Probability Distributions

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The probability distribution is often written as $P(X = x) =$ (some function of x).

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The expected value and sd describe the *center* and *width* of the distribution respectively.

Shorthand notation

$P(X = x)$ is cumbersome. x is also just a dummy variable. Common shorthand notation: Use $p(x)$ instead, equivocate between the quantity itself and the dummy variable. E.g.:

$$\mathbb{E}(x) = \sum x p(x) \tag{1}$$

Numerical handling

Numerical handling of discrete probability distributions for a single quantity:

```
xs = np.arange(5, 21) # Grid of possibilities
ps = xs**2            # Not normalised
ps = ps/ps.sum()      # Normalise it
plt.bar(xs, ps)       # Plot it

# Expected value and variance
ex = np.sum(xs*ps)
variance = np.sum(ps*(xs - ex)**2)
np.sum(ps[xs >= 10]) # P(x >= 10)
```

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- Poisson (how many occurrences of rare event?)

Astronomy uses for Discrete Uniform

The number of emission lines in this spectrum is somewhere from 0 to 100, and I don't know the number.

Astronomy uses for Binomial

Suppose it is known (or hypothesised) that 30% of stars of a particular type exhibit a certain kind of oscillation signal. In a new sample of $N = 100$ such stars, let x be the number that have the oscillation.

Then $x \sim \text{Binomial}(100, 0.3)$.

I set a probability equal to a frequency here. What implicit assumption am I making?

Astronomy uses for Poisson

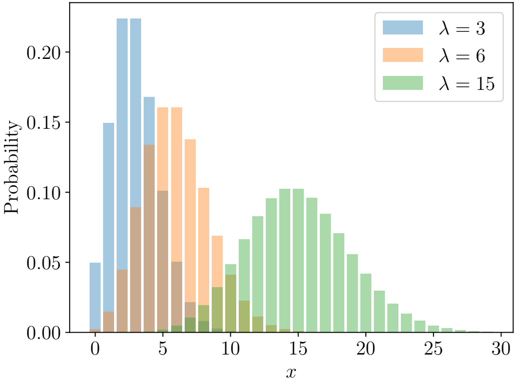
The intensity of an X-ray source is such that you would expect to detect λ photons per minute. Let x be the actual number of photons you observe in a minute.

$$x|\lambda \sim \text{Poisson}(\lambda) \tag{2}$$

$$p(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \tag{3}$$

where $\lambda \geq 0$ and $x \in \{0, 1, 2, 3, ...,\}$.

Three Poisson Distributions



Continuous distributions

These are characterised by a continuous hypothesis space (e.g. “all real numbers”) and a *probability density function* (PDF).

For example, normal/gaussian distributions:

$$X|\mu, \sigma \sim \text{Normal}(\mu, \sigma^2) \tag{4}$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] \tag{5}$$

$f_X(x)$ is the full notation favoured by many statisticians. You can also just write $f(x)$ or $p(x)$ (and not having any upper-case X).

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Numerical handling of continuous probability distributions for a single quantity:

```
xs = np.linspace(-10.0, 10.0, 10001)  # Grid of possibilities
ps = np.exp(-0.5*((xs - 2.0)/1.5)**2) # Not normalised
ps = ps/np.trapz(ps, x=xs) # Normalise it (integral, not sum)
plt.plot(xs, ps)           # Plot it

# Expected value and variance - integrals, not sums
ex = np.trapz(xs*ps, x=xs)
variance = np.trapz(ps*(xs - ex)**2, x=xs)
np.trapz(ps*(xs >= 5.0), x=xs) # P(x >= 5)
```

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- Gamma (nice two-parameter family for a positive quantity)
- Exponential (how long between subsequent events?)
- Pareto ('power law')

Astronomy uses for Uniform

There is an asteroid somewhere in an image. Where?

Astronomy uses for Normal/Gaussian

A very common and popular model for how far ‘noise’ will cause a measurement to depart from the true value of what is being measured.

Astronomy uses for t

I recommend using the t-distribution instead of the gaussian distribution for noise if you think a heavier-tailed distribution might be appropriate.

Higher dimensions

Joint probability distributions can be created using the product rule. e.g.,

$$p(x, y) = p(x)p(y|x) \tag{6}$$

The joint distribution allows you to calculate the probability of statements about the *pair* (x, y).

$$P((x, y) \in R) = \int_R p(x, y)\, dx\, dy \tag{7}$$

Marginalisation

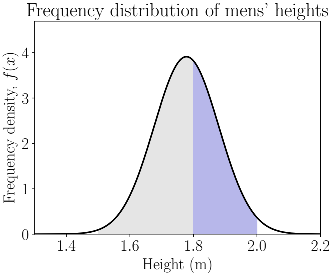
If you have a joint distribution but only care about statements about one of the quantities, you can find the *marginal distribution*:

$$p(x) = \int_{-\infty}^{\infty} p(x, y)\, dy \tag{8}$$

Bayesian and frequentist uses

Probability distributions are used in both Bayesian and frequentist senses.

A frequency distribution...



$$
\begin{aligned}
\text{Frac}\left(x \geq 1.8 | x \in [1.5, 2.0]\right) &= \frac{\text{Frac}\left(x \geq 1.8, x \in [1.5, 2.0]\right)}{\text{Frac}\left(x \in [1.5, 2.0]\right)} \\
&= \frac{\int_{1.8}^{2.0} f(x)\,dx}{\int_{1.5}^{2.0} f(x)\,dx}
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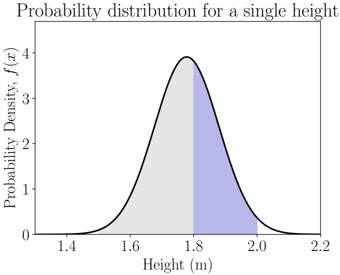
A probability distribution...



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&= \frac{\int_{1.8}^{2.0} f(x)\,dx}{\int_{1.5}^{2.0} f(x)\,dx}
\end{aligned}
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Opinion

Important!

The biggest source of confusion in statistics is the failure to distinguish between **frequency distributions** which describe *populations*, and **probability distributions** which describe *uncertainty about a single quantity*[a].

[a] Could be a single non-scalar quantity, such as (3.2, 1.7).

Exercises

See `questions2.pdf`