INTERFERENCE CONTINUED

Let $i = 1, \ldots, N, \ldots, N_T$ enumerate a population of N_T units. Suppose the first $N \leq N_T$ are eligible to receive treatment ($\mathcal{E}$). The remaining $N_T - N$ units are ineligible to receive treatment ($\mathcal{I}$).

For sub-population $\mathcal{I}$, the treatment assignment is always $Z_i = 0$. However, the outcomes of these units may be affected by the assignments of eligible units. i.e. these units may experience spillover effects.

e.g. in the MTO study, researchers discussed the possibility of spillovers onto the existing residents of neighborhoods where recipients of housing vouchers might move. But the study did not collect data on any of these units.

INTERFERENCE CONTINUED

Let $\mathbf{z} = (z_1, ..., z_N, \mathbf{0}'_{N_T-N})'$ denote the "allocation" of units to treatment groups. Again, eligible units can receive treatment $z_i = 1$ or $z_i = 0$. $\mathbf{0}_{N_T-N}$ is a vector of 0's of size $N_T - N$ denoting the assignment of the $N_T - N$ ineligible units.

Let $Y_i(\mathbf{z}) \equiv Y_i(z_i, \mathbf{z}_{-i})$ denote i's potential outcome under the allocation $\mathbf{z}$. We decompose $\mathbf{z}$ into z_i, the assignment of unit i, and $\mathbf{z}_{-i}$, the assignment of all other units. Each unit has 2^N potential outcomes. The observed outcome is $Y_i(\mathbf{Z}) = \sum_{\mathbf{z}} Y_i(\mathbf{z})1(\mathbf{Z} = \mathbf{z})$.

Consider a particular allocation $\mathbf{Z}$ of the N units with N_1 treated units, N_0 control units. One can compute

(a) the mean outcome $\bar{Y}_I(\mathbf{Z}) = \frac{\sum_{\mathbf{i=N+1}}^{\mathbf{N_T}} \mathbf{Y_i(Z)}}{\mathbf{N_T - N}}$ for units ineligible for assignment,

(b) the mean outcome $\bar{Y}_1(\mathbf{Z}) = N_1^{-1} \sum_{i:Z_i=1} Y_i(\mathbf{Z})$ for the treatment units,

(c) the mean outcome $\bar{Y}_0(\mathbf{Z}) = N_0^{-1} \sum_{i:Z_i=0, i \in \mathcal{E}} Y_i(\mathbf{Z})$ for the control units.

INTERFERENCE CONTINUED

Suppose the observed allocation is a completely randomized experiment with N_1 treatment units and N_0 control units. Let $R(N_0, N_1)$ denote the set of all possible randomizations and $|R(N_0, N_1)|$ denote the cardinality of the set.

For $i = 1, ..., N$, let $E(Y_i(z_i, \mathbf{Z}_{-i}))$ denote i's expected outcome over the 2^{N-1} allocations in which i is assigned treatment z_i. For $i = N+1, ..., N_T$, $Z_i = 0$ for all allocations. The expectation is taken over all 2^N possible allocations.

Averaging these over the ineligibles gives the average potential outcome

$$E_{\mathcal{I}} E(Y_i(0, \mathbf{Z})) = \frac{\sum_{i=N+1}^{N_T} E(Y_i(0, \mathbf{Z}))}{N_T - N}.$$

Similarly, averaging these over the eligibles gives $\mathrm{E}_{\mathcal{E}} E(Y_i(0, \mathbf{Z}_{-i}))$ and $\mathrm{E}_{\mathcal{E}} E(Y_i(1, \mathbf{Z}_{-i}))$.

INTERFERENCE CONTINUED

Taking expectations over the randomization set (Sobel, 2006), shows

(a) $\bar{Y}_I(\mathbf{Z}) = \frac{\sum_{\mathbf{i=N+1}}^{\mathbf{N_T}} \mathbf{Y_i(Z)}}{\mathbf{N_T - N}}$

(b) $\bar{Y}_1(\mathbf{Z}) = N_1^{-1} \sum_{i:Z_i=1} Y_i(\mathbf{Z})$

(c) the mean outcome $\bar{Y}_0(\mathbf{Z}) = N_0^{-1} \sum_{i:Z_i=0, i \in \mathcal{E}} Y_i(\mathbf{Z})$

are unbiased for

(a) $E_{\mathcal{I}} E(Y_i(0, \mathbf{Z}))$

(b) $\mathrm{E}_{\mathcal{E}} E(Y_i(0, \mathbf{Z}_{-i}))$

(c) $\mathrm{E}_{\mathcal{E}} E(Y_i(1, \mathbf{Z}_{-i}))$

Thus, $\bar{Y}_1(\mathbf{Z}) - \bar{Y}_0(\mathbf{Z})$ is unbiased for

$$E_{\mathcal{E}} E(Y_i(1, \mathbf{Z}_{-i})) - E_{\mathcal{E}} E(Y_i(0, \mathbf{Z}_{-i})).$$

INTERFERENCE CONTINUED

In the MTO studies, economists and policy researchers assumed SUTVA held. Then

$$E_{\mathcal{E}} E(Y_i(1, \mathbf{Z}_{-i})) - E_{\mathcal{E}} E(Y_i(0, \mathbf{Z}_{-i}))$$

reduces to the ATE. But if SUTVA does not hold, the expectation reflects both treatment and spillovers.

To see this, it is necessary to compare the allocations with N_0 control units and N_1 treatment units with some other set of allocations.

Sobel (2006) used the hypothetical baseline study "o" in which no units are treated. This is a natural choice, but were a different baseline chosen the values of the treatment and spillover effects would change. Rosenbaum (2007) proposes the uniformity trial.

INTERFERENCE CONTINUED

Let $Y_i(o)$, $i = 1, ...N_T$ denote i's outcome in this study. Let $E_{\mathcal{E}}(Y_i(o))$ and $E_{\mathcal{I}}(Y_i(o))$ denote the average values of the outcome for the eligibles and ineligibles, respectively.

$$E_{R(N_0,N_1)}(\bar{Y}_1(\mathbf{Z}) - \bar{Y}_0(\mathbf{Z})) = [E_{\mathcal{E}} E(Y_i(1, \mathbf{Z}_{-i})) - E_{\mathcal{E}}(Y_i(o))] - [E_{\mathcal{E}} E(Y_i(0, \mathbf{Z}_{-i})) - E_{\mathcal{E}}(Y_i(o$$

The second term on the right compares the average potential outcomes of the N eligible units when they are not treated with the average potential outcomes of these N units in the baseline study where no units are treated.

This is a pure spillover effect. The first term is then the average effect of treatment. If the spillover is 0, the same result would be obtained if SUTVA is assumed; otherwise this is not the case.

INTERFERENCE CONTINUED

No assumptions about the nature of interference between units are made. However, the results are not so useful for inference because we only see outcomes in the one study that has been conducted.

Progress depends on the investigator's ability to make reasonable assumptions about the pattern of interference in special cases. e.g. that units are grouped into strata with interference within, but not across strata as in the cluster randomized experiment.

The results depend on the relative proportion of treatment and control units. One might expect in many social contexts that spillovers are larger when a greater proportion of interfering units are treated. In these cases, one assumes there is interference, but that it is limited or "partial" (Sobel 2006). Hong and Raudenbush (2006) use both types of assumptions in their study of the effects of retention policies in schools.

INTERFERENCE CONTINUED

Hudgens and Halloran (2008) group the N units into clusters that are presumed to be non-interfering, with $i = 1..., N_j$ units per cluster. Let $Y_{ij}(z_{ij}, \mathbf{z}_{-ij})$ denote the potential response of unit i in cluster j when i receives treatment z_{ij} and the remaining units receive the allocation $\mathbf{z}_{-ij}$.

The difference

$$Y_{ij}(1, \mathbf{z}_{-ij}) - Y_{ij}(0, \mathbf{z}^*_{-ij}) = [Y_{ij}(1, \mathbf{z}_{-ij}) - Y_{ij}(0, \mathbf{z}_{-ij})] + [Y_{ij}(0, \mathbf{z}_{-ij}) - Y_{ij}(0, \mathbf{z}^*_{-ij})]$$

may be decomposed as the sum of a direct effect of treatment for unit i in cluster j when the remaining units in cluster j are allocated to $\mathbf{z}_{-ij}$ and the spillover comparing $\mathbf{z}_{-ij}$ with $\mathbf{z}^*_{-ij}$ for unit i when untreated.

The decomposition is not unique since also

$$Y_{ij}(1, \mathbf{z}_{-ij}) - Y_{ij}(0, \mathbf{z}^*_{-ij}) = [Y_{ij}(1, \mathbf{z}_{-ij}) - Y_{ij}(1, \mathbf{z}^*_{-ij})] + [Y_{ij}(1, \mathbf{z}^*_{-ij}) - Y_{ij}(0, \mathbf{z}^*_{-ij})]$$

INTERFERENCE CONTINUED

The first decomposition is natural if one wants to know how units that are untreated are affected by treated units.

When $(1, \mathbf{z}_{-ij})$ and $(0, \mathbf{z}^*_{-ij})$ belong to the same randomization set, and these allocations are held fixed, averaging over units in the two allocations gives the average direct effect of treatment in allocation $\mathbf{z}_{-ij}$ and the average spillover for the pair $(\mathbf{z}_{ij}, \mathbf{z}^*_{-ij})$.

If instead unit i is fixed, and the average of the decomposition is taken over the randomization set one obtains the average effect of treatment for unit i in cluster j and a spillover effect of 0.

INTERFERENCE CONTINUED

In Hudgens and Halloran (2008), the allocation $\mathbf{z}_{ij}$ belongs to the randomization set ψ and the allocation $\mathbf{z}^*_{-ij}$ to a randomization set ϕ. Then averaging the decomposition over ψ and ϕ for a fixed unit i in cluster j gives the unit average direct effect over ψ and the unit average spillover for the pair (ψ, ϕ).

Averaging these over the units in cluster j gives the average direct effects in cluster j over ψ and the average spillover in cluster j for the pair (ψ, ϕ), and averaging these over clusters gives population average direct effects over ψ and spillover effects for the pair (ψ, ϕ).

INTERFERENCE CONTINUED

A notable feature is their consideration of standard errors of estimates. The problem is non-trivial using randomization based inference: within each cluster, a single systematic sample is taken.

It is necessary to make additional assumptions to obtain variance estimates (Lohr, 2010). To that end, HH use a strong version of the assumption that a unit's potential outcomes depend only on the proportion of units receiving treatment. Specifically, suppose that in cluster j, $K_j = 1, ..., N_j - 1$ units receive treatment.

Hudgens and Halloran assume that for every value of K_j, a unit's potential outcomes depend only on whether the unit receives treatment or not: in other words, within allocations where K_j units are treated, SUTVA holds. HH call this assumption "stratified interference": the assumption has proved important in the subsequent literature on interference.

INTERFERENCE CONTINUED

Under the stratified interference assumption and the assumption that treatment effects are additive, i.e. the same for all units, Hudgens and Halloran derive unbiased variance estimators for various direct and spillover effects.

If the treatment effects are heterogeneous, the estimators are conservative. It is worth pointing out that the stratified interference assumption is quite strong and will not be plausible in many realistic cases where clusters consist of interacting units.

e.g. in the MTO study, if unit i moves only if unit i' is also assigned to the treatment group and moves, then for $K_j \geq 2$, the "stratified interference" assumption cannot hold.

More recent work on interference takes off from the starting points above. See Hong (2015) and VanderWeele (2015).