

Math 20: Probability

Homework 5 Solution

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Problem 1

4 pts

Chapter 5.1 Exercise 6

Let X_1, X_2, \dots, X_n be n mutually independent random variables, each of which is uniformly distributed on the integers from 1 to k . Let Y denote the minimum of the X_i 's. Find the distribution of Y .

4 pts

We have

$$\begin{aligned} P(Y \leq y) &= 1 - P(Y > y) \\ &= 1 - P(\min\{X_1, X_2, \dots, X_n\} > y) \\ &= 1 - P(X_1 > y, X_2 > y, \dots, X_n > y) \\ &= 1 - \left(\frac{k-y}{k}\right)^n. \end{aligned}$$

Similarly, $P(Y \leq y-1) = 1 - \left(\frac{k-y+1}{k}\right)^n$.

Hence

$$m(y) = P(Y = y) = P(Y \leq y) - P(Y \leq y-1) = \frac{(k-y+1)^n - (k-y)^n}{k^n}.$$

Problem 2

3 pts

Chapter 5.1 Exercise 7

A dice is rolled until the first time T that a six turns up.

- (a) What is the probability distribution for T ?
- (b) Find $P(T > 3)$.
- (c) Find $P(T > 6 | T > 3)$.

1 pts

$$P(T = n) = \left(\frac{5}{6}\right)^{n-1} \frac{1}{6},$$

for $n = 1, 2, \dots$.

1 pts

$$P(T > 3) = \left(\frac{5}{6}\right)^3 = \frac{125}{216}.$$

1 pts

$$P(T > 6 | T > 3) = P(T > 3) = \frac{125}{216}.$$

Problem 3

3 pts

Chapter 5.1 Exercise 13

The Poisson distribution with parameter $\lambda = 0.3$ has been assigned for the outcome of an experiment. Let X be the outcome function. Find $P(X = 0)$, $P(X = 1)$, and $P(X > 1)$.

1 pts

$$P(X = 0) = e^{-0.3} \approx 0.7408.$$

1 pts

$$P(X = 1) = 0.3e^{-0.3} \approx 0.2222.$$

1 pts

$$P(X > 1) = 1 - P(X = 0) - P(X = 1) \approx 0.0370.$$

Problem 4

3 pts

Chapter 5.1 Exercise 18

A baker blends 600 raisins and 400 chocolate chips into a dough mix and, from this, makes 500 cookies.

- (a) Find the probability that a randomly picked cookie will have no raisins.
- (b) Find the probability that a randomly picked cookie will have exactly two chocolate chips.
- (c) Find the probability that a randomly chosen cookie will have at least two bits (raisins or chips) in it.

1 pts

We know that

$$\lambda = np = 600 \times \frac{1}{500} = \frac{6}{5}.$$

Hence

$$P(X = 0) = e^{-\lambda} = e^{-6/5} = 0.3012.$$

1 pts

Similarly, we know that

$$\lambda = np = 400 \times \frac{1}{500} = \frac{4}{5}.$$

Hence

$$P(X = 2) = \frac{\lambda^2}{2!} e^{-\lambda} = \frac{8}{25} e^{-4/5} = 0.1438.$$

1 pts

Now we have

$$\lambda = np = (600 + 400) \times \frac{1}{500} = 2.$$

Hence

$$\begin{aligned} P(X \geq 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - e^{-\lambda} - \lambda e^{-\lambda} \\ &= 1 - e^{-2} - 2e^{-2} \\ &= 0.5940. \end{aligned}$$

Problem 5

4 pts

Chapter 5.1 Exercise 28

An airline finds that 4 percent of the passengers that make reservations on a particular flight will not show up. Consequently, their policy is to sell 100 reserved seats on a plane that has only 98 seats. Find the probability that every person who shows up for the flight will find a seat available.

4 pts

In this problem, we have

$$\lambda = np = 100 \times 4\% = 4.$$

And the probability is

$$\begin{aligned} P(X \geq 100 - 98) &= P(X \geq 2) \\ &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - e^{-\lambda} - \lambda e^{-\lambda} \\ &= 1 - (1 + 4)e^{-4} \\ &= 0.9084. \end{aligned}$$

Problem 6

4 pts

Chapter 5.1 Exercise 32

It is often assumed that the auto traffic that arrives at the intersection during a unit time period has a Poisson distribution with expected value m . Assume that the number of cars X that arrive at an intersection from the north in unit time has a Poisson distribution with parameter $\lambda = m$ and the number Y that arrive from the west in unit time has a Poisson distribution with parameter $\lambda = \bar{m}$. If X and Y are independent, show that the total number $X + Y$ that arrive at the intersection in unit time has a Poisson distribution with parameter $\lambda = m + \bar{m}$.

4 pts

$$\begin{aligned}
 P(X + Y = k) &= \sum_{i=0}^k P(X = i)P(Y = k - i) \\
 &= \sum_{i=0}^k \frac{m^i e^{-m}}{i!} \frac{\bar{m}^{k-i} e^{-\bar{m}}}{(k-i)!} \\
 &= \sum_{i=0}^k \frac{m^i}{i!} \frac{\bar{m}^{k-i}}{(k-i)!} e^{-(m+\bar{m})} \\
 &= \frac{e^{-(m+\bar{m})}}{k!} \sum_{i=0}^k \frac{k!}{i!(k-i)!} m^i \bar{m}^{k-i} \\
 &= \frac{e^{-(m+\bar{m})}}{k!} \sum_{i=0}^k \binom{k}{i} m^i \bar{m}^{k-i} \\
 &= \frac{(m + \bar{m})^k e^{-(m+\bar{m})}}{k!}
 \end{aligned}$$

Therefore, the total number $X + Y$ follows a Poisson distribution with parameter $\lambda = m + \bar{m}$.

Problem 7

4 pts

Chapter 5.1 Exercise 37

There are an unknown number of moose on Isle Royale (a National Park in Lake Superior). To estimate the number of moose, 50 moose are captured and tagged. Six months later 200 moose are captured and it is found that 8 of these

were tagged. Estimate the number of moose on Isle Royale from these data (no need to verify your guess by computer program).

4 pts

We have

$$N, \quad k = 50, \quad n = 200, \quad x = 8.$$

Assume that

$$\frac{k}{N} = \frac{x}{n}.$$

It follows that

$$N = \frac{kn}{x} = \frac{50 \times 200}{8} = 1250.$$

Problem 8

4 pts

Chapter 5.1 Exercise 38

A manufactured lot of buggy whips has 20 items, of which 5 are defective. A random sample of 5 items is chosen to be inspected. Find the probability that the sample contains exactly one defective item

- (a) if the sampling is done with replacement.
- (b) if the sampling is done without replacement.

2 pts

$$P(X = 1) = B(n, p, k) = B(5, \frac{1}{4}, 1) = \binom{5}{1} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^4 = \frac{405}{1024} = 0.3955.$$

2 pts

$$P(X = 1) = h(N, k, n, x) = h(20, 5, 5, 1) = \frac{\binom{5}{1} \binom{15}{4}}{\binom{20}{5}} \approx 0.4402.$$

Problem 9

4 pts

Chapter 5.1 Exercise 39

Suppose that N and k tend to ∞ in such a way that k/N remains fixed. Show that

$$h(N, k, n, x) \rightarrow b(n, k/N, x).$$

4 pts

$$\begin{aligned}
 h(N, k, n, x) &= \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}} \\
 &= \frac{k!(N-k)!n!(N-n)!}{x!(k-x)!(n-x)!(N-k-n+x)!N!} \\
 &= \frac{n!}{x!(n-x)!} \frac{k!(N-k)!(N-n)!}{(k-x)!(N-k-n+x)!N!} \\
 &= \binom{n}{x} \frac{(k)_x (N-k)!(N-n)!}{(N-k-n+x)!N!} \\
 &= \binom{n}{x} \frac{(k)_x (N-k)_{n-x} (N-n)!}{N!} \\
 &= \binom{n}{x} \frac{(k)_x (N-k)_{n-x}}{N_n} \\
 &\approx \binom{n}{x} \frac{k^x (N-k)^{n-x}}{N^n} \\
 &= \binom{n}{x} \left(\frac{k}{N}\right)^x \left(\frac{N-k}{N}\right)^{n-x}.
 \end{aligned}$$