# AMS 206 - Winter 2017
# Quiz 1 - Solution

1. **[30 Points]** Show that the Beta family is a conjugate family to a negative binomial likelihood with unknown success probability and known number of successes, i.e., if $x \mid \theta \sim \mathsf{NegBin}(k, \theta)$ so that the probability mass function is

$$p(x \mid \theta) = \binom{x-1}{k-1} \theta^k (1-\theta)^{x-k} \qquad x = k, k+1, k+2, \dots$$

and $\theta \sim \mathsf{Beta}(a, b)$ with density

$$p(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1} \qquad \theta \in [0, 1]$$

then $\theta \mid x \sim \mathsf{Beta}(a^*, b^*)$ for a pair of parameters $a^*, b^* > 0$. Find an expression for $a^*$ and $b^*$.

**Solution:** From Bayes theorem

$$\begin{aligned} p(\theta \mid x) &\propto \binom{x-1}{k-1} \theta^k (1-\theta)^{x-k} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1} \\ &\propto \theta^{k+a-1} (1-\theta)^{x-k+b-1} \qquad \theta \in [0, 1] \end{aligned}$$

which is proportional to beta kernel with parameters $a^* = a + k$ and $b^* = x - k + b$. Hence,

$$p(\theta \mid x) = \frac{\Gamma(a+b+x)}{\Gamma(a+k)\Gamma(b+x-k)} \theta^{a+k-1} (1-\theta)^{x-k+b-1} \qquad \theta \in [0, 1]$$

2. **[70 points]** Let $x_1, \dots, x_n$ be an independent sample with $x_i \mid \theta, s_i \sim \mathsf{Uni}[0, s_i\theta]$, where the values $s_1, \dots, s_n$ are known covariates. If $\theta$ is assigned a Pareto prior with density

$$p(\theta) = \begin{cases} \frac{ab^a}{\theta^{a+1}} & \theta > b \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Find the posterior distribution for $\theta$. *Hint: the prior is conjugate!*

   **Solution:** Note that

$$\begin{aligned} p(x_1, \dots, x_n \mid \theta) &= \begin{cases} \left\{\prod_{i=1}^n \frac{1}{s_i}\right\} \frac{1}{\theta^n} & x_1 < s_1\theta, \dots, x_n < s_n\theta \\ 0 & \text{otherwise} \end{cases} \\ &\propto \begin{cases} \frac{1}{\theta^n} & \theta > \max_{1 \le i \le n} \left\{\frac{x_i}{s_i}\right\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Hence,

$$
\begin{aligned}
p(\theta \mid x_1, \ldots, x_n) &\propto \begin{cases} \frac{1}{\theta^n} \frac{1}{\theta^{a+1}} & \theta > \max_{1 \le i \le n} \left\{ \frac{x_i}{s_i} \right\}, \theta > b \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{\theta^{n+a+1}} & \theta > \max \left\{ b, \max_{1 \le i \le n} \left\{ \frac{x_i}{s_i} \right\} \right\} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

.

This corresponds to the kernel of a Pareto distribution with parameters $a^* = a + n$ and $b^* = \max \left\{ b, \max_{1 \le i \le n} \left\{ \frac{x_i}{s_i} \right\} \right\}$. Hence

$$
p(\theta \mid x_1, \ldots, x_n) = \begin{cases} \frac{a^* (b^*)^{a^*}}{\theta^{a^*+1}} & \theta > b^* \\ 0 & \text{otherwise} \end{cases}.
$$

(b) Find a closed-form expressions for the posterior mean and $\alpha\%$ HPD credible intervals for $\theta$.

**Solution:** The posterior expectation is

$$
\begin{aligned}
\mathsf{E}\{\theta \mid x_1, \ldots, x_n\} &= \int_{b^*}^{\infty} \theta \frac{a^* (b^*)^{a^*}}{\theta^{a^*+1}} \mathrm{d}\theta \\
&= a^* (b^*)^{a^*} \frac{1}{a^* - 1} \left[ -\frac{1}{\theta^{a^*-1}} \right]_{b^*}^{\infty} \\
&= \frac{a^*}{a^* - 1} (b^*)^{a^*} \frac{1}{(b^*)^{a^*-1}} = \frac{a^*}{a^* - 1} b^*
\end{aligned}
$$

For the posterior HPD, note that the Pareto has a decreasing density function. Therefore, the interval is of the form $(b^*, u)$ where $u$ satisfies

$$
\int_{b^*}^{u} \frac{a^* (b^*)^{a^*}}{\theta^{a^*+1}} \mathrm{d}\theta = \alpha
$$

Solving the integral we have

$$
\begin{aligned}
\int_{b^*}^{u} \frac{a^* (b^*)^{a^*}}{\theta^{a^*+1}} \mathrm{d}\theta &= (b^*)^{a^*} \left[ -\frac{1}{\theta^{a^*}} \right]_{b^*}^{u} \\
&= (b^*)^{a^*} \left[ \frac{1}{(b^*)^{a^*}} - \frac{1}{u^{a^*}} \right] = 1 - \left( \frac{b}{u} \right)^{a^*}
\end{aligned}
$$

Hence

$$
1 - \left( \frac{b}{u} \right)^{a^*} = \alpha \quad \Rightarrow \quad u = \frac{b}{(1 - \alpha)^{1/a^*}}.
$$

(c) Find the predictive distribution for a new observation $x^*$ with associated covariate $s^*$,

$$p(x^* \mid s^*, x_1, \ldots, x_n, s_1, \ldots, s_n, a, b)$$

**Solution:** Note that

$$p(x^* \mid s^*, x_1, \ldots, x_n, s_1, \ldots, s_n, a, b)$$
$$\int p(x^* \mid \theta, s^*, x_1, \ldots, x_n, s_1, \ldots, s_n, a, b) p(\theta \mid s^*, x_1, \ldots, x_n, s_1, \ldots, s_n, a, b) \mathrm{d}\theta$$

In this setting

$$p(x^* \mid \theta, s^*, x_1, \ldots, x_n, s_1, \ldots, s_n, a, b) = p(x^* \mid \theta, s^*)$$

and

$$p(\theta \mid s^*, x_1, \ldots, x_n, s_1, \ldots, s_n, a, b) = p(\theta \mid x_1, \ldots, x_n, s_1, \ldots, s_n, a, b)$$

so

$$\begin{aligned}
p(x^* \mid s^*, x_1, \ldots, x_n, s_1, \ldots, s_n, a, b) &= \int p(x^* \mid \theta, s^*) p(\theta \mid x_1, \ldots, x_n, s_1, \ldots, s_n, a, b) \mathrm{d}\theta \\
&= \int_{\max\{b^*, x^*/s^*\}}^{\infty} \frac{1}{s^* \theta} \frac{a^* \, (b^*)^{a^*}}{\theta^{a^*+1}} \mathrm{d}\theta \\
&= \frac{a^* \, (b^*)^{a^*}}{s^*} \int_{\max\{b^*, x^*/s^*\}}^{\infty} \frac{\mathrm{d}\theta}{\theta^{a^*+2}} \\
&= \frac{a^* \, (b^*)^{a^*}}{s^*(a^* + 1)} \left[ -\frac{1}{\theta^{a^*+1}} \right]_{\max\{b^*, x^*/s^*\}}^{\infty} \\
&= \frac{1}{s^*} \frac{a^*}{(a^* + 1)} \frac{(b^*)^{a^*}}{(\max\{b^*, x^*/s^*\})^{a^*+1}} \qquad 0 \leq x^* < \infty
\end{aligned}$$