Problem Set 4

Econ 211C

Question 1 .... *50 points*

Hamilton, problem 13.2.

Solution: For any $ARMA(p,q)$ process, let $r = \max(p, q+1)$. [13.1.22] and [13.1.23] in Hamilton tell us that the state equation and the observation equation are respectively,

$$\begin{aligned}
\boldsymbol{\xi}_{t+1} &= \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{r-1} & \phi_r \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \boldsymbol{\xi}_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (1) \\
y_t &= \mu + \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_{r-1} \end{bmatrix} \boldsymbol{\xi}_t, \quad (2)
\end{aligned}$$

where

$$\boldsymbol{\xi}_t = \begin{bmatrix} (1-\phi_1 L - \phi_2 L^2 - \cdots - \phi_r L^r)^{-1} \varepsilon_t \\ (1-\phi_1 L - \phi_2 L^2 - \cdots - \phi_r L^r)^{-1} \varepsilon_{t-1} \\ \vdots \\ (1-\phi_1 L - \phi_2 L^2 - \cdots - \phi_r L^r)^{-1} \varepsilon_{t-r+1} \end{bmatrix}.$$

If y_t is an $AR(p)$ process, then we have $r = \max(p, q+1) = p$ and $\theta_1 = \theta_2 = \cdots = \theta_{p-1} = 0$. In general, an $AR(p)$ process can be expressed as

$$\left(1-\phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p\right)(y_t - \mu) = \varepsilon_t, \quad \forall t,$$

which implies

$$\left(1-\phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p\right)^{-1} \varepsilon_t = y_t - \mu, \quad \forall t.$$

Substituting this last expression into $\boldsymbol{\xi}_t$ in Equations (1) and (2) gives us

$$\begin{aligned}
\begin{bmatrix} y_{t+1}-\mu \\ y_t - \mu \\ \vdots \\ y_{t-p+2}-\mu \end{bmatrix} &= \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t-\mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t-p+1}-\mu \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\
y_t &= \mu + \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_t-\mu \\ y_{t-1} - \mu \\ \vdots \\ y_{t-p+1}-\mu \end{bmatrix},
\end{aligned}$$

which are [13.1.14] and [13.1.15], respectively. If y_t is an $MA(1)$ process, we have $r = \max(p, q+1) = q+1 = 2$ and $\phi_1 = \phi_2 = 0$. Now,

$$\left(1 - \phi_1 L - \phi_2 L^2\right)^{-1} \varepsilon_t = \left(1 - 0 \times L - 0 \times L^2\right)^{-1} \varepsilon_t = \varepsilon_t, \quad \forall t.$$

Substituting this last expression into $\boldsymbol{\xi}_t$ in Equations (1) and (2) gives us

$$\begin{aligned} \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_t \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}, \\ y_t &= \mu + \begin{bmatrix} 1 & \theta \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{bmatrix}, \end{aligned}$$

which are [13.1.17] and [13.1.18], respectively.

Question 2 .... *50 points*

Hamilton, problem 13.4.

Solution:

Given the model specified in [13.4.3] and [13.4.4], it's clear that $\boldsymbol{F} = \mathbf{0}$. As a result

$$\boldsymbol{K}_t \equiv \boldsymbol{F}\boldsymbol{P}_{t|t-1}\boldsymbol{H}\left(\boldsymbol{H}'\boldsymbol{P}_{t|t-1}\boldsymbol{H} + \boldsymbol{R}\right)^{-1} = \mathbf{0}.$$

The initial conditions of the Kalman recursions are

$$\begin{aligned} \hat{\boldsymbol{\xi}}_{1|0} &= \mathrm{E}\left(\boldsymbol{\xi}_1\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \boldsymbol{P}_{1|0} &= \mathrm{E}\left[\left(\boldsymbol{\xi}_1 - \mathrm{E}\left(\boldsymbol{\xi}_1\right)\right)\left(\boldsymbol{\xi}_1 - \mathrm{E}\left(\boldsymbol{\xi}_1\right)\right)'\right] = \boldsymbol{Q} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \end{aligned}$$

and $\forall t \geq 1$,

$$\begin{aligned} \hat{\boldsymbol{\xi}}_{t+1|t} &= \boldsymbol{F}\hat{\boldsymbol{\xi}}_{t|t-1} + \boldsymbol{K}_t\left(\boldsymbol{y}_t - \boldsymbol{A}'\boldsymbol{x}_t - \boldsymbol{H}'\hat{\boldsymbol{\xi}}_{t|t-1}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \boldsymbol{P}_{t+1|t} &= \boldsymbol{F}\boldsymbol{P}_{t|t}\boldsymbol{F}' + \boldsymbol{Q} = \boldsymbol{Q} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}. \end{aligned}$$

And in turn we have

$$\begin{aligned}\left|\boldsymbol{H}'\boldsymbol{P}_{t|t-1}\boldsymbol{H}+\boldsymbol{R}\right| &= \left|\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}+\mathbf{0}\right| = \sigma_1^2+\sigma_2^2,\\ \boldsymbol{y}_t-\boldsymbol{A}'\boldsymbol{x}_t-\boldsymbol{H}'\hat{\boldsymbol{\xi}}_{t|t-1} &= y_t - 0 = y_t.\end{aligned}$$

Substituting into [13.4.1] gives us

$$\begin{aligned}& f_{\boldsymbol{Y}_t|\boldsymbol{X}_t,\boldsymbol{\mathcal{Y}}_{t-1}}\left(\boldsymbol{y}_t|\boldsymbol{x}_t,\boldsymbol{\mathcal{Y}}_{t-1}\right)\\ =\; & (2\pi)^{-n/2}\left|\boldsymbol{H}'\boldsymbol{P}_{t|t-1}\boldsymbol{H}+\boldsymbol{R}\right|^{-1/2}\\ & \times\exp\left\{-\frac{1}{2}\left(\boldsymbol{y}_t-\boldsymbol{A}'\boldsymbol{x}_t-\boldsymbol{H}'\hat{\boldsymbol{\xi}}_{t|t-1}\right)'\left(\boldsymbol{H}'\boldsymbol{P}_{t|t-1}\boldsymbol{H}+\boldsymbol{R}\right)^{-1}\left(\boldsymbol{y}_t-\boldsymbol{A}'\boldsymbol{x}_t-\boldsymbol{H}'\hat{\boldsymbol{\xi}}_{t|t-1}\right)\right\}\\ =\; & (2\pi)^{-1/2}\left(\sigma_1^2+\sigma_2^2\right)^{-1/2}\exp\left[-\frac{y_t^2}{2\left(\sigma_1^2+\sigma_2^2\right)}\right].\end{aligned}$$

Hence, the sample log-likelihood [13.4.4] becomes

$$\begin{aligned}\sum_{t=1}^{T}\log f_{\boldsymbol{Y}_t|\boldsymbol{X}_t,\boldsymbol{\mathcal{Y}}_{t-1}}\left(\boldsymbol{y}_t|\boldsymbol{x}_t,\boldsymbol{\mathcal{Y}}_{t-1}\right) &= \sum_{t=1}^{T}\log\left\{(2\pi)^{-1/2}\left(\sigma_1^2+\sigma_2^2\right)^{-1/2}\exp\left[-\frac{y_t^2}{2\left(\sigma_1^2+\sigma_2^2\right)}\right]\right\}\\ &= -\frac{T}{2}\log\left(2\pi\right)-\frac{T}{2}\log\left(\sigma_1^2+\sigma_2^2\right)-\frac{\sum_{t=1}^{T}y_t^2}{2\left(\sigma_1^2+\sigma_2^2\right)},\end{aligned}$$

which is exactly [13.4.5].