

Essentials of Applied Data Analysis

IPSA-USP Summer School 2017

Handout - The Basics of Probability Theory - III

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jan/17

Introduction to Probability - Part III

Basic Notions of probability, part III.

This handout is of EXTREME importance, so don't rush.

Independence

The event B is said to be independent of the event A if the probability that B occurs is not influenced by whether A has or has not occurred.

IMPORTANT: Independence is a *key* concept to us

Independence - Examples

- Toss a coin twice. The result of the second toss is independent of the result in the first toss.
- Take one card from a deck. Don't put it back. Take a second card. The result of the second card is not independent of the first.
- A legislator support to a government (or being a woman, or being from some part of the country) is not independent of her party.

Axioms and theorems of probability (5)

- If A and B are independent events, then $P(A \cap B) = P(A) * P(B)$

Independence - Coin

1) Toss a coin twice. What's is the probability of getting heads (H) twice?

$$P(H_1 \cap H_2) = P(H_1) * P(H_2) = P(H) * P(H) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

$$\text{Sample Space} = \{H,H\}, \{H,T\}, \{T,H\}, \{T,T\}$$

2) Toss a coin 50 times. What's is the probability of getting 50 times head (H)?

$$P(H_1 \cap H_2 \cap \dots \cap H_{50}) = (P(H))^{50} = \left(\frac{1}{2}\right)^{50} \simeq 8.8 * 10^{-16}$$

3) Toss a coin 10 times. What's is the probability of getting at least one Tail (T)?

$$\begin{aligned} P(1 \text{ Tail or more}) &= P(\text{exactly 1 Tail}) + P(\text{exactly 1 Tails}) + \dots + P(\text{exactly 10 Tails}) = \\ &= \{P(T \cap H \cap \dots \cap H) + P(H \cap T \cap H \cap \dots \cap H) + P(H \cap \dots \cap H \cap T)\} + \\ &\quad + P(\text{exactly 2 Tails}) + \dots + P(\text{exactly 10 Tails}) = \end{aligned}$$

OMG!!! How do I compute that?

Use the fact that the complementary of “getting at least one Tail” is “not getting any Tails”, which is the same as “getting 10 heads”. And “getting 10 heads” is much easier to compute:

$$P(1 \text{ Tail or more}) = 1 - P(\text{not getting any Tails}) = 1 - P(10 \text{ Hs}) = 1 - (1/2)^{10} \simeq 0.9990234$$

Independence - Cards

Take a card from a deck (52 cards).

What's is the probability of getting a Queen (Q)?

$$P(Q) = \frac{4}{52} = \frac{1}{13}$$

What's is the probability of getting a Hearts (H)?

$$P(H) = \frac{13}{52} = \frac{1}{4}$$

What's is the probability of getting the Queen of Hearts ($Q \cap H$)?

$$P(Q \cap H) = P(Q) * P(H) = \frac{1}{13} * \frac{1}{4} = \frac{1}{52}$$

If the card deck is complete, the suit and the number are independent of each other, so we can apply $P(A \cap B) = P(A) * P(B)$

Conditional Probability

Let A be an event that is **not independent** of B , and $P(B) > 0$. The probability that an event A occurs once B has occurred ($P(A|B)$), which can be read as *conditional probability* of A given B is **not** the probability of A :

$$P(A|B) \neq P(A)$$

Axioms and theorems of probability (6)

- If A and B are not independent events, then the conditional probability of A given B , $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability

Intuitively, we can think as the conditional probability of A given B as:

$$P(A|B) = \frac{\frac{\# \text{ of outcomes in A and B}}{\# \text{ of outcomes in the sample space}}}{\frac{\# \text{ of outcomes in B}}{\# \text{ of outcomes in the sample space}}} = \frac{\# \text{ outcomes in event A and B}}{\# \text{ outcomes in B}}$$

Conditional Probability - dice

Roll a dice. Define event B as "getting a number less than or equal to 3". What is the probability of getting a 1 (event A) given that B occurred?

Answer:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{\# \text{ of outcomes in A and B}}{\# \text{ of outcomes in the sample space}}}{\frac{\# \text{ of outcomes in B}}{\# \text{ of outcomes in the sample space}}} = \frac{\frac{\#\{1\}}{\#\{1,2,3,4,5,6\}}}{\frac{\#\{1,2,3\}}{\#\{1,2,3,4,5,6\}}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{2}{6} = \frac{1}{3}$$

Alternatively, you can read this problem as "I already know I got 1, 2 or 3. Given that, what is the probability that I got a 1?"

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\# \text{ outcomes in event A and B}}{\# \text{ outcomes in B}} = \frac{\#\{1\}}{\#\{1,2,3\}} = \frac{1}{3}$$

Conditional Probability - family

1) A family has two kids. What is the probability that both kids be are girls?

2) A family has two kids. The first is a girl. What is the probability that the other one is also a girl?

3) A family has two kids. One is a girl (event B). What is the probability that the other kid is a girl (event $G = 2$ girls)?

4) (SUPER HARD EXTRA QUESTION) A family has two kids. One is a girl called Arya (event B). What is the probability that the other kid is a girl (event $G = 2$ girls)?

Conditional Probability - family - answers

1) A family has two kids. What is the probability that both kids be are girls?

$$P(G \cap G) = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \text{ (Independent events)}$$

2) A family has two kids. The first is a girl. What is the probability that the other one is also a girl?

$$P(G) = \frac{1}{2} \text{ (Independent events)}$$

3) A family has two kids. One is a girl (event B). What is the probability that the other kid is a girl (event $G = 2$ girls)?

Let's look at the sample space: $\{(B,B), (B,G), (G,B) \text{ and } (G,G)\}$ form the complete sample space. The conditionality is that one of the kids is a girl and the combination (B,B) is no longer possible. Hence, our "new" sample space is: $\{(B,G), (G,B) \text{ and } (G,G)\}$. So we have

$$P(G|B) = \frac{P(G \cap G)}{P(G)} = \frac{\frac{\#\{(G,G)\}}{\#\{(B,B), (B,G), (G,B), (G,G)\}}}{\frac{\#\{(B,G), (G,B), (G,G)\}}{\#\{(B,B), (B,G), (G,B), (G,G)\}}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Or, more simply,

$$P(G|B) = \frac{P(G \cap G)}{P(G)} = \frac{\#\{(G,G)\}}{\#\{(B,G), (G,B), (G,G)\}} = \frac{1}{3}$$

4) (SUPER HARD EXTRA QUESTION) A family has two kids. One is **a girl called Arya** (event B). What is the probability that the other kid is a girl (event $G = 2$ girls)?

Answers can be found at Mlodinov's book.

Axioms and theorems of probability (7)

- If A and B are independent events, then $P(A|B) = P(A)$ and the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

can be rewritten as (as previously seen)

$$P(A \cap B) = P(A) * P(B)$$

- If A and B are mutually exclusive, then

$$P(A \cap B) = 0$$

hence

$$P(A|B) = P(B|A) = 0$$

It's all about conditionality

Conditionality is central to statistics. Let's consider some well known social sciences problems.

- Race/Gender wage gap: we are often interested in measuring gender and/or race wage gap *given* other individual characteristics (e.g. - education, family environment, etc).
- Likelihood of voting for the incumbent *given* economy performance (GDP, unemployment, etc)
- Probability of finishing high school *given* income.

And so on...