(iii) uniform prior $[-1,1]$ for ρ . . .

Full conditional . . .

$$f(\rho \mid \mu, \sigma^2, y) \propto \prod_{t=2}^{N} N\left(y_T \mid (y_{T-1}-\mu)\rho+\mu, \sigma^2\right) \cdot \pi(\rho)$$

* $z_T = y_T - \mu \Rightarrow (z_{T-1} = y_{T-1} - \mu)$

$$f(\rho \mid \mu, \sigma^2, y) \propto N(z_T \mid z_{T-1}, \rho, \sigma^2) \cdot (-1 < \rho < 1)$$

$$\propto \exp\left[-\frac{1}{2\sigma^2}\sum_{T=2}^{N}\left(z_T - \rho z_{T-1}\right)^2\right] \cdot (-1 < \rho < 1)$$

$$\propto \exp\left[-\frac{1}{2\sigma^2}\left(\rho^2\sum_{T=2}^{N} z_T^2 - 2\rho\sum_{T=2}^{N} z_T z_{T-1} + \sum z_T^2\right]$$

$$a = \sum_{t=2}^{N} z_T^2$$

$$b = \sum_{t=2}^{N} z_T z_{T-1}$$

$$\propto \exp\left[-\frac{1}{2\sigma^2}(a\rho^2 - 2b\rho)\right] \Rightarrow \exp\left[-\frac{1}{2\sigma^2}a\left(\rho^2 - \frac{2b\rho}{a} + \frac{b^2}{a^2}\right) + \frac{b^2}{a^2}\cdot\frac{1}{2\sigma^2}\right]$$

$$\propto \exp\left[\frac{1}{2\sigma^2}a\left(\rho - \frac{b}{a}\right)^2\right] \Rightarrow \exp\left[-\frac{1}{2}\left(\frac{a}{\sigma^2}\right)\left(\rho - \frac{b}{a}\right)^2\right] \Rightarrow \exp\left[-\frac{1}{2}\left(\frac{\rho - \frac{b}{a}}{\frac{1}{\sqrt{\frac{a}{\sigma^2}}}}\right)^2\right]$$

$$\sim N\left(\frac{b}{a}, \frac{a}{\sigma^2}\right) \Rightarrow \boxed{\sim N\left(\frac{\sum_{T=2}^{N} z_T z_{T-1}}{\sum_{T=1}^{N-1} z_T^2}, \frac{\sigma^2}{\sum_{T=1}^{N-1} z_T^2}\right)}$$

2.) $p(x_i|\theta) = h(x_i)\exp\{\theta_T(x_i) - c(\theta)\}$

(a.) Find expression for Jeffreys Prior

$$\pi(\theta) \propto \sqrt{I(\theta)} \Rightarrow \sqrt{I(\theta)} = -E\sqrt{\frac{\partial^2}{\partial\theta^2}\left[\log(L(\theta|x_1 \ldots x_n))\right]}$$

$$I(\theta) = -nE\left[\frac{\partial^2}{\partial\theta^2}\log(L(\theta|x))\right]$$

$$I(\theta) = -nE\left[\frac{\partial^2}{\partial\theta^2}n\left[\log(h(x))\right] + \theta T(x) - c(\theta)\right]$$

$$I(\theta) = -nE\left[\frac{\partial}{\partial\theta}T(x) - c(\theta)'\right]$$

$$I(\theta) = -nE\left[-c(\theta)''\right]$$

$$\Rightarrow \boxed{\pi(\theta) \propto \sqrt{nc(\theta)''}}$$

b.) iid Poisson $\Rightarrow$ likelihood $= \frac{\theta^x e^\theta}{x_i!} \Rightarrow \frac{1}{x_i!}\exp[x\log(\theta) - \theta]$

$c(\theta) = \exp(\theta)$

$$\hookrightarrow C'(\theta) = \exp(\theta)d\theta$$

$$c'(\theta) = \frac{1}{\lambda}\exp(\theta)$$

$$c''(\theta) = \frac{1}{\lambda}d\theta\ \exp(\theta)$$

$$= \frac{1}{\lambda^2}\exp(\theta)$$

$$= \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$I(\lambda) = -nE\left[-c(\theta)''\right] = nE\left(\frac{1}{\lambda}\right] \Rightarrow \sqrt{\frac{n}{\lambda}}$$

$$\pi(\lambda) \propto \sqrt{I(\lambda)} = \boxed{\frac{1}{\sqrt{\lambda}}}$$

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AMS 206
Midterm 2

1.) $y_1 \ldots y_n$ generated from...

$$y_T = \mu_T + \rho(y_{T-1} - \mu) + \varepsilon_T \qquad \varepsilon_T \sim N(0, \theta^2)$$

$y_T = \log(r_T) \qquad y_T = \mu_T + \rho y_{T-1} - \rho\mu \Rightarrow y_T - \rho y_{T-1} = \mu(1-\rho)$

$$\Rightarrow \mu = \frac{y_T - \rho y_{T-1}}{1-\rho}$$

(a) Full conditional for precision...

(i) $f(\theta^2 \mid \mu^T, \rho^T, y)$ — Gamma prior w/ parameters a and b

$Z_T = y_T - \mu \Rightarrow Z_{T-1} = y_{T-1} - \mu$

$$f(\theta^2 \mid \mu^T, \rho^T, y) \propto \prod_{T=2}^{N} (Z_T \mid Z_{T-1}, \rho, \theta^2) \cdot IG(\theta^2 \mid a^2, b^2)$$

$$\propto (\theta^2)^{-\frac{n-1}{2}} \exp\left\{-\frac{1}{2\theta^2}\sum_{T=2}^{N}(Z_T - \rho Z_{T-1})^2\right\} (\theta^2)^{(-a-1)} \exp\left\{\frac{-b}{\theta^2}\right\}$$

$\longrightarrow$ Full Conditional for precision follows Inverse Gamma

$$\boxed{\sim IG\left(a + \frac{n-1}{2},\ b + \frac{1}{2}\sum_{t=2}^{N}(Z_T - \rho Z_{T-1})^2\right)}$$

(ii) Long term Average μ — Gaussian prior Mean ξ and variance τ^2

$$p(\mu \mid \theta^2, y) \propto p(y \mid \mu, \theta^2) \cdot \pi(\mu)$$

$$p(y_i \mid y_{i-1}, \mu, \theta^2, \rho) \propto \prod_{i=1}^{N} \exp\left[-\frac{(y_i - \mu)^2}{2\theta^2}\right] \cdot \pi(\mu)$$

$$\propto \exp\left[-\frac{1}{2\theta^2}\sum_{i=1}^{N}(y_i - \mu)^2\right] \cdot \pi(\mu)$$

$$\propto \exp\left[-\frac{1}{2\theta^2}\left(n\mu^2 - 2\sum_{i=1}^{N} y_i\mu + \sum_{i=1}^{N} y_i^2\right)\right]$$

$$\propto \exp\left[-\frac{1}{\frac{2\theta^2}{n}}\left(\mu - \sum_{i=1}^{N} y_i/n\right)^2 + A\right] \longrightarrow \bar{y}_i = \frac{\sum_{i=1} y_i}{n}$$

$$p(\mu \mid \theta^2, \rho, y) \sim N\left(\mu \mid \bar{y}, \frac{\theta^2}{n}\right)$$

$$f(\mu \mid \theta^2, \rho, y) \propto \prod_{t=2}^{N} f(y_T \mid \mu(1-\rho) + \rho y_{T-1}, \theta^2) \cdot \pi(\mu) \propto \prod_{t=2}^{N} f\left(\frac{y_T - \rho y_{T-1}}{1-\rho} \,\middle|\, \mu, \left(\frac{\theta}{1-\rho}\right)^2\right) \cdot \pi(\mu)$$

$$y_T^* = \frac{y_T - \rho y_{T-1}}{1-\rho} \qquad \bar{y}^* = \frac{\sum_{t=2}^{N} y_T^*}{n-1} \qquad \theta^* = \frac{\theta}{(1-\rho)}$$

$$f(\mu \mid \theta^2, \rho, y) \propto N\left(\bar{y}^* \mid \mu, \frac{\theta^*}{n-1}\right) \cdot N(\xi, \tau^2)$$

$$\boxed{\Longrightarrow \sim N\left[\frac{\frac{\bar{y}^*}{\frac{\theta^2}{(n-1)}} + \frac{\xi}{\tau^2}}{\frac{1}{\frac{\theta^2}{n-1}} + \frac{1}{\tau^2}},\ \left(\frac{n-1}{\theta^2} + \frac{1}{\tau^2}\right)^{-1}\right]}$$