GSERM 2017
Regression III
Linear Regression Review

June 19, 2017 (morning session)

“Regression III”

- “Regression” course
- Texts: Weisberg (2013) and others
- Course materials at the github repo: `https://github.com/PrisonRodeo/GSERM-2017-git`
- Software: R > Stata
- “Introduction to R and RStudio” today after the break...
- Grading: Two homework assignments plus a final examination

The Course (Probably)

Day One:

- Brief Review of Linear (and other) Regression Models
- Linear Model Specification: Interactions, Polynomials, and More

Day Two:

- Non-Linearity: Data Transformations
- Non-Linearity: Nonlinear Models

Day Three:

- Outliers: Detection and Influence
- Robust Inference: Bootstrapping and Other Resampling Methods

Day Four:

- Varying Coefficients and Mixture Models

Day Five:

- Introduction to Maximum Likelihood and Generalized Linear Models

Regression

"Regression," conceptually:

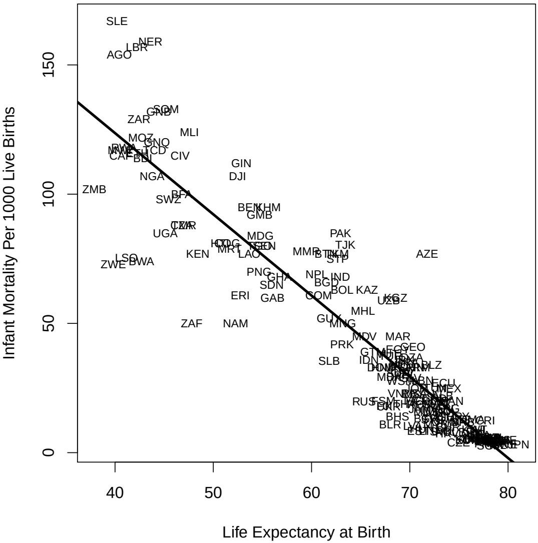
$$\Pr(Y|\mathbf{X}) = f(\mathbf{X})$$

Two important things:

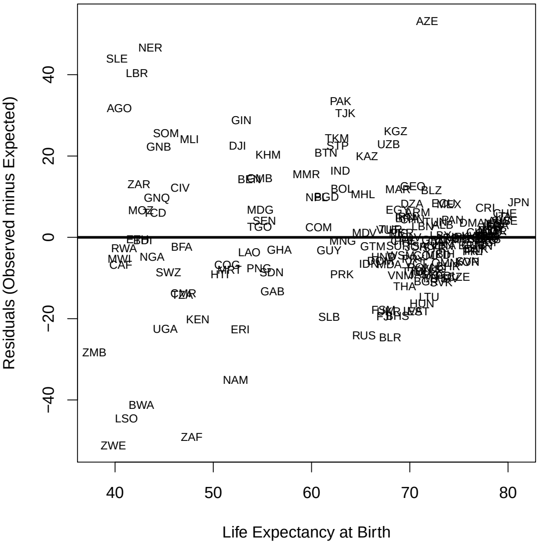
- The distribution of Y is *conditional on all variables in* $\mathbf{X}$, and
- The conditional distribution of Y is conditional on the *joint distribution* of the elements of $\mathbf{X}$.

→ Regression is <u>hard</u>...

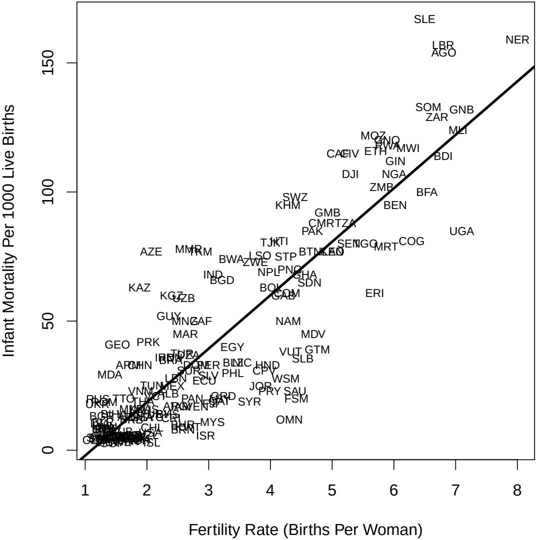
Infant Mortality and Life Expectancy



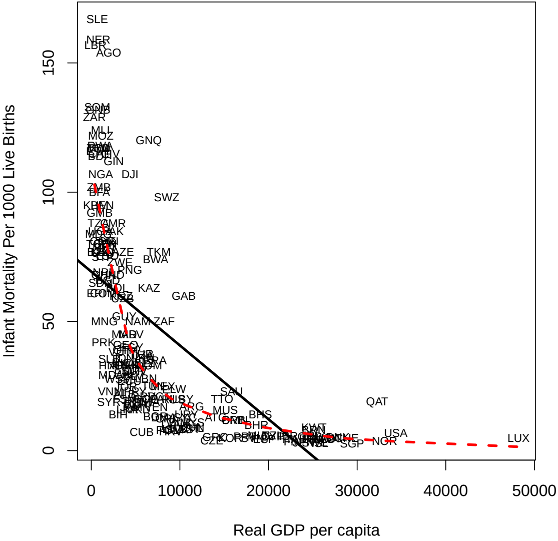
Infant Mortality and Life Expectancy: “Residuals”



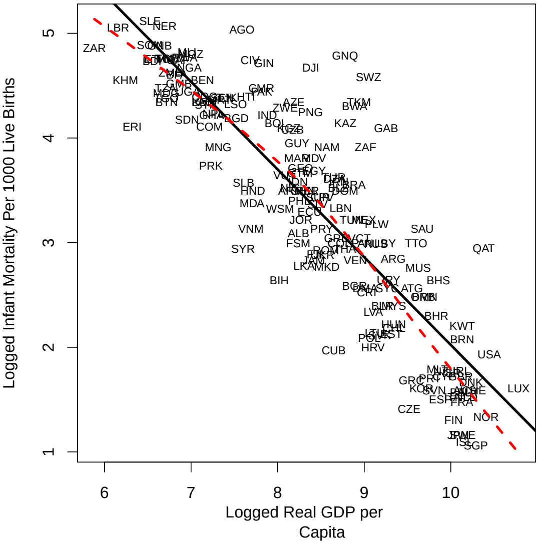
Infant Mortality and Fertility



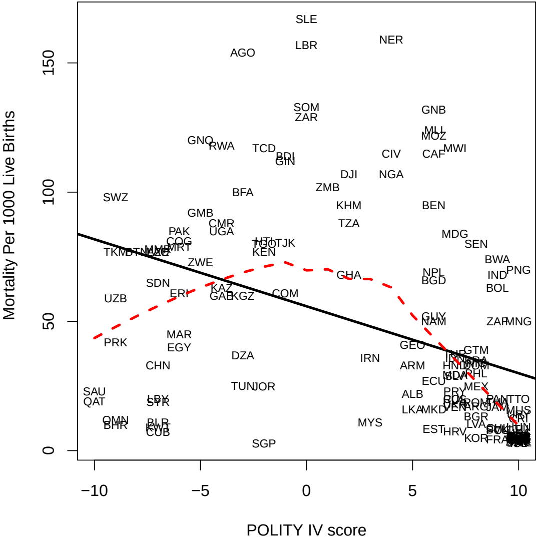
Infant Mortality and Wealth



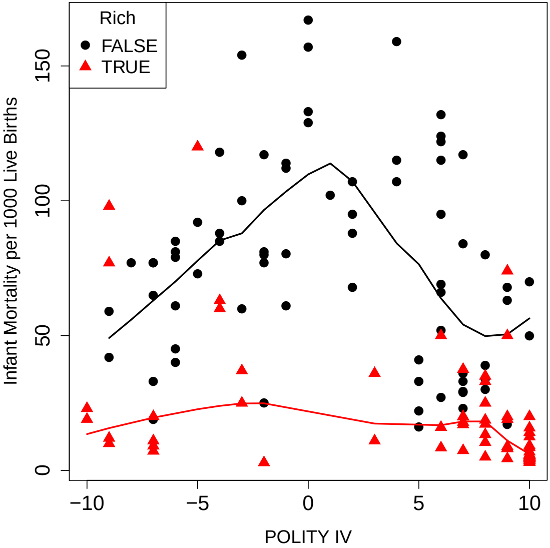
(Logged) Infant Mortality and (Logged) Wealth



Infant Mortality and Democracy



Infant Mortality, (Dichotomized) Wealth, and Democracy



Wherefore Regression?

	Description	**Explanation**	**Prediction**
Task	Summarize data	Correlation/causation	Forecast OOS / future data
Emphasis	Data	Theory / Hypotheses	Outcomes
Focus	Univariate	Multivariate	Multivariate
Typical Application	Summarize / "reduce" data	Discuss marginal associations between predictors and an outcome of interest	Optimize out-of-sample predictive power / minimize prediction error

Linear Regression

$$Y_i = \mu + u_i \tag{1}$$

$$\mu_i = \beta_0 + \beta_1 X_i$$

so:

$$Y_i = \beta_0 + \beta_1 X_i + u_i \tag{2}$$

Goals:

- Estimate $\hat{\beta}_0$ and $\hat{\beta}_1$
- Estimate the *variability* $\hat{\beta}_0$ and $\hat{\beta}_1$

Linear Regression (continued)

If we have $\hat{\beta}_0$ and $\hat{\beta}_1$, then:

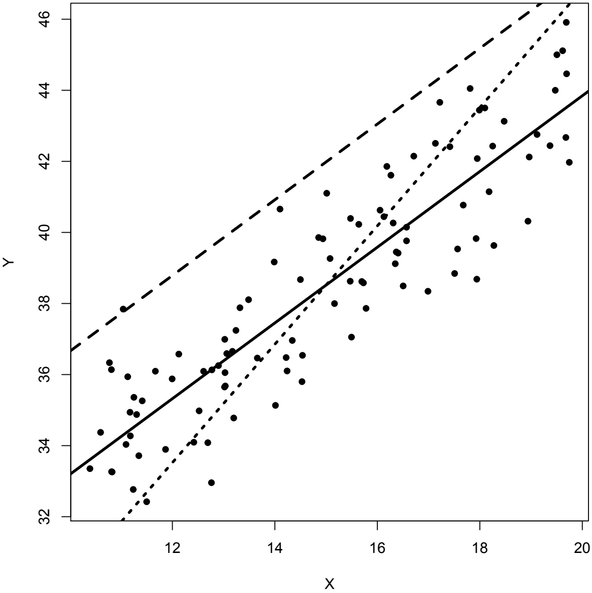
$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i \tag{3}$$

and

$$\begin{aligned} \hat{u}_i &= Y_i - \hat{Y}_i \\ &= Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \end{aligned} \tag{4}$$

Q: How to estimate $\hat{\beta}_0$ and $\hat{\beta}_1$?

Scatterplot: X and Y (with regression lines)



Ordinary Least Squares

Choose $\hat{\beta}_0$ and $\hat{\beta}_1$ to minimize $\hat{S} = \sum_{i=1}^{N} \hat{u}_i^2$.

$$\begin{aligned}
\hat{S} &= \sum_{i=1}^{N} (Y_i - \hat{Y}_i)^2 \\
&= \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 \\
&= \sum_{i=1}^{N} (Y_i^2 - 2Y_i\hat{\beta}_0 - 2Y_i\hat{\beta}_1 X_i + \hat{\beta}_0^2 + 2\hat{\beta}_0\hat{\beta}_1 X_i + \hat{\beta}_1^2 X_i^2)
\end{aligned}$$

OLS (continued)

Differentiate:

$$\begin{aligned}\frac{\partial \hat{S}}{\partial \hat{\beta}_0} &= \sum_{i=1}^{N}(-2Y_i + 2\hat{\beta}_0 + 2\hat{\beta}_1 X_i) \\ &= -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) \\ &= -2\sum_{i=1}^{N}\hat{u}_i\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \hat{S}}{\partial \hat{\beta}_1} &= \sum_{i=1}^{N}(-2Y_i X_i + 2\hat{\beta}_0 X_i + 2\hat{\beta}_1 X_i^2) \\ &= -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)X_i \\ &= -2\sum_{i=1}^{N}\hat{u}_i X_i\end{aligned}$$

OLS (continued)

Yields:

$$\sum_{i=1}^{N} Y_i = N\hat{\beta}_0 + \hat{\beta}_1 \sum_{i=1}^{N} X_i$$

and

$$\sum_{i=1}^{N} Y_i X_i = \hat{\beta}_0 \sum_{i=1}^{N} X_i + \hat{\beta}_1 \sum_{i=1}^{N} X_i^2$$

OLS (continued)

Solving yields:

$$\begin{aligned}\hat{\beta}_1 &= \frac{\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{N}(X_i - \bar{X})^2} \\ &= \frac{\text{Covariance of } X \text{ and } Y}{\text{Variance of } X}\end{aligned} \tag{5}$$

and

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \tag{6}$$

$\text{Var}(\hat{\beta}_1)$

$$u_i \sim \text{i.i.d. } N(0, \sigma^2)$$

meaning:

$$\text{Var}(Y|X, \beta) = \sigma^2$$

so:

$$\begin{aligned}
\text{Var}(\hat{\beta}_1) &= \text{Var}\left[\frac{\sum_{i=1}^{N}(X_i - \bar{X})Y_i}{\sum_{i=1}^{N}(X_i - \bar{X})^2}\right] \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2\, \text{Var}(Y) \\
&= \left[\frac{1}{\sum(X_i - \bar{X})^2}\right]^2 \sum(X_i - \bar{X})^2\, \sigma^2 \\
&= \frac{\sigma^2}{\sum(X_i - \bar{X})^2}.
\end{aligned}$$

Var($\hat{\beta}_0$) and Cov($\hat{\beta}_0, \hat{\beta}_1$)

Similarly:

$$\text{Var}(\hat{\beta}_0) = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \sigma^2$$

and :

$$\text{Cov}(\hat{\beta}_0, \hat{\beta}_1) = \frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \sigma^2$$

Important Things

- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto \sigma^2$
- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -\sum(X_i - \bar{X})$
- $\text{Var}(\hat{\beta}_0)$ and $\text{Var}(\hat{\beta}_1) \propto -N$
- $\text{sign}[\text{Cov}(\hat{\beta}_0, \hat{\beta}_1)] = \text{sign}(\bar{X})$

Inference

If $u_i \sim N(0, \sigma^2)$, then:

$$\hat{\beta}_0 \sim N[\beta_0, \text{Var}(\hat{\beta}_0)]$$

and

$$\hat{\beta}_1 \sim N[\beta_1, \text{Var}(\hat{\beta}_1)]$$

Means:

$$\begin{aligned} z_{\hat{\beta}_1} &= \frac{(\hat{\beta}_1 - \beta_1)}{\sqrt{\text{Var}(\hat{\beta}_1)}} \\ &= \frac{(\hat{\beta}_1 - \beta_1)}{\text{s.e.}(\hat{\beta}_1)} \\ &= \sim N(0,1) \end{aligned}$$

A Small Problem...

$\sigma^2 =$???

Solution: use

$$\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{N - k}$$

Gives:

$$\widehat{\text{Var}(\hat{\beta}_1)} = \frac{\hat{\sigma}^2}{\sum (X_i - \bar{X})^2},$$

and

$$\widehat{\text{Var}(\hat{\beta}_0)} = \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \hat{\sigma}^2$$

Inference (continued)

$$\begin{aligned}
\widehat{\text{s.e.}(\hat{\beta}_1)} &= \sqrt{\widehat{\text{Var}(\hat{\beta}_1)}} \\
&= \sqrt{\frac{\hat{\sigma}^2}{\sum(X_i - \bar{X})^2}} \\
&= \frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}
\end{aligned}$$

implies:

$$\begin{aligned}
t_{\hat{\beta}_1} \equiv \frac{(\hat{\beta}_1 - \beta_1)}{\widehat{\text{s.e.}(\hat{\beta}_1)}} &= \frac{(\hat{\beta}_1 - \beta_1)}{\frac{\hat{\sigma}}{\sqrt{\sum(X_i - \bar{X})^2}}} \\
&= \frac{(\hat{\beta}_1 - \beta_1)\sqrt{\sum(X_i - \bar{X})^2}}{\hat{\sigma}} \\
&\sim t_{N-k}
\end{aligned}$$

Predictions and Variance

Point prediction:

$$\hat{Y}_k = \hat{\beta}_0 + \hat{\beta}_1 X_k$$

Y_k is unbiased:

$$\begin{aligned} \mathrm{E}(\hat{Y}_k) &= \mathrm{E}(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\ &= \mathrm{E}(\hat{\beta}_0) + X_k \mathrm{E}(\hat{\beta}_1) \\ &= \beta_0 + \beta_1 X_k \\ &= \mathrm{E}(Y_k) \end{aligned}$$

Variability:

$$\begin{aligned} \mathrm{Var}(\hat{Y}_k) &= \mathrm{Var}(\hat{\beta}_0 + \hat{\beta}_1 X_k) \\ &= \frac{\sum X_i^2}{N \sum (X_i - \bar{X})^2} \sigma^2 + \left[\frac{\sigma^2}{\sum (X_i - \bar{X})^2} \right] X_k^2 + 2 \left[\frac{-\bar{X}}{\sum (X_i - \bar{X})^2} \sigma^2 \right] X_k \\ &= \sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right] \end{aligned}$$

Variability of Predictions

$$\text{Var}(\hat{Y}_k) = \sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2} \right]$$

means that $\text{Var}(\hat{Y}_k)$:

- Decreases in N
- Decreases in $\text{Var}(X)$
- Increases in $|X - \bar{X}|$

Predictions and Inference

Standard error of the prediction:

$$\widehat{\text{s.e.}(\hat{Y}_k)} = \sqrt{\sigma^2 \left[\frac{1}{N} + \frac{(X_k - \bar{X})^2}{\sum (X_i - \bar{X})^2}\right]}$$

$\rightarrow$ (e.g.) confidence intervals:

$$95\%\,\text{c.i.}(\hat{Y}_k) = \hat{Y}_k \pm [1.96 \times \widehat{\text{s.e.}(\hat{Y}_k)}]$$

Variation in Y

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(\hat{Y} + \hat{u}) \\
&= \mathrm{Var}(\hat{Y}) + \mathrm{Var}(\hat{u}) + 2\,\mathrm{Cov}(\hat{Y}, \hat{u}) \\
&= \mathrm{Var}(\hat{Y}) + \mathrm{Var}(\hat{u})
\end{aligned}$$

$$\begin{array}{ccccc}
\textbf{TSS} & = & \textbf{MSS} & + & \textbf{RSS} \\
\text{(``Total'')} & & \text{(``Estimated,'' or ``Model'')} & & \text{(``Residual'')}
\end{array}$$

Model Fit: R^2

$$\begin{aligned} R^2 &= \frac{\text{MSS}}{\text{TSS}} \\ &= \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} \\ &= 1 - \frac{\text{RSS}}{\text{TSS}} \\ &= 1 - \frac{\sum \hat{u}_i^2}{\sum(Y_i - \bar{Y})^2} \end{aligned}$$

R^2 Is...

R-squared:

- is "the proportion of variance explained"
- $\in [0, 1]$
  - $R^2 = 1.0 \equiv$ a "perfect (linear) fit"'
  - $R^2 = 0 \equiv$ no (linear) $X - Y$ association

For a single X,

$$\begin{aligned} R^2 &= \hat{\beta}_1^2 \frac{\sum(X_i - \bar{X})^2}{\sum(Y_i - \bar{Y})^2} \\ &= r_{XY}^2 \end{aligned}$$

Adjusted R^2

$$R^2_{adj.} = 1 - \frac{(1 - R^2)(N - c)}{(N - k)}$$

where $c = 1$ if there is a constant in the model and $c = 0$ otherwise.

$R^2_{adj.}$:

- $R^2_{adj.} \rightarrow R^2$ as $N \rightarrow \infty$
- $R^2_{adj.}$ can be > 1, or < 0...
- $R^2_{adj.}$ increases with model “fit,” but
- The extent of that increase is discounted by a factor proportional to the number of covariates.

R^2 Alternatives

- Standard Error of the Estimate:

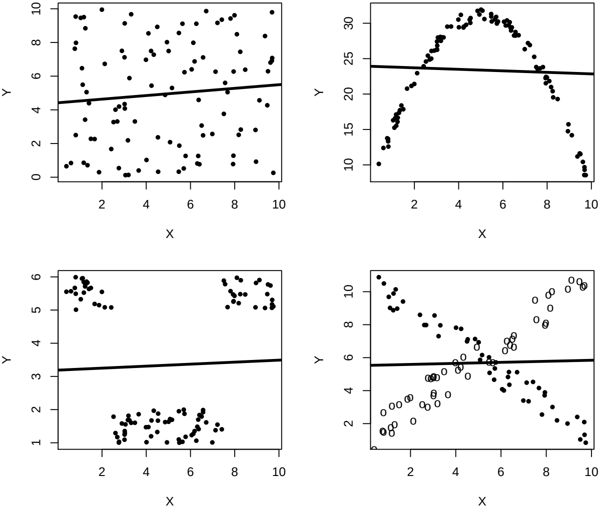
$$\text{SEE} = \sqrt{\frac{\text{RSS}}{N-k}}$$

- F-tests

- ROC / AUC

- Graphical methods

Caution: Different Ways to get $R^2 = 0$



Linear Regression: *k* Predictors

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_K X_{Ki} + u_i$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & X_{21} & \cdots & X_{K1} \\ 1 & X_{12} & X_{22} & \cdots & X_{K2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1N} & X_{2N} & \cdots & X_{KN} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}.$$

Estimating $\boldsymbol{\beta}$

Residuals:

$$\mathbf{u} = \mathbf{Y} - \mathbf{X}\beta$$

The inner product of **u**:

$$\begin{aligned}
\mathbf{u}'\mathbf{u} &= \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \\
&= u_1^2 + u_2^2 + ... + u_N^2 \\
&= \sum_{i=1}^{N} u_i^2
\end{aligned}$$

Estimating $\boldsymbol{\beta}$

$$\begin{aligned}\mathbf{u}'\mathbf{u} &= (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) \\ &= \mathbf{Y}'\mathbf{Y} - 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{Y}' + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}\end{aligned}$$

Now get:

$$\frac{\partial \mathbf{u}'\mathbf{u}}{\partial \boldsymbol{\beta}} = -2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta}$$

Solve:

$$\begin{aligned}-2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= 0 \\ -\mathbf{X}'\mathbf{Y} + \mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= 0 \\ \mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= \mathbf{X}'\mathbf{Y} \\ (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\ \boldsymbol{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}\end{aligned}$$

The Importance of $\mathbf{V}(\hat{\boldsymbol{\beta}})$

$$\begin{aligned}\mathbf{V}(\hat{\boldsymbol{\beta}}) &= \mathsf{E}[\hat{\boldsymbol{\beta}} - \mathsf{E}(\hat{\boldsymbol{\beta}})]^2 \\ &= \mathsf{E}\{[\hat{\boldsymbol{\beta}} - \mathsf{E}(\hat{\boldsymbol{\beta}})][\hat{\boldsymbol{\beta}} - \mathsf{E}(\hat{\boldsymbol{\beta}})]'\}\end{aligned}$$

Rewrite:

$$\begin{aligned}\mathbf{V}(\hat{\boldsymbol{\beta}}) &= \mathsf{E}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \\ &= \mathsf{E}\{[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}][(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}]'\} \\ &= \mathsf{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\mathbf{u}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}]\end{aligned}$$

The Importance of $\mathbf{V}(\hat{\beta})$

Taking expectations:

$$\begin{aligned}
\mathbf{V}(\hat{\beta}) &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{E}(\mathbf{uu}')\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}$$

Estimating $\mathbf{V}(\hat{\beta})$

Empirical estimate:

$$\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{N - K}$$

Yields:

$$\widehat{\mathbf{V}(\hat{\beta})} = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$$

OLS Assumptions

1. Zero Expectation Disturbances

$$\mathsf{E}(\mathbf{u}) = \mathbf{0}$$

OLS Assumptions

2. Homoscedasticity / No Error Correlation

$$\begin{aligned}
\mathbf{uu}' &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix} \\
&= \begin{bmatrix} u_1^2 & u_1 u_2 & \cdots & u_1 u_N \\ u_2 u_1 & u_2^2 & \cdots & u_2 u_N \\ \vdots & \vdots & \ddots & \vdots \\ u_N u_1 & u_N u_2 & \cdots & u_N^2 \end{bmatrix}
\end{aligned}$$

Expectation must be:

$$\mathrm{E}(\mathbf{uu}') = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}$$

OLS Assumptions

3. "Fixed" $\mathbf{X}$...

- No *measurement error* in the $\mathbf{X}$s, and
- $\text{Cov}(\mathbf{X}, \mathbf{u}) = \mathbf{0}$.

4. $\mathbf{X}$ is full column rank.

Means:

- no exact linear relationship among $\mathbf{X}$, and
- $K < N$.

5. Normal Disturbances

$$\mathbf{u} \sim \mathbf{N}(\mathbf{0}, \sigma^2 \mathbf{I})$$

OLS: Properties

Under these assumptions, the OLS estimate of $\hat{\boldsymbol{\beta}}$ is:

- **Unbiased**
- **Fully Efficient**

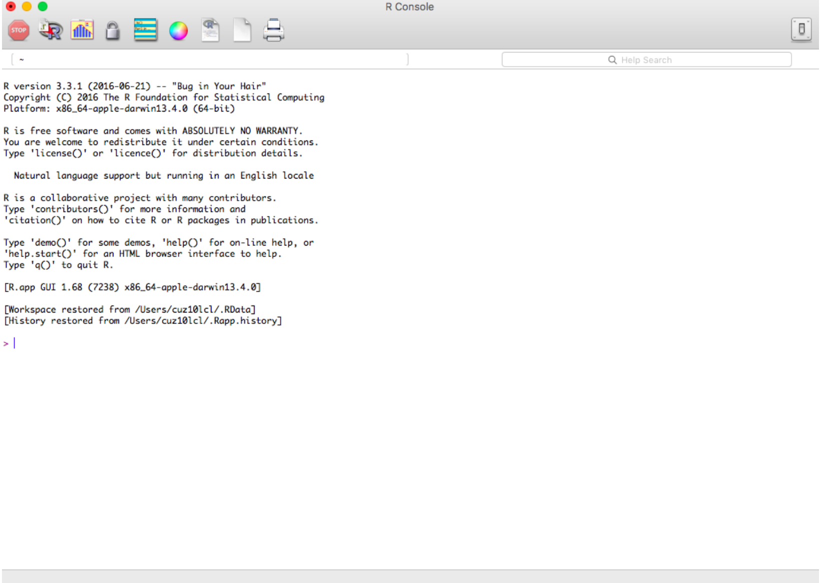
(i.e., **“BLUE”**)

BREAK

Software

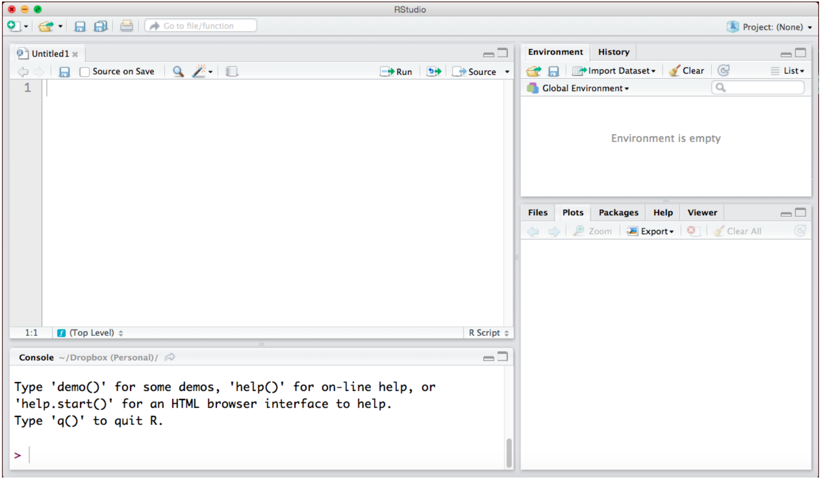
- Preferred: R + RStudio
- Also viable: Stata
- Highly discouraged: SAS, SPSS/PSPP, etc.

R



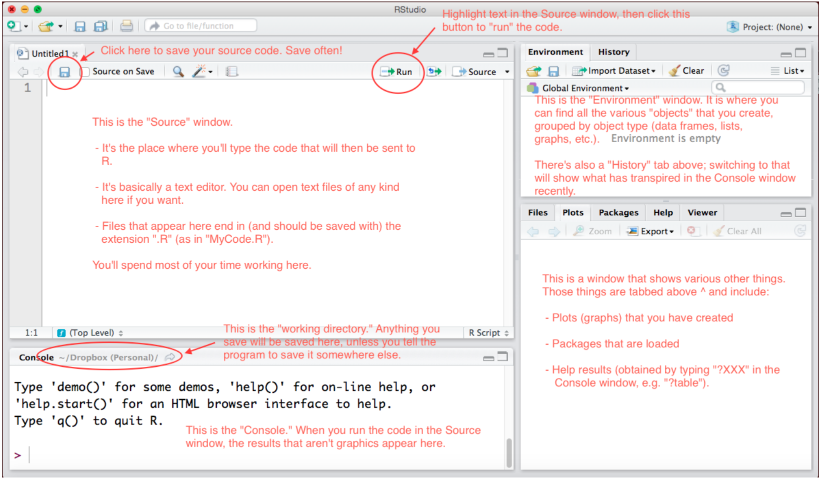
```
R version 3.3.1 (2016-06-21) -- "Bug in Your Hair"
Copyright (C) 2016 The R Foundation for Statistical Computing
Platform: x86_64-apple-darwin13.4.0 (64-bit)

R is free software and comes with ABSOLUTELY NO WARRANTY.
You are welcome to redistribute it under certain conditions.
Type 'license()' or 'licence()' for distribution details.

  Natural language support but running in an English locale

R is a collaborative project with many contributors.
Type 'contributors()' for more information and
'citation()' on how to cite R or R packages in publications.

Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

[R.app GUI 1.68 (7238) x86_64-apple-darwin13.4.0]

[Workspace restored from /Users/cuz10lcl/.RData]
[History restored from /Users/cuz10lcl/.Rapp.history]

> 
```

RStudio



```
Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

>
```

RStudio (annotated)



Highlight text in the Source window, then click this button to "run" the code.

Click here to save your source code. Save often!

This is the "Source" window.

- It's the place where you'll type the code that will then be sent to R.

- It's basically a text editor. You can open text files of any kind here if you want.

- Files that appear here end in (and should be saved with) the extension ".R" (as in "MyCode.R").

You'll spend most of your time working here.

This is the "working directory." Anything you save will be saved here, unless you tell the program to save it somewhere else.

```
Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

>
```

This is the "Console." When you run the code in the Source window, the results that aren't graphics appear here.

This is the "Environment" window. It is where you can find all the various "objects" that you create, grouped by object type (data frames, lists, graphs, etc.).

There's also a "History" tab above; switching to that will show what has transpired in the Console window recently.

This is a window that shows various other things. Those things are tabbed above ^ and include:

- Plots (graphs) that you have created

- Packages that are loaded

- Help results (obtained by typing "?XXX" in the Console window, e.g. "?table").

Inside the Source Window

This:

```
> table(df$X)
```

... means “Type the phrase ‘table(df$X)’ on the command line,” or – equivalently – “Type the phrase ‘table(df$X)’ into your Source code, and then run it.”

Inside the Source Window

More often, you'll see:

```
with(df, plot(Y˜X,pch=19,col="red")) # draw a scatterplot
abline(h=0,lty=2) # add a horizontal line at zero
abline(v=0,lty=2) # add a vertical line at zero
text(df$X,df$Y,labels=df$names,pos=1) # add labels
```

... which means "Put this block of text into your Source code, and then run it."

Note:

- R / RStudio ignores line breaks
- Anything to the right of a "#" is a comment

Very basic R examples...

(see `GSERM-2017-R-Intro.R` in the github repo)

Help For Learning R(Studio)

In rough order of preference:

- Quick-R (`http://www.statmethods.net/`)
- The “Level-Zero” R Tutorial (doesn’t integrate RStudio, but is otherwise very good)
- Statistics with R
- The Do It Yourself Introduction to R
- Also be sure to consult the Regression III “Useful R Resources” guide (on GitHub).

Example Data: Infant Mortality

```
> url <- getURL("https://raw.githubusercontent.com/PrisonRodeo/
    GSERM-2017-git/master/Data/CountryData2000.csv")
> Data <- read.csv(text = url) # read the "votes" data
> rm(url)
>
> # Summary statistics
>
> # install.packages("psych") <- Install psych package, if necessary
> library(psych)

> with(Data, describe(infantmortalityperK))
  vars   n  mean    sd median trimmed   mad min max range skew kurtosis   se
1    1 179 43.83 40.39     29   38.38 34.26 2.9 167 164.1    1     0.06 3.02

> with(Data, describe(DPTpct))
  vars   n  mean    sd median trimmed   mad min max range  skew kurtosis    se
1    1 181 81.71 19.77     90   85.23 11.86  24  99    75 -1.31     0.57 1.47
```

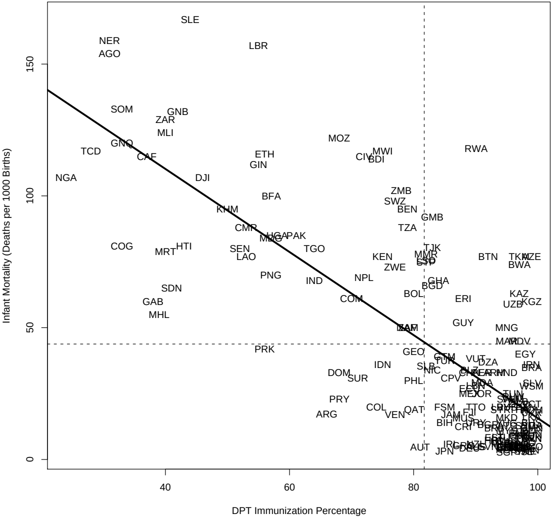
OLS Regression

```
> IMDPT<-lm(infantmortalityperK~DPTpct,data=Data,na.action=na.exclude)
> summary.lm(IMDPT)

Call:
lm(formula = infantmortalityperK ~ DPTpct, data = Data)

Residuals:
    Min       1Q  Median       3Q      Max
-56.801 -16.328  -5.105  11.777  86.590

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 173.2771     8.4893   20.41   <2e-16 ***
DPTpct       -1.5763     0.1009  -15.62   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 26.19 on 175 degrees of freedom
  (14 observations deleted due to missingness)
Multiple R-squared:  0.5824,Adjusted R-squared:   0.58
F-statistic: 244.1 on 1 and 175 DF,  p-value: < 2.2e-16
```

Analysis of Variance

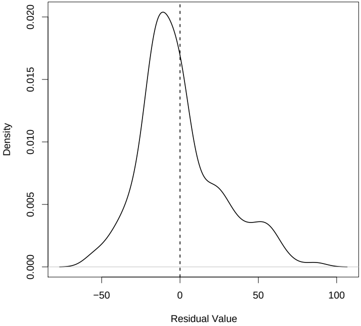
```
> anova(IMDPT)
Analysis of Variance Table

Response: infantmortalityperK
           Df Sum Sq Mean Sq F value    Pr(>F)
DPTpct      1 167423  167423  244.09 < 2.2e-16 ***
Residuals 175 120033     686
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

Regression of Infant Mortality on DPT Immunization Rates



Fitted Values, Residuals, etc.

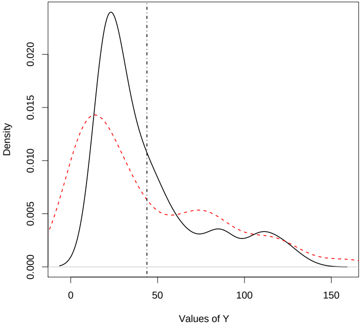
```
> # Residuals (u):
> Data$IMDPTres <- with(Data, residuals(IMDPT))
> describe(Data$IMDPTres)

  var   n mean    sd median   mad   min   max range skew kurtosis   se
1   1 177    0 26.12   -5.1 19.42 -56.8 86.59 143.4 0.75     0.44 1.96
```



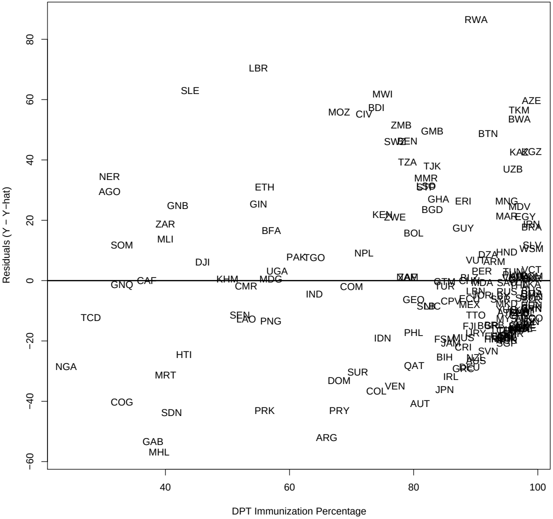
Fitted Values

```
> # Fitted Values:
> Data$IMDPThat<-fitted.values(IMDPT)
> describe(Data$IMDPThat)

  var   n  mean    sd median  mad   min   max range skew kurtosis   se
1   1 177 44.26 30.84  31.41 18.7 17.22 135.4 118.2  1.3     0.59 2.32
```

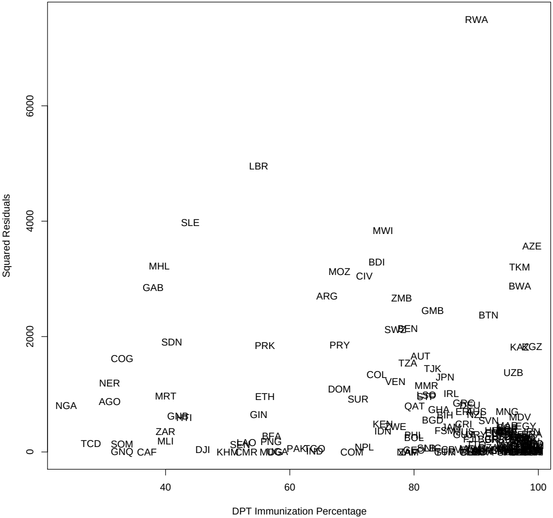
Density Plot: Actual (Y) and Fitted Values ($\hat{Y}$)



Regression Residuals ($\hat{u}$) vs. DPT Percentage



Squared Residuals vs. DPT Percentage



Inference

Var($\hat{\beta}$):

```
> vcov(IMDPT)
```

```
            (Intercept)   DPTpct
(Intercept)     72.0677 -0.83317
DPTpct          -0.8332  0.01018
```

95 percent c.i.s:

```
> confint(IMDPT)
```

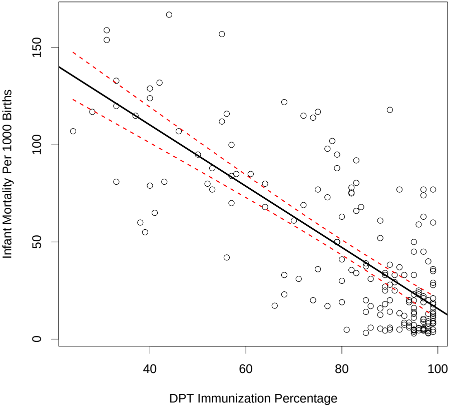
```
              2.5 %  97.5 %
(Intercept) 156.523 190.032
DPTpct       -1.775  -1.377
```

Predictions

```
> SEs<-predict(IMDPT,interval="confidence")
> SEs
      fit    lwr    upr
1    25.10  20.53  29.68
3    17.22  12.05  22.40
4    23.53  18.84  28.21
.
.
<rows omitted>
.
.
189  21.95  17.15  26.75
190  39.29  35.36  43.23
191  17.22  12.05  22.40
```

A Plot, With CIs

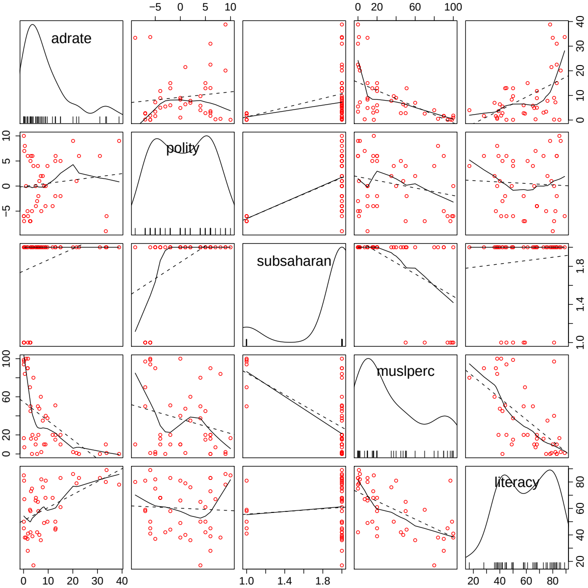
Scatterplot of Infant Mortality and DPT Immunizations, along with Least-Squares Line and 95% Prediction Confidence Intervals



Multivariate Example: Africa Data

```
> library(RCurl)
> temp<-getURL("https://raw.githubusercontent.com/PrisonRodeo/GSERM-2017-git/master/Data/africa2001.csv")
> Data<-read.csv(text=temp, header=TRUE)
> Data<-with(Data, data.frame(adrate,polity,
+           subsaharan=as.numeric(subsaharan),muslperc,literacy))

> summary(Data)
     adrate           polity           subsaharan      muslperc          literacy
 Min.   : 0.100   Min.   :-9.0000   Min.   :1.00   Min.   :  0.00   Min.   :17.00
 1st Qu.: 2.700   1st Qu.:-4.5000   1st Qu.:2.00   1st Qu.: 10.00   1st Qu.:43.00
 Median : 6.000   Median : 0.0000   Median :2.00   Median : 20.00   Median :61.00
 Mean   : 9.365   Mean   : 0.5116   Mean   :1.86   Mean   : 35.96   Mean   :60.07
 3rd Qu.:12.900   3rd Qu.: 5.5000   3rd Qu.:2.00   3rd Qu.: 55.50   3rd Qu.:78.50
 Max.   :38.800   Max.   :10.0000   Max.   :2.00   Max.   :100.00   Max.   :89.00

> cor(Data)
               adrate      polity  subsaharan   muslperc    literacy
adrate      1.0000000  0.11794182  0.33129420 -0.5709233  0.51489444
polity      0.1179418  1.00000000  0.52819844 -0.2391715 -0.05079354
subsaharan  0.3312942  0.52819844  1.00000000 -0.5772513  0.09472968
muslperc   -0.5709233 -0.23917151 -0.57725134  1.0000000 -0.61960385
literacy    0.5148944 -0.05079354  0.09472968 -0.6196039  1.00000000
```

Africa Data



adrate
polity
subsaharan
muslperc
literacy
−5
0
5
10
0
20
60
100
0
10
20
30
40
10
5
0
−5
1.0
1.4
1.8
100
60
20
0
80
60
40
20
0
10
20
30
40
1.0
1.4
1.8
20
40
60
80

A Regression

```
> model<-lm(adrate˜polity+subsaharan+muslperc+literacy,data=Data)
> summary(model)

Call:
lm(formula = adrate ˜ polity + subsaharan + muslperc + literacy,
    data = Data)

Residuals:
     Min       1Q   Median       3Q      Max
-15.4681  -4.3947  -0.5251   3.4246  22.9358

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -4.39843   14.94744  -0.294   0.7702
polity      -0.01390    0.27969  -0.050   0.9606
subsaharan   3.72969    5.43093   0.687   0.4964
muslperc    -0.08689    0.06282  -1.383   0.1747
literacy     0.16575    0.09433   1.757   0.0869 .
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 8.264 on 38 degrees of freedom
Multiple R-squared:  0.3771,Adjusted R-squared:  0.3115
F-statistic: 5.751 on 4 and 38 DF,  p-value: 0.001013
```

Variance-Covariance Matrix of $\hat{\beta}$

```
> options(digits=4)
> vcov(model)

            (Intercept)    polity subsaharan  muslperc  literacy
(Intercept)    223.4259  1.088030   -72.2628 -0.771309 -1.002421
polity           1.0880  0.078229    -0.6642 -0.000293  0.001968
subsaharan     -72.2628 -0.664212    29.4950  0.206067  0.171765
muslperc        -0.7713 -0.000293     0.2061  0.003946  0.004098
literacy        -1.0024  0.001968     0.1718  0.004098  0.008898
```

Inference: Tests...

Test $H_0 : \beta_{\texttt{polity}} = \beta_{\texttt{subsaharan}} = 0$:

```
> library(lmtest)
> modelsmall<-lm(adrate~muslperc+literacy,data=Data)
> waldtest(model,modelsmall)

Wald test

Model 1: adrate ~ polity + subsaharan + muslperc + literacy
Model 2: adrate ~ muslperc + literacy
  Res.Df Df    F Pr(>F)
1     38
2     40 -2 0.27   0.76
```

More tests...

Test $H_0 : \beta_{\texttt{muslperc}} = 0.1$:

```
> library(car)
> linearHypothesis(model,"muslperc=0.1")

Linear hypothesis test

Hypothesis:
muslperc = 0.1

Model 1: restricted model
Model 2: adrate ~ polity + subsaharan + muslperc + literacy

  Res.Df  RSS Df Sum of Sq    F Pr(>F)
1     39 3200
2     38 2595  1       605 8.85 0.0051 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

More tests...

Test $H_0 : \beta_{\texttt{literacy}} = \beta_{\texttt{muslperc}}$:

```
> linearHypothesis(model,"literacy=muslperc")

Linear hypothesis test

Hypothesis:
- muslperc  + literacy = 0

Model 1: restricted model
Model 2: adrate ~ polity + subsaharan + muslperc + literacy

  Res.Df  RSS Df Sum of Sq    F  Pr(>F)
1     39 3534
2     38 2595  1       938 13.7 0.00067 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```