1a.) $H_0: \lambda_1 \leq \lambda_2 \quad H_a: \lambda_1 > \lambda_2$

Anthony Fontana
AMS 206
Final Exam
(Quiz)

$$P(H_0 | X, y, \lambda_1, \lambda_2) = \frac{1}{1 + \frac{P(X,y | H_a \lambda_1 \lambda_2) P_r(H_a)}{P(X,y | H_0, \lambda_1, \lambda_2) P_r(H_0)}}$$

<u>likelihood:</u>

$$\lambda_1^n \exp\left\{-\lambda_1 \sum_{i=1}^{n} x_i\right\} \cdot \lambda_2^n \exp\left\{-\lambda_n \sum_{i=1}^{n} y_i\right\} = (\lambda_1 \lambda_2)^n \exp\left\{-\lambda_1 n\bar{x} - \lambda_2 n\bar{y}\right\}$$

<u>Joint Prior:</u>

$$\frac{b^a}{\Gamma(a)} \lambda_1^{-a-1} \exp\left\{\frac{-b}{\lambda_1}\right\} \cdot \frac{b^a}{\Gamma(a)} \lambda_2^{-a-1} \exp\left\{\frac{-b}{\lambda_2}\right\} = \left(\frac{b^a}{\Gamma(a)}\right)^2 (\lambda_1 \lambda_2)^{-a-1} \exp\left\{\frac{-b}{\lambda_1} - \frac{b}{\lambda_2}\right\}$$

<u>$P(x, y | H_0 \lambda_1 \lambda_2) =$</u>

$$\int_0^\infty \int_0^{\lambda_2} \left(\frac{b^a}{\Gamma(a)}\right)^2 (\lambda_1 \lambda_2)^{n-a-1} \exp\left\{\frac{-b}{\lambda_1} - \frac{b}{\lambda_2}\right\} \cdot \exp\left\{-\lambda_1 n\bar{x} - \lambda_2 n\bar{y}\right\} d\lambda_1 d\lambda_2$$

$$\Rightarrow \left(\frac{b^a}{\Gamma(a)}\right)^2 \int_0^\infty \int_0^{\lambda_2} (\lambda_1 \lambda_2)^{n-a-1} \exp\left\{\frac{-b}{\lambda_1}\right\} \exp\left\{\frac{-b}{\lambda_2}\right\} \exp\left\{-\lambda_1 n\bar{x}\right\} \exp\left\{-\lambda_2 n\bar{y}\right\} d\lambda_1 d\lambda_2$$

$$\Rightarrow \left(\frac{b^a}{\Gamma(a)}\right)^2 \int_0^\infty \int_0^{\lambda_2} (\lambda_1 \lambda_2)^{n-a-1} \exp\left\{-\frac{b}{\lambda_1} - \lambda_1 n\bar{x}\right\} \exp\left\{-\frac{b}{\lambda_2} - \lambda_2 n\bar{y}\right\} d\lambda_1 d\lambda_2$$

$$\Rightarrow \int_0^\infty \lambda_2^{n-a-1} \exp\left\{\frac{-b}{\lambda_2} - \lambda_2 n\bar{y}\right\} \underbrace{\int_0^{\lambda_2} \lambda_1^{n-a-1} \exp\left\{\frac{-b}{\lambda_1} - \lambda_1 n\bar{x}\right\} d\lambda_2}$$

Some ridiculously complicated IBP that I dont know how to do

$$\exp\left\{-\frac{b}{\lambda_1} - \lambda_1 n\bar{x}\right\} = \exp\left\{-\frac{b}{\lambda_1} - \frac{\lambda_1(\lambda_1 n\bar{x})}{\lambda_1}\right\}$$

$$= \exp\left\{\frac{-b - \lambda_1(\lambda_1 n\bar{x})}{\lambda_1}\right\}$$

No idea how to integrate that $\longrightarrow$ $$= \exp\left\{\frac{-b - (\lambda_1^2 \lambda_1 n \lambda_1 \bar{x})}{\lambda_1}\right\}$$

1b) $\lim_{\alpha,\beta\to 0} \quad (\lambda_1 \lambda_2) \sim \frac{\beta^\alpha}{\Gamma(\alpha)}(\lambda_1\lambda_2)^{-\alpha-1} \exp\left\{-\frac{\beta}{\lambda_1} - \frac{\beta}{\lambda_2}\right\}$

$$\propto \beta^\alpha(\lambda_1\lambda_2)^{-\alpha-1}\exp\left\{-\frac{\beta}{\lambda_1}-\frac{\beta}{\lambda_2}\right\}$$

$$\Rightarrow \lim_{\alpha\beta\to 0} \frac{1}{\lambda_1\lambda_2 \exp\left\{-\frac{\beta}{\lambda_1}-\frac{\beta}{\lambda_2}\right\}} = \boxed{\frac{1}{\lambda_1\lambda_2}}$$

1c) $\frac{P(X,y\mid H_a\,\lambda_1\lambda_2)}{P(X,y\mid H_o\,\lambda_1\lambda_2)}$

$$P(X,y\mid H_o,\lambda_1\lambda_2) = \int_0^\infty\int_0^{\lambda_2} \lambda_1^n \lambda_2^n \exp\{-\lambda_1 n\bar{x}\}\exp\{-\lambda_2 n\bar{y}\}\,d\lambda_1 d\lambda_2$$

$$= \int_0^{\lambda_2} \lambda_1^{n-1}\exp\{-\lambda_1 n\bar{x}\}\,d\lambda_1 \int_0^\infty \lambda_2^{n-1}\exp\{-\lambda_2 n\bar{y}\}\,d\lambda_2$$

Integrates to one, inverse of normalizing parameter $\rightarrow \frac{\Gamma(n)}{(n\bar{y})^n}\int_0^{\lambda_2}\lambda_1^{n-1}\exp\{-\lambda_1 n\bar{x}\}\,d\lambda_1$

$\rightarrow Pr(H_a) + Pr(H_o) = 1 \longrightarrow Pr(H_a) = 1 - Pr(H_o)$

$\rightarrow$ The probabilities make up the whole probability space so we know $1-Pr(\,)$ will have to equal the other; so $Pr(H_a) = 1 - Pr(H_o)$ which yields . . .

$$\frac{\cancel{\frac{\Gamma(n)}{(n\bar{y})^n}}\int_0^\infty \lambda_1^{n-1}\exp\{-\lambda_1 n\bar{x}\}\,d\lambda_1}{\cancel{\frac{\Gamma(n)}{(n\bar{y})^n}}\int_0^{\lambda_2}\lambda_1^{n-1}\exp\{-\lambda_1 n\bar{x}\}\,d\lambda_1}$$

$$\boxed{= \frac{\int_0^\infty \lambda_1^{n-1}\exp\{-\lambda_1 n\bar{x}\}\,d\lambda_1}{\int_0^{\lambda_2}\lambda_1^{n-1}\exp\{-\lambda_1 n\bar{x}\}\,d\lambda_1}}$$

1d.) See Code

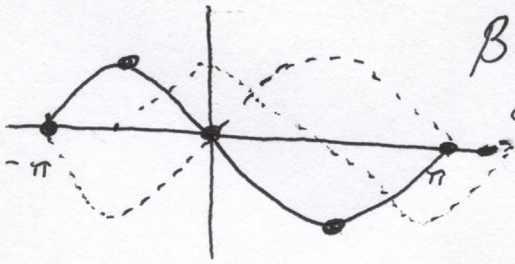
→ evidence in favor of null $\lambda_1 < \lambda_2$.

$$\frac{1}{1+\frac{alt}{null}} = .87$$

2a.) option 1: $\beta > 0$ and $\delta \in [-\pi, \pi]$

option 2: β not restricted $\delta \in [-\pi/2, \pi/2]$

→ Justification – The parameters will not be identifiable if not restricted. There will be infinitely many ways to fit a sinusoid function to a set of data, so no unique parameters can be found. This can be seen by graphing a sin(x)...



$\beta \not> 0$ ... can be negative

$\delta \in [-\pi, \pi]$ ... not holding

→ essentially, w/out the functions restricted they wont be completely invertible. This causes problems for identifiability.

- → Choosing the first restriction is easier for computation.
- → I would choose uniform prior for δ. This is primarily because I have no background knowlege of the physics to make an informed judgement about the appropriate parameter values. Maybe if I were a phycisist I would know.
- → I would probably choose a normal prior for β. There doesnt seem to be a reason why amplitude will be non-normal.

2b.) $y_i = \alpha + \beta\sin(\omega x_i + \delta) + \varepsilon_i \qquad \varepsilon_i \sim N(0, \sigma^2)$

Show: $y_i = \alpha + \beta\sin(\omega x_i + \delta) = \alpha + \gamma_1(\sin(\omega x_i)) + \gamma_2\cos(\omega x_i) + \varepsilon$

use: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$

$$= \alpha + \underbrace{\beta\cos(\delta)}_{\gamma_1}\sin(\omega x_i) + \underbrace{\beta\sin(\delta)}_{\gamma_2}\cos(\omega x_i) + \varepsilon_i$$

$$= \alpha + \gamma_1\sin(\omega x_i) + \gamma_2\cos(\omega x_i) + \varepsilon_i$$

$\longrightarrow \gamma_1 + \gamma_2 = \beta\cos(\delta) + \beta\sin(\delta)$

want to use pythagorean identities to flesh out relationship between gammas

$$(\gamma_1 + \gamma_2)^2 = (\beta\cos(\delta) + \beta\sin(\delta))^2$$

$$\gamma_1^2 + \gamma_2^2 = \beta^2\cos^2(\delta) + \beta^2\sin^2(\delta)$$

$$\gamma_1^2 + \gamma_2^2 = \beta^2\underbrace{(\cos^2(\delta) + \sin^2(\delta))}_{= 1}$$

so $\sqrt{\gamma_1^2 + \gamma_2^2} = \sqrt{\beta^2}$

$$\boxed{\beta = \sqrt{\gamma_1^2 + \gamma_2^2} = \sqrt{\beta\cos^2(\delta) + \beta\sin^2(\delta)}}$$

use Tangent identity to derive δ

$$\mathrm{Tan}(\delta) = \frac{\gamma_2}{\gamma_1} = \frac{\beta\sin(\delta)}{\beta\cos(\delta)}$$

$$\boxed{\delta = \arctan\left(\frac{\gamma_2}{\gamma_1}\right)}$$

2c.) Note* Multivariate so $D = (1, \cos(\omega x_i), \sin(\omega x_i))$
2D.)
→ Full conditional for amplitude β

$$P(\beta | \omega, \theta^2, y) \propto (\beta, y | \omega, \theta^2) = p(y | \beta, \theta^2, \omega) \cdot p(\beta / \omega, \theta^2)$$

$$\propto P(y | \beta, \omega, \theta^2_\omega) \propto N(((D^T D)^{-1} D^T y), \theta^2 (D^T D)^{-1})$$

* also note these parameters are now vectors of observations

→ Full Conditional for noise

$$P(\theta^2 | \omega, \beta, y) \propto p(\theta^2 y | \omega, \beta) = p(y | \omega, \beta, \theta^2) p(\theta^2)$$

$$\propto N(x_i, D\beta, \theta^2_N) \theta^2$$

$$\propto IG\left(\frac{n}{2}, \underbrace{\frac{1}{2}(y - D\beta)^T (y - D\beta)}\right)$$

→ Frequency → needs Monte Carlo Random Walk Metropolis Hasting

$$P(\omega | \beta, \theta^2, y) \propto \exp\left\{-\frac{1}{2\theta^2}(y - D\beta)^T (y - D\beta)\right\}$$

extra Credit: $\gamma_1, \gamma_2 \sim N(0, \theta^2)$ iid ...

$$\sum_{i=1}^{n} \gamma_i \sim \chi^2 \text{ [chi-square ; DF = 2]}$$

The sum of a bunch of iid normals yeilds χ^2

$\beta = \sqrt{\gamma_1^2 + \gamma_2^2} = \sqrt{\chi^2}$ Square root of some chi-square, call this variable "x"

$$\beta = \sqrt{x} \implies \beta^2 = x$$

→ Problem boils down to finding the pdf of a function of a random variable.

→

2c Continued)

2d

This problem needs a Jacobian transformation.

Suppose X is continuos and Random and $g : \mathbb{R} \to \mathbb{R}$ is strictly monotonic and differentiable ... $Y = g(X)$

(in our case $X \sim \chi^2$)

$$f_y(y) = \begin{cases} \frac{f_x(x_1)}{|g'(x_1)|} = f_x(x_1) \cdot \left|\frac{dx_1}{dy}\right| & \text{where } g(x_1) = y \\ 0 & \text{if } g(x) = y \text{ no sol} \end{cases}$$

$$|J| = \frac{\partial x}{\partial \beta} = |2\beta| \qquad \chi^2 = \frac{1}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \qquad \underline{k=2}$$

$$= \frac{1}{2\Gamma(1)} \beta^{\frac{2}{2}-1} e^{-\frac{2}{2}}$$

But it can be shown for $k=2$, this special case leads to Rayleigh

$$\cdots \frac{1}{2\Gamma(1)} (\beta^2)^{1-\frac{1}{2}} \exp\left\{-\frac{\beta^2}{2}\right\} |2\beta| = \beta \exp\left\{-\frac{\beta^2}{2}\right\}, \beta > 0$$

if $X \sim$ Rayleigh(1) Then $X^2 \sim \chi^2(2)$

$\to$ Now for δ $\qquad \gamma_1 = \beta\cos(\delta) \quad \gamma_2 = \beta\sin(\delta)$

$$f_\delta(\delta) = \frac{1}{2\pi} \quad \delta \in [-\pi, \pi]$$

$\longrightarrow$

$Y = \cos(\delta) \qquad Y \sim \text{Uni}[-1, 1]$

$$\delta_y(y) = \sum_{\cos(\delta)=y} f_\delta(\cos^{-1}(Y)) \left| \frac{\partial \delta}{\partial y} \right| =$$

$$= \sum_{\cos(\delta)=y} f_\delta(\cos^{-1}(y)) \left| \frac{1}{-\sin(\cos^{-1}(y))} \right|$$

$$= \frac{2}{2\pi} \cdot \frac{1}{\sin(\cos^{-1}(y))} \quad \forall \cos^{-1}(y) \in [0, \pi]$$

$$f_y(y) = \frac{1}{\pi \sin(\cos^{-1}(y))} \qquad y \in [-1, 1]$$

c) See Code for simulations

d.) See Code for MCMC

e.) See output on other document

f.) . . . .

g.) No clue

h.) Yes, I found it by visual inspection