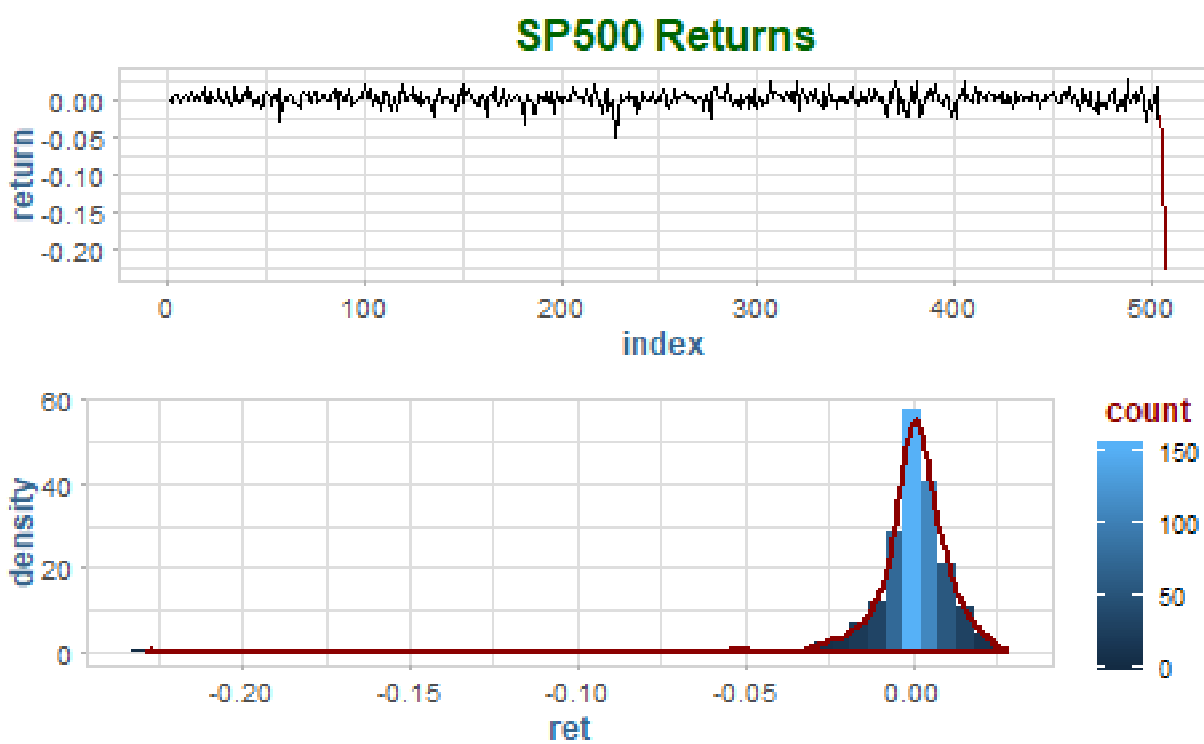
Problem Background

On Black Monday, the return on the S&P500 was *-22.8%*.

In this lab we are going to look at GARCH models and how they relate to predicting extreme events in financial markets.

```
data(SP500, package = "Ecdat")

ret.blm <- SP500$r500[1805] # Black Monday is the 1805 obs.
x <- SP500$r500[(1804 - 2*253+1 ): 1804]
returns <- data.table(ret = c(x, ret.blm))[, x := .I]

suppressWarnings({
   p1 <- ggplot(returns, aes( x = x)) +
      geom_line(aes(x = x, y = ifelse( x < 505, ret, NA))) +
      geom_line(aes(x = x, y = ifelse( x >= 505, ret, NA)), col = "darkred") +
      labs(title = "SP500 Returns", x = "index", y = "return")

   p2 <- ggplot(returns, aes(ret, y = ..density..)) +
      geom_histogram(aes(fill = ..count..), bins = 50) +
      geom_density(aes(y = ..density..), col = "darkred", lwd = 1)

   grid.arrange(p1, p2, nrow = 2)
})
```



Now, we fit the GARCH model.

Below, we will fit a AR(1) + GARCH(1, 1) model for the data 2 years prior to Black Monday (assuming 253 trading days/year).

```
spec <- ugarchspec( mean.model = list( armaOrder = c(1, 0) ),
                    variance.model = list( garchOrder = c(1, 1)),
                    distribution.model = "std")

fit <- ugarchfit(data = x, spec = spec)

dfhat <- coef(fit)[6]

forecast <- ugarchforecast(fit, data = x, n.ahead = 1)
forecast
```

```
*------------------------------------*
*       GARCH Model Forecast         *
*------------------------------------*
Model: sGARCH
Horizon: 1
Roll Steps: 0
Out of Sample: 0

0-roll forecast [T0=1971-05-21 20:00:00]:
       Series   Sigma
T+1 -0.003204 0.01208
```

The parameter estimates for the model are below:

Table 1: Parameter Estimates

mu	ar1	omega	alpha1	beta1	shape
0.001333	0.082048	0	1.2e-05	0.99898	3.292482

Problem 1.)

What is the conditional probability of a return less than or equal to -0.228 on Black Monday?

```
ret.pred <- fitted(forecast)
ret.sd <- sigma(forecast)

z_score <- as.numeric( ( ret.blm - ret.pred) / ret.sd )

bm.prob <- pt(z_score, dfhat, lower.tail = T)
```

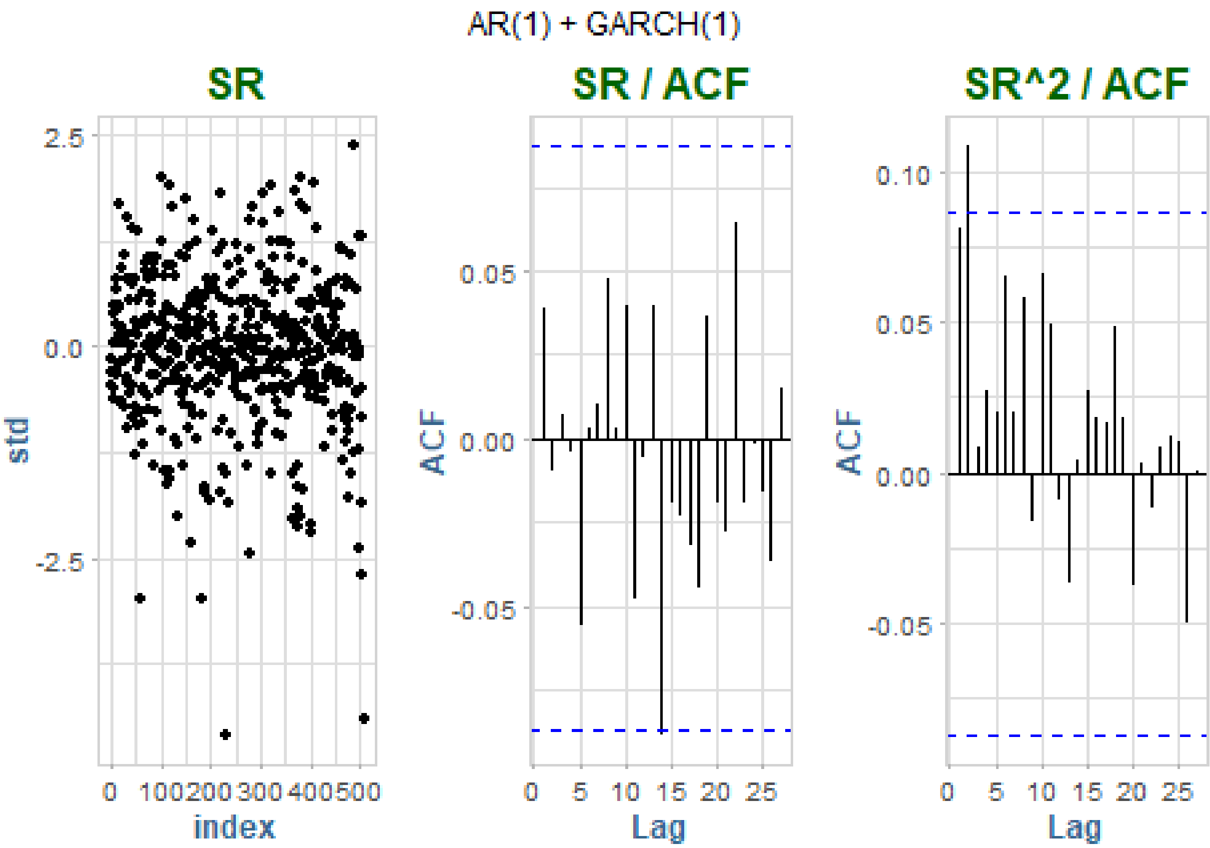
The probability of a return less than or equal to that of Black Monday is: 0.00009449865

Problem 2.)

Model Diagnostics

Compute and plot the standardized residuals. Also, plot the ACF of the standardized residuals and their squares.

```
plot_residuals <- function(res, sd, mname) {

   residual <- data.table(
      res = res,
      std = res / sd)[,
        res_sq := res*res][,
                          index := .I]

   p1 <- ggplot(residual, aes(x = index, y = std)) +
      geom_point() +
      labs(title = "SR")

   p2 <- ggAcf(residual$std) +
      labs(title = "SR / ACF")

   p3 <- ggAcf(residual$res_sq) +
      labs(title = "SR^2 / ACF")

   grid.arrange(p1, p2, p3, nrow = 1, top = mname)
}

plot_residuals(forecast@model$modeldata$residuals,
               forecast@model$modeldata$sigma,
               "AR(1) + GARCH(1)")
```



Does the model fit adequately?

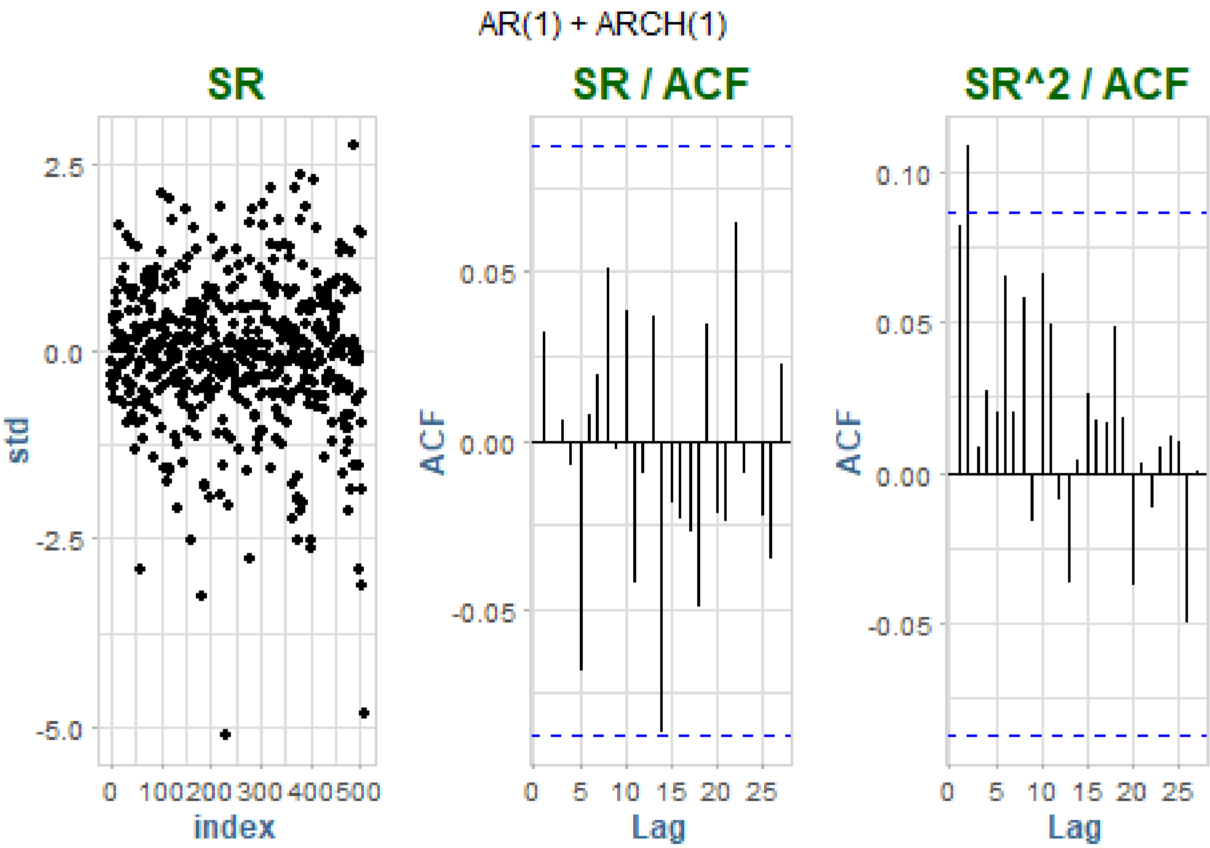
In the residual plot above, we see what appears to be white noise for the residual plots. Neither of the ACF plots display significant residual auto-correlation, so we would say this model fits reasonably well outside of a few points (specifically at lag 14 in the SR ACF and lag 2 in the SR^2 ACF). There is volatility clustering in the SR^2 between lags 5-10 and 15-20, but nothing significant.

Problem 3.)

AR(1) + ARCH(1)

Would an AR(1) + ARCH(1) model provide an adequate fit?

```
spec2 <- ugarchspec( mean.model = list( armaOrder = c(1, 0) ),
                     variance.model = list( garchOrder = c(1, 0)),
                     distribution.model = "std")

fit2 <- ugarchfit(data = x, spec = spec2)

plot_residuals(as.vector(residuals(fit2)),
               as.vector(sigma(fit2)),
               "AR(1) + ARCH(1)")
```



The residual diagnostics for the AR(1) + ARCH(1) are almost identical to the AR(1) + ARCH(1).

```
ic.data <- as.data.table(cbind(infocriteria(fit), infocriteria(fit2)),
                         keep.rownames = T)[,
                                                  D := V1 - V2]
colnames(ic.data) <- c("Information Criterion", "fit1", "fit2", "Delta")

pretty_kable(ic.data, "Model Fit Comparison", dig = 4)
```

Table 2: Model Fit Comparison

Information Criterion	fit1	fit2	Delta
Akaike	-6.5205	-6.5177	-0.0028
Bayes	-6.4704	-6.4759	0.0055
Shibata	-6.5208	-6.5179	-0.0029
Hannan-Quinn	-6.5008	-6.5013	0.0005

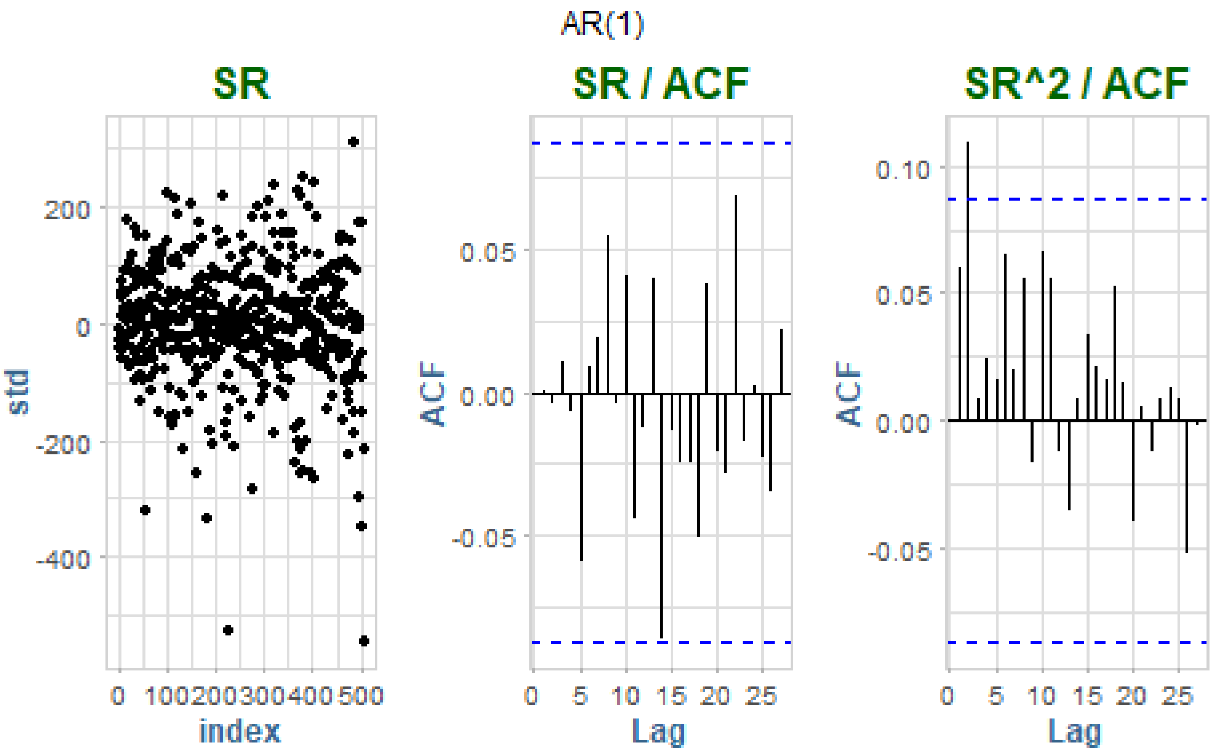
If we compare the information criterion for the model fits, we see that the AR(1) + GARCH(1) has lower AIC, but the AR(1) + ARCH(1) has a lower BIC.

Depending on the IC, we would choose different models, however, the differences are rather small.

Problem 4.)

Does an AR(1) model with a Gaussian conditional distribution prove an adequate fit?

```
fit3 <- arima( x, order = c(1, 0, 0))

suppressMessages({
  plot_residuals(fit3$residuals,
                 fit3$sigma2,
                 "AR(1)")
})
```




```
# normalize AIC/BIC to compare to previous models
pretty_kable(data.table(AIC = AIC(fit3) / fit3$nobs,
                        BIC = BIC(fit3) / fit3$nobs), "Information Criteron AR(1)", dig = 4)
```

Table 3: Information Criteron AR(1)

AIC	BIC
-6.4171	-6.392

Looking at the residual plots, we see similar characteristics of the previous models. There is some volatility clustering and the pronouced lags at 14 in the SR and 2 in the SR^2, and in general the lags are more pronounced in the Gaussian model. The information criterion is quite a bit higher in both AIC and BIC. I would say this model is not fully adequate, and would strongly prefer either of the two previous models.

Problem 5.)

The conditional variance of an AR(1) + GARCH(1, 1) process is:

$Y_t - \mu = \phi(Y_{t-1} - \mu) + a_t$

$a_t = \sigma_t \epsilon_t$

is

$\sigma_t^2 = \omega + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2$

Where $\mu, \phi, \omega, \alpha, \beta$ are constant parameters.

a.) What is $\mathbb{E}[Y_t - u]$?

b.) What is $\mathbb{E}[\epsilon(t)$?

c.) What is $\mathbb{E}[\epsilon(t)^2]$?

d.) What is $\mathbb{E}[\epsilon(t)\epsilon(t-1)]$?

e.) What is the *unconditional* variance of the process?

f.) Show that this correctly reduces to Ruppert Eq. 12.8 for unconditional variance of an AR(1) process.