REGRESSION ANALYSIS

In the previous lesson, we introduced model based inference for various randomized experiments.

We now reformulate model based inference in the linear regression framework:

REGRESSION ANALYSIS

Consider the causal regression models for the potential outcomes: for $i = 1, \ldots, n$, and $z = 0, 1$

$$Y_i(z) = \alpha + \tau z + \varepsilon_i(z)$$

where for identification $E(\varepsilon_i(z)) := 0$

Note that these models relate the two potential outcomes across all experimental units. In other words, for $z = 0$, we characterize $Y_i(0)$ for all n subjects, and for $z = 1$, we characterize all $Y_i(1)$. Of course, the two models are only hypothetical since we only observe one of the potential outcomes for each case.

It is clear that $\alpha = E(Y(0))$ and $\tau = E(Y(1) - Y(0))$

REGRESSION ANALYSIS

The researcher can only estimate the regression model for the observed outcomes $Y_i = Y_i(Z_i)$:

$$Y_i(Z_i) = \alpha^* + \tau^* Z_i + v_i$$

To identify α^* and τ^* in this regression model we force $E(v_i \mid Z_i) = 0$ which leads to

$$\alpha^* = E(Y_i \mid Z_i = 0) \text{ and } \tau^* = E(Y_i \mid Z_i = 1) - E(Y_i \mid Z_i = 0)$$

REGRESSION ANALYSIS

What is the relationship between the parameters of the causal regression

$$Y_i(z) = \alpha + \tau z + \varepsilon_i(z)$$

and the observed data model

$$Y_i(Z_i) = \alpha^* + \tau^* Z_i + v_i$$

?

Recall that $Y_i = Y_i(Z_i) = Z_i Y_i(1) + (1 - Z_i)(Y_i(0))$ gives

$$Y_i(Z_i) = \alpha + \tau Z_i + \varepsilon_i(Z_i) \equiv \alpha^* + \tau^* Z_i + v_i$$

where $\varepsilon_i(Z_i) = Z_i \varepsilon_i(1) + (1 - Z_i)\varepsilon_i(0)$

Thus,

$$E(Y_i(Z_i) \mid Z_i = 0) = \alpha + E(\varepsilon_i(Z_i) \mid Z_i = 0) = \alpha + E(\varepsilon_i(0) \mid Z_i = 0) = \alpha^*$$

Similarly,

$$\tau^* = E(Y_i(Z_i) \mid Z_i = 1) - E(Y_i(Z_i) \mid Z_i = 0)$$
$$= \tau + E(\varepsilon_i(1) \mid Z_i = 1) - E(\varepsilon_i(0) \mid Z_i = 0)$$

REGRESSION ANALYSIS

This means $\alpha^* \neq \alpha$, $\tau^* \neq \tau$ in general.

But equality holds if the potential outcomes (equivalently the potential errors $\varepsilon_i(0)$ and $\varepsilon_i(1)$) are independent of treatment assignment Z_i

That is to say, the parameters α and τ of the causal regression (**??**) are identified from the parameters α^* and τ^* of the observed data model if treatment assignment is independent of the potential outcomes (potential errors), as in a completely randomized experiment.

REGRESSION ANALYSIS

In a completely randomized study, treatment assignment is independent of the potential outcomes (potential errors) whereas in an observational study, persons might have some sense of whether or not the drug will be helpful.

To see how this might work out in the observational study, consider the following example: A depressed person chooses whether or not to take a medication to improve their mood.

MEDICATION EXAMPLE

Suppose subjects who take the drug ($Z_i = 1$) do so because they correctly believe the drug will be helpful to them and subjects who do not take the drug ($Z_i = 0$) correctly believe the drug will not help them.

Further suppose that half of the subjects take the drug, half do not, and that subjects who take treatment improve from a score of 5 to a score of 10, while subjects who do not take treatment receive a score of 8.

The treatment effect is thus 2.5; the investigator underestimates the effect at 2.

Now suppose that instead of improving from 5 to 10, the subjects improve from 5 to 15. In that case, the effect is 5, but the estimate is 7.

REGRESSION ANALYSIS

A slightly different way to see this is to start with the ordinary least squares (OLS) estimator in the observed data model: $\hat{\alpha}^* = \bar{Y}_0$, $\hat{\tau}^* = \bar{Y}_1 - \bar{Y}_0$

This is easy to see from the fact that, as you know, the residuals e_i and the weighted residuals sum to 0: $\sum_{i=1}^n e_i = 0$, $\sum_{i=1}^n Z_i e_i = 0$

We know from the previous modules that $\hat{\tau}^*$ is unbiased for $E(Y(1) \mid Z = 1) - E(Y(0) \mid Z = 0)$

In general this is not equal to $E(Y(1) - Y(0))$, but when treatment assigment Z_i is independent of potential outcomes, as in a completely randomized experiment,

$$E(Y(1) \mid Z = 1) - E(Y(0) \mid Z = 0) = E(Y(1) - Y(0))$$

REGRESSION ANALYSIS

The results extend readily to the case where the investigator is also interested in effects within strata defined by covariates

$$Y_{si'}(z) = \alpha_s + \tau_s z + \varepsilon_{si'}(z)$$

$$\text{where } E(\varepsilon_{si'}(z) \mid S = s) = 0,\ z = 0, 1$$

τ_s is the stratum average treatment effect $E(Y(1) - Y(0) \mid S = s)$

The analogous regression model is

$$Y_{si'}(Z_{si'}) = \alpha_s^* + \tau_s^* Z_{si'} + \varepsilon_{si'}(Z_{si'}) \equiv \alpha_s^* + \tau_s^* Z_{si'} + v_{si'}$$

$$\text{where } E(v_{si'} \mid Z_{si'}, S_i) = 0$$

REGRESSION ANALYSIS

It is easy to see from the stratified model

$$Y_{si'}(z) = \alpha_s + \tau_s z + \varepsilon_{si'}(z)$$

that in either a completely randomized experiment or a block randomized experiment, $\alpha_s = \alpha_s^*$, $\tau_s = \tau_s^*$

This is also true for an observational study, where an investigator forms strata from the covariates and correctly assumes that treatment assignment is unconfounded within strata, as would be the case in the analogous block randomized study.

It is easy to obtain the ATE by averaging the τ_s over the distribution of S.