Design of experiment - Homework 05

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※ Exercise1

Write down the pdf form of a multivariate normal distribution.

$\mathbf{X} \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$ or $\mathbf{X} \sim \mathrm{N}_k(\boldsymbol{\mu}, \Sigma)$ 標示出 *k*-dimensional

Mean $\boldsymbol{\mu} = \mathrm{E}[\mathbf{X}] = [[\mathrm{E}[X_1], \mathrm{E}[X_2], \mathrm{E}[X_3], \ldots \mathrm{E}[X_{k-1}], \mathrm{E}[X_k],]^{\mathrm{T}}$,

Covariance matrix $\Sigma = \begin{pmatrix} var(X_1) & \cdots & cov(X_1,\ X_k) \\ \vdots & \ddots & \vdots \\ cov(X_k,\ X_1) & \cdots & var(X_k) \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \cdots & \rho_{1k}\sigma_1\sigma_k \\ \vdots & \ddots & \vdots \\ \rho_{k1}\sigma_k\sigma_1 & \cdots & \sigma_k^2 \end{pmatrix}$

The pdf of multivariate normal:

$$f_{\mathbf{X}}(x_1, x_2, x_3, \ldots, x_{k-1}, x_k) = \frac{\exp\left(-\frac{1}{2}(\mathbf{X}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{X}-\boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k|\Sigma|}},\ \mathbf{X} \in \mathbb{R}.$$

※Question A.1

Let y be an N by 1 random vector with Ey = Xβ, and Var(y) = $\sigma^2 I_N$, where X is N by p and β is p by 1. Let Y = Py, where P is a projection (not necessarily orthogonal) onto the range of X. (a) Find the mean and (co)varianceof Y and y−Y. (b)Prove that Cov(Y, y−Y) is 0 if and only if P is an orthogonal projection.

(a) Find the mean and (co)varianceof Y and y−Y.

E(Y) = E(Py) = PE(y) = PXβ = Xβ;

$\mathrm{Var(Y) = Var(Py) = PVar(y)P'} = \mathrm{P}\sigma^2\mathrm{I_N}\mathrm{P}' = \mathrm{PP}'\sigma^2$

$\mathrm{E(y - Y) = E(y - Py) = E((I_N - P)y) = (I_N - P)E(y) = (I_N - P)X\beta = (X - PX)\beta = X\beta - PX\beta}$

Var(y - Y) = Var(y) + Var(Y) – 2Cov(y,Y) = $\sigma^2\mathrm{I_N} + \mathrm{P}\sigma^2\mathrm{P}' - \mathrm{cov(y, Py)} - \mathrm{cov(Py, y)}$

$= \sigma^2 + PP'\sigma^2 - \mathrm{P}'\sigma^2 - \mathrm{P}\sigma^2$

(b) Prove that Cov(Y, y−Y)is 0 if and only if P is an orthogonal projection.

$\Rightarrow \mathrm{cov}(Y, y - Y) = \mathrm{cov}(Py, (I_N - P)y) = P\mathrm{var}(y)(I_N - P)' = PI_N\sigma^2(I_N' - P') = (PI_N' - PP')\sigma^2,$

(If P is orthogonal projection then $P = P', PP = P$) $(PI_N' - PP')\sigma^2 = (P - PP)\sigma^2 = (P - P)\sigma^2 = 0,$

only when P is an orthogonal projection. Therefore, P is an orthogonal projection.

$\Leftarrow$ P is an orthogonal projection, then $P' = P,\ PP = P,\ P - P'P = P - PP = P - P = 0$

$0 = (P - P'P)\sigma^2 I_N = (P - PP)\sigma^2 I_N = P(I_N - P')\sigma^2 I_N = P(I'_N - P')\sigma^2 I_N = P\sigma^2 I_N(I'_N - P')$

$= P\,\mathrm{var}(y)((I'_N - P') = P\mathrm{cov}(y, y)(I'_N - P') = P\,\mathrm{cov}(y, y)(I_N - P)' = \mathrm{cov}(Py, (I - P)y) = \mathrm{cov}(Y, y - Y)$

Therefore, $\mathrm{cov}(Y, y - Y) = 0$

※Question A.2

Let $y = X\beta + \epsilon$, where ϵ is iid $N(0, I_N\sigma^2)$; y is N by 1, X is N by p, and β is p by 1. Let g be any N by 1 vector. What is the distribution of $(g'\mathbf{y})^2$? What, if anything, changes when $g'\mathbf{X}$ is zero?

Let $E(\mathbf{y}) = E(\mathbf{X}\beta + \epsilon) = E(\mathbf{X}\beta) + E(\epsilon) = E(\mathbf{X}\beta) = \boldsymbol{\mu}$

$\mathrm{Var}(\mathbf{y}) = \mathrm{Var}(\mathbf{X}\beta + \epsilon) = \mathrm{Var}(\epsilon) = I_N\sigma^2$

$$\mathbf{y} \sim N(\mu, I_N\sigma^2)$$

$$g'\mathbf{y} \sim N(g'\mu, g'I_N\sigma^2 g)$$

(a) What is the distribution of $(g'y)^2$

$$\frac{g'\mathbf{y}}{(g'I_N\sigma^2 g)^{\frac{1}{2}}} \sim N\left(\frac{g'\mu}{(g'I_N\sigma^2 g)^{\frac{1}{2}}}, 1\right)$$

$\Rightarrow \left(\frac{g'\mathbf{y}}{(g'I_N\sigma^2 g)^{\frac{1}{2}}}\right)\left(\frac{g'\mathbf{y}}{(g'I_N\sigma^2 g)^{\frac{1}{2}}}\right) \sim$ noncentral $\chi^2(1)$ and noncentrality parameter $\Omega = \frac{\mu' g g' \mu}{2}$

(b) $g'\mathbf{X} = 0$

$$g'\mathbf{X} = 0 \Rightarrow g'\mu = g'\mathbf{X}\beta = 0 \Rightarrow \frac{g'\mathbf{y}}{(g'I_N\sigma^2 g)^{\frac{1}{2}}} \sim N(0,1)$$

$$\Rightarrow \left(\frac{g'\mathbf{y}}{(g'I_N\sigma^2 g)^{\frac{1}{2}}}\right)\left(\frac{g'\mathbf{y}}{(g'I_N\sigma^2 g)^{\frac{1}{2}}}\right) \sim \chi^2(1) \Rightarrow (g'\mathbf{y})\,(g'\mathbf{y}) \sim \mathrm{Gamma}\left(\frac{1}{2}, 2(g'I_N\sigma^2 g)\right)$$