Computing Bayes factors using the normal approximation

How to multiply Gaussian densities

Here is the "crucial lemma" about gaussian distributions shown in lectures. (We used it to figure out how to implement fixed-effect meta-analysis).

It says that the pointwise multiple of Gaussian densities is proportional to another Gaussian density. It also computes the mean and variance of this density. And it works out the constant of proportionality, which turns out to be computable as yet another Gaussian density (but this time evaluated using the original distribution means, not the point of evaluation x).

Lemma 1. *If μ_1,μ_2 are two means and Σ_1, Σ_2 are covariance matrices then:*

$$MVN(x;\mu_1,\Sigma_1)\times MVN(x;\mu_2,\Sigma_2) = \mathit{const}\times MVN(x;a,A)$$

where

$$A = \left(\Sigma_1^{-1}+\Sigma_2^{-1}\right)^{-1} \qquad \text{and} \qquad a = A\left(\Sigma_1^{-1}\mu_1+\Sigma_2^{-1}\mu_2\right)$$

(if the matrices are invertible). Moreover the constant is also computable as a multivariate normal density,

$$\mathit{const} = MVN\left(\mu_1-\mu_2;0,\Sigma_1+\Sigma_2\right)$$

Application to Bayes factors

We can apply this to Bayesian analysis, by using a Gaussian prior $MVN(\mu,\Sigma)$ and a Gaussian likelihood $MVN(\hat{\beta},V)$, where V is the variance-covariance matrix of the loglikelihood. (As discussed in lectures, this is approximately applicable in fairly general settings when data quantities are large). In particular suppose we have a vector of true effects β and want to compare the model $\beta\sim MVN(\mu_1,\Sigma_1)$ to a model where all the effects are zero. Having seen some data, we want to know how probable M_1 is relative to M_0. This can be figured out using Bayes theorem written:

$$\begin{aligned}\frac{P(M_1|\text{data})}{P(M_0|\text{data})} &= \frac{P(\text{data}|M_1)}{P(\text{data}|M_0)}\cdot\frac{P(M_1)}{P(M_0)}\\ &= \frac{\int_\beta P(\text{data}|\beta,M_1)P(\beta|M_1)}{P(\text{data}|\beta=0)}\cdot\frac{P(M_1)}{P(M_0)}\end{aligned}$$

The first term is the Bayes factor, given the change odds in the data for model 1 relative to the null model. The last term is the prior odds of model 1 relative to the null model. (We must choose this).

The term being integrated is a product of Gaussians so we can use the lemma. The integral works out to be the constant term in the lemma = i.e.

$MVN\left(\hat{\beta} - \mu; 0, \Sigma + V\right)$. Hence the BF can be computed as:

$$\text{Bayes factor} = \frac{MVN(\hat{\beta} - \mu; 0, \Sigma + V)}{MVN(0; \hat{\beta}; V)}$$

This is one Bayesian analysis that is easy to implement !