First order approximation of stochastic models

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Outline

General problem

$$E_t\left\{f(y_{t+1}, y_t, y_{t-1}, u_t)\right\} = 0$$

$$\begin{aligned} E(u_t) &= 0 \\ E(u_t u_t') &= \Sigma_u \\ E(u_t u_\tau') &= 0 \quad t \neq \tau \end{aligned}$$

y : vector of endogenous variables

u : vector of exogenous stochastic shocks

Computation of first order approximation

- Perturbation approach: recovering a Taylor expansion of the solution function from a Taylor expansion of the original model.
- A first order approximation is nothing else than a standard solution thru linearization.
- A first order approximation in terms of the logarithm of the variables provides standard log-linearization.

Timing assumptions

$$E_t \{f(y_{t+1}, y_t, y_{t-1}, u_t)\} = 0$$

- shocks u_t are observed at the beginning of period t,
- decisions affecting the current value of the variables y_t, are function of
  - the previous state of the system, y_{t-1},
  - the shocks u_t.

The stochastic scale variable

$$E_t \{f(y_{t+1}, y_t, y_{t-1}, u_t)\} = 0$$

- At period t, the only unknown stochastic variable is y_{t+1}, and, implicitly, u_{t+1}.
- We introduce the *stochastic scale variable*, σ and the auxiliary random variable, ϵ_t, such that

$$u_{t+1} = \sigma \epsilon_{t+1}$$

The stochastic scale variable (continued)

$$E(\epsilon_t) = 0 \tag{1}$$
$$E(\epsilon_t \epsilon_t') = \Sigma_\epsilon \tag{2}$$
$$E(\epsilon_t \epsilon_\tau') = 0 \quad t \neq \tau \tag{3}$$

and

$$\Sigma_u = \sigma^2 \Sigma_\epsilon$$

Remarks

$$E_t \left\{ f(y_{t+1}, y_t, y_{t-1}, u_t) \right\} = 0$$

- The exogenous shocks may appear only at the current period
- There is no deterministic exogenous variables
- Not all variables are necessarily present with a lead and a lag
- Generalization to leads and lags on more than one period: similar to deterministic case, but more complicated for lead variables (because of expectancy operator)

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Solution function

$$y_t = g(y_{t-1}, u_t, \sigma)$$

where σ is the stochastic scale of the model. If $\sigma = 0$, the model is deterministic. For $\sigma > 0$, the model is stochastic.

Under some conditions, the existence of $g()$ function is proven via an implicit function theorem. See H. Jin and K. Judd "Solving Dynamic Stochastic Models" (`http://bucky.stanford.edu/papers/PerturbationMethodRatEx.pdf`)

Solution function (continued)

Then,

$$\begin{aligned} y_{t+1} &= g(y_t, u_{t+1}, \sigma) \\ &= g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma) \end{aligned}$$

Let's:

$$\begin{aligned} &F(y_{t-1}, u_t, \epsilon_{t+1}, \sigma) \\ &\qquad = f(g(g(y_{t-1}, u_t, \sigma), \sigma\epsilon_{t+1}, \sigma), g(y_{t-1}, u_t, \sigma), y_{t-1}, u_t) \end{aligned}$$

So that the problem is redefined as:

$$E_t \left\{ F(y_{t-1}, u_t, \epsilon_{t+1}, \sigma) \right\} = 0$$

The perturbation approach

- Obtain a Taylor expansion of the unkown solution function in the neighborhood of a problem that we know how to solve.
- The problem that we know how to solve is the deterministic steady state.
- One obtains the Taylor expansion of the solution for the Taylor expansion of the original problem.
- One consider two different perturbations:
  1. points in the neighborhood from the steady sate,
  2. from a deterministic model towards a stochastic one (by increasing σ from a zero value).

The perturbation approach (continued)

- The Taylor approximation is taken with respect to y_{t-1}, u_t and σ, the arguments of the solution function

$$y_t = g(y_{t-1}, u_t, \sigma).$$

- At the deterministic steady state, all derivatives are deterministic as well.

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Steady state

A deterministic steady state, $\bar{y}$, for the model satisfies

$$f(\bar{y}, \bar{y}, \bar{y}, 0) = 0$$

A model can have several steady states, but only one of them will be used for approximation.
Furthermore,

$$\bar{y} = g(\bar{y}, 0, 0)$$

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First order approximation

Around $\bar{y}$:

$$
\begin{aligned}
E_t\left\{F^{(1)}(y_{t-1},u_t,\epsilon_{t+1},\sigma)\right\} = & \\
& E_t\Big\{f(\bar{y},\bar{y},\bar{y},0)+f_{y_+}\left(g_y\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+g_u\sigma\epsilon'+g_\sigma\sigma\right) \\
& +f_{y_0}\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+f_{y_-}\hat{y}+f_u u\Big\} \\
= \quad & 0
\end{aligned}
$$

with $\hat{y}=y_{t-1}-\bar{y}$, $u=u_t$, $\epsilon'=\epsilon_{t+1}$, $f_{y_+}=\frac{\partial f}{\partial y_{t+1}}$, $f_{y_0}=\frac{\partial f}{\partial y_t}$, $f_{y_-}=\frac{\partial f}{\partial y_{t-1}}$, $f_u=\frac{\partial f}{\partial u_t}$, $g_y=\frac{\partial g}{\partial y_{t-1}}$, $g_u=\frac{\partial g}{\partial u_t}$, $g_\sigma=\frac{\partial g}{\partial \sigma}$.

Taking the expectation

$$
\begin{aligned}
E_t\left\{F^{(1)}(y_{t-1},u_t,\epsilon_{t+1},\sigma)\right\} = & \\
& f(\bar{y},\bar{y},\bar{y},0) + f_{y_+}\left(g_y\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+g_\sigma\sigma\right) \\
& +f_{y_0}\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+f_{y_-}\hat{y}+f_u u\Big\} \\
= & \left(f_{y_+}g_y g_y + f_{y_0}g_y + f_{y_-}\right)\hat{y} + \left(f_{y_+}g_y g_u + f_{y_0}g_u + f_u\right)u \\
& +\left(f_{y_+}g_y g_\sigma + f_{y_0}g_\sigma\right)\sigma \\
= & \; 0
\end{aligned}
$$

Recovering g_y

$$\left(f_{y_+} g_y g_y + f_{y_0} g_y + f_{y_-}\right) \hat{y} = 0$$

Structural state space representation:

$$\begin{bmatrix} 0 & f_{y_+} \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} g_y \hat{y} = \begin{bmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} \hat{y}$$

or

$$\begin{bmatrix} 0 & f_{y_+} \\ I & 0 \end{bmatrix} \begin{bmatrix} y_t - \bar{y} \\ y_{t+1} - \bar{y} \end{bmatrix} = \begin{bmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{bmatrix} \begin{bmatrix} y_{t-1} - \bar{y} \\ y_t - \bar{y} \end{bmatrix}$$

Structural state space representation

$$Dx_{t+1} = Ex_t$$

with

$$x_{t+1} = \left[\begin{array}{c} y_t - \bar{y} \\ y_{t+1} - \bar{y} \end{array} \right] \qquad x_t = \left[\begin{array}{c} y_{t-1} - \bar{y} \\ y_t - \bar{y} \end{array} \right]$$

- There are multiple solutions but we want a unique stable one.
- Problem when D is singular.

Real generalized Schur decomposition

Taking the real generalized Schur decomposition of the pencil $< E, D >$:

$$
\begin{aligned}
D &= QTZ \\
E &= QSZ
\end{aligned}
$$

with T, upper triangular, S quasi-upper triangular, $Q'Q = I$ and $Z'Z = I$.

Generalized eigenvalues

λ_i solves

$$\lambda_i D x_i = E x_i$$

For diagonal blocks on S of dimension 1 x 1:

- $T_{ii} \neq 0$: $\lambda_i = \frac{S_{ii}}{T_{ii}}$
- $T_{ii} = 0$, $S_{ii} > 0$: $\lambda_i = +\infty$
- $T_{ii} = 0$, $S_{ii} < 0$: $\lambda_i = -\infty$
- $T_{ii} = 0$, $S_{ii} = 0$: $\lambda_i \in \mathbb{C}$

A pair of complex eigenvalues

When a diagonal block of matrix S is a 2x2 matrix of the form $\begin{bmatrix} S_{ii} & S_{i,i+1} \\ S_{i+1,i} & S_{i+1,i+1} \end{bmatrix}$:

- the corresponding block of matrix T is a diagonal matrix,
- $(S_{i,i}T_{i+1,i+1} + S_{i+1,i+1}T_{i,i})^2 < -4S_{i+1,i}S_{i+1,i}T_{i,i}T_{i+1,i+1}$,
- there is a pair of conjugate complex eigenvalues

$$\lambda_i, \lambda_{i+1} =$$

$$\frac{S_{ii}T_{i+1,i+1} + S_{i+1,i+1}T_{i,i} \pm \sqrt{(S_{i,i}T_{i+1,i+1} - S_{i+1,i+1}T_{i,i})^2 + 4S_{i+1,i}S_{i+1,i}T_{i,i}T_{i+1,i+1}}}{2T_{i,i}T_{i+1,i+1}}$$

Applying the decomposition

$$
\begin{aligned}
D \begin{bmatrix} I \\ g_y \end{bmatrix} g_y \hat{y} &= E \begin{bmatrix} I \\ g_y \end{bmatrix} \hat{y} \\
\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} g_y \hat{y} & \\
&= \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} \hat{y}
\end{aligned}
$$

where rows and columns are re-ordered such that:

- (T_{11}, S_{11}) contain stable generalized eigenvalues (modulus ≤ 1)
- (T_{22}, S_{22}) contain explosive generalized eigenvalues (modulus > 1)

Selecting the stable trajectory

To exclude explosive trajectories, one imposes

$$Z_{21} + Z_{22} g_y = 0$$

$$g_y = -Z_{22}^{-1} Z_{21}$$

A unique stable trajectory exists if Z_{22} is square and non-singular: there are as many roots larger than one in modulus as there are forward-looking variables in the model (Blanchard and Kahn condition) and the rank condition is satisfied.

An alternative algorithm: Cyclic reduction

- Solving

$$A_0 + A_1 X + A_2 X^2 = 0$$

- Iterate

$$\begin{aligned}
A_0^{(k+1)} &= -A_0^{(k)}(A_1^{(k)})^{-1}A_0^{(k)},\\
A_1^{(k+1)} &= A_1^{(k)} - A_0^{(k)}(A_1^{(k)})^{-1}A_2^{(k)} - A_2^{(k)}(A_1^{(k)})^{-1}A_0^{(k)},\\
A_2^{(k+1)} &= -A_2^{(k)}(A_1^{(k)})^{-1}A_2^{(k)},\\
\widehat{A}_1^{(k+1)} &= \widehat{A}_1^{(k)} - A_2^{(k)}(A_1^{(k)})^{-1}A_0^{(k)}.
\end{aligned}$$

for $k = 1, \ldots$ with $A_0^{(1)} = A_0$, $A_1^{(1)} = A_1$, $A_2^{(1)} = A_2$, $\widehat{A}_1^{(1)} = A_1$ and until $||A_0^{(k)}||_\infty < \epsilon$ and $||A_2^{(k)}||_\infty < \epsilon$.

- Then

$$X \approx -(\widehat{A}_1^{(k+1)})^{-1}A_0$$

Recovering g_u

$$f_{y_+} g_y g_u + f_{y_0} g_u + f_u = 0$$

$$g_u = -\left(f_{y_+} g_y + f_{y_0}\right)^{-1} f_u$$

Hong Lan & Alexander Meyer-Gohde, 2012. "Existence and Uniqueness of Perturbation Solutions to DSGE Models," SFB 649 Discussion Papers, Humboldt University, show that $f_{y_+} g_y + f_{y_0}$ is an invertible matrix under standard regularity and saddle stability assumptions.

Recovering g_σ

$$f_{y_+} g_y g_\sigma + f_{y_0} g_\sigma = 0$$

$$g_\sigma = 0$$

Yet another manifestation of the certainty equivalence property of first order approximation.

First order approximated decision function

$$y_t = \bar{y} + g_y \hat{y} + g_u u$$

$$\begin{aligned} E\left\{y_t\right\} &= \bar{y} \\ \Sigma_y &= g_y \Sigma_y g_y' + \sigma^2 g_u \Sigma_\epsilon g_u' \end{aligned}$$

The variance is solved for with an algorithm for Lyapunov equations.

Outline

A simple RBC model

Consider the following model of an economy.

- Representative agent preferences

$$U = \sum_{t=1}^{\infty} \left(\frac{1}{1+\rho} \right)^{t-1} E_t \left[\log \left(C_t \right) - \frac{L_t^{1+\gamma}}{1+\gamma} \right].$$

  The household supplies labor and rents capital to the corporate sector.

  - L_t is labor services
  - $\rho \in (0, \infty)$ is the rate of time preference
  - $\gamma \in (0, \infty)$ is a labor supply parameter.
  - C_t is consumption,
  - w_t is the real wage,
  - r_t is the real rental rate

RBC Model (continued)

- The household faces the sequence of budget constraints

$$K_t = K_{t-1}(1-\delta) + w_t L_t + r_t K_{t-1} - C_t,$$

where
  - K_t is capital at the end of period
  - $\delta \in (0,1)$ is the rate of depreciation
- The production function is given by the expression

$$Y_t = A_t K_{t-1}^{\alpha}\left((1+g)^t L_t\right)^{1-\alpha}$$

where $g \in (0,\infty)$ is the growth rate and α and β are parameters.
- A_t is a technology shock that follows the process

$$A_t = A_{t-1}^{\lambda}\exp(e_t),$$

where e_t is an i.i.d. zero mean normally distributed error with standard deviation σ_1 and $\lambda \in (0,1)$ is a parameter.

The household problem

Lagrangian

$$L = \max_{C_t, L_t, K_t} \sum_{t=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{t-1} E_t \Big[\log\left(C_t\right) - \frac{L_t^{1+\gamma}}{1+\gamma} \\ - \mu_t \left(K_t - K_{t-1}\left(1-\delta\right) - w_t L_t - r_t K_{t-1} + C_t\right) \Big]$$

First order conditions

$$\frac{\partial L}{\partial C_t} = \left(\frac{1}{1+\rho}\right)^{t-1} \left(\frac{1}{C_t} - \mu_t\right) = 0$$

$$\frac{\partial L}{\partial L_t} = \left(\frac{1}{1+\rho}\right)^{t-1} \left(L_t^{\gamma} - \mu_t w_t\right) = 0$$

$$\frac{\partial L}{\partial K_t} = -\left(\frac{1}{1+\rho}\right)^{t-1} \mu_t + \left(\frac{1}{1+\rho}\right)^{t} E_t\left(\mu_{t+1}(1-\delta+r_t)\right) = 0$$

First order conditions

Eliminating the Lagrange multiplier, one obtains

$$\begin{aligned} L_t^{\gamma} &= \frac{w_t}{C_t} \\ \frac{1}{C_t} &= \frac{1}{1+\rho} E_t \left(\frac{1}{C_{t+1}} (r_{t+1} + 1 - \delta) \right) \end{aligned}$$

The firm problem

$$\max_{L_t, K_{t-1}} A_t K_{t-1}^{\alpha} \left((1+g)^t L_t \right)^{1-\alpha} - r_t K_{t-1} - w_t L_t$$

First order conditions:

$$r_t = \alpha A_t K_{t-1}^{\alpha-1} \left((1+g)^t L_t \right)^{1-\alpha}$$
$$w_t = (1-\alpha) A_t K_{t-1}^{\alpha} \left((1+g)^t \right)^{1-\alpha} L_t^{-\alpha}$$

Goods market equilibrium

$$K_t + C_t = K_{t-1}(1-\delta) + A_t K_{t-1}^{\alpha}\left((1+g)^t L_t\right)^{1-\alpha}$$

Dynamic Equilibrium

$$
\begin{aligned}
\frac{1}{C_t} &= \frac{1}{1+\rho} E_t \left(\frac{1}{C_{t+1}} (r_{t+1} + 1 - \delta) \right) \\
L_t^{\gamma} &= \frac{w_t}{C_t} \\
r_t &= \alpha A_t K_{t-1}^{\alpha-1} \left((1+g)^t L_t \right)^{1-\alpha} \\
w_t &= (1-\alpha) A_t K_{t-1}^{\alpha} \left((1+g)^t \right)^{1-\alpha} L_t^{-\alpha} \\
K_t + C_t &= K_{t-1}(1-\delta) + A_t K_{t-1}^{\alpha} \left((1+g)^t L_t \right)^{1-\alpha}
\end{aligned}
$$

Existence of a balanced growth path

There must exist a growth rates g_c and g_k so that

$$(1+g_k)^t K_1 + (1+g_c)^t C_1 = \frac{(1+g_k)^t}{1+g_K} K_1 (1-\delta) + A \left(\frac{(1+g_k)^t}{1+g_k} K_1 \right)^{\alpha} \left((1+g)^t L_t \right)^{1-\alpha}$$

So,

$$g_c = g_k = g$$

Stationarized model

Let's define

$$\widehat{C}_t = C_t/(1+g)^t$$
$$\widehat{K}_t = K_t/(1+g)^t$$
$$\widehat{w}_t = w_t/(1+g)^t$$

Stationarized model (continued)

$$\frac{1}{\widehat{C}_t(1+g)^t} = \frac{1}{1+\rho}E_t\left(\frac{1}{\widehat{C}_{t+1}(1+g)(1+g)^t}(r_{t+1}+1-\delta)\right)$$

$$L_t^\gamma = \frac{\widehat{w}_t(1+g)^t}{\widehat{C}_t(1+g)^t}$$

$$r_t = \alpha A_t\left(\widehat{K}_{t-1}\frac{(1+g)^t}{1+g}\right)^{\alpha-1}\left((1+g)^t L_t\right)^{1-\alpha}$$

$$\widehat{w}_t(1+g)^t = (1-\alpha)A_t\left(\widehat{K}_{t-1}\frac{(1+g)^t}{1+g}\right)^{\alpha}\left((1+g)^t\right)^{1-\alpha}L_t^{-\alpha}$$

$$\left(\widehat{K}_t+\widehat{C}_t\right)(1+g)^t = \widehat{K}_{t-1}\frac{(1+g)^t}{1+g}(1-\delta) + A_t\left(\widehat{K}_{t-1}\frac{(1+g)^t}{1+g}\right)^{\alpha}\left((1+g)^t L_t\right)^{1-\alpha}$$

Stationarized model (continued)

$$
\begin{aligned}
\frac{1}{\widehat{C}_t} &= \frac{1}{1+\rho} E_t \left(\frac{1}{\widehat{C}_{t+1}(1+g)} (r_{t+1} + 1 - \delta) \right) \\
L_t^{\gamma} &= \frac{\widehat{w}_t}{\widehat{C}_t} \\
r_t &= \alpha A_t \left(\frac{\widehat{K}_{t-1}}{1+g} \right)^{\alpha-1} L_t^{1-\alpha} \\
\widehat{w}_t &= (1-\alpha) A_t \left(\frac{\widehat{K}_{t-1}}{1+g} \right)^{\alpha} L_t^{-\alpha} \\
\widehat{K}_t + \widehat{C}_t &= \frac{\widehat{K}_{t-1}}{1+g}(1-\delta) + A_t \left(\frac{\widehat{K}_{t-1}}{1+g} \right)^{\alpha} L_t^{1-\alpha}
\end{aligned}
$$

Dynare implementation

```
var C K L w r A;
varexo e;

parameters rho delta gamma alpha lambda g;

alpha = 0.33;
delta = 0.1;
rho = 0.03;
lambda = 0.97;
gamma = 0;
g = 0.015;
```

Dynare implementation (continued)

```
model;
1/C=1/(1+rho)*(1/(C(+1)*(1+g)))*(r(+1)+1-delta);
L^gamma = w/C;
r = alpha*A*(K(-1)/(1+g))^(alpha-1)*L^(1-alpha);
w = (1-alpha)*A*(K(-1)/(1+g))^alpha*L^(-alpha);
K+C = (K(-1)/(1+g))*(1-delta)
     +A*(K(-1)/(1+g))^alpha*L^(1-alpha);
log(A) = lambda*log(A(-1))+e;
end;
```

Dynare implementation (continued)

```
steady_state_model;
A = 1;
r = (1+g)*(1+rho)+delta-1;
L = ((1-alpha)/(r/alpha-delta-g))*r/alpha;
K = (1+g)*(r/alpha)^(1/(alpha-1))*L;
C = (1-delta)*K/(1+g)
     +(K/(1+g))^alpha*L^(1-alpha)-K;
w = C;
end;

steady;
```

Dynare implementation (continued)

```
shocks;
var e; stderr 0.01;
end;

check;

stoch_simul(order=1);
```

Alternative implementation

Let Dynare detrend equations for you

```
parameters g;
trend_var(growth_factor=1+g) Z; // Productivity trend
var(deflator = Z) C K w;
var L r A;
varexo e;

parameters rho delta gamma alpha lambda;

model;
// Declare non-detrended model equations
end;
```

Decision and transition functions

Dynare output:

```
POLICY AND TRANSITION FUNCTIONS
                C           K           L           w           r           A
Constant   1.003043    3.125296    0.906526    1.003043    0.145450    1.000000
K(-1)      0.144433    0.779746   -0.105500    0.144433   -0.042523           0
A(-1)      0.757723    1.149948    0.589451    0.757723    0.204452    0.970000
e          0.781158    1.185514    0.607681    0.781158    0.210776    1.000000
```

$$C_t = 1.003 + 0.144\left(K_{t-1} - \bar{K}\right) + 0.758\left(A_{t-1} - \bar{A}\right) + 0.781 e_t$$

Outline

Dating variables in Dynare

Dynare will automatically recognize predetermined and non-predetermined variables, but you must observe a few rules:

- period t variables are set during period t on the basis of the state of the system at period $t-1$ and shocks observed at the beginning of period t.
- therefore, stock variables must be on an end-of-period basis: investment of period t determines the capital stock at the end of period t.

Note: with the `predetermined_variables` command, one can use a beginning-of-period convention for stocks when writing the model. However, the IRFs and other output will still be at end-of-stock convention.

Log-linearization

- Taking a log-linear approximation of a model is equivalent to take a linear approximation of a model with respect to the logarithm of the variables.
- In practice, it is sufficient to replace all occurences of variable X with $exp(LX)$ where $LX = \log X$.
- It is possible to make the substitution for some variables and not anothers. You wouldn't want to take a log approximation of a variable whose steady state value is negative ...
- There is no evidence that log-linearization is more accurate than simple linearization. In a growth model, it is often more natural to do a log-linearization.

Outline

The role of the Dynare preprocessor

- Dynare solves generic problems
- the preprocessor reads your `*.mod` file and translates it in specific MATLAB/Octave files
- *filename*.m: main MATLAB/Octave script for the model
- *filename*_static.m: static model
- *filename*_dynamic.m: dynamic model
- *filename*_steadystate2.m: steady state function
- *filename*_set_auxiliary_variables.m: auxiliary variables function

Overall design of Dynare