STRUCTURAL NESTED MEAN MODELS AND G-ESTIMATION

With the g-formula and IPTW, we estimated effects such as $E(Y_t(\bar{z}_t)) - E(Y_t(\bar{z}_t^*))$ or $E(Y_t(\bar{z}_t) - Y_t(\bar{z}_t^*) \mid \mathbf{X}_1)$.

G-estimation is used to estimate treatment effects conditional on past treatment and confounder history.

e.g. for $t = 1, ..., T$:

$$E(Y_{t+k}(\bar{z}_t, \mathbf{0}_k) - Y_{t+k}(\bar{z}_{t-1}, \mathbf{0}_{k+1}) \mid \bar{\mathbf{X}}_t = \bar{\mathbf{x}}_t, \bar{Z}_t = \bar{z}_t) = \gamma_{tk}(\bar{x}_t, \bar{z}_t; \boldsymbol{\psi}^*)$$

for $k = 0, ..., T - t$, where $\mathbf{0_k}$ is used to denote a sequence of k periods in which treatment is withheld, and $\gamma_{tk}(\bar{x}_t, \bar{z}_t; \boldsymbol{\psi}^*)$ is a known function of $\bar{x}_t$ and $\bar{z}_t$ depending on parameters $\boldsymbol{\psi}$ with true value $\boldsymbol{\psi}^*$.

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In some instances the outcome is only measured in period T or interest resides only in the effects at time T.

We focus on structural nested mean models (SNMM) with effects defined on the difference scale. However, g-estimation can be used to estimate treatment effects on other scales e.g. odds-ratios.

In addition, other kinds of structural models, e.g. for distributions, can be defined and estimated using g-estimation. See Vansteelandt and Joffe (2014).

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Each estimand in

$$E(Y_{t+k}(\bar{z}_t, \mathbf{0}_k) - Y_{t+k}(\bar{z}_{t-1}, \mathbf{0}_{k+1}) \mid \bar{\mathbf{X}}_t = \bar{\mathbf{x}}_t, \bar{Z}_t = \bar{z}_t) = \gamma_{tk}(\bar{x}_t, \bar{z}_t; \psi^*)$$

is the effect in period $t + k$ of treatment in period t followed by no further treatment versus no treatment in period t, also followed by no further treatment, for units with covariates $\bar{\mathbf{x}}_t$ that actually followed the sub-regimen $\bar{z}_t$ through period t.

For example, if $T = 2$, for $t = 1$, the effects are
$E(Y_1(1,0) - Y_1(0,0) \mid Z_1 = 1, \mathbf{X}_1 = \mathbf{x}_1) = \gamma_{10}(\mathbf{x}_1, z_1; \psi^*)$ and
$\mathrm{E}(Y_2(1,0) - Y_2(0,0) \mid Z_1 = 1, \mathbf{X}_1 = \mathbf{x}_1) = \gamma_{11}(\mathbf{x}_1, z_1; \psi^*)$, and, for $t = 2$,
$\mathrm{E}(Y_2(0,1) - Y_2(0,0) \mid Z_1 = 0, Z_2 = 1, \bar{\mathbf{X}}_2 = \bar{\mathbf{x}}_2) = \gamma_{20}(\bar{\mathbf{x}}_2, \bar{z}_2; \psi^*)$ and
$E(Y_2(1,1) - Y_2(0,0) \mid Z_1 = 1, Z_2 = 1, \bar{\mathbf{X}}_2 = \bar{\mathbf{x}}_2) = \gamma_{20}(\bar{\mathbf{x}}_2, \bar{z}_2; \psi^*)$.

STRUCTURAL NESTED MEAN MODELS AND G-ESTIMATION

SNMM's are "structural" because they model potential outcomes. They are "nested" because in each period, effects are conditioned on the history prior to current treatment.

Vansteelandt and Joffe (2014) describe various models, including structural distribution models, structural models for survival data, etc.

We follow their exposition but consider the most elementary issues.

STRUCTURAL NESTED MEAN MODELS AND G-ESTIMATION

Consider the case $T = 1$, with $Z_1 = 1$ for treatment, $Z_1 = 0$ for the absence of treatment. A structural mean model (SMM) contrasts treatment with its absence among subjects with covariates $\mathbf{X}_1$:

$$E(Y_1(z_1) - Y_1(0) \mid \mathbf{X}_1 = \mathbf{x}_1, Z_1 = z_1) = \gamma^*(\mathbf{x}_1, z_1, \boldsymbol{\psi}^*),$$

where $\gamma^*(\mathbf{x}_1, z_1, \boldsymbol{\psi}^*)$ is a known function equal to 0 if $z_1 = 0$, and $\boldsymbol{\psi}$ is a vector of parameters with true value $\boldsymbol{\psi}^*$. e.g. $\gamma^*(\mathbf{x}_1, z_1, \boldsymbol{\psi}^*) = \psi^* z_1$ for the case where the effect does not depend on the covariates $\mathbf{X}_1$.

The SMM is a model for ATT$(\mathbf{X}_1)$, the effect of treatment on the treated for units with covariates $\mathbf{X}_1$.

Recall that to identify the ATT, it was only necessary to assume that $Y_1(0) \perp\!\!\!\perp Z_1 \mid \mathbf{X}_1$, as $E(Y_1 \mid \mathbf{X}_1 = \mathbf{x}_1, Z_1 = 1) = E(Y_1(1) \mid \mathbf{X}_1 = \mathbf{x}_1, Z_1 = z)$. This parameter can be estimated multiple ways. e.g. inverse probability treatment weighting, matching, etc. However, these procedures will not extend to the more general estimands.

STRUCTURAL NESTED MEAN MODELS AND G-ESTIMATION

To estimate $\boldsymbol{\psi}$ using g-estimation, a variable

$$U^*(\boldsymbol{\psi}) = Y - \gamma^*(\mathbf{X}_1, Z_1, \boldsymbol{\psi})$$

is defined. It follows immediately

$$E(U^*(\boldsymbol{\psi}^*) \mid \mathbf{X}_1 = \mathbf{x}_1, Z_1 = z_1)) = E(Y(0) \mid \mathbf{X}_1 = \mathbf{x}_1, Z_1 = z_1),$$

and using the unconfoundedness condition $Y(0) \perp\!\!\!\perp Z_1 \mid \mathbf{X}_1$ gives

$$E(U^*(\boldsymbol{\psi}^*) \mid \mathbf{X}_1 = \mathbf{x}_1, Z_1 = z_1) = E(U^*(\boldsymbol{\psi}^*) \mid \mathbf{X}_1 = \mathbf{x}_1).$$

STRUCTURAL NESTED MEAN MODELS AND G-ESTIMATION

$\boldsymbol{\psi}^*$ can be estimated using estimating equations. Assume a sample of size n,

$$\mathbf{0} = \sum_{i=1}^{n} [\mathbf{d}^*(Z_i, \mathbf{X}_{i1}) - E(\mathbf{d}^*(Z_i, \mathbf{X}_{i1}) \mid \mathbf{X}_{i1})][U_i^*(\boldsymbol{\psi}) - E(U_i^*(\boldsymbol{\psi}) \mid \mathbf{X}_{i1})],$$

where $\mathbf{d}^*$ is a vector of known functions with q components, where q is the number of parameters in $\boldsymbol{\psi}$.

e.g. for the case $\gamma^*(\boldsymbol{x}, z_1, \psi^*) = \psi^* z$, d is 1 dimensional, and one might take $d(Z_{i1}, \mathbf{X}_{i1}) = Z_{i1}$.

Those not familiar with estimating equations might wish to consult Bickel and Doksum (2015). To implement this approach, Vansteelandt and Joffe (2014) point out that it is necessary to model the propensity score $\Pr(Z_1 = 1 \mid \mathbf{X}_1)$, which is needed for $\mathrm{E}(\mathbf{d}(Z_1, \mathbf{X}_1) \mid \mathbf{X}_1)$, also $E(U^*(\boldsymbol{\psi}) \mid \mathbf{X}_1)$. They also discuss efficient estimation and double robustness.

STRUCTURAL NESTED MEAN MODELS AND G-ESTIMATION

To estimate the effects defined in

$$E(Y_{t+k}(\bar{z}_t, \mathbf{0}_k) - Y_{t+k}(\bar{z}_{t-1}, \mathbf{0}_{k+1}) \mid \bar{\mathbf{X}}_t = \bar{\mathbf{x}}_t, \bar{Z}_t = \bar{z}_t) = \gamma_{tk}(\bar{x}_t, \bar{z}_t; \boldsymbol{\psi}^*),$$

the idea is to extend the treatment above.

For $t = 1, ..., T$, define variables $U^*_{tk}(\psi)$ for $k = 0, ..., T - t$ such that $\mathrm{E}(U^*_{tk}(\psi^*) \mid \bar{\mathbf{X}}_t = \bar{\mathbf{x}}_t, \bar{Z}_{t-1} = \bar{z}_{t-1}, Z_t) = E(Y_{t+k}(\bar{z}_{t-1}, \mathbf{0}) \mid \bar{\mathbf{X}}_t = \bar{\mathbf{x}}_t, \bar{Z}_{t-1} = \bar{z}_{t-1}, Z_t)$; then, using the sequential randomization assumption, this reduces to $\mathrm{E}(U^*_{tk}(\psi^*) \mid \bar{\mathbf{X}}_t = \bar{\mathbf{x}}_t, \bar{Z}_{t-1} = \bar{z}_{t-1})$, which can then be used to construct estimating equations. See Vansteelandt and Joffe (2014).