Esssentials of Applied Data Analysis IPSA-USP Summer School 2018

Linear Regression

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Correlation and linear model

As we have seen in class, we can exam the relationship between two continuous variables, X and Y by calculating the covariance, or, even better, it's "standardized" version, the correlation.

Another way to represent the same relationship is by a *linear model*, characterized by the following equation:

$$Y = \alpha + \beta X + \epsilon$$

Where α is the *intercept* coefficient, *beta* is the slope coefficient and ϵ is an error term. We will carefully look the two coefficients and forget about the error term a little bit.

Slope

Let's say we want to ask the following question: how much we expect Y – that will be called the *dependent variable* or *outcome* from now on – to vary when we vary X – *independent variable* or *predictor*?

That can be answered by looking at β. Pay attention to the formula: if you increase X by 1 (and ignore the error term) Y will increase $\beta \times X$.

It is easy to answer what is the value of β when we are dealing with two variables that have been previously standardized ($z_x = (x_i - E[x])/\sigma$): it is exactly the correlation between the two variables!

We don't need, however, to transform the variables to calculate the β, but we already have a hint: it has something to do with correlation and, consequently, with covariance.

In fact, the formula to calculate β is quite easy to interpret. Let's take a look at 3 ways we can write it:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(X - \bar{X})(Y - \bar{Y})}{\sum_{i=1}^{n}(X - \bar{X})^2} = \frac{E[(X - E[X])(Y - E[Y])]}{E(X - E[X])^2} = \frac{Cov(X, Y)}{Var(X)}$$

The last two expressions should be familiar to us by now. The last one, in particular, can directly interpreted: β is the covariance of X and Y divided by the variance of Y. In other words, it is how much of the covariance is "explained" by the variance of X.

So, on way to describe what we are doing in a linear regression model is: we are explaining the variation of the dependent variable (Y) by the variation in the independent variable (X).

Going back to the probability fundamentals: we are examining the joint distribution of the two variables and Y as a linear function of X.

Note that we are using a hat over β. This is because we are getting an estimate of β in the same way we were getting estimates of the mean and variance of variable in a random sample. $\hat{\beta}$ and $\hat{\alpha}$ are the *parameters* we estimate using a linear regression, or, more precisely, the *ordinary least squares* method.

Intercept

α, the intercept is even easier to interpret. When $X = 0$, α is the value for Y. Graphically, α is where the regression line intercepts the vertical axis. If $\alpha = 0$, the line gos through the origin.

The formula o calculate $\hat{\alpha}$ is:

$$\hat{\alpha} = E[Y] - \hat{\beta}E[X]$$

Fitted Values

The values of Y that can be obtained with regression line are called fitted values (because they fit the linear model perfectly). If you forget about the error term, Fitted values $\hat{Y}$ are calculated in a sample by using the estimated parameters $\hat{\alpha}$ and $\hat{\beta}$:

$$\hat{Y} = \hat{\alpha} + \hat{\beta}X$$

A nice way to interpret the fitted values is: they are the expected values of Y given especfic values of $X = X_i$, or $E[Y|X = x_i]$. This interpretation is directly connected to what we have discussed about joint and conditional probability, or joint distribution. The regression line is what we would expected of Y given X once we learn something about the relationship about Y and X.

Residuals

To estimate a regression line, we choose the line that has the least sum of residual squares:

$$SSR = \sum_{i=1}^{n} \hat{\epsilon}_i^2$$

This line also has the following property: the sum of the residuals (not the squares residuals, just the residuals) should be zero:

$$\sum_{i=1}^{n} \hat{\epsilon}_i = 0$$

Residuals are, graphically, the vertical distance of each observation to the regression line. Mathematically, they can be obtained by subtratcting the fitted values from the observed values:

$$\hat{\epsilon}_i = Y_i - \hat{Y}$$

or, by substiting $hatY$ by it's formula:

$$\hat{\epsilon}_i = Y_i - (\hat{\alpha} + \hat{\beta} X_i)$$

A consequence of estimating a regression line through ordinary least squares is the line always go through the center of the data, the point ($\bar{X}$ and $\bar{Y}$).

Take a look at the nice figure from chapter 4 of Imai's book and check if you can locate all of the parameters we have talked about

R-square - the coefficient of determination

How good is a model? The more of the variance in the outcome Y and X is explained by the predictor X, the best is the model. How can we measure it?

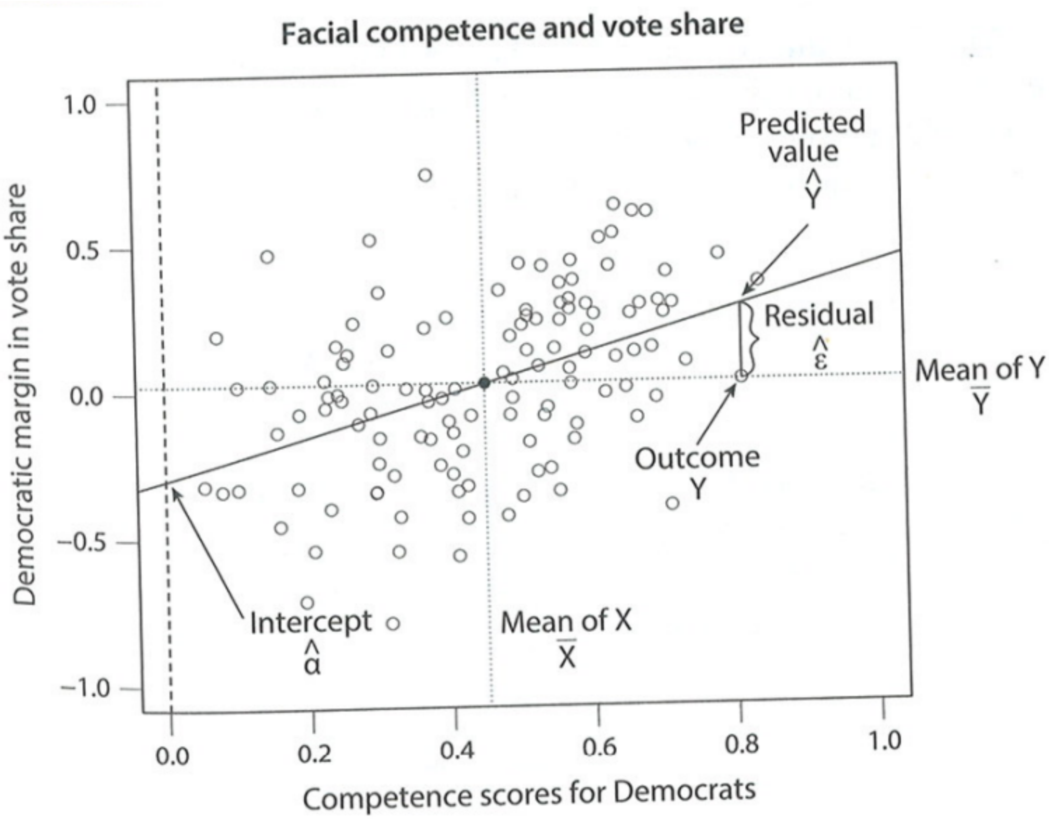
If you didn't have any information on the relationship of X and Y, and you want to guess the value of Y, what would be a good guess? The mean of Y.

With a regression model, you can guess better than just the mean value. You can use the conditional values of Y on X – $E[Y|X]$ to improve you guess.

The difference between this two guesses – the plain Y expected value and the conditional expected value on X – is exactly the variation of Y explained by the model and is represented by the model sum os squares:

$$MSS = \sum_{i=1}^{n} (\hat{Y} - \bar{Y})^2$$

The total variation in Y is the total sum of squares:



Facial competence and vote share
Predicted value
$\hat{Y}$
Residual
$\hat{\varepsilon}$
Mean of Y
$\bar{Y}$
Outcome
Y
Intercept
$\hat{\alpha}$
Mean of X
$\bar{X}$
1.0
0.5
0.0
−0.5
−1.0
Democratic margin in vote share
0.0
0.2
0.4
0.6
0.8
1.0
Competence scores for Democrats

Figure 1:

$$TSS = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

and it can be divided into two parts, MSS and SSR, or into a part that is explained by the model and a part that it is not. Remember that the Squared Sum of Residuals (SSR) i:

$$SSR = \sum_{i=1}^{n}(Y_i - \hat{Y})^2$$

The coefficient of determination, R^2, is the answer to the fist question. It is a measure of mdoel fit and represents the portion of Y explained by X. It can be calculated using the summations described above:

$$R^2 = \frac{MSS}{TSS} = \frac{TSS - SSR}{TSS} = 1 - \frac{SSR}{TSS}$$