

# Math 20: Probability

## Homework 2 Solution

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### Problem 1

4 pts

#### Chapter 1.2 Exercise 6

A dice is loaded in such a way that the probability of each face turning up is proportional to the number of dots on that face. (For example, a six is three times as probable as a two.) What is the probability of getting an even number in one throw?

Assume that the probability of getting 1 is  $p$ . Then the probability of getting  $n$  is  $np$ , where  $n \in \{1, 2, 3, 4, 5, 6\}$ .

Notice that  $\sum_{n=1}^6 np = 1$ . That is,  $21p = 1$  and we get  $p = \frac{1}{21}$ .

To get an even number, we need  $n = 2, 4$  or  $6$ . Therefore, the probability is

$$(2 + 4 + 6) \cdot \frac{1}{21} = \frac{4}{7}.$$

### Problem 2

4 pts

#### Chapter 1.2 Exercise 14

Let  $X$  be a random variable with distribution function  $m_X(x)$  defined by

$$m_X(-1) = \frac{1}{5}, \quad m_X(0) = \frac{1}{5}, \quad m_X(1) = \frac{2}{5}, \quad m_X(2) = \frac{1}{5}.$$

- (a) Let  $Y$  be the random variable defined by the equation  $Y = X + 3$ . Find the distribution function  $m_Y(y)$  of  $Y$ .
- (b) Let  $Z$  be the random variable defined by the equation  $Z = X^2$ . Find the distribution function  $m_Z(z)$  of  $Z$ .

(a)

$$m_Y(y) = \begin{cases} \frac{1}{5}, & Y = 2, 3 \text{ or } 5 \\ \frac{2}{5}, & Y = 4 \end{cases}$$

(b)

$$m_Z(z) = \begin{cases} \frac{1}{5}, & Z = 0 \\ \frac{3}{5}, & Z = 1 \\ \frac{1}{5}, & Z = 4 \end{cases}$$

### Problem 3

4 pts

#### Chapter 1.2 Exercise 22

A dice is rolled until the first time that a six turns up. We shall see that the probability that this occurs on the  $n$ th roll is  $(\frac{5}{6})^{n-1} \frac{1}{6}$ . Using this fact, describe the appropriate infinite sample space and distribution function for the experiment of rolling a dice until a six turns up for the first time. Verify that for your distribution function  $\sum_{\omega} m(\omega) = 1$ .

- Sample space:  $\Omega = \{1, 2, \dots, +\infty\}$
- Distribution function:  $m(\omega) = (\frac{5}{6})^{\omega-1} \frac{1}{6}$

$$\sum_{\omega=1}^{+\infty} m(\omega) = \sum_{\omega=1}^{+\infty} (\frac{5}{6})^{\omega-1} \frac{1}{6} = \frac{1}{6} \cdot \frac{1}{1 - \frac{5}{6}} = 1.$$

## Problem 4

4 pts

### Chapter 1.2 Exercise 26

Two cards are drawn successively from a deck of 52 cards. Find the probability that the second card is higher in rank than the first card. Hint: Show that  $1 = P(\text{higher}) + P(\text{lower}) + P(\text{same})$  and use the fact that  $P(\text{higher}) = P(\text{lower})$ .

The possible outcomes of drawing two cards from a deck of 52 cards are

- the first card is higher in rank than the second card,
- the second card is higher in rank than the first card, and
- the ranks of the two cards are the same.

Therefore,  $1 = P(\text{higher}) + P(\text{lower}) + P(\text{same})$ .

By symmetry, we know that  $P(\text{higher}) = P(\text{lower})$ . That is, to get the probability that the second card is higher in rank than the first card, we only need to figure out  $P(\text{same})$ , which is

$$13 \cdot \frac{4 \cdot 3}{52 \cdot 51} = \frac{1}{17}.$$

And hence

$$P(\text{higher}) = P(\text{lower}) = \frac{1}{2} \cdot \left(1 - \frac{1}{17}\right) = \frac{8}{17}.$$

## Problem 5

5 pts

### Chapter 2.2 Exercise 2

Suppose you choose a real number  $X$  from the interval  $[2, 10]$  with a density function of the form

$$f(x) = Cx,$$

where  $C$  is a constant.

- Find  $C$ .
- Find  $P(E)$ , where  $E = [a, b]$  is a subinterval of  $[2, 10]$ .

(c) Find  $P(X > 5)$ ,  $P(X < 7)$ , and  $P(X^2 - 12X + 35 > 0)$ .

(a) Given that

$$\int_2^{10} f(x)dx = \int_2^{10} Cx dx = \frac{C}{2} x^2 \Big|_2^{10} = 48C = 1,$$

we obtain  $C = \frac{1}{48}$ .

(b)

$$P(E) = P([a, b]) = \int_a^b f(x)dx = \int_a^b \frac{1}{48} x dx = \frac{b^2 - a^2}{96}.$$

(c)

$$P(X > 5) = \int_5^{10} f(x)dx = \int_5^{10} \frac{1}{48} x dx = \frac{75}{96} = \frac{25}{32}.$$

$$P(X < 7) = \int_2^7 f(x)dx = \int_2^7 \frac{1}{48} x dx = \frac{45}{96} = \frac{15}{32}.$$

Solving the inequality  $X^2 - 12X + 35 = (X - 5)(X - 7) > 0$  yields  $X < 5$  or  $X > 7$ . Hence

$$\begin{aligned} P(X^2 - 12X + 35 > 0) &= P(X < 5) + P(X > 7) \\ &= 1 - P(X > 5) + 1 - P(X < 7) \\ &= 1 - \frac{25}{32} + 1 - \frac{15}{32} \\ &= \frac{3}{4}. \end{aligned}$$

## Problem 6

4 pts

### Chapter 2.2 Exercise 5

Suppose you are watching a radioactive source that emits particles at a rate described by the exponential density

$$f(t) = \lambda e^{-\lambda t},$$

where  $\lambda = 1$ , so that the probability  $P(0, T)$  that a particle will appear in the next  $T$  seconds is  $P([0, T]) = \int_0^T \lambda e^{-\lambda t} dt$ . Find the probability that a particle (not necessary the first) will appear

- (a) within the next second.
- (b) within the next 3 seconds.
- (c) between 3 and 4 seconds from now.
- (d) after 4 seconds from now.

(a)

$$P([0, 1]) = \int_0^1 \lambda e^{-\lambda t} dt = -e^{-\lambda t} \Big|_0^1 = 1 - e^{-\lambda} = 1 - e^{-1} = 1 - \frac{1}{e}.$$

(b)

$$P([0, 3]) = \int_0^3 \lambda e^{-\lambda t} dt = -e^{-\lambda t} \Big|_0^3 = 1 - e^{-3\lambda} = 1 - e^{-3} = 1 - \frac{1}{e^3}.$$

(c) Notice the phrasing “**not necessary the first**”.

$$P([0, 4 - 3]) = P([0, 1]) = 1 - \frac{1}{e}.$$

(d) Similar to the last part,

$$P([0, +\infty - 4]) = P([0, +\infty)) = 1.$$

## Problem 7

4 pts

### Chapter 2.2 Exercise 6

Assume that a new light bulb will burn out after  $t$  hours, where  $t$  is chosen from  $[0, \infty)$  with an exponential density

$$f(t) = \lambda e^{-\lambda t}.$$

In this context,  $\lambda$  is often called the **failure rate** of the bulb.

(a) Assume that  $\lambda = 0.01$ , and find the probability that the bulb will not burn out before  $T$  hours. This probability is often called the **reliability** of the bulb.

(b) For what  $T$  is the reliability of the bulb  $= \frac{1}{2}$ .

(a)

$$P([T, +\infty)) = \int_T^{+\infty} f(t) dt = \int_T^{+\infty} \lambda e^{-\lambda t} dt = e^{-\lambda T} = e^{-T/100}.$$

(b) Since  $P([T, +\infty)) = e^{-T/100} = \frac{1}{2}$ , we have  $T = 100 \ln(2)$ .

## Problem 8

4 pts

### Chapter 5.2 Exercise 2

Choose a number  $U$  from the interval  $[0, 1]$  with uniform distribution. Find the cumulative distribution and density for the random variables

(a)  $Y = \frac{1}{U+1}$ .

(b)  $Y = \log(U + 1)$ .

(a) The cumulative distribution function is

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P\left(\frac{1}{U+1} \leq y\right) \\ &= P\left(U \geq \frac{1}{y} - 1\right) \\ &= 2 - \frac{1}{y}, \end{aligned}$$

where  $\frac{1}{2} \leq y \leq 1$ .

Further we have

$$f(y) = \frac{d}{dy} F_Y(y) = \frac{1}{y^2}.$$

(b) The cumulative distribution function is

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\log(U + 1) \leq y) \\ &= P(U \leq e^y - 1) \\ &= e^y - 1, \end{aligned}$$

where  $0 \leq y \leq \log(2)$ .

Further we have

$$f(y) = \frac{d}{dy} F_Y(y) = e^y.$$

## Problem 9

4 pts

### Chapter 5.2 Exercise 10

Let  $U, V$  be random numbers chosen independently from the interval  $[0, 1]$ . Find the cumulative distribution and density for the random variables

(a)  $Y = \max(U, V)$ .

(b)  $Y = \min(U, V)$ .

(a)

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\max(U, V) \leq y) \\ &= P(U \leq y \text{ and } V \leq y) \\ &= P(U \leq y \cap V \leq y) \\ &= P(U \leq y) \cdot P(V \leq y) \\ &= y^2, \end{aligned}$$

where  $0 \leq y \leq 1$ .

Further we have

$$f(y) = \frac{d}{dy} F_Y(y) = 2y.$$

(b)

$$\begin{aligned} 1 - F_Y(y) &= P(Y > y) = P(\min(U, V) > y) \\ &= P(U > y \text{ and } V > y) \\ &= P(U > y \cap V > y) \\ &= P(U > y) \cdot P(V > y) \\ &= (1 - y)^2, \end{aligned}$$

where  $0 \leq y \leq 1$ . And hence

$$F_Y(y) = 1 - (1 - y)^2.$$

Further we have

$$f(y) = \frac{d}{dy} F_Y(y) = 2(1 - y).$$