Exercise 2 in advanced image analysis 02503:

The Ising Model

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The purpose of this exercise is to become acquainted with the physical properties of the Ising model by sampling configurations according to the Gibbs distribution.

Background material is found in the course material in section 1.3 of Li's textbook, and on page 84-94 in JMC's PhD thesis.

This exercise will not do analysis of observed images, but rather study the dynamics of the Ising model as a physical system. The Ising model can be used to describe the prior that enters in a complete Markov random Field image analysis system. In the exercise 3 such a complete Markov random field vision analysis will be made based on the Potts model which is a generalization of the Ising model. So we are warming up.

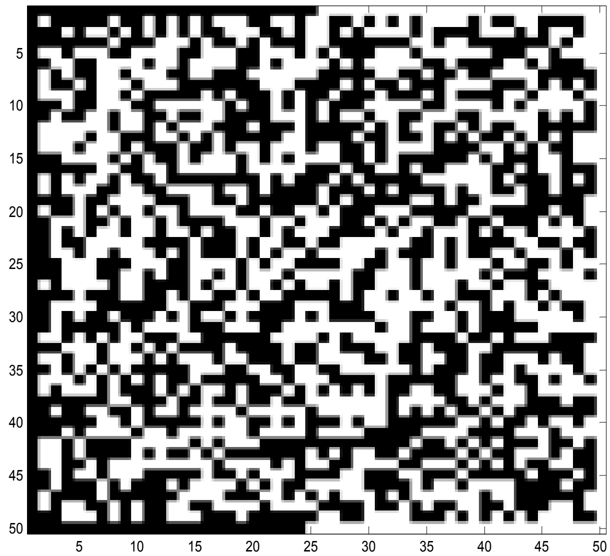
Task 1: *The conditional probability* of the Ising model from Li's equation (1.40) is

$$P(f_i \mid f_{Ni}) = Z^{-1} \exp(\alpha_i f_i + \sum_{i' \in Ni} \beta_{ii'} f_i f_{i'})$$

The first task is to write this more specifically: The four nearest neighbours are used, and the model is homogenous, and f_i takes values 0 and 1. Write the normalization constant Z explicitly – it has only two terms – and write the sum over the four neighbours, and introduce β_1, β_2 for the coupling horizontally and vertically respectively.
First you should set α, and to do this, consider a pixel i and assume that the north and east neighbours are 0, while the south and west neighbours are 1. Show that setting $\alpha = -\beta_1 - \beta_2$ leads to the two f_i values 0 and 1 being equally probable – corresponding to zero external magnetic field in case of a physical system. Then write the final expression for the conditional probability.

Task 2: Program *a Gibbs sampler*. The Gibbs sampler is described in Li page 60, Figure 2.8. Think about the order in which the sites are updated. It is not advisable to simply sweep over the image and update adjacent pixels – instead use two sweeps and update every second pixel in the first sweep (corresponding to the white fields of a chess-board) and then update the remaining half in the second sweep (the black fields of the chess-board). This is called *one iteration*.

Use a field of size 50 by 50. The initial configuration of the interior is chosen randomly. Half of the boundary is set to ones and the other half to zeroes as shown in the figure below. *The pixels on the boundary should never be updated with the Gibbs sampler*. Thus the boundary pixels are clamped to the initial values. This is a trick to avoid that the field falls into purely zeroes or ones. It also avoids the complications of defining the neighbours at the boundary of the field.



Write a piece of Matlab code that initialises the field and updates it.
Try different betas and get acquainted with the system (read: debug it). It is useful to write a function that executes a number of iterations given as a parameter.

Task 3: Perform 8 experiments with fields for at least 100 iterations with beta-values as in this table:

Experiment	β_1	β_2
1	3	3
2	3	-3
3	-3	3
4	-3	-3
5	1	1
6	1	-1
7	-1	1
8	-1	-1

Discuss the results in relation to Figure 3.8 in JMC's thesis. Do you observe peculiar structures in experiment 2, 3 and 4?

Task 4: Write code that computes the correlation between pixel values and the values 3 steps to the left, and similarly a function that computer the vertical spatial correlation over a distance of three (avoid using the clamped pixels in this computation).

Simulate 2 fields for at least 100 iterations with values

Experiment	β_1	β_2
9	0.5	1.5
10	0.5	3

Discuss the results, and use the computed correlations to quantify your observations.

Task 5: Finally study the phase transition for the isotropic Ising model, $\beta = \beta_1 = \beta_2$ for the set of β values given in the table below. Use at least 100 iterations.

Experiment	β
11	0.0
12	0.5
13	1.0
14	1.5
15	1.6
16	1.7
17	1.8
18	1.9
19	2.0
20	3.0
21	5.0
22	10.0

For each experiment, plot the correlation coefficient (use the mean of the horizontal and vertical correlations, since the model is now isotropic) *as a function of the iteration number* (from 1 to 100). Discuss the fluctuations and how fast equilibrium seems to be achieved for different experiments.

Also compute the average correlation over the last 20 sweeps for each experiment, and plot this versus β, and verify that the change is steepest at the expected critical β (cf. figure 3.7 in JMC's thesis).

Display the twelve final configurations of these experiments as a cartoon (use `subplot(3,4,n)`) and discuss the results.