Esssentials of Applied Data Analysis IPSA-USP Summer School 2018

The Basics of Probability Theory - Independence and conditional probability

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jan/18

Introduction to Probability - Part III

Basic Notions of probability, part III.

Independence

The event B is said to be independent of the event A if the probability that B occurs is not influenced by whether A has or has not ocurred.

IMPORTANT: Independence is a *key* concept to us

Independence - Examples

- Toss a coin twice. The result of the second toss is independent of the result in the first toss.
- Take one card from a deck. Don't put it back. Take a second card. The result of the second card is *not* independent of the first.

- In general, a legislator support to a government (or being a woman, or being from some part of the country) is not independent of her party.

Axioms and theorems of probability (5)

- If A and B are independent events, then $P(A \cap B) = P(A) * P(B)$

Independence - Coin

1) Toss a coin twice. What's is the probability of getting heads (H) twice?

$$P(H_1 \cap H_2) = P(H_1) * P(H_2) = P(H) * P(H) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$

$$\text{Sample Space} = \{\text{H,H}\},\{\text{H,T}\},\{\text{T,H}\},\{\text{T,T}\}$$

2) Toss a coin 50 times. What's is the probability of getting 50 times head (H)?

$$P(H_1 \cap H_2 \cap ... \cap H_{50}) = (P(H))^{50} = \left(\frac{1}{2}\right)^{50} \simeq 8.8 * 10^{-16}$$

3) Toss a coin 10 times. What's is the probability of getting at least one Tail (T)?

$P(\text{1 Tail or more}) = P(\text{exactly 1 Tail}) + P(\text{exactly 2 Tails}) + ... + P(\text{exactly 10 Tails}) =$

$$= \{P(T \cap H \cap ... \cap H) + P(H \cap T \cap H \cap ... \cap H) + P(H \cap ... \cap H \cap T)\} +$$
$$+ P(\text{exactly 2 Tails}) + ... + P(\text{exactly 10 Tails}) =$$

OMG!!! How do I compute that?

Use the fact that the complementary of "getting at least one Tail" is "not getting any Tails", which is the same as "getting 10 heads". And "getting 10 heads" is much easier to compute:

$P(\text{1 Tail or more}) = 1 - P(\text{not getting any Tails}) = 1 - P(\text{10 Hs}) = 1 - (1/2)^{10} \simeq 0.9990234$

Independence - Cards

Take a card from a deck (52 cards).

What's is the probability of getting a Queen (Q)?

$$P(Q) = \frac{4}{52} = \frac{1}{13}$$

What's is the probability of getting a Hearts (H)?

$$P(Q) = \frac{13}{52} = \frac{1}{4}$$

What's is the probability of getting the Queen of Hearts ($Q \cap H$)?

$$P(Q \cap H) = P(Q) * P(H) = \frac{1}{13} * \frac{1}{4} = \frac{1}{52}$$

If the card deck is complete, the suit and the number are independent of each other, so we can apply $P(A \cap B) = P(A) * P(B)$

Conditional Probability

Let A be an event that is **not independent** of B, and $P(B) > 0$. The probability that an event A occurs once B has ocurred – $P(A|B)$, which can be read as *conditional probability* of A given B – is **not** equal to the probabilty of A:

$$P(A|B) \neq P(A)$$

Axioms and theorems of probability (6)

- If A and B are not independent events, then the conditional probability of A given B, $P(A|B)$, is defined as the joint probability of A and B divided by the marginal probability of B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability

Intuitively, we can think as the conditional probability of A given B as:

$$P(A|B) = \frac{\frac{\text{\# of outcomes in A } \textbf{and} \text{ B}}{\text{\# of outcomes in the sample space}}}{\frac{\text{\# of outcomes in B}}{\text{\# of outcomes in the sample space}}} = \frac{\text{\# outcomes in event A } \textbf{and} \text{ B}}{\text{\# outcomes in B}}$$

Conditional Probability - dice

Roll a dice. Define event B as "gettting a number less than or equal to 3". What is the probability of getting a 1 (event A) given that B occurred?

Answer:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{\text{\# of outcomes in A } \textbf{and} \text{ B}}{\text{\# of outcomes in the sample space}}}{\frac{\text{\# of outcomes in B}}{\text{\# of outcomes in the sample space}}} = \frac{\frac{\#\{1\}}{\#\{1,2,3,4,5,6\}}}{\frac{\#\{1,2,3\}}{\#\{1,2,3,4,5,6\}}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{2}{6} = \frac{1}{3}$$

Alternatively, you can read this problem as "I already know I got 1, 2 or 3. Given that, what is the probability that I got a 1"?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} == \frac{\text{\# outcomes in event A } \textbf{and} \text{ B}}{\text{\# outcomes in B}} = \frac{\#\{1\}}{\#\{1,2,3\}} = \frac{1}{3}$$

Conditional Probability - family - answers

1) A family has two kids. What is the probability that both kids be are girls?

$$P(G \cap G) = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \text{ (Independent events)}$$

2)A family has two kids. The first is a girl. What is the probability that the other one is also a girl?

$$P(G_2) = \frac{1}{2} \text{ (Independent events)}$$

3) A family has two kids. One is a girl (event B). What is the probability that the other kid is a girl (event A = 2 girls)?

Let's look at the sample space: {(B,B), (B,G), (G,B) and (G,G)} form the complete sample space. The conditionality is that one of the kids is a girl and the combination (B,B) is no longer possible. Hence, our "new" sample space is: {(B,G), (G,B) and (G,G)}. So we have

$$P(A|B) = \frac{P(G \cap G)}{P(G)} = \frac{\frac{\#\{(G,G)\}}{\#\{(B,B),(B,G),(G,B),(G,G)\}}}{\frac{\#\{(B,G),(G,B),(G,G)\}}{\#\{(B,B),(B,G),(G,B),(G,G)\}}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Or, more simply,

$$P(A|B) = \frac{P(G \cap G)}{P(G)} = \frac{\#\{(G,G)\}}{\#\{(B,G),(G,B),(G,G)\}} = \frac{1}{3}$$

4) (SUPER HARD EXTRA QUESTION) A family has two kids. One is **a girl called Arya** (event B). What is the probability that the other kid is a girl (event A = 2 girls)?

Answer can be found at Imai's or Mlodinov's books.

Axioms and theorems of probability (7)

- If A and B are independent events, then $P(A|B) = P(A)$ and the formula
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
can be rewritten as (as previously seen)
$$P(A \cap B) = P(A) * P(B)$$

- If A and B are mutually exclusive, then

$$P(A \cap B) = 0$$

  hence

$$P(A|B) = P(B|A) = 0$$

- Law of Total Probability note that we can write the probability of A and B ($A \cap B$ in two different forms:

$$P(A \cap B) = P(A|B) * P(B) = P(B|A) * P(A)$$

  From that we can derive the law of total probability which states that the probability of A can be separated into two parts: the conditional probability of A on B times the probability of B and the conditional probability of A on the complement of B (B^c) times the probability of B^C

$$P(A) = P(A|B) * P(B) + P(A|B^c) * P(B^c)$$

It's all about conditionality

Conditionality is central to statistics. Let's consider some well know social sciences problems.

- Race/Gender wage gap: we are often interested in measuring gender and/or race wage gap *given* other individual characteristics (e.g. - education, family enviroment, etc).
- Likelihood of voting for the incumbent *given* economy performance (GDP, unempolyment, etc)
- Probability of finshing high school *given* income.

Hey, let's discuss it for a few minutes.