Estimation of Production Function 2

Proxy (Control Function) Approach

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Proxy Approach in Production Function Estimation

- de facto standard in the literature.
- Basic idea: rely on firm's optimal decisions to control for unobserved productivity.
  - Need to explicitly thinkn about underlying firms' decision problem.
- I cover four important papers in the literature
  1. OP: Olley and Pakes (1996, EMA)
  2. LP: Levinsohn and Petrin (2003, RES)
  3. ACF: Ackerberg, Caves, and Frazer (2015, EMA)
  4. GNR: Gandhi, Navarro, and Rivers (2017, forthcoming JPE)

Olley and Pakes (1996, Econometrica)

- One of the most influential and important papers in empirical IO
  - Citation in Google scholar: 5531 (as of 10/14/2018)
- Methodological contribution: Propose an estimation strategy for production function.
- Empirical contribution: Evolution of prodcutivity in telecommunication equipment industry, mainly due to
  1. liberalization in the regulatory environment in telecommunication sector (e.g., the divestiture of AT&T)
  2. technological change
- Finding: aggregate productivity growth is mainly due to **reallocation of capital**.
  - We will see application in the next week.

Model

- Value-added production function

$$y_{jt} = \alpha l_{jt} + \beta k_{jt} + \omega_{jt} + \epsilon_{jt} \tag{1}$$

  - ω_{jt} : unobserved productivity (source of endogeneity),
  - ϵ_{jt} : ex-post shock (or measurement error)
  - y_{jt}: value-added. No intermediate input in the RHS.

- Assumption 1: 1st order Markov process of ω_{jt}

$$E[\omega_{jt}|\{\omega_{j\tau}\}_{\tau=1}^{t-1}] = E[\omega_{jt}|\omega_{jt-1}],$$

implying that

$$\omega_{jt} = \underbrace{E[\omega_{jt}|\omega_{jt-1}]}_{\equiv g(\omega_{j,t-1})} + \xi_{jt},$$

  - ξ_{jt} is "innovation" term. Uncorrelated with past information.
  - Note 1: Markovian asumption excludes firm-level fixed effects.
  - Note 2: $g(\cdot)$ can be nonparametric.

Underlying Framework as guide for empirical analysis

- Based on Erickson and Pakes (1995, REStud)
- Bellman equation

$$V_t(\omega_{jt}, k_{jt}) = \max\{\Phi, \sup_{i_{jt}\geq 0} \pi_t(\omega_{jt}, k_{jt}) - IC(i_{jt}) + \beta E[V_{t+1}(\omega_{jt+1}, k_{jt+1})|\Omega_t$$

$$s.t. k_{jt+1} = (1-\delta)k_{jt} + i_{jt}$$

  - Φ : Scrap value when exit, $IC(i_{jt})$: investment costs, Ω_t: information set.
  - π_t is per-period payoff. For example, $\pi_t(\omega_{jt}, k_{jt}) = \max_{l_{jt}} P_t \cdot F(k_{jt}, l_{jt}, \omega_{jt}) - w_t l_{jt}$

- The solution to the above problem

$$\chi_{jt} = \begin{cases} 1 & if \ \omega_{jt} \geq \underline{\omega}_t(k_{jt}) \\ 0 & otherwise \end{cases}$$

$$i_{jt} = f_t(\omega_{jt}, k_{jt})$$

- Assumption 2: Timing of input choices
  - Labor l_{jt} is
    - non-dynamic (does not depend on the past l_{jt-1})
    - determined in period t
  - Capital k_{jt} is
    - dynamic: $k_{jt} = (1-\delta)k_{jt-1} + i_{i,t-1}$.
    - Determined at period $t-1$.

- Assumption 3: Monotonicity in investment choice

$$i^*_{j,t} = f_t(\omega_{jt}, k_{jt})$$

is strictly monotonic in ω_{jt}.

  - The index t includes market environment (demand, price, investment costs, etc) that affects investment decisions.
  - Do not need to impose functional form on $f_t()$
  - **Scaler unobservable assumption**
    - no unobserved heterogeneity at firm-and-time level such as input prices.

Step 1 (/2): Inversion of Productivity

- Using optimal investment equation:

$$i^*_{j,t} = f_t(\omega_{jt}, k_{jt})$$
$$\iff \omega_{jt} = f_t^{-1}(i_{jt}, k_{jt})$$

  - Strict monotonicity is important for inversion!

- Putting this into the production function:

$$y_{jt} = \alpha l_{jt} + \underbrace{\beta k_{jt} + f_t^{-1}(i_{jt}, k_{jt})}_{\equiv \Phi_t(k_{jt}, i_{jt})} + \epsilon_{jt}$$

- $f_t^{-1}(i_{jt}, k_{jt})$ is potentially complicated function (drived from the dynamic problem)
- We can treat $\Phi_t(k_{jt}, i_{jt})$ as **non-parametric function** of k_{jt}, i_{jt} and t.
- A **semi-parametric partially linear model** by Robinson (1987)

- The model:

$$y_{jt} = \alpha l_{jt} + \Phi_t(k_{jt}, i_{jt}) + \epsilon_{jt}$$

  where $\Phi_t(k_{jt}, i_{jt}) = \beta k_{jt} + f_t^{-1}(i_{jt}, k_{jt})$

- ϵ_{jt} is not correlated with inputs (since it is measurement error!!)

- We can estimate labor coefficient α and composite function $\Phi_t(k_{jt}, i_{jt})$.
  - Regress y_{jt} on l_{jt} and non-parametric function of (t, i_{jt}, k_{jt}).
  - OP uses 4th order polynomial approximation for $\Phi_t(k_{jt}, i_{jt})$.

Step 2 (/2): Estimate Remaining Parameters (w.o. sample selection)

- Remember that

$$\omega_{jt} = g(\omega_{jt-1}) + \xi_{jt}$$

- Recall that, from the step 1, $\hat{\omega}_{jt} = \hat{\Phi}_t(k_{jt}, i_{jt}) - \beta k_{jt}$
- Then,

$$\hat{\Phi}_t(k_{jt}, i_{jt}) = \beta k_{jt} + g\left(\hat{\Phi}_{t-1}(k_{jt-1}, i_{jt-1}) - \beta k_{jt-1}\right) + \xi_{jt}$$

- By timing assumption of k_{jt}, it is uncorrelated with ξ_{jt}.
  - ξ_{jt} is an innovation, serially-uncorrelated.
- Run nonlinear least squares to estimate $(\beta, g(\cdot))$
  - Again, we use polynomial approximation for $g(\cdot)$.

Correction of Sample Selection

- Data is unbalanced panel due to entry/exit.
  - Two-thirds of observations gone if focus on balanced panel in the OP data.
- Need to correct sample selection due to entry/exit through unobserved productivity.
- Overview:
  - Step 1: Same as before.
    - ω_{jt} is completely proxied by a nonparametric function of (i_{jt}, k_{jt}, t)
  - Step 1.5: Estimate exit/survival probability
  - Step 2: Modified procedure

How to modify Step 2

- Sample selection leads to

$$E[\Phi_t|\chi_{jt} = 1, \Omega_{jt}] = \beta k_{jt} + E[\omega_{jt}|\omega_{jt-1}, \chi_{jt} = 1]$$

  where $\chi_{jt} = 1$ if firm i survives in period t, and Ω_{jt} is the information set in period t
- Note that

$$\begin{aligned} E[\omega_{jt}|\omega_{jt-1}, \chi_{jt} = 1] &= E[\omega_{jt}|\omega_{jt-1}, \omega_{jt} \geq \underline{\omega}_t(k_{jt})] \\ &= \tilde{g}(\omega_{jt-1}, \underline{\omega}_t(k_{jt})) \end{aligned}$$

- We need to know the threshold $\underline{\omega}_t(k_{jt})$. Use exit decision to control for this!

Step 1.5: Estimate exit/survival probability

- Probability of continuing operation

$$\begin{aligned}\Pr[\chi_{jt}=1|\underline{\omega}_t(k_{jt}),\Omega_{jt-1}] &= \Pr[\omega_{jt}\geq\underline{\omega}_t(k_{jt})|\underline{\omega}_t(k_{jt}),\Omega_{jt-1}]\\ &= \Pr[\omega_{jt}\geq\underline{\omega}_t(k_{jt})|\omega_{j,t-1}]\\ &\equiv p_t(\omega_{j,t-1},\underline{\omega}_t(k_{jt}))\\ &\equiv P_{jt}\end{aligned}$$

- If we have an estimate of P_{jt}, we can invert $p(\cdot)$ to obtain

$$\underline{\omega}_t(k_{jt})=p_t^{-1}(\omega_{j,t-1},P_{jt})$$

- Then,

$$\begin{aligned}\tilde{g}(\omega_{jt-1},\underline{\omega}_t(k_{jt})) &= \tilde{g}(\omega_{jt-1},p_t^{-1}(\omega_{j,t-1},P_{jt}))\\ &\equiv G(\omega_{j,t-1},P_{jt})\end{aligned}$$

- Now, we need to estimate P_{jt}

- Remember that

$$P_{jt} = \Pr\left[\chi_{jt} = 1 | \underline{\omega}_t(k_{jt}), \Omega_{jt-1}\right]$$

  - $k_{jt} = (1-\delta)k_{jt-1} + i_{jt-1}$
  - ω_{jt-1} is a function of k_{jt-1} and $i_{j,t-1}$

- Run probit/logit with flexible polynomial functions of (k_{jt-1}, i_{jt-1}).

Going back to modified Step 2

- With sample correction,

$$\hat{\Phi}_t(k_{jt}, i_{jt}) = \beta k_{jt} + G\left(\hat{\Phi}_{t-1}(k_{jt-1}, i_{jt-1}) - \beta k_{jt-1}, \hat{P}_{jt}\right) + \xi_{jt}$$

- G should include the survival probability (propensity score) to control for sample selection.
- Almost same as before!

Empirical Analysis (Table VI)

ALTERNATIVE ESTIMATES OF PRODUCTION FUNCTION PARAMETERS[a]
(STANDARD ERRORS IN PARENTHESES)

Sample:	Balanced Panel		Full Sample[c, d]						
								Nonparametric F_ω	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Estimation Procedure	Total	Within	Total	Within	OLS	Only P	Only h	Series	Kernel
Labor	.851	.728	.693	.629	.628			.608	
	(.039)	(.049)	(.019)	(.026)	(.020)			(.027)	
Capital	.173	.067	.304	.150	.219	.355	.339	.342	.355
	(.034)	(.049)	(.018)	(.026)	(.018)	(.02)	(.03)	(.035)	(.058)
Age	.002	−.006	−.0046	−.008	−.001	−.003	.000	−.001	.010
	(.003)	(.016)	(.0026)	(.017)	(.002)	(.002)	(.004)	(.004)	(.013)
Time	.024	.042	.016	.026	.012	.034	.011	.044	.020
	(.006)	(.017)	(.004)	(.017)	(.004)	(.005)	(.01)	(.019)	(.046)
Investment	—	—	—	—	.13	—	—	—	—
					(.01)				
Other Variables	—	—	—	—	—	Powers of P	Powers of h	Full Polynomial in P and h	Kernel in P and h
# Obs.[b]	896	896	2592	2592	2592	1758	1758	1758	1758

Levinsohn and Petrin (2003, REStud)

- Propose an approach that uses **intermediate inputs** as a proxy for unobserved productivity.
- Observation: Over 50% of firms report "zero" investment, leading to loss of observations.
  - No bias (since unobserved productivity ω_{jt} is completely proxied),
  - To make inversion work, we need to focus on the sample with positive investments. Efficiency loss due to the smaller sample size.
- They apply their method to the census data in Chili.
- However, their approach (and OP) has been criticized in the recent literature (ACF and GNR).

Setting

- Cobb-Douglas function for **gross output**.

$$y_{jt} = \alpha l_{jt} + \beta k_{jt} + \gamma m_{jt} + \omega_{jt} + \epsilon_{jt}$$

  and Markov process of productivity

$$\omega_{jt} = g(\omega_{jt-1}) + \xi_{jt},\ E[\xi_{jt}|\omega_{jt-1}] = 0 \tag{2}$$

- Timing assumptions on inputs
  - K_{jt} is determined at time $t-1$. (quasi-fixed inputs)
  - L_{jt} and M_{jt} are determined at time t. (flexible inputs)
- Input demand equation: $m_{jt} = \mathbb{M}_t(k_{jt}, \omega_{jt})$
  - Assumption: strictly increasing in ω_{jt}.
  - Indexed by t since this incorporates market condition.

LP 1st stage

- By inverting intermediate input demand, we have

$$m_{jt} = \mathbb{M}_t(k_{jt}, \omega_{jt}) \Rightarrow \omega_{jt} = \mathbb{M}_t^{-1}(k_{jt}, m_{jt})$$

  and we could proxy ω_{jt} by a function of k_{jt} and m_{jt}
- 1st stage regression:

$$y_{jt} = \alpha l_{jt} + \underbrace{\beta k_{jt} + \gamma m_{jt} + \mathbb{M}_t^{-1}(k_{jt}, m_{jt})}_{\equiv \phi_t(k_{jt}, m_{jt})} + \epsilon_{jt} \tag{3}$$

  - They estimate this by approximating ϕ_t using polynomial.
- We can estimate $\hat{\alpha}, \hat{\phi}_t(l_{jt}, k_{jt}, m_{jt})$ and $\hat{\epsilon}_{jt}$.

LP 2nd Stage

- From the 1st stage:

$$\omega_{jt}(\beta,\gamma) = \hat{\phi}_{jt} - (\beta k_{jt} + \gamma m_{jt})$$

- Again,

$$\hat{\phi}_{jt} = \beta k_{jt} + \gamma m_{jt} + g\left(\hat{\phi}_{jt-1} - (\beta k_{jt-1} + \gamma m_{jt-1})\right) + \xi_{jt}.$$

- $(\hat{\phi}_{jt-1}, k_{jt-1}, m_{jt-1})$ uncorrelated with ξ_{jt}.
- Regarding the first two terms, moment conditions

$$E[\eta_{jt} k_{jt}] = 0, E[\eta_{jt} m_{jt-1}] = 0$$

  - k_{jt} is uncorrelated with η_{jt} since k_{jt} is determined at $t-1$.
  - m_{jt} is correlated with η_{jt}, but m_{jt-1} is not.
  - Function $g(\cdot)$ is approximated by polynomial.
  - Or, given (β,γ), we do nonparametric regression of $\omega_{jt}(\beta,\gamma)$ on $\omega_{jt-1}(\beta,\gamma)$ to evaluate $\eta_{jt}(\beta,\gamma)$.

Ackerberg, Caves, and Frazer (2015, Econometrica)

- ACF pointed out "collinearity problems" associated with LP and OP.
- They propose an alternative approach using "value-added" production function.

Collinearity Problem regarding flexible inputs in the 1st stage

- Remember 1st stage in LP;

$$y_{jt} = \alpha l_{jt} + \underbrace{\beta k_{jt} + \gamma m_{jt} + \mathbb{M}_t^{-1}(k_{jt}, m_{jt})}_{\equiv \phi_t(k_{jt}, m_{jt})} + \epsilon_{jt}$$

- Labor input is flexible in LP framework, implying that

$$\begin{aligned} l_{jt} &= \mathbb{L}_t(k_{jt}, \omega_{jt}) \\ &= \mathbb{L}_t(k_{jt}, \mathbb{M}_t^{-1}(k_{jt}, m_{jt})) \end{aligned}$$

Once we control for k_{jt}, m_{jt} and time t, no independent variation in l_{jt}.

- If the true data generating process satisfies the assumptions, we cannot identify α in the 1st stage !!

Would variation in input price help?

- Consider the firm-level input price

$$l_{jt} = \mathbb{L}_t(k_{jt}, \omega_{jt}, p^l_{jt})$$

where p^l_{jt} is the firm-and-year level wage (exogenous).

- This means that m_{jt} also depends on p^l_{jt}. The first stage is

$$y_{jt} = \alpha l_{jt} + \underbrace{\beta k_{jt} + \gamma m_{jt} + \mathbb{M}_t^{-1}(k_{jt}, m_{jt}, p^l_{jt})}_{\equiv \phi_t(k_{jt}, m_{jt}, p^l_{jt})} + \epsilon_{jt}$$

- We must observe the wage p^l_{jt} (otherwise the inversion not possible).
- Even with observed wage, we still have the same collinearity issue...
- Under the assumptions of OP-type framework, hard to estimate coefficients on flexible inputs (e.g., m_{jt} and l_{jt})
  - GNR (2018), Bond and Soberbom (2005).

"Value-added" Approach by ACF

- Point 1: Consider value added production function

$$va_{jt} = \alpha l_{jt} + \beta k_{jt} + \omega_{jt} + \epsilon_{jt}$$

  - $VA_{jt} = PY_{jt} - \rho M_{jt}$, $va_{jt} = \ln VA_{jt}$

- Point 2: Timing assumption on inputs. Either one of them is fine.
  1. (k_{jt}, l_{jt}) is determined $t-1$ (quasi-fixed) $\Rightarrow m_{jt}$ at t
  2. k_{jt} at time $t-1 \Rightarrow l_{jt}$ and m_{jt} at t (both flexible)
     - Note: The first assumption is more transparent in terms of identification.

- Point 3: Input demand equation is given by

$$m_{jt} = \mathbb{M}_t(k_{jt}, l_{jt}, \omega_{jt})$$

  and this is strictly increasing in ω_{jt}.

- m_{jt} depends on l_{jt} as well.
  - Includes both quasi-fixed (at $t-1$) and flexible (at t) l_{jt}.
- This allows:
  - labor is dynamic input (i.e., l_{jt} is a part of state variable) due to adjustment costs.
  - serially correlated **unobserved** cross-firm variation in wages
- Why? Because the inversion on m_{jt} is done **conditional on** l_{jt}.
- These variations are useful for identification of labor coefficient!!
  - GNR, Bond and Soberbom (2005).

1st Stage in ACF

- Using the inversion, $\omega_{jt} = \mathbb{M}_t^{-1}(k_{jt}, l_{jt}, m_{jt})$, we have

$$va_{jt} = \underbrace{\alpha l_{jt} + \beta k_{jt} + \mathbb{M}_t^{-1}(k_{jt}, l_{jt}, m_{jt})}_{=\phi_t(l_{jt}, k_{jt}, m_{jt})} + \epsilon_{jt}$$

- We get $\hat{\phi}_{jt} = \alpha l_{jt} + \beta k_{jt} + \mathbb{M}_t^{-1}(k_{jt}, l_{jt}, m_{jt})$ and $\hat{\epsilon_{jt}}$.
- Note 1: We no longer obtain α in the 1st stage.
- Note 2: The 1st stage allow us to purge the measurement error $\hat{\epsilon_{jt}}$.

2nd Stage in ACF

- In the 2nd stage,

$$\hat{\phi}_{jt} = \alpha l_{jt} + \beta k_{jt} + g\left(\hat{\phi}_{j,t-1}(l_{jt-1}, k_{jt-1}, m_{jt-1}) - (\alpha l_{jt-1} + \beta k_{jt-1})\right) + \xi_{jt}$$

  - We have no m_{jt} in the 2nd stage.
- Capital coefficient β can be identified by $E[\xi_{jt} k_{jt}] = 0$
- Identification of the coefficient on l_{jt}:
  - If L_{jt} is quasi-fixed, then $E[\xi_{jt} L_{jt}] = 0$.

Gandhi, Navarro, and Rivers (2017, forthcoming JPE)

- GNR formalizes non-identification of gross output production function in OP-type framework.
  - In particular, they show non-identification of the effect of "flexible" inputs (e.g., intermediate inputs) on the output.
- They propose an alternative strategy to estimate gross output production function using **optimality conditions for profit maximization**

Assumptions in GNR

1. Production function is given as

$$Y_{jt} = F(L_{jt}, K_{kt}, M_{jt})e^{\omega_{jt}}e^{\epsilon_{jt}}$$

   where functional form of F is not specified.
   - Remember: Capital letters are in levels.
2. Unobservables
   2.1 $\omega_{jt} = g(\omega_{jt-1}) + \xi_{jt}, E[\xi_{jt}|\omega_{jt-1}] = 0.$
   2.2 ϵ_{jt} is not observed at the timing of decision.
3. Timing assumptions
   3.1 K_{jt} and L_{jt} are quasi-fixed (determined at or prior to period $t-1$).
   3.2 M_{jt} is flexible (determined at time t).
4. Input demand equation

$$M_{jt} = \mathbb{M}_t(L_{jt}, K_{jt}, \omega_{jt})$$

   where $\mathbb{M}_t(L_{jt}, K_{jt}, \omega_{jt})$ is strictly increasing in ω_{jt} for any (L_{jt}, K_{Jt}).

(FYI) Non-identification theorem of $F(L_{jt}, K_{kt}, M_{jt})$

- Theorem: Under Assumption 1-4, F is not identified.
- (Sketch of proof) Using inversion , we can write

$$y_{jt} = \ln F(L_{jt}, K_{jt}, M_{jt}) + g\left(\mathbb{M}_{t-1}^{-1}(L_{j,t-1}, K_{j,t-1}, M_{jt-1})\right) + \xi_{jt} + \epsilon_{jt}$$

- Taking conditional expectation,

$$E[y_{jt}|\Gamma_{jt}] = E[\ln F(L_{jt}, K_{jt}, M_{jt})|\Gamma_{jt}] + g_t(L_{j,t-1}, K_{j,t-1}, M_{jt-1})$$

where $\Gamma_{jt} = \{L_{jt}, K_{jt}, \{L_{j\tau}, K_{j\tau}, M_{j\tau}\}_{\tau=1}^{t-1}\}$.
- Remember

$$M_{jt} = \mathbb{M}_t(L_{jt}, K_{jt}, h_t(L_{j,t-1}, K_{j,t-1}, M_{jt-1}) + \xi_{jt})$$

so $E[\ln F(L_{jt}, K_{jt}, M_{jt})|\Gamma_{jt}]$ only depends on $(L_{jt}, K_{jt}, L_{jt-1}, K_{jt-1}, M_{jt-1})$
- 5 exogenous variables, but 6 coordinates of functions. $\Rightarrow$ Not identified.

New Approach

- Production function

$$Y_{jt} = F(L_{jt}, K_{jt}, M_{jt}) e^{\omega_{jt}} e^{\epsilon_{jt}}$$

- We employ the FOC of firm's profit maximization.
- Let (ρ_t, P_t) be price of intermediate input and output.
- Under perfect competition, the FOC is given as

$$P_t \frac{\partial F}{\partial M_{j,t}} e^{\omega_{jt}} \mathcal{E} = \rho_t \tag{4}$$

where $\mathcal{E} = E[e^{\epsilon_{jt}}]$ (expected error)

- Combine FOC (4) and production function gives us

$$\ln \frac{\rho_t M_{jt}}{P_t Y_{jt}} = \ln G_t(L_{jt}, K_{jt}, M_{jt}) + \ln \mathcal{E} - \epsilon_{jt} \tag{5}$$

  - LHS is expenditure share of intermediate input.

$$\frac{\rho_t M_{jt}}{P_t Y_{jt}}$$

  - G_t is elasticity of production w.r.t intermediate input

$$G_t(\cdot) \equiv \frac{\partial F(\cdot)}{\partial M_{jt}} \frac{M_{jt}}{F(\cdot)}$$

  - $\mathcal{E} \equiv E[exp(\epsilon_{jt})]$.
- This identifies $G_t(L_{jt}, K_{jt}, M_{jt})$ and $\mathcal{E}$.

Example: Cobb-Douglas function

- Remember that $y_{jt} = \alpha l_{jt} + \beta k_{jt} + \gamma m_{jt} + \omega_{jt} + \epsilon_{jt}$
- Elasticity w.r.t intermediate product is γ!

$$\ln \frac{\rho_t M_{jt}}{P_t Y_{jt}} = \ln \gamma + \ln \mathcal{E} - \epsilon_{jt}. \tag{6}$$

  This identifies γ and $\mathcal{E}$.

- Using $\hat{\gamma}$ and $\hat{\epsilon}_{jt}$, we have

$$\omega_{jt}(\alpha, \beta) = (y_{jt} - \hat{\epsilon}_{jt}) - (\alpha l_{jt} + \beta k_{jt} + \hat{\gamma} m_{jt})$$

  And remember productivity growth process

$$\omega_{jt} = g(\omega_{jt-1}) + \xi_{jt}$$

- Use moment condition $E[\eta_{jt} l_{jt}] = 0$ and $E[\eta_{jt} k_{jt}] = 0$ to estimate (α, β).

(FYI) Nonparametric identification

- 1st stage: Remember

$$\ln \frac{\rho_t M_{jt}}{P_t Y_{jt}} = \ln G_t(L_{jt}, K_{jt}, M_{jt}) + \ln \mathcal{E} - \epsilon_{jt} \tag{7}$$

  This identifies $G_t(L_{jt}, K_{jt}, M_{jt})$, and $\mathcal{E}$.

- 2nd stage: $G_t(L_{jt}, K_{jt}, M_{jt})/M_{jt} = \partial \ln F_t(L_{jt}, K_{jt}, M_{jt})/\partial M_{jt}$. The fundamental theorem of calculus:

$$\int \frac{G_t(M_{jt}, L_{jt}, K_{jt})}{M_{jt}} dM_{jt} = \ln F(L_{jt}, K_{jt}, M_{jt}) + \mathcal{C}_t(L_{jt}, K_{jt}) \tag{8}$$

  LHS is observable. Need to identify the integration constant $\mathcal{C}_t(L_{jt}, K_{jt})$.

- Combine (8) and $y_{jt} = \ln F(L_{jt}, K_{jt}, M_{jt}) + \omega_{jt} + \epsilon_{jt}$;

$$\underbrace{y_{jt} - \int \frac{G_t(M_{jt}, L_{jt}, K_{jt})}{M_{jt}} dM_{jt} - \epsilon_{jt}}_{\mathcal{Y}_{jt}(observed)} = -\mathcal{C}_t(L_{jt}, K_{jt}) + \omega_{jt} \tag{9}$$

- Using $\omega_{jt} = h(\omega_{jt-1}) + \eta_{jt}$, we have

$$\mathcal{Y}_{jt} = -\mathcal{C}_t(L_{jt}, K_{jt}) + h\left(\mathcal{Y}_{jt-1} + \mathcal{C}_{t-1}(L_{jt-1}, K_{jt-1})\right) + \eta_{jt} \tag{10}$$

- Since $E[\eta_{jt}|L_{jt}] = 0$ and $E[\eta_{jt}|K_{jt}] = 0$, we can identify $\mathcal{C}_t(L_{jt}, K_{jt})$. $\Rightarrow$ Identify $F(L_{jt}, K_{jt}, M_{jt})$.

(FYI) Nonparametric Estimation

- Since identification is constructive, estimation is straightforward.
- They use Sieve approximation (e.g. Chen 2007) to approximate $G_t(M_{jt}, K_{jt}, L_{jt})$ and $\mathcal{C}_t(L_{jt}, K_{jt})$.
  - They recommend polynomial approximation to have a closed-form integral in the 2nd stage.
- See Chen et al (2018, Working Paper) for an empirical study that implements non-parametric estimator in the context of privatization in China