Math 20: Probability

Homework 8

August 29, 2020

Problem 1

4 pts

Chapter 7.1 Exercise 5

Consider the following two experiments: the first has outcome X taking on the values 0, 1, and 2 with equal probabilities; the second results in an (independent) outcome Y taking on the value 3 with probability $\frac{1}{4}$ and 4 with probability $\frac{3}{4}$. Find the distribution of

(a) $Y + X$.

The distribution of X is

$$p_X = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

And that of Y is

$$p_Y = \begin{pmatrix} 3 & 4 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

2 pts

The range of $Y+X$ is $\{3,4,5,6\}$. We have

$$P(Y+X=3)=P(X=0)P(Y=3)=\frac{1}{3}\frac{1}{4}=\frac{1}{12},$$

$$P(Y+X=4)=P(X=0)P(Y=4)+P(X=1)P(Y=3)=\frac{1}{3}\frac{3}{4}+\frac{1}{3}\frac{1}{4}=\frac{1}{3},$$

$$P(Y+X=5)=P(X=1)P(Y=4)+P(X=2)P(Y=3)=\frac{1}{3}\frac{3}{4}+\frac{1}{3}\frac{1}{4}=\frac{1}{3},$$

and

$$P(Y+X=6)=P(X=2)P(Y=4)=\frac{1}{3}\frac{3}{4}=\frac{1}{4}.$$

That is, the distribution of $Y+X$ is

$$p_{Y+X}=\begin{pmatrix} 3 & 4 & 5 & 6 \\ \frac{1}{12} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}.$$

(b) $Y-X$.

2 pts

The range of $Y-X$ is $\{1,2,3,4\}$. We have

$$P(Y-X=1)=P(X=2)P(Y=3)=\frac{1}{3}\frac{1}{4}=\frac{1}{12},$$

$$P(Y-X=2)=P(X=1)P(Y=3)+P(X=2)P(Y=4)=\frac{1}{3}\frac{1}{4}+\frac{1}{3}\frac{3}{4}=\frac{1}{3},$$

$$P(Y-X=3)=P(X=0)P(Y=3)+P(X=1)P(Y=4)=\frac{1}{3}\frac{3}{4}+\frac{1}{3}\frac{1}{4}=\frac{1}{3},$$

and

$$P(Y-X=4)=P(X=0)P(Y=4)=\frac{1}{3}\frac{3}{4}=\frac{1}{4}.$$

That is, the distribution of $Y-X$ is

$$p_{Y-X}=\begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{12} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}.$$

Problem 2

4 pts

Chapter 7.2 Exercise 5

Suppose that X and Y are independent and $Z = X + Y$. Find f_Z if

(a)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0. & \text{otherwise} \end{cases}$$

$$f_Y(x) = \begin{cases} \mu e^{-\mu x}, & \text{if } x > 0 \\ 0. & \text{otherwise} \end{cases}$$

Assume that $\lambda \neq \mu$.

2 pts

When $Z = X + Y > 0$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \\ &= \int_{0}^{\infty} f_X(z-y) f_Y(y) dy \\ &= \int_{0}^{z} f_X(z-y) f_Y(y) dy \\ &= \int_{0}^{z} \lambda e^{-\lambda(z-y)} \mu e^{-\mu y} dy \\ &= \lambda\mu e^{-\lambda z} \int_{0}^{z} e^{-(\mu-\lambda)y} dy \\ &= \frac{\lambda\mu e^{-\lambda z}}{\mu-\lambda}(1 - e^{-(\mu-\lambda)z}) \\ &= \frac{\lambda\mu}{\mu-\lambda}(e^{-\lambda z} - e^{-\mu z}). \end{aligned}$$

Therefore,

$$f_Z(x) = \begin{cases} \frac{\lambda\mu}{\mu-\lambda}(e^{-\lambda x} - e^{-\mu x}), & \text{if } x > 0 \\ 0. & \text{otherwise} \end{cases}$$

(b)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0. & \text{otherwise} \end{cases}$$

$$f_Y(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0. & \text{otherwise} \end{cases}$$

2 pts

When $0 < Z = X + Y \leq 1$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \\ &= \int_0^1 f_X(z-y) f_Y(y) dy \\ &= \int_0^z f_X(z-y) f_Y(y) dy \\ &= \int_0^z \lambda e^{-\lambda(z-y)} dy \\ &= \lambda e^{-\lambda z} \int_0^z e^{\lambda y} dy \\ &= e^{-\lambda z}(e^{\lambda z} - 1) \\ &= 1 - e^{-\lambda z}. \end{aligned}$$

When $Z > 1$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \\ &= \int_0^1 f_X(z-y) f_Y(y) dy \\ &= \int_0^1 \lambda e^{-\lambda(z-y)} dy \\ &= \lambda e^{-\lambda z} \int_0^1 e^{\lambda y} dy \\ &= e^{-\lambda z}(e^{\lambda} - 1) \\ &= e^{-\lambda(z-1)} - e^{-\lambda z}. \end{aligned}$$

Therefore,

$$f_Z(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } 0 < x \leq 1 \\ e^{-\lambda(x-1)} - e^{-\lambda x}, & \text{if } x > 1 \\ 0. & \text{otherwise} \end{cases}$$

Problem 3

4 pts

Chapter 7.2 Exercise 10

Let X_1, X_2, $\cdots$, X_n be n independent random variables each of which has an exponential density with mean μ. Let M be the minimum value of the X_j. Show that the density for M is exponential with mean $\frac{\mu}{n}$.

Hint: Use cumulative distribution functions.

4 pts

$$\begin{aligned} F_M(x) &= P(M \le x) = 1 - P(M > x) \\ &= 1 - P(\min(X_1, X_2, \cdots, X_n) > x) \\ &= 1 - P(X_1 > x)P(X_2 > x) \cdots P(X_n > x) \\ &= 1 - (e^{-x/\mu})^n \\ &= 1 - e^{-nx/\mu}. \end{aligned}$$

Therefore,

$$f_m(x) = \frac{d}{dx} F_M(x) = \frac{d}{dx}(1 - e^{-nx/\mu}) = \frac{n}{\mu} e^{-nx/\mu}.$$

That is to say, M is exponential with mean $\frac{\mu}{n}$.

Problem 4

3 pts

Chapter 8.2 Exercise 4

Let X be a continuous random variable with values exponentially distributed over $[0, +\infty)$ with parameter $\lambda = 0.1$.

(a) Find the mean and variance of X.

1 pts

$$E(X) = \frac{1}{\lambda} = \frac{1}{0.1} = 10.$$

$$V(X) = \frac{1}{\lambda^2} = \frac{1}{(0.1)^2} = 100.$$

(b) Using Chebyshev's Inequality, find an upper bound for the following probabilities: $P(|X-10| \geq 2)$, $P(|X-10| \geq 5)$, $P(|X-10| \geq 9)$, and $P(|X-10| \geq 20)$.

Note: Are the bounds all useful?

1 pts

$$P(|X-10| \geq 2) \leq \frac{100}{4} = 25.$$

$$P(|X-10| \geq 5) \leq \frac{100}{25} = 4.$$

$$P(|X-10| \geq 9) \leq \frac{100}{81}.$$

$$P(|X-10| \geq 20) \leq \frac{100}{400} = \frac{1}{4}.$$

For the first three probabilities Chebyshev's estimate is greater than 1, and so the best estimate is 1. For the last one Chebyshev's estimate gives an upper bound $\frac{1}{4}$.

(c) Calculate these probabilities exactly, and compare with the bounds in (b).

1 pts

$$P(|X-10| \geq 2) = 1 - \int_{8}^{12} \lambda e^{-\lambda x} dx \approx 0.852.$$

$$P(|X-10| \geq 5) = 1 - \int_{5}^{15} \lambda e^{-\lambda x} dx \approx 0.617.$$

$$P(|X-10| \geq 9) = 1 - \int_{1}^{19} \lambda e^{-\lambda x} dx \approx 0.245.$$

$$P(|X-10| \geq 20) = 1 - \int_{0}^{30} \lambda e^{-\lambda x} dx \approx 0.0498.$$

Comparing these Chebyshev's estimates with the exact values, we have:

$$(1, 0.852), \qquad (1, 0.617), \qquad (1, 0.245), \qquad (0.25, 0.0498).$$

Problem 5

3 pts

Chapter 8.2 Exercise 12

A share of common stock in the Pilsdorff beer company has a price Y_n on the nth business day of the year. Finn observes that the price change $X_n = Y_{n+1} - Y_n$ appears to be a random variable with mean $\mu = 0$ and variance $\sigma^2 = \frac{1}{4}$. If $Y_1 = 30$, find a lower bound for the following probabilities, under the assumption that the X_n's are mutually independent.

(a) $P(25 \leq Y_2 \leq 35)$.

1 pts

We know that $Y_2 = Y_1 + (Y_2 - Y_1) = Y_1 + X_1$. Hence

$$\begin{aligned} P(25 \leq Y_2 \leq 35) &= P(-5 \leq X_1 \leq 5) \\ &= P(|X_1| \leq 5) = 1 - P(|X_1| \geq 5) \\ &\geq 1 - \frac{1/4}{5^2} = \frac{99}{100}. \end{aligned}$$

(b) $P(25 \le Y_{11} \le 35)$.

1 pts

We know that

$$\begin{aligned} Y_{11} &= Y_1 + (Y_2 - Y_1) + (Y_3 - Y_2) + \cdots + (Y_{11} - Y_{10}) \\ &= Y_1 + X_1 + X_2 + \cdots + X_{10}. \end{aligned}$$

Hence

$$\begin{aligned} P(25 \le Y_{11} \le 35) &= P(-5 \le X_1 + X_2 + \cdots + X_{10} \le 5) \\ &= P(|X_1 + X_2 + \cdots + X_{10}| \le 5) \\ &= 1 - P(|X_1 + X_2 + \cdots + X_{10}| \ge 5) \\ &\ge 1 - \frac{10 \times 1/4}{5^2} = \frac{9}{10}. \end{aligned}$$

(c) $P(25 \le Y_{101} \le 35)$.

1 pts

We know that

$$\begin{aligned} Y_{101} &= Y_1 + (Y_2 - Y_1) + (Y_3 - Y_2) + \cdots + (Y_{101} - Y_{100}) \\ &= Y_1 + X_1 + X_2 + \cdots + X_{100}. \end{aligned}$$

Hence

$$\begin{aligned} P(25 \le Y_{101} \le 35) &= P(-5 \le X_1 + X_2 + \cdots + X_{100} \le 5) \\ &= P(|X_1 + X_2 + \cdots + X_{100}| \le 5) \\ &= 1 - P(|X_1 + X_2 + \cdots + X_{100}| \ge 5) \\ &\ge 1 - \frac{100 \times 1/4}{5^2} = 0. \end{aligned}$$

Problem 6

4 pts

Chapter 9.1 Exercise 3

A true-false examination has 48 questions. June has probability $\frac{3}{4}$ of answering a question correctly. April just guesses on each question. A passing score is 30 or more correct answers. Compare the probability that June passes the exam with the probability that April passes it.

Note: You do not need to get a specific number. It is good enough to use the $\mathrm{NA}(a, b)$ notation we have seen in class.

2 pts

For June, we need to consider the probability $i \leq S_n \leq j$ where $i = 30$ and $j = 48$ for a Binomial distribution with parameters $n = 48$ and $p = \frac{3}{4}$.

It is straightforward to see that $np = 36$, $npq = 9$ and $\sqrt{npq} = 3$. Further we obtain that

$$\frac{i - \frac{1}{2} - np}{\sqrt{npq}} = \frac{30 - \frac{1}{2} - 36}{3} = -\frac{13}{6},$$

and

$$\frac{j + \frac{1}{2} - np}{\sqrt{npq}} = \frac{48 + \frac{1}{2} - 36}{3} = -\frac{25}{6},$$

Therefore the probability is

$$P(30 \leq S_n \leq 48) \approx \mathrm{NA}(-\frac{13}{6}, \frac{25}{6}) \approx 0.985.$$

2 pts

For April, we need to consider the probability $i \le S_n \le j$ where $i = 30$ and $j = 48$ for a Binomial distribution with parameters $n = 48$ and $p = \frac{1}{2}$.

It is straightforward to see that $np = 24$, $npq = 12$ and $\sqrt{npq} = \sqrt{12}$. Further we obtain that

$$\frac{i - \frac{1}{2} - np}{\sqrt{npq}} = \frac{30 - \frac{1}{2} - 24}{\sqrt{12}} = \frac{11}{2\sqrt{12}},$$

and

$$\frac{j + \frac{1}{2} - np}{\sqrt{npq}} = \frac{48 + \frac{1}{2} - 24}{\sqrt{12}} = \frac{49}{2\sqrt{12}},$$

Therefore the probability is

$$P(30 \le S_n \le 48) \approx \mathrm{NA}(\frac{11}{2\sqrt{12}}, \frac{49}{2\sqrt{12}}) \approx 0.056.$$

Problem 7

5 pts

Chapter 9.2 Exercise 6

A bank accepts rolls of pennies and gives 50 cents credit to a customer without counting the contents. Assume that a roll contains 49 pennies 30 percent of the time, 50 pennies 60 percent of the time, and 51 pennies 10 percent of the time.

Note: You can simply use the $\mathrm{NA}(a, b)$ notation.

(a) Find the expected value and the variance for the amount that the bank loses on a typical roll.

1 pts

Let X be the amount that the bank loses on a typical roll.

$$E(X) = 1 \times 30\% + 0 \times 60\% - 1 \times 10\% = 0.2.$$

$$V(X) = E(X^2) - E^2(X) = 1^2 \times 30\% + 0^2 \times 60\% + (-1)^2 \times 10\% - (0.2)^2 = 0.36.$$

(b) Estimate the probability that the bank will lose more than 25 cents in 100 rolls.

1 pts

We need to consider the probability $c \leq S_n \leq d$ where $c = 26$ and $d = 100$ for an independent trials process with $n = 100$.

It is straightforward to see that $n\mu = 20$, $n\sigma^2 = 36$ and $\sqrt{n\sigma^2} = 6$. Further we obtain that

$$\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{26 - 20}{6} = 1,$$

and

$$\frac{d - n\mu}{\sqrt{n\sigma^2}} = \frac{100 - 20}{6} = \frac{40}{3}.$$

Therefore the probability is

$$P(26 \leq S_n \leq 100) \approx \text{NA}(1, \frac{40}{3}) \approx 0.1587.$$

(c) Estimate the probability that the bank will lose exactly 25 cents in 100 rolls.

1 pts

We need to consider the probability $S_n = k$ where $k = 25$ for an independent trials process with $n = 100$.

We obtain that

$$\frac{k - n\mu}{\sqrt{n\sigma^2}} = \frac{25 - 20}{6} = \frac{5}{6}.$$

Therefore the probability is

$$P(S_n = 25) \approx \frac{1}{6}\phi(\frac{5}{6}) \approx 0.047.$$

(d) Estimate the probability that the bank will lose any money in 100 rolls.

1 pts

We need to consider the probability $c \le S_n \le d$ where $c = 1$ and $d = 100$ for an independent trials process with $n = 100$.

Similarly, we obtain that

$$\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{1 - 20}{6} = -\frac{19}{6}.$$

Therefore the probability is

$$P(1 \le S_n \le 100) \approx \mathrm{NA}(-\frac{19}{6}, \frac{40}{3}) \approx 0.999.$$

(e) How many rolls does the bank need to collect to have a 99 percent chance of a net loss?

1 pts

We need to consider the probability $c \le S_n \le d$ where $c = 1$ and $d = 100$ for an independent trials process with n to be determined.

The probability is

$$P(1 \le S_n \le n) \approx \mathrm{NA}(\frac{1 - 0.2n}{0.6\sqrt{n}}, \frac{n - 0.2n}{0.6\sqrt{n}}).$$

Notice that the upper bound is $\frac{n-0.2n}{0.6\sqrt{n}} = \frac{4\sqrt{n}}{3}$. If n is large enough (say $n \ge 10$), it can be considered approximately the same as $+\infty$ for the standard normal distribution. Therefore,

$$P(1 \le S_n \le n) \approx \mathrm{NA}(\frac{1 - 0.2n}{0.6\sqrt{n}}, +\infty) \ge 0.99.$$

That is,

$$\mathrm{NA}(-\infty, \frac{1 - 0.2n}{0.6\sqrt{n}}) = 1 - \mathrm{NA}(\frac{1 - 0.2n}{0.6\sqrt{n}}, +\infty) \le 0.01.$$

Referring to a z table, we get

$$\frac{1 - 0.2n}{0.6\sqrt{n}} \le -2.33.$$

And the smallest n satisfying the above inequality is 59.

Problem 8

3 pts

Chapter 9.3 Exercise 11

The price of one share of stock in the Pilsdorff Beer Company (see Problem 5) is given by Y_n on the nth day of the year. Finn observes that the differences $X_n = Y_{n+1} - Y_n$ appears to be independent random variable with a common distribution having mean $\mu = 0$ and variance $\sigma^2 = \frac{1}{4}$. If $Y_1 = 100$, estimate the probability that Y_{365} is

Note: For part (a), the answer can be obtained without using a calculator. For part (b) and (c), you can simply use the $\mathrm{NA}(a, b)$ notation.

(a) ≥ 100.

1 pts

Same as Problem 5, we can rewrite Y_{365} as $Y_1 + \sum_{i=1}^{364} X_i$.

We need to consider the probability $c \leq S_n \leq d$ where $c = 0$ and $d = +\infty$ for an independent trials process with $n = 364$.
It is straightforward to see that $n\mu = 0$, $n\sigma^2 = 91$ and $\sqrt{n\sigma^2} = \sqrt{91}$. Further we obtain that

$$\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{0 - 0}{\sqrt{91}} = 0,$$

and

$$\frac{d - n\mu}{\sqrt{n\sigma^2}} = \frac{+\infty - 0}{\sqrt{91}} = \infty.$$

Therefore the probability is

$$P(0 \leq S_n \leq \infty) \approx \mathrm{NA}(0, \infty) = 0.5.$$

(b) ≥ 110.

1 pts

We need to consider the probability $c \leq S_n \leq d$ where $c = 10$ and $d = +\infty$ for an independent trials process with $n = 364$.
Now we have

$$\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{10 - 0}{\sqrt{91}} = \frac{10}{\sqrt{91}}.$$

Therefore the probability is

$$P(0 \leq S_n \leq \infty) \approx \text{NA}(\frac{10}{\sqrt{91}}, \infty) \approx 0.147.$$

(c) ≥ 120.

1 pts

We need to consider the probability $c \leq S_n \leq d$ where $c = 20$ and $d = +\infty$ for an independent trials process with $n = 364$.
Now we have

$$\frac{c - n\mu}{\sqrt{n\sigma^2}} = \frac{20 - 0}{\sqrt{91}} = \frac{20}{\sqrt{91}}.$$

Therefore the probability is

$$P(0 \leq S_n \leq \infty) \approx \text{NA}(\frac{20}{\sqrt{91}}, \infty) \approx 0.018.$$