionary taxes are levied proportionally on labor and capital income. We assume here that these tax rates are constant in time, and denote them τ_ℓ and τ_k respectively. Government spending is a given fraction of national income:

$$G_t = \zeta Y_t$$

We shall compute the optimal τ_ℓ and τ_k associated to ζ. We assume a Cobb-Douglas production function and hundred percent depreciation:

$$Y_t = AK_t^{\alpha}L_t^{1-\alpha}$$

$$K_{t+1} = I_t$$

The utility function is logarithmic:

$$U = \sum \beta^t LogC_t$$

We finally assume that the government budget is balanced in each period.

7.3.1 Questions

1. Derive the first order conditions.
2. Compute consumption and investment as a function of income and the tax rates.
3. Find the optimal tax rates τ_ℓ and τ_k.

7.3.2 Solution

1. First order conditions

Households maximize their intertemporal utility subject to their successive budget constraints, i.e.:

$$\text{Maximize} \sum \beta^t Log C_t \qquad \text{s.t.}$$

B.C.

$$C_t + K_{t+1} = (1-\tau_\ell)\,\omega_t L_t + (1-\tau_k)\,R_t K_t$$

where R_t, the rate of return on capital, is equal to:

$$R_t = \frac{\alpha Y_t}{K_t} \tag{7.22}$$

The first order conditions yield the following equation, which replaces (7) and (16):

$$\boxed{\frac{I_t}{C_t} = \alpha\beta\,(1-\tau_k)\,\frac{Y_{t+1}}{C_{t+1}}} \tag{7.23}$$

replace

Combining with $Y_t = C_t + I_t + G_t$ and $G_t = \zeta Y_t$ we obtain:

t → ∞

$$\boxed{\frac{I_t}{C_t} = \frac{\alpha\beta}{1-\zeta}\,(1-\tau_k) + \frac{\alpha\beta}{1-\zeta}\,(1-\tau_k)\,\frac{I_{t+1}}{C_{t+1}}} \tag{7.24}$$

S.S.

2. Consumption and investment

<1

Equation (24) can be integrated forward provided:

$$\alpha\beta\,(1-\tau_k) < 1-\zeta \tag{7.25}$$

and it yields:

$$\boxed{\frac{I_t}{C_t} = \frac{\alpha\beta\,(1-\tau_k)}{1-\zeta-\alpha\beta\,(1-\tau_k)}} \tag{7.26}$$

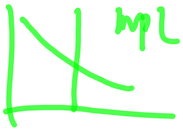
Y = I + G + C

So:

$$\frac{I_t}{\alpha\beta\,(1-\tau_k)} = \frac{C_t}{1-\zeta-\alpha\beta\,(1-\tau_k)} = \frac{I_t+C_t}{1-\zeta} = Y_t \tag{7.27}$$

G = ζY

$\text{Log}\, K_{t+1} = \text{Log}(\alpha\beta(1-\tau_k)) + \log Y_t$ / constant



$\sim + \alpha \log K_t + \text{Log}\, A L^{1-\alpha}$

L, A fix

no n, no x

We therefore have consumption and investment:

$$C_t = [1 - \zeta - \alpha\beta(1 - \tau_k)]\, Y_t \tag{7.28}$$

?

$$K_{t+1} = I_t = \alpha\beta(1 - \tau_k)\, Y_t \tag{7.29}$$

$\log Y_t = \log K_t^{\alpha}$

3. Optimal taxes

From the law of evolution of capital (29), and the production function $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$, we deduce that, up to a constant:

how

$$LogK_t = \frac{1-\alpha^t}{1-\alpha} Log\left[\alpha\beta(1-\tau_k)\right] + \alpha^t LogK_0 \tag{7.30}$$

/c

Now the government wants to maximize:

$$\sum \beta^t LogC_t = \sum \beta^t \left\{Log\left[1-\zeta-\alpha\beta(1-\tau_k)\right] + LogY_t\right\} \tag{7.31}$$

how

which becomes, in view of the Cobb-Douglas production function:

$$\sum \beta^t \left\{Log\left[1-\zeta-\alpha\beta(1-\tau_k)\right] + \alpha LogK_t\right\} \tag{7.32}$$

Let us insert (30) into (32), omitting the irrelevant terms in K_0. This gives the maximand:

$$\sum \beta^t \left\{Log\left[1-\zeta-\alpha\beta(1-\tau_k)\right] + \alpha\frac{1-\alpha^t}{1-\alpha} Log\left[\alpha\beta(1-\tau_k)\right]\right\} \tag{7.33}$$

/c_t

which is equal to:

$$\frac{Log\left[1-\zeta-\alpha\beta(1-\tau_k)\right]}{1-\beta} + \frac{\alpha\beta}{(1-\beta)(1-\alpha\beta)} Log\left[\alpha\beta(1-\tau_k)\right] \tag{7.34}$$

/c_t

Differentiating with respect to τ_k we find:

$$\tau_k = \zeta \tag{7.35}$$

$\frac{1}{1-\zeta-\alpha\beta(1-\tau_k)}$

Now the government's financing constraint is:

$$G_t = \tau_\ell \omega_t L_t + \tau_k R_t K_t \tag{7.36}$$

$- \frac{1}{(1-\tau_k)-\alpha\beta(1-\tau_k)}$

We further have:

$$G_t = \zeta Y_t \tag{7.37}$$

and:

$$Y_t = R_t K_t + \omega_t L_t \tag{7.38}$$

These last four equations yield:

$$\tau_\ell = \zeta \tag{7.39}$$