

**Problem 1**

Suppose the risk measure  $\mathfrak{R}$  is  $VaR(\alpha)$  for some  $\alpha$ .

Let  $P_1$  and  $P_2$  be two portfolios whose returns have a joint normal distribution with means  $\mu_1$  and  $\mu_2$ , standard deviations  $\sigma_1$  and  $\sigma_2$ , and correlation  $\rho$ .

Suppose the initial investments are  $S_1$  and  $S_2$ .

Show that  $\mathfrak{R}(P_1 + P_2) \leq \mathfrak{R}(P_1) + \mathfrak{R}(P_2)$  under joint normality.

**Solution:**

From **19.12**, we know that:

$$\begin{aligned}\widehat{VaR}_{P_i}^{norm}(\alpha) &= -S_i * \{\hat{\mu}_{P_i} + \phi^{-1}(\alpha)\hat{\sigma}_{P_i}\} \\ \dots &= -S_i * \{\hat{\mu}_{P_i} + Z_\alpha \hat{\sigma}_{P_i}\}\end{aligned}$$

Where  $S_i$  is the initial value of the portfolio  $P_i$  for  $i$  in 1, 2. Let the portfolio weights,  $w_i$ ,

$$w_1 = \frac{S_1}{S_1 + S_2}, w_2 = 1 - w_1$$

From **16.1** and **16.3**, we know that,

$$\begin{aligned}\mathbb{E}(R_p) &= \mu_P = w_1\mu_1 + w_2\mu_2 \\ \sigma_p^2 &= w_1\sigma_1^2 + w_2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2\end{aligned}$$

Where  $\rho \leq 1$ ,  $w_1 > 0$  and  $w_2 > 0$ .

$$\text{Let } \rho_P \leq w_1\sigma_1 + w_2\sigma_2$$

When  $\alpha < 0.5$  (reasonable for tail events), then  $Z_\alpha < 0$ , so that:

$$Z_\alpha \sigma_p \geq w_1 Z_\alpha \sigma_1 + w_2 Z_\alpha \sigma_2$$

Adding portfolio means from above,

$$\begin{aligned}\mu_p + Z_\alpha \sigma_p &\geq w_1\mu_1 + w_1 Z_\alpha \sigma_1 + w_2\mu_2 + w_2 Z_\alpha \sigma_2 \\ \dots &= w_1(\mu_1 + Z_\alpha \sigma_1) + w_2(\mu_2 + Z_\alpha \sigma_2)\end{aligned}$$

Multiplying both sides by  $-S$ ,

$$\begin{aligned}-S\{\mu_p + Z_\alpha \sigma_p\} &\leq -S w_1(\mu_1 + Z_\alpha \sigma_1) - S w_2(\mu_2 + Z_\alpha \sigma_2) \\ \dots &= -S_1(\mu_1 + Z_\alpha \sigma_1) - S_2(\mu_2 + Z_\alpha \sigma_2)\end{aligned}$$

Again using **19.12**,

$$VaR_p^{norm}(\alpha) \leq VaR_{p_1}^{norm}(\alpha) + VaR_{p_2}^{norm}(\alpha)$$

**Problem 2**

```
data <- read.csv(paste0(data.dir, "Stock_Bond.csv"),
                 header = T)

prices <- as.matrix(data[1:500, c(3, 5, 7, 9, 11)])

n <- nrow(prices)

returns <- apply(prices, 2, FUN = function( p ){
  (p[-1] / p[1:n-1]) - 1
})
```

- What are the sample mean vector and sample covariance matrix of the 499 returns on these stocks?

```
stock.means <- colMeans(returns)
stock.cov <- cov(returns)
```

Table 1: Stock Mean Returns

	mu
GM_AC	0.00098
F_AC	0.00156
UTX_AC	0.00005
CAT_AC	0.00123
MRK_AC	0.00086

Table 2: Covariance Matrix

	GM_AC	F_AC	UTX_AC	CAT_AC	MRK_AC
GM_AC	0.00041	0.00029	0.00019	0.00023	0.00019
F_AC	0.00029	0.00045	0.00020	0.00028	0.00022
UTX_AC	0.00019	0.00020	0.00037	0.00022	0.00017
CAT_AC	0.00023	0.00028	0.00022	0.00050	0.00023
MRK_AC	0.00019	0.00022	0.00017	0.00023	0.00031

- How many shares of each stock should one buy to invest \$50 million in an equally weighted portfolio?

```
capital <- 5e7

n.stocks <- ncol(prices)
n.shares <- as.data.frame((capital/n.stocks)/prices[500,])

pretty_kable(data.table(Stock = rownames(n.shares),
                        Shares = comma(n.shares[,1])), "Equal Weighted Shares")
```

Table 3: Equal Weighted Shares

Stock	Shares
GM_AC	594,530.32
F_AC	2,232,142.86
UTX_AC	2,923,976.61
CAT_AC	1,821,493.62
MRK_AC	1,754,385.96

```
VaR_N = function( alpha, mu, v, S ) {
  - S * ( mu + qnorm(alpha) * sqrt(v) )
}

stock.weight <- 1/n.stocks
port.weights <- rep( stock.weight, n.stocks )

port.ret <- sum(stock.weight * stock.means)
port.var <- t(port.weights) %*% stock.cov %*% port.weights
```

- What is the one-day  $\text{VaR}(0.01)^*$  for this equally weighted portfolio?

```
var1d <- VaR_N(0.1, port.ret, port.var, capital)
```

One-day  $\text{VaR}(0.01) = 984,181.54$

- What is the five-day  $\text{VaR}(0.01)^*$  for this portfolio?

```
port.ret <- 5 * sum( stock.weight * stock.means )
port.var <- 5 * sum( stock.weight**2 * diag(stock.cov) )

var5d <- VaR_N(0.1, port.ret, port.var, capital)
```

Five-day  $\text{VaR}(0.01) = 1,062,887.10$

\* VaR estimates assume normality of returns.