Problem 1

Suppose the risk measure $\Re$ is $VaR(\alpha)$ for some α.

Let P_1 and P_2 be two portfolios whos returns have a joint normal distribution with means μ_1 and μ_2, standard deviations σ_1 and σ_2, and correlation ρ.

Suppose the initial investments are S_1 and S_2.

Show that $\Re(P_1 + P_2) <= \Re(P_1) + \Re(P_2)$ under joint normality.

Solution:

From **19.12**, we know that:

$$\widehat{VaR}_{P_i}^{norm}(\alpha) = -S_i \, * \{\hat{\mu}_{P_i} + \phi^{-1}(\alpha)\hat{\sigma}_{P_i}\}$$

Where S_i is the initial value of the portfolio P_i for i in 1, 2. Let the portfolio weights, w,

$$w_1 = \frac{S_1}{S_1+S_2}$$
$$w_2 = \frac{S_2}{S_1+S_2}$$

Such that,

$$\mu_P = w_1\mu_1 + w_2\mu_2$$
$$\sigma_p^2 = w_1\sigma_1^2 + w_2\sigma_1^2 + 2w_1w_2\sigma_1\sigma_2\rho$$

Where ρ < 1, w_1 > 0 and w_2 > 0.

Let $\rho_P < w_1\sigma 1 + w_2\sigma_2$

When $\alpha < 0.5$, then $\Phi^{-1}(\alpha) < 0$, so that

$$\Phi^{-1}(\alpha)\sigma_p > w_1\Phi^{-1}(\alpha)\sigma_1 + w_2\Phi^{-1}(\alpha)\sigma_2$$

Adding portfolio means from above,

$$\mu_p + \Phi^{-1}(\alpha)\sigma_p > w_1\mu_1 + w_1\Phi^{-1}(\alpha)\sigma_1 + w_2\mu_2 + w_2\Phi^{-1}(\alpha)\sigma_2$$
$$\ldots = w_1(\mu_1 + \Phi^{-1}(\alpha)\sigma_1) + w_2(\mu_2 + \Phi^{-1}(\alpha)\sigma_2)$$

Multiplying both sides by -S,

$$-S\{\mu_p + \Phi^{-1}(\alpha)\sigma_p\} < -Sw_1(\mu_1 + \Phi^{-1}(\alpha)\sigma_1) - Sw_2(\mu_2 + \Phi^{-1}(\alpha)\sigma_2)$$
$$\ldots = -S_1(\mu_1 + \Phi^{-1}(\alpha)\sigma_1) - S_2(\mu_2 + \Phi^{-1}(\alpha)\sigma_2)$$

Again using **19.12**,

$$VaR_p^{norm}(\alpha) < VaR_{p_1}^{norm}(\alpha) + VaR_{p_2}^{norm}(\alpha)$$

Problem 2