SENSITIVITY ANALYSES FOR MEDIATION

We omit the subscript i in this lesson to ease the notational burden. Denote the observed confounders, $\mathbf{X}$, an unobserved confounder, U, treatment assignment, $Z \in \{0, 1\}$, and outcome, Y. The goal is to estimate the average treatment effect at the value $\mathbf{X}$, denoted ATE($\mathbf{X}$).

Treatment assignment is assumed unconfounded given $(\mathbf{X}, U)$. However, the investigator incorrectly assumes unconfoundedness, given $\mathbf{X}$.

The bias from ignoring U is

$$\mathrm{bias}(\mathbf{X}) = [E(Y \mid \mathbf{X}, Z = 1) - E(Y \mid \mathbf{X}, Z = 0)] - [E(Y(1) \mid \mathbf{X}) - E(Y(0) \mid \mathbf{X})].$$

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Suppose the unobserved confounder, U, is continous. The bias at $\mathbf{X} = \mathbf{x}$ can be written as:

$$\int_{-\infty}^{\infty} E(Y \mid \mathbf{X}, Z = 1, U = u)[f(u \mid \mathbf{X}, Z = 1) - f(u \mid \mathbf{X})]du -$$
$$\int_{-\infty}^{\infty} E(Y \mid \mathbf{X}, Z = 0, U = u)[f(u \mid \mathbf{X}, Z = 0) - f(u \mid \mathbf{X})]du.$$

Conditional bias, bias($\mathbf{X}$), depends on two unknown conditional expectations and two unknown conditional distributions of U.

Integrating bias($\mathbf{X}$) over the distribution of $\mathbf{X}$ gives the unconditional bias.

SENSITIVITY ANALYSES FOR MEDIATION

Sensitivity analysis hinges on whether an investigator can make reasonable assumptions about these unknown expectations/distributions.

e.g. VanderWeele and Arah (2011) assume

$$\begin{aligned} &E(Y \mid \mathbf{X}, Z=1, U=u) - E(Y \mid \mathbf{X}, Z=1, U=u') \\ &\quad = E(Y \mid \mathbf{X}, Z=0, U=u) - E(Y \mid \mathbf{X}, Z=0, U=u'). \end{aligned}$$

When U is binary, a little arithmetic gives

$$\begin{aligned} \mathit{bias}(\mathbf{X}) = {} & [E(Y \mid \mathbf{X}, Z=z, U=1) - E(Y \mid \mathbf{X}, Z=z, U=0)] \times \\ & [\Pr(U=1 \mid \mathbf{X}, Z=1) - \Pr(U=1 \mid \mathbf{X}, Z=0)]. \end{aligned}$$

Sensitivity analysis proceeds under varying assumptions about the magnitude of these two components. Similar results can be obtained for risk ratios.

n.b. the assumption

$$
\begin{aligned}
&E(Y \mid \mathbf{X}, Z = 1, U = u) - E(Y \mid \mathbf{X}, Z = 1, U = u') \\
&\quad = E(Y \mid \mathbf{X}, Z = 0, U = u) - E(Y \mid \mathbf{X}, Z = 0, U = u').
\end{aligned}
$$

is strong in practice.

Rearranging and using the unconfoundedness assumption gives

$$E(Y(1) - Y(0) \mid \mathbf{X}, U = u) = E(Y(1) - Y(0) \mid \mathbf{X}, U = u').$$

A sufficient condition is $Y(1) - Y(0) \perp\!\!\!\perp U \mid \mathbf{X}$. i.e. the difference in potential outcomes does not depend on the unobserved confounder given the covariates.

This is more credible than assuming the potential outcomes are independent of U given the observed confounders. However, an investigator who is unwilling to make this assumption is likely unwilling to make the weaker assumption.

SENSITIVITY ANALYSES FOR MEDIATION

Turning to mediation, suppose the assumptions we made to identify these effects hold with U added to the conditioning set.

$E(Y \mid Z = z, M = m, \mathbf{X} = \mathbf{x}, U = u) = E(Y(z, m) \mid \mathbf{X} = \mathbf{x}, U = u)$ and the bias for the CDE $E(Y(z, m) - Y(z^*, m^*) \mid \mathbf{X})$ can be written

$$\int_{-\infty}^{\infty} E(Y \mid \mathbf{X}, Z = z, M = m, U = u) \times [f(u \mid \mathbf{X}, Z = z, M = m) - f(u \mid \mathbf{X}, M = m)] du$$
$$- \int_{-\infty}^{\infty} E(Y \mid \mathbf{X}, Z = z^*, M = m^*, U = u) \times [f(u \mid \mathbf{X}, Z = z^*, M = m^*) - f(u \mid \mathbf{X}, M = m^*)] du.$$

Integrating over the distribution of $\mathbf{X}$ gives the unconditional bias.

VanderWeele (2015) considers controlled direct effects $E(Y(1, m) - Y(0, m)$.

Assume

$$\begin{aligned} &E(Y \mid \mathbf{X}, Z = 1, M = m, U = u) - E(Y \mid \mathbf{X}, Z = 1, M = m, U = u') \\ &\quad = E(Y \mid \mathbf{X}, Z = 0, M = m, U = u) - E(Y \mid \mathbf{X}, Z = 0, M = m, U = u'). \end{aligned}$$

In the special where U is binary, a little arithmetic gives bias($\mathbf{X}$):

$$\begin{aligned} &[E(Y \mid \mathbf{X}, Z = 1, M = m, U = 1) - E(Y \mid \mathbf{X}, Z = 1, M = m, U = 1)] \\ &\quad \times [\Pr(U = 1 \mid \mathbf{X}, Z = 1, M = m) - \Pr(U = 1 \mid \mathbf{X}, Z = 0, M = m)] \end{aligned}$$

SENSITIVITY ANALYSES FOR MEDIATION

Sensitivity analysis proceeds under varying assumptions about the magnitude of the two components of

$$
\begin{aligned}
bias(\mathbf{X}) = [E(Y \mid \mathbf{X}, Z = z, U = 1) - E(Y \mid \mathbf{X}, Z = z, U = 0)][\Pr(U = 1 \mid \mathbf{X}, Z = 1) \\
- \Pr(U = 1 \mid \mathbf{X}, Z = 0)].
\end{aligned}
$$

The condition

$$
\begin{aligned}
&E(Y \mid \mathbf{X}, Z = 1, M = m, U = u) - E(Y \mid \mathbf{X}, Z = 1, M = m, U = u') \\
&\quad = E(Y \mid \mathbf{X}, Z = 0, M = m, U = u) - E(Y \mid \mathbf{X}, Z = 0, M = m, U = u'),
\end{aligned}
$$

can be rewritten as:

$$
\begin{aligned}
&E(Y(1, m) \mid \mathbf{X}, U = u) - E(Y(1, m) \mid \mathbf{X}, Z = 1, M = m, U = u') \\
&\quad = E(Y(0, m) \mid \mathbf{X}, U = u') - E(Y(0, m) \mid \mathbf{X}, U = u').
\end{aligned}
$$

As before this condition may be too strong in practice.

SENSITIVITY ANALYSES FOR MEDIATION

The bias due to ignoring U can also be obtained for average direct and indirect effects.

Readers who are interested in this should consult the book by VanderWeele (2015).

For the case of linear models, Imai et al. (2010) provide nice results.

SENSITIVITY ANALYSES FOR MEDIATION

In a randomized study where U is not a confounder for the relationship between outcomes and treatment assignment Z, the confounder U is needed to make the relationship between the potential outcomes $Y(z, m)$ and the mediator M conditionally unconfounded.

U is either a pre-treatment covariate or a post-treatment variable not affected by Z. The more likely case of confounding by post-treatment variables affected by Z is not covered.