Introduction to Bayesian Estimation of DSGE Models

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Introduction

- DSGE models are the workhorse models in Macroeconomics
  - Real Business Cycle Models
  - **New Keynesian Models**

- Two possible approaches to obtaining meaningful results
  - Calibration techniques
  - Econometric estimation procedures

<u>Goal</u>: Cover basic Bayesian techniques and their applications to a medium scale New-Keynesian model

Introduction

- Why should you use Bayesian techniques?
  1) Medium between calibration and MLE
  2) Straigtforward model comparison
  3) Likelihood function is highly nonlinear
  4) Asymptotic theory hold in computation space

- It all starts here:

$$P(\vartheta|\boldsymbol{X}) = \frac{\overbrace{p(\boldsymbol{X}|\vartheta)}^{\textit{likelihood}}\overbrace{p(\vartheta)}^{\textit{prior}}}{\underbrace{\int_{\vartheta} p(\boldsymbol{X}|\vartheta)p(\vartheta)d\vartheta}_{\textit{marginal}}}$$

Model

NK Model: Households

- Unit measure of households who discount the future at rate β
- Subject to "habit stock", $\frac{C_t}{A_t}$, supply labor, H_t
- Pays lump taxes, T_t, and hold real money balance, $\frac{M_t}{P_t}$
- Access to government bonds, B_t, and state-contingent securities, SC_t
- Receive residual firm profits, D_t

NK Model: Households

- Representative agent's problem:

$$\max \ \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \left(\frac{(C_{t+s}/A_{t+s})^{1-\tau} - 1}{1-\tau} + \ln\left(\frac{M_{t+s}}{P_{t+s}}\right) - H_{t+s} \right) \right]$$

subject to the B.C.

$$P_t C_t + B_t + M_t + T_t = P_t W_t H_t \ldots$$

$$+ P_t SC_t + R_{t-1} B_{t-1} + M_{t-1} + P_t D_t$$

NK Model: Final Goods

- Final goods market is perfectly competitive

- CES production aggregates unit measure of intermediate goods

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}}$$

- Firms problem,

$$\max \ \Pi_t = P_t \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}} - \int_0^1 P_t(j) Y_t(j) dj$$

NK Model: Intermediate Goods

- Intermediate goods market is monopolistically competitive
- Labor market is frictionless and perfectly competitive
- Linear technology subject to productivity shocks,

$$Y_t(j) = A_t N_t(j)$$

- Quadratic price adjustment cost,

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \pi\right)^2 Y_t(j)$$

- Firms problem,

$$\max \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s Q_{t+s|t} \left(\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j) - W_{t+s} N_{t+s}(j) - AC_{t+s}(j)\right)\right]$$

NK Model: Government

- Monetary policy subject to interest feedback rule

$$\ln R_t = (1-\rho_R)\ln R_t^* + \rho_R \ln R_{t-1} + \epsilon_{R,t}$$

- Target nominal rate, R_t^*, given by
  1) Steady-state real interest rate, r
  2) Gross and target inflation rate, π^*
  3) Level-output gap due to nominal rigidities

$$\ln R_t^* = \ln r + (1-\psi_1)\ln \pi^* + \psi_1 \ln \pi_t + \psi_2 \ln\left(\frac{Y_t}{Y_t^*}\right)$$

- Fiscal policy given by government B.C.,

$$P_t G_t + R_{t-1}B_{t-1} + M_{t-1} = T_t + B_t + M_t$$

NK Model: Exogenous Processes

- Aggregate productivity given by random walk with drift,

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln z_t, \quad \ln z_t = \rho_z \ln z_{t-1} + \epsilon_{z,t}$$

- Define $g_t = \frac{1}{1-\zeta_t}$ where ζ_t is fraction of Y_t purchased by government,

$$\ln g_t = (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t}$$

- Shocks are mean zero and uncorrelated,

$$\begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{z,t} \\ \epsilon_{g,t} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_R^2 & 0 & 0 \\ 0 & \sigma_z^2 & 0 \\ 0 & 0 & \sigma_g^2 \end{bmatrix} \right)$$

Likelihood Function

Solving the Model

- Use the following steps to solve the model:
  1) Solve for equilibrium conditions
  2) Find non-stochastic steady state by setting $\sigma = 0$
  3) Transform model variables by $x_t = \frac{X_t}{A_t}$
  4) Rewrite equilibrium conditions as deviations from s.s.: $\hat{x}_t = \ln\left(\frac{x_t}{x_{ss}}\right)$
  5) Find 1^{st} order approximation about steady state

Linear Rational Expectations System

- Linearized equilibrium conditions,

$$\hat{y}_t = \mathbb{E}_t(\hat{y}_{t+1}) - \frac{1}{\tau}\left(\hat{R}_t - \mathbb{E}_t(\hat{\pi}_{t+1}) - \overbrace{\mathbb{E}_t(\hat{z}_{t+1})}^{\rho_z \hat{z}_t}\right) + \hat{g}_t - \overbrace{\mathbb{E}_t(\hat{g}_{t+1})}^{\rho_g \hat{g}_t}$$

$$\hat{\pi}_t = \beta\mathbb{E}_t(\hat{\pi}_{t+1}) + \underbrace{\tau\frac{1-\nu}{\nu\pi^2\phi}}_{\kappa}(\hat{y}_t - \hat{g}_t)$$

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)\psi_1\hat{\pi}_t + (1-\rho_r)\psi_2(\hat{y}_t - \hat{g}_t) + \epsilon_{R,t}$$

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \epsilon_{g,t}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z,t}$$

Quasi-Schur Decomposition

- Need to impose the transverasltiy condition

- Define the following
  1) $s_t = [\hat{y}_t, \hat{\pi}_t, \hat{R}_t, \hat{y}_{t-1}, \hat{g}_t, \hat{z}_t, \mathbb{E}_t(\hat{y}_{t+1}), \mathbb{E}_t(\hat{\pi}_t)]'$
  2) $\epsilon_t = [\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{R,t}]'$
  3) $\eta_t = [\hat{y}_t - \mathbb{E}_{t-1}(\hat{y}_t), \hat{\pi}_t - \mathbb{E}_{t-1}(\hat{\pi}_t)]'$

$$\Rightarrow \Gamma_0 s_t = \Gamma_1 s_{t-1} + \Psi \epsilon_t + \Pi \eta_t$$

- Apply Quasi-Schur decomposition,

$$\Gamma_0 = Q' \Omega Z' \qquad \Gamma_1 = Q' \Lambda Z'$$

where, Λ and Ω are triangular and $Q'Q = Z'Z = I$

Quasi-Schur Decomposition

- Generalized eigenvalues of Γ_0 and Γ_1 are given by $\frac{\lambda_{ii}}{\omega_{ii}}$
- BPremultiply system by Q, define $w_t = Z' s_t$, and substitute for Γ_0, Γ_1:

$$\Omega w_t = \Lambda w_{t-1} + Q(\Psi \epsilon_t + \Pi \eta_t)$$

- Partition system into nonexplosive block (1) and explosive block (2)
  1) Need $w_{2,0} = 0$ $\forall$ structural shocks
  2) Need shocks and expectation errors of (2) to cancel $\forall\, t$

Quasi-Schur Decomposition

- Solve for $w_{2,t-1}$ and iterate explosive block forward,

$$w_{2,t-1} = \lim_{T\to\infty} (\Lambda_2^{-1}\Omega_2)^T w_{2,t+T} - \sum_{i=1}^{\infty} \Lambda_2^{-i} Q_2(\Psi\epsilon_{t+i} + \Pi\eta_{t+i})$$

- Diagonal elements of $\Lambda_2^{-1}\Omega_2$ are inverse e-vals, $\frac{\omega_{ii}}{\lambda_{ii}} < 1$

$$\Rightarrow \lim_{T\to\infty} \left(\frac{w_{nn}}{\lambda_{nn}}\right)^T = 0 \Rightarrow w_{2,t}^{(n)} = 0 \Rightarrow w_{2,t}^{(n-1)} = 0 \ldots$$

- Sufficient condition for req. 2 is

$$\eta_t = -(Q_2\Pi)^{-1} Q_2 \Psi \epsilon_t$$

State-Space Form

- Substitute expectation errors and solve for s_t

$$s_t = \Phi_1 s_{t-1} + \Phi_2 \epsilon_t$$

- Define observer equation

$$\zeta_t = \Upsilon_1 + \Upsilon_t s_t + u_t$$

- Avoid stochastic singularity by
  1) **Limiting number of observables**
  2) Adding measurement errors

State-Space Form

- Define the following set of observables:
  1) Quarterly GDP growth rate

$$GR_t = \gamma^Q + 100(\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t)$$

  2) Annualized quarterly inflation rate

$$INF_t = \pi^A + 400\hat{\pi}_t$$

  3) Annualized nominal interest rate

$$INT_t = \pi^A + r^A + 4\gamma^Q + 400\hat{R}_t$$

- State-space model implies joint likelihood function

$$p(\zeta_{1:t}, s_{1:t}|\theta) = \prod_{t=1}^{T} p(\zeta_t|s_t, \theta)p(s_t|s_{t-1}, \theta)$$

Kalman-Filter

- Bayesian (and frequentist) inference can only be done using observables

- Integrate out unobservables using Linear (Kalman) Filter
  - Requires normally distributed shocks
  - Initiate at stationary distribution

- Stept to Kalman-Filter:
  1) Predict distribution of unobservables
  2) Obtain distribution of unobservables using prediction
  3) Update prediction using distribution of observables

Likelihood Function

Algorithm

Computation	Distribution	Mean/Variance
$p(s_t \mid \zeta_{1:t-1}, \theta)$	$\mathcal{N}(\bar{s}_{t-1\mid t-1}, P_{t-1\mid t-1})$	$\bar{s}_{t\mid t-1} = \Phi_1 \bar{s}_{t-1\mid t-1}$ $P_{t\mid t-1} = \Phi_1 P_{t-1\mid t-1} \Phi_1' + \Phi_\epsilon \Sigma_\epsilon \Phi_\epsilon'$
$p(\zeta_t \mid \zeta_{1:t-1}, \theta)$	$\mathcal{N}(\bar{\zeta}_{t\mid t-1}, F_{t\mid t-1})$	$\bar{\zeta}_{t\mid t-1} = \Upsilon_0 + \Upsilon_2 \bar{s}_{t\mid t-1}$ $F_{t\mid t-1} = \Upsilon_2 P_{t\mid t-1} \Upsilon_2' + \Sigma_u$
$p(s_t \mid \zeta_{1:t}, \theta)$	$\mathcal{N}(\bar{s}_{t\mid t}, P_{t\mid t})$	$\bar{s}_{t\mid t} = \bar{s}_{t\mid t-1} + P_{t\mid t-1} \Upsilon_2' F_{t\mid t-1}^{-1} (\zeta_t - \bar{\zeta}_{t\mid t-1})$ $P_{t\mid t} = P_{t\mid t-1} - P_{t\mid t-1} \Upsilon_2' F_{t\mid t-1}^{-1} \Upsilon_2 P_{t\mid t-1}$

repeat

- Find likelihood function using collection of step 2,

$$p(\zeta \mid \theta) = \prod_{t=1}^{T} p(\zeta_t \mid \zeta_{1:t-1}, \theta)$$

Prior

Prior Specification

- Conduct literature review of parameter micro-level estimates
- Partition data into pre- & post-sample and calculate statistics using pre-sample
- Calculate statistics of interest using joint-prior
  1) Cross-correlations
  2) Impulse response functions

Prior Specification

- Priors taken from Lubik and Schorfheide (2004)

Prior Distributions

Param	Domain	Dist	Mean	StnDev
ρ_g	$[0,1]$	Uniform	0.5	0.29
ρ_z	$[0,1]$	Uniform	0.5	0.29
κ	$[0,1]$	Uniform	0.5	0.29
ρ_R	$[0,1]$	Uniform	0.5	0.29
r^A	$\mathbb{R}^+$	Gamma	0.5	0.5
π^A	$\mathbb{R}^+$	Gamma	7.0	2.0
τ	$\mathbb{R}^+$	Gamma	2.0	0.5
ψ_1	$\mathbb{R}^+$	Gamma	1.5	0.24
ψ_2	$\mathbb{R}^+$	Gamma	0.50	0.25
σ_R	$\mathbb{R}^+$	InvGamma	0.40	4.0
σ_G	$\mathbb{R}^+$	InvGamma	1.0	4.0
σ_z	$\mathbb{R}^+$	InvGamma	0.50	4.0
γ^A	$\mathbb{R}$	Normal	0.40	0.20

Posterior

Metropolis-Hastings

- Recall Baye's Law,

$$P(\vartheta|\boldsymbol{X}) = \frac{p(\boldsymbol{X}|\vartheta)p(\vartheta)}{\int\limits_{\vartheta} p(\boldsymbol{X}|\vartheta)p(\vartheta)d\vartheta}$$

- Joint distribution may not have closed form integral
  - No analytical solution
- Conditional distributions may take no known functional form
  - No Gibbs sampling
- Turn to Metropolis-Hastings algorithms

Metrpolois-Hastings

- Draws samples from $P(x)$ using only $f(x)$ where $P(x) \propto f(x)$ and proposal distribution
  - Set $P(x) = P(\vartheta|\boldsymbol{X})$
  - Set $f(x) = p(\boldsymbol{X}|\vartheta)p(\vartheta)$
- Numerous algorithms with different proposal distributions, $q(\vartheta^*|\vartheta^{(s-1)})$

Algorithm

1) Choose $\vartheta^{(0)}$ mostly arbitrarily
2) Draw proposal from $q(\vartheta^*|\vartheta^{(s-1)})$
3) Set $\vartheta^{(s)} = \vartheta^*$ with probabilty $\alpha(\vartheta^*, \vartheta^{(s-1)}) = \min\{\frac{p(\vartheta^*|\boldsymbol{X})/q(\vartheta^*|\vartheta^{(s-1)})}{p(\vartheta^{(s-1)}|\boldsymbol{X})/q(\vartheta^{(s-1)}|\vartheta^*)}, 1\}$
4) Repeat until convergence of means

Matrpolis-Hastings

- Unclear direction for $\vartheta^{(0)}$ to move$\Rightarrow$ use Random Walk-MH
- Proposal distribution given by,

$$\vartheta^* = \vartheta^{(s-1)} + z \quad \Leftrightarrow \quad q(\vartheta^*|\vartheta^{(s-1)}) = \mathcal{N}(\vartheta^{(s-1)}, c^2\hat{\Sigma})$$

$$\Rightarrow \alpha(\vartheta^*, \vartheta^{(s-1)}) = \min\{\frac{p(\vartheta^*|\boldsymbol{X})}{p(\vartheta^{(s-1)}|\boldsymbol{X})}, 1\}$$

- Many ways to choose $\hat{\Sigma}$
  1) Inverse Hessian of log-posterior at mode
  2) **Covariance matrix of pre-estimation RWMH**
- Set tuning constnat, c, to get 20%-40% acceptance

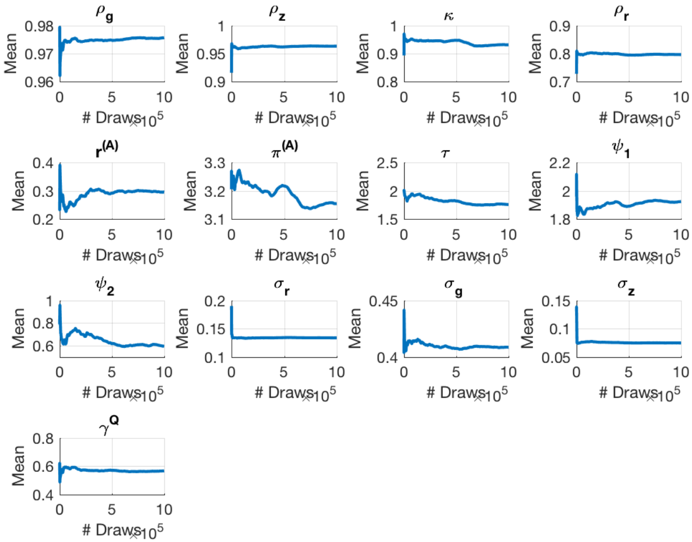
Results: Algorithm

- 10,000 iteration pre-estimation RWMH to obtion $\hat{\Sigma}$ using $\Sigma = I$

- Collect FRED data from 1983-2003
  1) GR_t from real GDP and adult civilian pop.
  2) INF_t from CPI for all urban goods
  3) INT_t from effective federal funds rate

- 1,000,000 posterior samples with 500,000 burn in

Results: Convergence



Results: Posterior