Problem Set 4

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Question 1 (50 points)

Suppose the current value of an asset is S_0 and that the continuously-compounded annual risk-free interest rate is r. You are interested in valuing a put option on the asset with strike price X. Recall the Black-Scholes-Merton option pricing formula for a put:

$$P = Xe^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$

$$d_1 = \frac{\log(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\log(\frac{S_0}{X}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

where σ is the volatility of price increments and T is the time to expiry.

a. (20 points)

Consider the case where $\sigma \to 0$. What is the value of the put when $S_0 > Xe^{-rT}$. Prove your answer mathematically.

$$S_0 > Xe^{-rT}$$

$$\log(S_0) > \log(Xe^{-rT})$$

$$\log(S_0) > \log(X) + \log(e^{-rT})$$

$$\log(S_0) > \log(X) - rT\log(e)$$

$$\log(S_0) > \log(X) - rT$$

$$\log(S_0) - \log(X) + rT > 0$$

$$\log(\frac{S_0}{X}) + rT > 0$$

$$\lim_{\sigma\to 0} d_1 = \lim_{\sigma\to 0} \frac{\log(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$\lim_{\sigma\to 0} d_1 = \infty$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$d_2 = \infty - \sigma\sqrt{T}$$

$$d_2 = \infty$$

$$P = Xe^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$

$$P = Xe^{-rT}\Phi(-\infty) - S_0\Phi(-\infty)$$

$$P = Xe^{-rT}(0) - S_0(0)$$

$$P = 0$$

b. (20 points)

Consider the case where $\sigma \to 0$. What is the value of the put when $S_0 < Xe^{-rT}$. Prove your answer mathematically.

$$S_0 < Xe^{-rT}$$
$$\log(S_0) < \log(Xe^{-rT})$$
$$\log(S_0) < \log(X) + \log(e^{-rT})$$
$$\log(S_0) < \log(X) - rT\log(e)$$
$$\log(S_0) < \log(X) - rT$$
$$\log(S_0) - \log(X) + rT < 0$$
$$\log(\frac{S_0}{X}) + rT < 0$$
$$\lim_{\sigma \to 0} d_1 = \lim_{\sigma \to 0} \frac{\log(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$
$$\lim_{\sigma \to 0} d_1 = -\infty$$
$$d_2 = d_1 - \sigma\sqrt{T}$$
$$d_2 = -\infty - \sigma\sqrt{T}$$
$$d_2 = -\infty$$
$$P = Xe^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$
$$P = Xe^{-rT}\Phi(-(-\infty)) - S_0\Phi(-(-\infty))$$
$$P = Xe^{-rT}(1) - S_0(1)$$
$$P = Xe^{-rT} - S_0$$

c. (10 points)

Interpret your solutions to parts (a) and (b).

In part (a), $S_0 > Xe^{-rT}$, meaning the underlying asset price is greater than the continuously-compounded strike price. This means that the option to sell the put option at the strike price is worthless. For example, if the market price is greater than the strike price, you wouldnt utilize the put option since you could make more if the market price is higher. This is called *out of the money.*

In part (b), $S_0 < Xe^{-rT}$, meaning the underlying asset price is less than the continuously-compounded strike price. This means that the option to sell the put option at the strike price is worth more than the market price. The seller can make more money by excercising the option than by taking the market price. This is called *in the money.*

Question 2 (50 points)

Suppose that a stock price S follows geometric Brownian motion with expected return μ and volatility σ. What is the process followed by the variable $t^2 S^3$?

$$G = t^2 S^3$$
$$\frac{G}{t^2} = S^3$$
$$dG = (\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial G}{\partial S}\sigma S dZ$$
$$\frac{\partial G}{\partial S} = 3t^2 S^2$$
$$\frac{\partial^2 G}{\partial S^2} = 6t^2 S$$
$$\frac{\partial G}{\partial t} = 2tS^3$$
$$dG = (3t^2 S^2 \mu S + 2tS^3 + \frac{1}{2}6t^2 S\sigma^2 S^2)dt + 3t^2 S^2 \sigma S dzZ$$
$$dG = (3t^2 S^3 \mu + 2tS^3 + 3t^2 S^3 \sigma^2)dt + 3t^2 S^3 \sigma S dZ$$
$$dG = \{3G(\mu + \sigma^2) + 2tS^3\}dt + 3G\sigma dZ$$
$$dG = \{3G(\mu + \sigma^2) + \frac{2G}{t}\}dt + 3G\sigma dZ$$