MATH/STAT 429/629
Spring 2019
Final Exam
05/15/19
Time Limit: 110 Minutes

Name (Print): ____________________

Signature ____________________

This exam contains 10 pages (including this cover page) and 31 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use two pages of hand-written notes (front and back), scratch paper, and a scientific calculator on this exam.

You are required to show your work on each free-response problem on this exam (you are not required to show work on multiple choice). The following rules apply:

- **Write the letter corresponding to your choice on the blank** to answer multiple choice questions.
- **Organize your work** for free-response problems, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit. If you need extra space write the **solution or answer** on the page and attach supporting work by stapling extra pages to the exam.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Please do not write in the table to the right or on the footer at the bottom each page.

Page	Points	Score
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	15	
9	15	
10	10	
Total:	100	

1 Multiple-choice questions

1. (2 points) Solve the system by using elementary row operations on the equations. Follow the elimination procedure. The order of answer choices is (x_1, x_2).

$$\begin{aligned} x_1 + 3x_2 &= -1 \\ 4x_1 + 5x_2 &= -18 \end{aligned}$$

A. $(-7, 2)$ B. $(0, 0)$ C. $(-1, -18)$ D. $(-3, 3)$

1. ____________

2. (2 points) Find the general solution of the system whose **augmented** matrix is given below.

$$\begin{bmatrix} 3 & -5 & 2 & 0 \\ 9 & -15 & 6 & 0 \\ 12 & -20 & 8 & 0 \end{bmatrix}$$

A. $\begin{cases} x_1 = -5x_2 \\ x_2 = 3x_3 \\ x_3 \text{ is free} \end{cases}$ B. $\begin{cases} x_1 = \frac{5}{3}x_2 - \frac{2}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$ C. $\begin{cases} x_1 = 5 \\ x_2 = -3 \\ x_3 = 2 \end{cases}$ D. No solution(s).

2. ____________

3. (2 points) Compute $\mathbf{u} - 2\mathbf{v}$ when $\mathbf{u} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

A. $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ B. $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} -6 \\ -3 \end{bmatrix}$ D. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

3. ____________

4. (2 points) Use the definition of $A\mathbf{x}$ to write the vector equation as a matrix equation.

$$x_1 \begin{bmatrix} -8 \\ 7 \\ -5 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -9 \\ -4 \\ -8 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -6 \\ 0 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 7 \\ 5 \end{bmatrix}$$

A. $\begin{bmatrix} -8 & -9 & -6 \\ 7 & -4 & 0 \\ -5 & -8 & 6 \\ 6 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 7 \\ 5 \end{bmatrix}$ B. $\begin{bmatrix} -8 & -9 & -6 \\ 7 & -4 & 0 \\ -5 & -8 & 6 \\ 6 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 7 \\ 5 \end{bmatrix}$

C. $\begin{bmatrix} -8 & -9 & -6 \\ 7 & -4 & 0 \\ -5 & -8 & 6 \\ 6 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 7 \\ 5 \end{bmatrix}$ D. The matrices are not compatible.

4. ____________

5. (2 points) Determine the nature of solutions to the following system.

$$\begin{aligned} 6x_1 - 3x_2 + 15x_3 &= 0 \\ -6x_1 - 9x_2 - 6x_3 &= 0 \\ 12x_1 + 6x_2 + 21x_3 &= 0 \end{aligned}$$

A. It is impossible to determine. B. The system has only a trivial solution.
C. The system has a unique, nontrivial solution. D. The system has infinite solutions.

5. ____________

Points earned out of a possible 10 points on the page: ____________

6. (2 points) Determine by inspection whether the vectors are linearly independent.
$\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \end{bmatrix}$

   A. The set is linearly independent because at least one of the vectors is a multiple of another vector.

   B. The set is linearly dependent because at least one of the vectors is a multiple of another vector.

   C. The set is linearly dependent because there are four vectors but only two entries in each vector.

   D. The set is linearly independent because there are four vectors in the set but only two entries in each vector.

6. ____________

7. (2 points) Let $T : \Re^2 \to \Re^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the image under T of $3\mathbf{u}+2\mathbf{v}$.

   A. $\begin{bmatrix} -16 \\ -9 \end{bmatrix}$ B. $\begin{bmatrix} 16 \\ 9 \end{bmatrix}$ C. $\begin{bmatrix} -9 \\ 16 \end{bmatrix}$ D. $\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

7. ____________

8. (2 points) If a matrix A is 8×5 and the product AB is 8×3, what is the size of B?

   A. 3×5 B. 5×5 C. 5×3 D. 3×3

8. ____________

9. (2 points) Find the inverse of the matrix. $\begin{bmatrix} 9 & 3 \\ -8 & -3 \end{bmatrix}$

   A. The matrix is not invertible. B. $\begin{bmatrix} 1 & -1 \\ \frac{8}{3} & -3 \end{bmatrix}$ C. $\begin{bmatrix} 3 & 3 \\ -8 & -9 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 1 \\ -\frac{8}{3} & -3 \end{bmatrix}$

9. ____________

10. (2 points) Determine if the matrix below is invertible. Use as few calculations as possible. Justify your answer.
$\begin{bmatrix} 5 & 0 & 0 \\ -4 & -5 & 0 \\ 7 & 4 & -2 \end{bmatrix}$

   A. The matrix is not invertible. If the given matrix is A, the equation $A\mathbf{x} = \mathbf{b}$ has no solution for some $\mathbf{b}$ in $\Re^3$.

   B. The matrix is invertible. The given matrix has three pivot positions.

   C. The matrix is invertible. If the given matrix is A, there is a 3×3 matrix C such that $CI = A$.

   D. The matrix is not invertible. The given matrix has two pivot positions.

10. ____________

Points earned out of a possible 10 points on the page: ____________

11. (2 points) Compute the determinant of the following matrix.
$$\begin{bmatrix} 1 & w & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
A. w B. 3 C. 1 D. Cannot say.

11. ____________

12. (2 points) Find the determinant below, where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4.$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$$
A. 4 B. 20 C. -4 D. Cannot say

12. ____________

13. (2 points) Let H be the set of all vectors of the form $\begin{bmatrix} 5t \\ 2t \\ 4t \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\Re^3$ such that $H = \text{Span}\{\mathbf{v}\}$.

A. $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ B. $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$ C. $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ D. No such $\mathbf{v}$ exists.

13. ____________

14. (2 points) Determine if $\mathbf{w} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ is in Nul A, where $A = \begin{bmatrix} 3 & -5 & 4 \\ 5 & -3 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

A. No, because $A\mathbf{w} = \begin{bmatrix} -15 \\ -4 \\ 15 \end{bmatrix}$ B. No, the Nul A is empty.

C. Yes, because $A\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ D. Cannot say.

14. ____________

15. (2 points) Determine if the set of vectors shown to the right is a basis for $\Re^3$.
$$\left\{ \begin{bmatrix} 2 \\ 3 \\ -9 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 12 \end{bmatrix} \right\}.$$
A. No, the set does not span $\Re^3$. B. Yes, the set is a basis for $\Re^3$.
C. No, the set is linearly dependent. D. Cannot say.

15. ____________

Points earned out of a possible 10 points on the page: ____________

16. (2 points) State the dimension for the subspace below.
$$\left\{ \begin{bmatrix} p-9q \\ 5p+8r \\ -9q+8r \\ -6p+12r \end{bmatrix} : p, q, r \text{ in } \Re \right\}.$$
A. 0 B. 2 C. 3 D. 4

16. ______________

17. (2 points) If the null space of a 6×8 matrix A is 4-dimensional, what is the dimension of the column space of A?

A. 4 B. 6 C. 8 D. 0

17. ______________

18. (2 points) Is $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ an eigenvalue of $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}$? If so, find the eigenvalue λ.

A. Yes, and $\lambda = 2$. B. No, $\mathbf{v}$ is not an eigenvalue of A.
C. Yes, and $\lambda = -2$. D. Yes, and $\lambda = 0$.

18. ______________

19. (2 points) Find the eigenvalues of $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$.

A. 3, 3 B. 2, 6 C. -3, -3 D. 2, 4

19. ______________

20. (2 points) Identify a nonzero 2×2 matrix that is invertible but not diagonalizable.

A. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

20. ______________

Points earned out of a possible 10 points on the page: ______________

21. (2 points) Compute $||\mathbf{w}||$ using $\mathbf{w} = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$.

A. $2\sqrt{6}$ B. 24 C. -16 D. 1

21. ____________

22. (2 points) Determine whether the set of vectors is orthogonal.
$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$

A. No, the zero vector is in the set.

B. Yes, each pair of vectors are orthogonal to each other.

C. No, the second and third vectors are not orthogonal.

D. Cannot say.

22. ____________

23. (2 points) Find the orthogonal projection of $\mathbf{y}$ onto Span$\{\mathbf{u_1}, \mathbf{u_2}\}$.
$\mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$

A. $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$ B. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ C. $\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ D. $\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$

23. ____________

24. (2 points) Determine if the matrix $A = \begin{bmatrix} -5 & 9 \\ 4 & -9 \end{bmatrix}$ is symmetric.

A. The matrix is not symmetric, since $A^T = \begin{bmatrix} -5 & 4 \\ 9 & -9 \end{bmatrix}$.

B. The matrix is symmetric, since $A^T = \begin{bmatrix} -5 & 9 \\ 4 & -9 \end{bmatrix}$.

C. The matrix is not symmetric, since the matrix is not invertible.

D. Cannot say.

24. ____________

25. (2 points) Compute the quadratic form $\mathbf{x}^T A\mathbf{x}$ for $A = \begin{bmatrix} 2 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

A. 161 B. $2x_1^2 + \frac{1}{9}x_1x_2 + x_2^2$ C. $4x_1^2 + \frac{2}{3}x_1x_2 + x_2^2$ D. $2x_1^2 + \frac{2}{3}x_1x_2 + x_2^2$

25. ____________

2 Free-response questions

26. (5 points) Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -15 \\ 19 \end{bmatrix}$.

Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

27. (5 points) Let $\{\mathbf{x}_1, \mathbf{x}_2\}$ be a basis for a subspace W, where $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

Use the Gram-Schmidt process to construct an orthonormal basis for W.

28. (5 points) Orthogonally diagonalize the matrix $A = \begin{bmatrix} -3 & 3 \\ 3 & 5 \end{bmatrix}$ by giving an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

29. Consider $A = \begin{bmatrix} 1 & 2 & 0 & -1 & 20 \\ 1 & 1 & -1 & 1 & 7 \\ -2 & -1 & 3 & -3 & -5 \end{bmatrix}$.

(a) (5 points) What is the rank of A? (Hint: find an echelon form)

(b) (5 points) What is the dimension of the null space of A? (Write a short justification).

30. Find the singular value decomposition $U\Sigma V^T$ for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$ by completing the following steps:

(a) (5 points) Find an orthogonal diagonalization of $A^T A$.

(b) (5 points) Set up V by using the normalized eigenvectors from above and Σ by using the singular values of A and zeros where appropriate.

(c) (5 points) Construct U (hint: $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$).

31. Consider $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

(a) (5 points) Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution? (Justify with a calculation).

(b) (5 points) If not, find the least squares solution $\hat{\mathbf{x}}$ by solving the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$.