INSTRUMENTAL VARIABLES WITH A CONTINUOUS "TREATMENT"

A special case of mediation occurs when the treatment assignment variable Z affects the outcome only through the mediator M. i.e Z does not directly affect the outcome Y.

This assumption can be reasonable in a in a randomized double blinded encouragement study. The assumption is less likely to hold in randomized studies where subjects know which treatment group they belong to.

e.g. Let Z denotes assignment to a job training program or the control group and M denotes whether or not the treatment is actually taken. Suppose a subject assigned to treatment who does not take up treatment may feel demoralized and therefore more depressed (Y) than had the subject been assigned to the control group.

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Natural experiments are observational studies where an event that is essentially randomized is used to assign units to a treatment or control group.

The assignment is presumed to affect an intermediate outcome that in turn affects the outcome Y. It is often assumed that assignment does not directly affect Y.

e.g. Angrist, Imbens and Rubin (1996) used the draft lottery as an instrument to study the effect of service on excess civilian mortality.

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Under this assumption (no direct effect/exclusion restriction), $Y_i(0, m) = Y_i(1, m)$. i.e. Z affects response Y only through intermediate outcome M.

The controlled direct effects, $E(Y_i(1, m) - Y_i(0, m) \mid \mathbf{X})$, and the natural direct effects, $E(Y_i(1, M_i(1)) - Y_i(0, M_i(1)) \mid \mathbf{X}_i)$, are 0.

The average indirect effects do not depend on how $Y_i(1, M(1)) - Y_i(0, M(0))$ is decomposed. The average indirect effect is equal to the total effect, $E(Y_i(1, M_i(1)) - Y_i(0, M_i(0)) \mid \mathbf{X}_i)$.

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Under the assumption,

$$M_i(0), M_i(1), \{Y_i(0, m), Y_i(1, m) : m \in \Omega_M\} \perp\!\!\!\perp Z_i \mid \mathbf{X}_i = \mathbf{x},$$

and the average indirect effect is identified.

If the exclusion restriction holds, it is not necessary to make additional assumptions about the unconfoundedness of mediator-outcome relationships. However, it is often the case that such indirect effects are not of primary interest.

e.g. the controlled direct effects $E(Y_i(z, m) - Y_i(z, m*) \mid \mathbf{X}_i)$ capture the direct effect of the mediator on the outcome. But the exclusion restriction and the condition: $E(M_i(1) - M_i(0) \mid \mathbf{X}_i, Z_i) \neq 0$ do not suffice to identify these effects.

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Z is an instrument with respect to the outcome if Y if

$$E(M_i(1) - M_i(0) \mid \mathbf{X}_i, Z_i) \neq 0,$$

and both

$$M_i(0), M_i(1), \{Y_i(0, m), Y_i(1, m) : m \in \Omega_M\} \perp\!\!\!\perp Z_i \mid \mathbf{X}_i = \mathbf{x}$$

and

$$Y_i(0, m) = Y_i(1, m)$$

hold.

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Angrist, Imbens and Rubin (1996) require also that an instrument satisfy a monotonicity condition: $M_i(1) \geq M_i(0)$.

In the econometrics literature an instrument is often defined using notation for observed outcomes only, and instrumental variables are used in other, non-causal contexts. See Greene (2017) and Imbens (2014).

Since $Y_i(1, m) = Y_i(0, m)$, we write the potential outcomes as $Y_i(m)$. We consider three prominent cases:

1. Z is binary and M continuous,
2. both Z and M are binary,
3. Z is continuous and M is binary.

INSTRUMENTAL VARIABLES WITH A CONTINUOUS "TREATMENT"

We begin with case where Z is binary and M is continuous, using a linear model for the potential outcomes to illustrate:

$$M_i(z) = \alpha_M^{(c)} + \beta_M^{(c)} z + \lambda_M'^{(c)} \mathbf{X}_i + \varepsilon_i(z),$$
$$Y_i(m) = \alpha_Y^{(c)} + \beta_Y^{(c)} m + \lambda_Y'^{(c)} \mathbf{X}_i + \varepsilon_i(m),$$

where $\mathrm{E}(\varepsilon_i(z) \mid \mathbf{X}_i = \mathbf{x}) = \mathrm{E}(\varepsilon_i(m) \mid \mathbf{X}_i = \mathbf{x}) = 0$;

n.b. model parameters are denoted with the superscript "c" to indicate that these are the parameters of the model for the potential outcomes, not the parameters for the model for the observed data.

INSTRUMENTAL VARIABLES WITH A CONTINUOUS "TREATMENT"

Under

$$M_i(z) = \alpha_M^{(c)} + \beta_M^{(c)} z + \lambda_M'^{(c)} \mathbf{X}_i + \varepsilon_i(z),$$
$$Y_i(m) = \alpha_Y^{(c)} + \beta_Y^{(c)} m + \lambda_Y'^{(c)} \mathbf{X}_i + \varepsilon_i(m),$$

and assuming $E(M_i(1) - M_i(0) \mid \mathbf{X}_i, Z_i) \neq 0$ gives
$\beta_M^{(c)} = E(M_i(1) - M_i(0) \mid \mathbf{X}_i)$ and

$$\tau^{(c)}(\mathbf{X}) \equiv E(Y_i(M_i(1)) - Y_i(M_i(0)) \mid \mathbf{X}_i) = \beta_Y^{(c)} \beta_M^{(c)} + E(\varepsilon_i(M_i(1)) - \varepsilon_i(M_i(0)) \mid \mathbf{X}_i).$$

Under the additional assumption $E(M_i(1) - M_i(0) \mid \mathbf{X}_i, Z_i) \neq 0$ and a (strong) constant effect assumption $\varepsilon_i(m) = \varepsilon_i$ for all m, the ratio $\tau_c(\mathbf{X})/\beta_Y^{(c)}$, the IV estimand is the average controlled direct effect of M on Y for all $\mathbf{X}$.

INSTRUMENTAL VARIABLES WITH A CONTINUOUS "TREATMENT"

The econometric literature refers to the assumption of no treatment heterogeneity. To obtain this result when treatment effect heterogeneity is present, the assumption $\mathrm{E}(\varepsilon_i(M_i(1)) - \varepsilon_i(M_i(0)) \mid \mathbf{X}_i) = 0$ is required.

Sobel (2008) gives sufficient conditions for this to hold. The conditions are weaker than in the mediation literature, where the mediator outcome relationship is assumed unconfounded given the covariates and treatment assignment: $Y_i(z, m) \perp\!\!\!\perp M_i \mid \mathbf{X}_i = \mathbf{x}, Z_i = z$.

The assumption of constant effect is neither stronger nor weaker than the unconfoundedness assumption. Related conditions may be found in the econometrics literature. See Wooldridge (2010). Econometricians have also studied IVs in the context of nonlinear, semi-parametric, and non-parametric models. See Imbens (2014) for further citations.