Estimation of Single-Agent Dynamic Discrete Choice Model 2

Two-Step Estimator

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Introduction

- Last week: Full solution approach
  - Given parameter θ, solve the DP to obtain model prediction, calculate the objective function.
  - Search θ that minimize (or maximize) the objective function.
- This approach is computationally demanding.
- This week: Two-step estimator
  - Idea: Correspondence between value function and policy function (optimal choice).
  - Step1: We estimate policy function from the data.
  - Step 2: Use the estimated policy function to construct the value function, which fed into calculating the objective function in estimation.
  - This avoids solving the dynamic programming in estimation.
- Plan
  1. Introduce two-step estimator (Hotz and Miller 1993)
  2. A quick preview of problem set 5

Recap on Rust (1987)

- Bellman equation

$$V(x_t, \epsilon_t) = \max_{d_t} \underbrace{u(i, x_t, \theta_1)}_{\tilde{u}(d_t, x_t, \theta_1) + \epsilon_t(d_t)} + \beta EV(x_t, \epsilon_t, d_t)$$

where

$$EV(x_t, \epsilon_t, d_t) = \int V(x', \epsilon') dP(x'|x_t, d_t) dP(\epsilon)$$

and P is the transition probability from (x, ϵ) to (x', ϵ').

- Define the expected value $V(x) = \int V(x,\epsilon)dP(\epsilon)$, then

$$V(x_t) = \max_d \tilde{u}(d_t, x_t, \theta_1) + \beta \underbrace{\mathbb{E}_{x_{t+1}}[V(x_{t+1})|x_t, d_t]}_{\equiv EV(x_t, d_t)} + \epsilon_t(d_t)$$

- Under the logit assumption on ϵ

$$P(d|x) = \frac{\exp(v(x,d))}{\sum_j \exp(v(x,j))}$$

where

$$v(x,d) \equiv \tilde{u}(d,x,\theta_1) + \beta EV(x,d)$$

is called the choice-specific value

- Also,

$$V(x) = \log\left(\sum_d \exp(v(x,d))\right)$$

Hotz and Miller (1993)

- A disadvantage of Rusts approach is that it can be computationally intensive.
- With a richer state space, solving value function (inner fixed point) can take a very long time, which means estimation will take a very, very long time
- Hotz and Millers idea is to use observable data to form an estimate of (differences in) the value function from conditional choice probabilities (CCPs).
- Hotz and Miller (1993) is quite general and difficult to digest. I focus on what we can do with the Hotz-Miller' idea in the contect of Rust-type model.

Hotz-Miller inversion

- Notice: The CCPs do not change by subtracting some constant from every conditional value.

- Consider

$$dv(x,d) \equiv v(x,d) - v(x,0)$$

where $d = 0$ is a reference action.

- Consider the mapping Q such that

$$Q : R^{|I|-1} \longmapsto \Delta^{|I|-1}$$

where $|I|$ is the number of actions.

- Hotz-Miller inversion: Q is invertible.

HM inversion with logit errors

- CCP with logit error

$$P(d|x) = \frac{\exp(v(x,d))}{\sum_j \exp(v(x,j))}$$

- The HM inversion follows by taking logs and differencing across actions

$$\log P(d|x) - \log P(0|x) = v(x,d) - v(x,0)$$

- If you know the choice probability, you can get the utility difference!
- This is essentially the same as Berry (1994)'s inversion!

HM Estimation Overview

1. (Preliminary) Estimate the state transition of x_t (same as Rust 1987)
2. (Step 1) Estimate the conditional choice probability $P(d|x)$ from the data
3. (Step 2) Recover value functions from estimated CCP using the HM inversion.
4. Construct objective function using the estimated value functions.

- Procedure 3 (step 2) depends on the setting.
  - Case 1: Terminal action
  - Case 2: Reset action as in Rust (1987) $\rightarrow$ Problem set 5.

Case 1: Terminal action

- Suppose that $d = 0$ is a terminal action $EV(x, 0) = 0$ and thus

$$v(x, 0, \theta) = \tilde{u}(x, 0, \theta)$$

  - Example: Exit from the market, retirement, etc.

- In the logit case (γ is the Euler constant)

$$\begin{aligned} EV(x, d) &= \int V(x')dP(x'|x, d) \\ &= \int \log\left(\sum_{d'} \exp(v(x', d'))\right) dP(x'|x, d) + \gamma \\ &= \int \log\left(\sum_{d'} \exp\left(dv(x', d') + \tilde{u}(x', 0, \theta)\right)\right) dP(x'|x, d) + \gamma \end{aligned}$$

- You can get $dv(x, d) = v(x, d) - v(x, 0)$ by the HM inversion!

HM inversion

- Plug in the esimate from the HM inversion

$$\tilde{dv}(x, d) = \log \hat{P}(d|x) - \log \hat{P}(0|x)$$

- With this, you can construct

$$\tilde{EV}(x, d) = \int \log \left(\sum_{d'} \exp \left(\tilde{dv}(x', d') + \tilde{u}(x', 0, \theta) \right) \right) dP(x'|x, d) + \gamma$$

HM inversion

- Remember the choice probability

$$P(d|x) = \frac{\exp(v(x,d))}{\sum_j \exp(v(x,j))}$$

where

$$v(x,d) \equiv \tilde{u}(d,x,\theta_1) + \beta EV(x,d)$$

- With estimates of $\tilde{EV}$ from the HM inversion, you can calculate the choice probability!
- Can construct the likelihood and form the moment condition to estimate the parameter θ_1!

Further Readings

- I would recommend you to read Agguiregabiria and Mira (2002, Econometrica) for the further details of the two-step estimators.
  - AM (2002) provides a framework that nests Rust (1987) and Hotz Miller (1993) as special cases.

- Other important papers for estimation of single-agent dynamic discrete choice:
  - Hotz, Miller, Sanders and Smith (1994, RES): Forward simulation using the estimated CPP to obtain the value function.
  - Bajari, Benkard, and Levine (2007, EMA): Forward simulation in dynamic games. Useful in single-agent setting as well.
  - Arcidiacono and Miller (2011, EMA): Two-step estimator with unobserved heterogeneity.

Problem Set 5

- Study a simplified version of the bus-replacement problem.
- Observed state variable: bus milleage $x_t \in \{0, 1, \ldots, 10\}$
  - It increases by 1 with probability λ and remains the same with probability $1 - \lambda$. If $x_t = 10$, it does not change.
  - If you replace, the mileage is immediately set to 0. The transition in the next period is the same as above.
  - Set $\lambda = 0.7$ in this problem set.
- The per-period utility (omit subscript t)

$$u^0(x, \theta) = -(\theta_1 x + \theta_2 x^2) + \epsilon_0 \equiv \tilde{u}_0(x, \theta) + \epsilon_0$$
$$u^1(x, \theta) = -RC + \epsilon_1 \equiv \tilde{u}_1(x, \theta) + \epsilon_1$$

  - Parameter $\theta = (\theta_1, \theta_2, RC)$.
  - (ϵ_0, ϵ_1) are i.i.d. and distributed type1 extreme value.
- Discount factor set to $\beta = 0.95$

Question 1-(a)

- Solve the dynamic programming problem for $\theta = (0.3, 0, 4)$

- The Bellman equation

$$V(x,\epsilon) = \max\left\{\tilde{u}_0(x,\theta) + \epsilon_0 + \beta \int V(x',\epsilon')dP(\epsilon')dP(x'|x),\right.$$
$$\left.\tilde{u}_1(x,\theta) + \epsilon_1 + \beta \int V(x',\epsilon')dP(\epsilon')dP(x'|0)\right\}$$

  - (x,ϵ): state variable today, (x',ϵ') : state variables tomorrow.

- Transition probability: $P(x'|x)$
  - probability that the mileage changes from x to x'
  - Note: if replace, the transition is $P(x'|0)$ because the mileage is reset to 0.
- $P(\epsilon)$: density function of (ϵ_0, ϵ_1), type I extreme value distribution.

- Define $V(x) \equiv \int V(x,\epsilon)dP(\epsilon)$
- Using the property of logit error

$$V(x) = \ln\left[exp(v_0(x)) + exp(v_1(x))\right] + \gamma, \tag{1}$$

where

$$v_0(x) = -(\theta_1 x + \theta_2 x^2) + \beta \int V(x')dP(x'|x) \tag{2}$$

$$v_1(x) = -RC + \beta \int V(x')dP(x'|0) \tag{3}$$

and γ is the Euler constant (which is irrelevant in the analysis)

How to numerically solve the Bellman equation

- Consider a vector of value $\mathbf{V} = (V(0), \cdots, V(11))$
- Solve the Bellman equation using contraction mapping property to get the numerical value $\mathbf{V}$
- Given the value $\mathbf{V}^{(k)}$ at k-th iteration,
  1. Calculate $v_0(x)$ and $v_1(x)$ using equations (2) and (3)
     1.1 Use the state transition $P(x'|x)$.
  2. Use these to value to obtain $\mathbf{V}$ using equation (1). Call this $\mathbf{V}^{(k+1)}$
  3. Repeat this iteration until $\mathbf{V}^{(k)}$ and $\mathbf{V}^{(k+1)}$ are sufficiently close.
     3.1 Stop the iteration if $\max_{x\in\{0,1,\cdots,11\}}|V^{(k)}(x) - V^{(k+1)}(x)| < 1e-6$ (or sufficiently small number)

Conditional Choice Probability

- Once you get the convergence on $V(x)$, you can use the value to calculate the conditional choice probability using logit formura

$$P(replacement|x) = \frac{exp(v_1(x))}{\exp(v_0(x)) + \exp(v_1(x))}$$

- Report choice probabilities for each x!

Question 1-(b) and (c)

- The likelihood function

$$L(\theta) = \prod_i \underbrace{\left(\frac{e^{v_0(x_i)}}{e^{v_0(x_i)} + e^{v_1(x_i)}} \right)^{1\{d_i=0\}} \left(\frac{e^{v_1(x_i)}}{e^{v_0(x_i)} + e^{v_1(x_i)}} \right)^{1\{d_i=1\}}}_{\equiv L_i(\theta):\text{individual likelihood}},$$

  where i is an index for observation in the data.
  - Data: mileage x_i and replacement decision d_i

- Given θ,
  - Solve the DP as above.
  - Calculate the choice probability $P(replacement|x)$.
  - Use it to calculate the log-likelihood function $\log L(\theta)$
- Search for θ to maximize the log-likelihood

Programming Tips

- Recommend to prepare two functions in programming.
- 1: Main code:
  - call optimization routine to maximize the likelihood, which calls the function below.
- 2: function for likelihood:
  - input is parameter θ, output is likelihood
  - call the function below that solves DP, calculate the choice-probabilities and the likelihood given θ
- 3: function for solving DP
  - input is parameter θ, output is the value
  - Solve the Bellman equation by contraction mapping to obtain the value $V(x)$.

Standard Error

- Use the formula in the problem set.
- Calculate the derivative numerically:

$$f'(x) \approx \frac{f(x+\tau) - f(x)}{\tau}$$

  for a sufficiently small τ.
  - See online sources for the details.

Question 3-(a)

- Calculate

$$Prob(d_i = 1|x)$$

- This should be straightforward!! Count the frequency for each x!!

Question 3-(b)

- Let $p_d(x)$ be the choice-probability for d when the mileage is x (conditional choice probability)
  - Note: We estimate this in Question 3-(a) !!
- Replacement probability is

$$p_1(x) = \frac{\exp(v_1(x))}{\exp(v_1(x)) + \exp(v_0(x))}$$
$$\Longleftrightarrow \log(\exp(v_1(x)) + \exp(v_0(x))) = \log(\exp(v_1(x))) - \log(p_1(x))$$

- Goal: Rewrite $v_0(x)$ and $v_1(x)$ in terms of the choice probability.

$$\begin{aligned}
v_0(x) &= -\theta_1 x - \theta_2 x^2 + \beta\lambda V(x+1) + \beta(1-\lambda)V(x) \\
&= -\theta_1 x - \theta_2 x^2 + \beta\lambda \ln[e^{v_1(x+1)} + e^{v_0(x+1)}] \\
&\quad + \beta(1-\lambda)\ln[e^{v_1(x)} + e^{v_0(x)}] + \beta\gamma \\
&= -\theta_1 x - \theta_2 x^2 + \beta\lambda\{\ln e^{v_1(x+1)} - \ln(p_1(x+1))\} \\
&\quad + \beta(1-\lambda)\{\ln e^{v_1(x)} - \ln(p_1(x))\} + \beta\gamma
\end{aligned}$$

- And similary,

$$\begin{aligned}
v_1(x) &= -RC + \beta\lambda \ln[e^{v_1(1)} + e^{v_0(1)}] \\
&\quad + \beta(1-\lambda)\ln[e^{v_1(0)} + e^{v_0(0)}] + \beta\gamma \\
&= -RC + \beta\lambda\{\ln e^{v_1(1)} - \ln(p_1(1))\} \\
&\quad + \beta(1-\lambda)\{\ln e^{v_1(0)} - \ln(p_1(0))\} + \beta\gamma.
\end{aligned}$$

- Important observation: $v_1(x) = v_1(x')$ for $x = x'$ because the mileage gets reset with replacement!

- Therefore,

$$
\begin{aligned}
v_0(x) - v_1(x) &= -\theta_1 x - \theta_2 x^2 + \beta\lambda\{-\ln(p_1(x+1))\} \\
&\quad + \beta(1-\lambda)\{-\ln(p_1(x))\} \\
&\quad - (-RC + \beta\lambda\{-\ln(p_1(1))\} + \beta(1-\lambda)\{-\ln(p_1(0))\})
\end{aligned}
$$

for $x < \max X (= 10)$ and

$$
\begin{aligned}
v_0(x) - v_1(x) &= -\theta_1 x - \theta_2 x^2 + \beta\{-\ln(p_1(x))\} \\
&\quad - (-RC + \beta\lambda\{-\ln(p_1(1))\} + \beta(1-\lambda)\{-\ln(p_1(0))\})
\end{aligned}
$$

for $x = \max X (= 10)$.

- Once we have the estimated probability $\hat{p}_1(x)$, we can calculate $v_0 - v_1$ without solving the DP!

- Calculate the likelihood as above.
- The only difference is that $v_0(x) - v_1(x)$ can be obtained from $\hat{p}_1(x)$ without solving the DP!