Problem 1

Suppose the risk measure $\Re$ is $VaR(\alpha)$ for some α.

Let P_1 and P_2 be two portfolios whos returns have a joint normal distribution with means μ_1 and μ_2, standard deviations σ_1 and σ_2, and correlation ρ.

Suppose the initial investments are S_1 and S_2.

Show that $\Re(P_1 + P_2) \leq \Re(P_1) + \Re(P_2)$ under joint normality.

Solution:

From **19.12**, we know that:

$$\widehat{VaR}_{P_i}^{norm}(\alpha) = -S_i \ * \{\hat{\mu}_{P_i} + \phi^{-1}(\alpha)\hat{\sigma}_{P_i}\}$$
$$\ldots = -S_i \ * \{\hat{\mu}_{P_i} + Z_\alpha \hat{\sigma}_{P_i}\}$$

Where S_i is the initial value of the portfolio P_i for i in 1, 2. Let the portfolio weights, w_i,

$w_1 = \frac{S_1}{S_1+S_2}$, $w_2 = 1 - w_1$, so that $w_1 + w_2 = 1$.

From **16.1** and **16.3**, we know that,

$$\mathbb{E}(R_p) = \mu_P = w_1\mu_1 + w_2\mu_2$$
$$\sigma_p^2 = w_1\sigma_1^2 + w_2\sigma_1^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

Where $\rho \leq 1$, w_1 > 0 and w_2 > 0.

Let $\rho_{P_{12}} \leq w_1\sigma_1 + w_2\sigma_2$ and $\alpha < 0.5$*,

then $Z_\alpha < 0$, so that:

$$Z_\alpha\sigma_p \geq w_1 Z_\alpha \sigma_1 + w_2 Z_\alpha \sigma_2$$

Adding portfolio means from above,

$$\mu_p + Z_\alpha\sigma_p \geq w_1\mu_1 + w_1 Z_\alpha\sigma_1 + w_2\mu_2 + w_2 Z_\alpha\sigma_2$$
$$\ldots = w_1(\mu_1 + Z_\alpha\sigma_1) + w_2(\mu_2 + Z_\alpha\sigma_2)$$

Multiplying both sides by -S,

$$-S\{\mu_p + Z_\alpha\sigma_p\} \leq -Sw_1(\mu_1 + Z_\alpha\sigma_1) - Sw_2(\mu_2 + Z_\alpha\sigma_2)$$
$$\ldots = -S_1(\mu_1 + Z_\alpha\sigma_1) - S_2(\mu_2 + Z_\alpha\sigma_2)$$

Again using **19.12**,

$$VaR_p^{norm}(\alpha) \leq VaR_{p_1}^{norm}(\alpha) + VaR_{p_2}^{norm}(\alpha)$$

*Since we are interested in tail events.

Problem 2

```
data <- read.csv(paste0(data.dir, "Stock_Bond.csv"),
                 header = T)

prices <- as.matrix(data[1:500, c(3, 5, 7, 9, 11)])

n <- nrow(prices)

returns <- apply(prices, 2, FUN = function( p ){
   p[-1] / p[-n] - 1
})
```

- What are the sample mean vector and sample covariance matrix of the 499 returns on these stocks?

```
stock.means <- colMeans(returns)
stock.cov <- cov(returns)
```

Table 1: Stock Mean Returns

	mu
GM_AC	0.00098
F_AC	0.00156
UTX_AC	0.00005
CAT_AC	0.00123
MRK_AC	0.00086

Table 2: Covariance Matrix

	GM_AC	F_AC	UTX_AC	CAT_AC	MRK_AC
GM_AC	0.00041	0.00029	0.00019	0.00023	0.00019
F_AC	0.00029	0.00045	0.00020	0.00028	0.00022
UTX_AC	0.00019	0.00020	0.00037	0.00022	0.00017
CAT_AC	0.00023	0.00028	0.00022	0.00050	0.00023
MRK_AC	0.00019	0.00022	0.00017	0.00023	0.00031

- How many shares of each stock should one buy to invest $50 million in an equally weighted portfolio?

```
capital <- 5e7

n.stocks <- ncol(prices)
n.shares <- as.data.frame((capital/n.stocks)/prices[500,])

pretty_kable(data.table(Stock = rownames(n.shares),
                        Shares = comma(n.shares[,1])), "Equal Weighted Shares")
```

Table 3: Equal Weighted Shares

Stock	Shares
GM_AC	594,530.32
F_AC	2,232,142.86
UTX_AC	2,923,976.61
CAT_AC	1,821,493.62
MRK_AC	1,754,385.96

```
VaR_N = function( alpha, mu, v, S ) {
   - S * ( mu + qnorm(alpha) * sqrt(v)  )
}

stock.weight <- 1/n.stocks
port.weights <- rep( stock.weight, n.stocks )

port.ret <- sum(stock.weight * stock.means)
port.var <- t(port.weights) %*% stock.cov %*% port.weights
```

- What is the one-day VaR(0.1)* for this equally weighted portfolio?

```
var1d <- VaR_N(0.1, port.ret, port.var, capital)
```

One-day *VaR(.01)* = 984,181.54

- What is the five-day VaR(0.1)* for this portfolio?

```
port.ret <- 5 * sum( stock.weight * stock.means )
port.var <- 5 * sum( stock.weight**2 * diag(stock.cov) )

var5d <- VaR_N(0.1, port.ret, port.var, capital)
```

Five-day *VaR(.01)* = 1,062,887.10

* VaR estimates assume normality of returns.