Problem 1

Assume that a stock's log returns at any time scale have normal distribution.

- Suppose that its average annual log return is 100%, and its annual standard deviation ("volatility") of log returns is 200%.

What are its average (mu) and standard deviation (sigma) of daily log returns, assuming a year has 250 trading days?

```
trading.days <- 250

annual.ret <- 1
annual.vol <- 2

mu <- annual.ret / trading.days
sigma <- annual.vol / sqrt(trading.days)

prob1 <- list(mu = round(mu, 5), sigma = round(sigma, 5))
```

Average: **0.004**

Standard Deviation: **0.12649**

Problem 2

Simulate 250 instances of the daily log returns described in 1.) with random seed set.seed(2015).

- Compute the net returns of these instances, and compute their average and standard deviation.

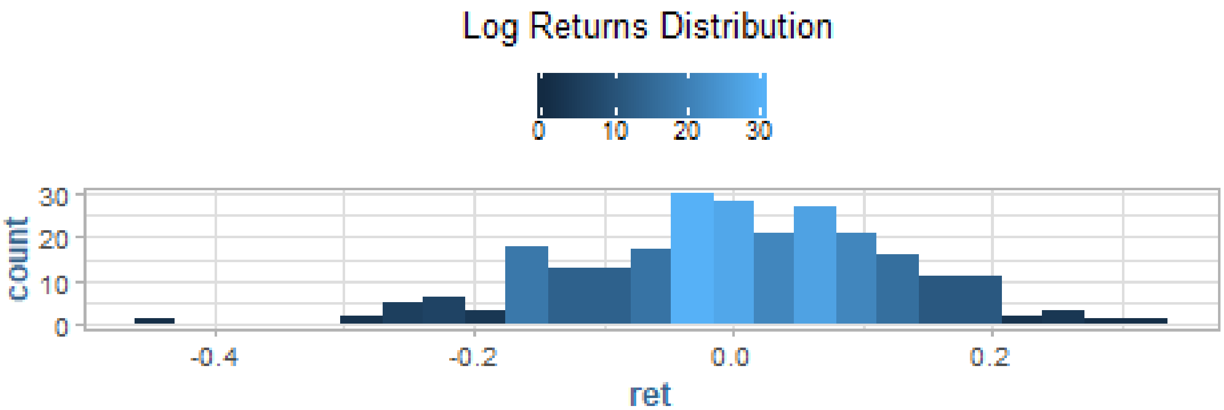
```
set.seed(2015)

logRet <- rnorm(250, mu, sigma)
netRet <- exp(logRet) - 1

m <- round(mean(netRet), 5)
s <- round(sd(netRet), 5)
```

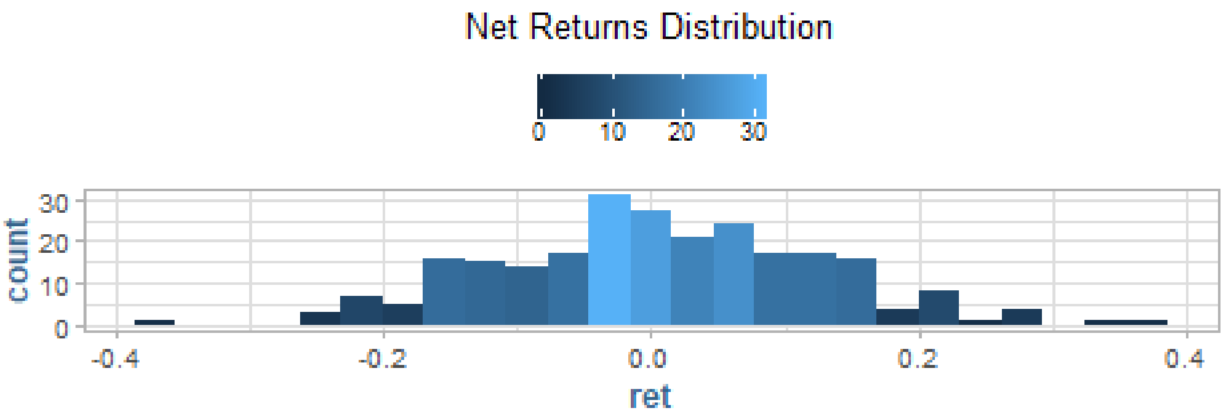
Average Net Return: **0.00695**

Standard Deviation of Net Returns: **0.12108**

Are the average (m) and standard deviation (s) of net returns same as the average and standard deviation of log returns computed in 1.)?

Answer: While not identical, they are close enough for practical purposes.





Problem 3

- Compute the quantity $m - \frac{s^2}{2}$.

```
prob3.a <- round(m - s^2/2, 5)
```

- $m - \frac{s^2}{2}$ = -0.00038

How does this compare with the average log return mu computed in part 1.)?

Answer: This is a close approximation to the original value of mu. We should note the extremely small sample size (250), and that a larger sample size (2.5m) produces a closer approximation:

```
set.seed(2015)

logRet <- rnorm(2500000, mu, sigma)
netRet <- exp(logRet) - 1

prob3.b <- round(mean(netRet) - sd(netRet)^2/2, 5)
```

2.5m samples = 0.00399

Their equality can be proven analytically through a mathematical theorem called Ito's Lemma that lies at the foundation of Black-Scholes options pricing formula. Their numerical equality is not that good here because Ito's Lemma assumes we can divide a period into infinitesimally small sub-periods.

So divide a year into 25,000 sub-periods (think of these sub-periods roughly as minutes) instead.

```
set.seed(2015)

trading.periods <- 25e3

min_from_days <- function(days) {
  60 * 6.5 * days
}

min.avg <- annual.ret / trading.periods
min.sd <- annual.vol / sqrt(trading.periods)

min.logRet <- rnorm( min_from_days(250), min.avg, min.sd)
min.netRet <- exp(min.logRet) - 1

new.m <- mean(min.netRet)
new.s <- sd(min.netRet)

new.mu <- mean(min.logRet)
new.sigma <- sd(min.logRet)

prob3.c <- round(new.m - new.s^2/2, 5)
```

Compare the new mu (average log return per minute) with the new $m - \frac{s^2}{2}$ (m is now the average net return per minute).

- New mu = 0.00009
- New $m - \frac{s^2}{2}$ = 0.00009

Also, compare the new sigma (standard deviation of log returns per minute) with the new s (standard deviation of net return per minute).

- New sigma: **0.01267**
- New s: **0.01268**

Problem 4

- If we assume that the stock's initial price is $1, what is the expected value of its log price log(P(t)) after t minutes expressed in terms of *mu*?

From A9.4 we know that if P is lognormal(μ, σ), then μ is the expected value of log(P). Sampling for t periods, we have:

$$log(\overline{P}) \sim \mathsf{N}(\mu, \frac{\sigma^2}{t})$$

Let $R \sim \mathsf{N}(\mu, \sigma^2)$, be a return that is IID standard normal.

$\mathbb{E}P_t$ = exp($\sigma R_t + \mu$), where R_t is IID standard normal.

Taking the log of both sides, we get:

$$log(P_t) = \sigma R_t + \mu$$

Subtracting σR_t from both sides:

$$\mu = log(P_t) - \sigma R_t$$

- And what is the expected value of its price P(t) expressed in terms of *mu* and *sigma*?

From the previous problem, we have:

$$\mu = log(P_t) - \sigma R_t$$

Multiply both sides by $\frac{1}{\sigma}$:

$$\frac{\mu}{\sigma} = log(P_t) - R_t$$

- Finally, express these expected values in terms of m and s instead.

Problem 5

The continuously compounded rate of growth of a stock is log(P(t))/t.

- What is the expected continuously compounded rate of growth of the stock in part **4**?

$$1 * \sum_{n=1}^{t} log(\frac{P_n}{P_{n-1}})$$

```r
set.seed(2015)

initial.price <- 1

t <- 250

logRet = rnorm( min_from_days(t), new.mu, new.sigma)
logPrice = c(initial.price, initial.price * cumsum(logRet))

ggplot(data.table(y = logPrice)[, x := .I], aes( x, y)) +
  geom_line(color = "cornflowerblue") +
  labs(title = "Log Stock Price P(t)", x = "Time", y = "Log Price") +
  theme(plot.title = element_text(hjust = 0.5))
```