Problem 1

Assume that a stock's log returns at any time scale have normal distribution.

- Suppose that its average annual log return is 100%, and its annual standard deviation ("volatility") of log returns is 200%.

What are its average (mu) and standard deviation (sigma) of daily log returns, assuming a year has 250 trading days?

```r
trading.days <- 250

annual.ret <- 1
annual.vol <- 2

mu <- annual.ret / trading.days
sigma <- annual.vol / sqrt(trading.days)

prob1 <- list(mu = round(mu, 5), sigma = round(sigma, 5))
```

Average: **0.004**

Standard Deviation: **0.12649**

Problem 2

Simulate 250 instances of the daily log returns described in 1.) with random seed set.seed(2015).

- Compute the net returns of these instances, and compute their average and standard deviation.

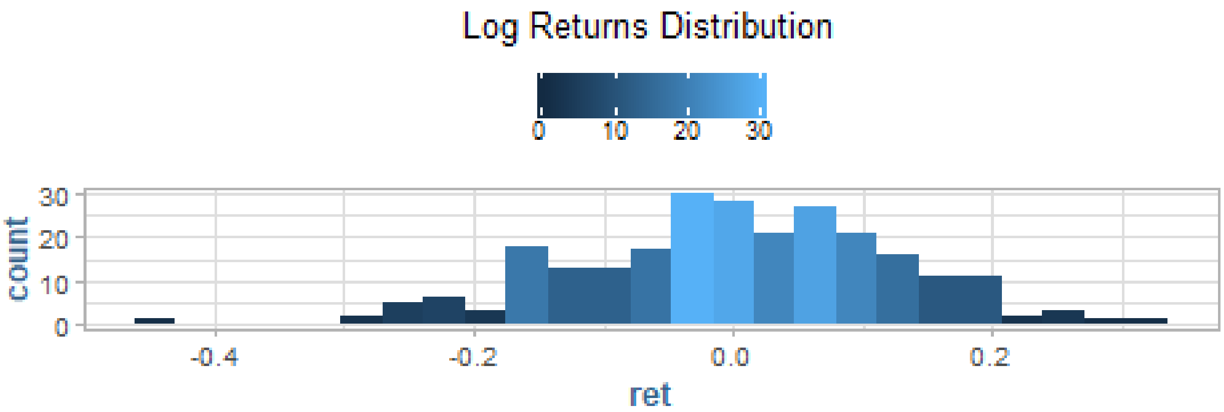
```r
set.seed(2015)

logRet <- rnorm(250, mu, sigma)
netRet <- exp(logRet) - 1

m <- round(mean(netRet), 5)
s <- round(sd(netRet), 5)
```

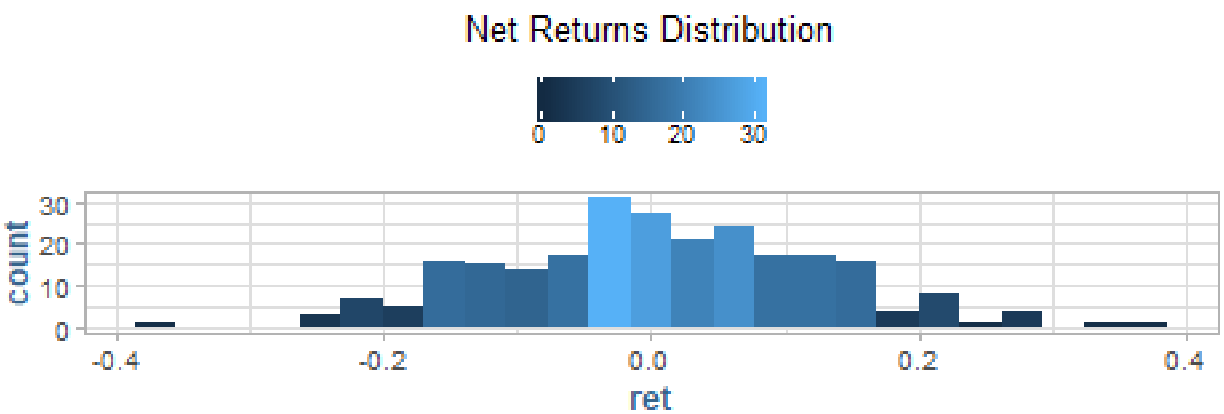
Average Net Return: **0.00695**

Standard Deviation of Net Returns: **0.12108**

Are the average (m) and standard deviation (s) of net returns same as the average and standard deviation of log returns computed in 1.)?

Answer: While not identical, they are close enough for practical purposes.





Problem 3

- Compute the quantity $m - \frac{s^2}{2}$.

```
prob3.a <- round(m - s^2/2, 5)
```

- $m - \frac{s^2}{2}$ = -0.00038

How does this compare with the average log return mu computed in part 1.)?

Answer: This is a close approximation to the original value of mu. We should note the extremely small sample size (250), and that a larger sample size (2.5m) produces a closer approximation:

```
set.seed(2015)

logRet <- rnorm(2500000, mu, sigma)
netRet <- exp(logRet) - 1

prob3.b <- round(mean(netRet) - sd(netRet)^2/2, 5)
```

2.5m samples = 0.00399

Their equality can be proven analytically through a mathematical theorem called Ito's Lemma that lies at the foundation of Black-Scholes options pricing formula. Their numerical equality is not that good here because Ito's Lemma assumes we can divide a period into infinitesimally small sub-periods.

So divide a year into 25,000 sub-periods (think of these sub-periods roughly as minutes) instead.

```r
set.seed(2015)

trading.periods <- 25e3

min_from_days <- function(days) {
  60 * 6.5 * days
}

min.avg <- annual.ret / trading.periods
min.sd <- annual.vol / sqrt(trading.periods)

min.logRet <- rnorm( min_from_days(250), min.avg, min.sd)
min.netRet <- exp(min.logRet) - 1

new.m <- mean(min.netRet)
new.s <- sd(min.netRet)

new.mu <- mean(min.logRet)
new.sigma <- sd(min.logRet)

prob3.c <- round(new.m - new.s^2/2, 5)
```

Compare the new mu (average log return per minute) with the new $m - \frac{s^2}{2}$ (m is now the average net return per minute).

- New mu = 0.00009
- New $m - \frac{s^2}{2}$ = 0.00009

Also, compare the new sigma (standard deviation of log returns per minute) with the new s (standard deviation of net return per minute).

- New sigma: **0.01267**
- New s: **0.01268**

Problem 4

- If we assume that the stock's initial price is $1, what is the expected value of its log price log(P(t)) after t minutes expressed in terms of *mu*?

Assuming:

μ is the average daily log return,

σ is the standard deviation of the daily log return

There are x minutes in a day, then 1 day = x minutes and 1 minute = $\frac{1}{x}$ day

For some minute i,

$$\mathbb{E}(r/minute) = \mathbb{E}(r_i) = \frac{\mu}{\sigma}$$
$$\mathbb{V}(r/minute) = \mathbb{V}(r_i) = \frac{\sigma^2}{x^2}$$
$$\sigma(r/minute) = \frac{\sigma}{x}$$

The expected return after t minutes is,

$$P_0 = 1$$
$$P_t = P_0(1 + R_1) ... (1 + R_t)$$
$$log(P_t) = (logP_0) + log(1 + R_1) + log(1 + R_t)$$
$$P_t = log(P_0) + r_1 + r_2 + r_t$$
Since $log(1) = 0$
$$P_t = \sum_{i=1}^{t} r_i$$
$$\mathbb{E}(P_t) = \mathbb{E}(\sum_{i=1}^{t} r_i),$$
$$\mathbb{E}(r_i) = \frac{\mu}{x}$$

- And what is the expected value of its price P(t) expressed in terms of *mu* and *sigma*?

$$\mathbb{V}(P_t) = \mathbb{V}(\sum_{i=1}^{t} r_i)) = t\frac{\sigma^2}{x^2}$$
$$\mathbb{E}(P_t) = t\frac{\mu}{x}$$

- Finally, express these expected values in terms of m and s instead.

$$\sigma P_t = \frac{\sigma}{x}\sqrt{t}$$

Problem 5

The continuously compounded rate of growth of a stock is log(P(t))/t.

- What is the expected continuously compounded rate of growth of the stock in part **4**?

$$\frac{P_t}{t} = \textstyle\sum_{i=1}^{t} \frac{r_i}{t}$$

$$\mathbb{E}[\frac{P_t}{t}] = \mathbb{E}[\textstyle\sum \frac{r_t}{t}] = t\frac{\frac{\mu}{x}}{t}$$

$$\mathbb{E}[\frac{P_t}{t}] = \frac{\mu}{x}$$