Math 20: Probability

Homework 4 Solution

July 17, 2020

Problem 1

4 pts

Chapter 4.1 Exercise 3

A dice is rolled twice. What is the probability that the sum of the faces is greater than 7, given that

(a) the first outcome was a 4?

Notice that

$$4+4=8>7, \qquad 4+5=9>7, \qquad 4+6=10>7.$$

Therefore,

$$P(X+Y>7\,|\,X=4)=\frac{3}{6}=\frac{1}{2}.$$

(b) the first outcome was greater than 3?

Similarly,

$$4+4=8, \qquad 4+5=9, \qquad 4+6=10,$$

$$5+3=8, \qquad 5+4=9, \qquad 5+5=10, \qquad 5+6=11,$$

and

$$6+2=8, \qquad 6+3=9, \qquad 6+4=10, \qquad 6+5=11, \qquad 6+6=12.$$

Out of the $3\times 6=18$ options, there are $3+4+5=12$ pairs of X and Y with $X+Y>7$. Therefore,

$$P(X+Y>7\,|\,X>3)=\frac{12}{18}=\frac{2}{3}.$$

(c) the first outcome was a 1?

Applying the same method as the first two parts,

$$P(X+Y>7 \mid X=1) = \frac{0}{6} = 0.$$

(d) the first outcome was less than 5?

Applying the same method as the first two parts,

$$P(X+Y>7 \mid X<5) = \frac{0+1+2+3}{4\times 6} = \frac{1}{4}.$$

Problem 2

4 pts

Chapter 4.1 Exercise 14

If $P(\tilde{B}) = \frac{1}{4}$ and $P(A \mid B) = \frac{1}{2}$, what is $P(A\cap B)$?

Since $P(\tilde{B}) = \frac{1}{4}$, we know that $P(B) = 1 - \frac{1}{4} = \frac{3}{4}$. Then we have

$$P(A\cap B) = P(A \mid B)P(B) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}.$$

Problem 3

4 pts

Chapter 4.1 Exercise 17

Prove that if A and B are independent so are

(a) A and $\tilde{B}$.

If A and B are independent, then $P(A\cap B) = P(A)P(B)$. Hence,

$$P(A\cap\tilde{B}) = P(A)-P(A\cap B) = P(A)-P(A)P(B) = P(A)(1-P(B)) = P(A)P(\tilde{B}).$$

That is, A and $\tilde{B}$ are independent.

(b) $\tilde{A}$ and $\tilde{B}$.

Since we have proven that A and $\tilde{B}$ are independent, we can use this as a fact to prove this part.

Now consider $P(\tilde{A} \cap \tilde{B})$, which can be rewritten as

$$P(\tilde{B}) - P(A \cap \tilde{B}) = P(\tilde{B}) - P(A)P(\tilde{B}) = (1 - P(A))P(\tilde{B}) = P(\tilde{A})P(\tilde{B}).$$

Therefore, $\tilde{A}$ and $\tilde{B}$ are independent.

Problem 4

4 pts

Chapter 4.1 Exercise 18

A doctor assumes that a patient has one of three diseases d_1, d_2 or d_3. Before any test, he assumes that an equal probability for each disease. He carries out a test that will be positive with probability 0.8 if the patient has d_1, 0.6 if he has disease d_2 and 0.4 if he has disease d_3. Given that the outcome of the test was positive, what probabilities should the doctor now assign to the three possible diseases?

We are after

$$P(d_1 \,|\, +), \qquad P(d_2 \,|\, +), \qquad P(d_3 \,|\, +).$$

And what are given to us are

$$P(d_1) = P(d_2) = P(d_3) = \frac{1}{3},$$

$$P(+ \,|\, d_1) = 0.8, \qquad P(+ \,|\, d_2) = 0.6, \qquad P(+ \,|\, d_3) = 0.4.$$

Thus we can obtain

$$P(+) = P(+ \,|\, d_1)P(d_1) + P(+ \,|\, d_2)P(d_2) + P(+ \,|\, d_3)P(d_3) = \frac{1}{3} \times (0.8 + 0.6 + 0.4) = 0.6.$$

Therefore,

$$P(d_1 \,|\, +) = \frac{P(d_1)P(+ \,|\, d_1)}{P(+)} = \frac{\frac{1}{3} \times 0.8}{0.6} = \frac{4}{9},$$

$$P(d_2 \,|\, +) = \frac{P(d_2)P(+ \,|\, d_2)}{P(+)} = \frac{\frac{1}{3} \times 0.6}{0.6} = \frac{1}{3},$$

and

$$P(d_3 \,|\, +) = \frac{P(d_3)P(+ \,|\, d_3)}{P(+)} = \frac{\frac{1}{3} \times 0.4}{0.6} = \frac{2}{9}.$$

Problem 5

4 pts

Chapter 4.1 Exercise 29

A student is applying to Harvard and Dartmouth. He estimates that he has a probability of 0.5 of being accepted at Dartmouth and 0.3 of being accepted at Harvard. He further estimates the probability that he will be accepted by both is 0.2. What is the probability that he is accepted by Dartmouth if he is accepted by Harvard? Is the event "accepted at Harvard" independent of the event "accepted at Dartmouth"?

We know that

$$P(\textbf{Dartmouth}) = 0.5, \qquad P(\textbf{Harvard}) = 0.3, \qquad P(\textbf{Dartmouth} \cap \textbf{Harvard}) = 0.2.$$

Hence

$$P(\textbf{Dartmouth} \,|\, \textbf{Harvard}) = \frac{P(\textbf{Dartmouth} \cap \textbf{Harvard})}{P(\textbf{Harvard})} = \frac{2}{3}.$$

Notice that $0.5 \neq \frac{2}{3}$. That is,

$$P(\textbf{Dartmouth}) \neq P(\textbf{Dartmouth} \,|\, \textbf{Harvard}).$$

The two events are not independent.

Problem 6

4 pts

Chapter 4.1 Exercise 36

A dice is thrown twice. Let X_1 and X_2 denote the outcomes. Define $X = \min(X_1, X_2)$. Find the distribution of X.

The possible values for X are $1, 2, 3, 4, 5$ and 6.

Notice that X_1 and X_2 are independent!

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$$\begin{aligned} P(X = 1) &= P(X_1 = 1 \cup X_2 = 1) \\ &= P(X_1 = 1) + P(X_2 = 1) - P(X_1 = 1 \cap X_2 = 1) \\ &= \frac{1}{6} + \frac{1}{6} - \frac{1}{36} \\ &= \frac{11}{36} \end{aligned}$$

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$$\begin{aligned} P(X = 2) &= P(X_2 \geq 2 \,|\, X_1 = 2)P(X_1 = 2) + P(X_2 = 2 \,|\, X_1 \geq 3)P(X_1 \geq 3) \\ &= P(X_2 \geq 2)P(X_1 = 2) + P(X_2 = 2)P(X_1 \geq 3) \\ &= \frac{5}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{4}{6} \\ &= \frac{9}{36} = \frac{1}{4}. \end{aligned}$$

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$$\begin{aligned} P(X = 3) &= P(X_2 \geq 3 \,|\, X_1 = 3)P(X_1 = 3) + P(X_2 = 3 \,|\, X_1 \geq 4)P(X_1 \geq 4) \\ &= P(X_2 \geq 3)P(X_1 = 3) + P(X_2 = 3)P(X_1 \geq 4) \\ &= \frac{4}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{3}{6} \\ &= \frac{7}{36}. \end{aligned}$$

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$$\begin{aligned} P(X = 4) &= P(X_2 \geq 4 \,|\, X_1 = 4)P(X_1 = 4) + P(X_2 = 4 \,|\, X_1 \geq 5)P(X_1 \geq 5) \\ &= P(X_2 \geq 4)P(X_1 = 4) + P(X_2 = 4)P(X_1 \geq 5) \\ &= \frac{3}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{2}{6} \\ &= \frac{5}{36}. \end{aligned}$$

- $$\begin{aligned}P(X=5) &= P(X_2 \geq 5 \,|\, X_1 = 5)P(X_1 = 5) + P(X_2 = 5 \,|\, X_1 \geq 6)P(X_1 \geq 6) \\ &= P(X_2 \geq 5)P(X_1 = 5) + P(X_2 = 5)P(X_1 \geq 6) \\ &= \frac{2}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} \\ &= \frac{3}{36} = \frac{1}{12}.\end{aligned}$$

- $$P(X=6) = P(X_1 = 6 \cap X_2 = 6) = P(X_1 = 6)P(X_2 = 6) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}.$$

  Therefore, the distribution of X is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{11}{36} & \frac{9}{36} & \frac{7}{36} & \frac{5}{36} & \frac{3}{36} & \frac{1}{36} \end{pmatrix}.$$

Problem 7

4 pts

Chapter 4.2 Exercise 1

Pick a point x at random (with uniform density) in the interval $[0, 1]$. Find the probability that $x > \frac{1}{2}$, given that

(a) $x > \frac{1}{4}$.

$$P(x > \frac{1}{2} \,|\, x > \frac{1}{4}) = \frac{1 - \frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}.$$

(b) $x < \frac{3}{4}$.

$$P(x > \frac{1}{2} \,|\, x < \frac{3}{4}) = \frac{\frac{3}{4} - \frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}.$$

(c) $|x - \frac{1}{2}| < \frac{1}{4}$.

$$P(x > \frac{1}{2} \,|\, |x - \frac{1}{2}| < \frac{1}{4}) = \frac{1}{2}.$$

(d) $x^2 - x + \frac{2}{9} < 0$.

Solving the above inequality $(x - \frac{1}{3})(x - \frac{2}{3}) < 0$ yields $\frac{1}{3} < x < \frac{2}{3}$.

$$P(x > \frac{1}{2} \,|\, \frac{1}{3} < x < \frac{2}{3}) = \frac{\frac{2}{3} - \frac{1}{2}}{\frac{2}{3} - \frac{1}{3}} = \frac{1}{2}.$$

Problem 8

4 pts

Chapter 4.2 Exercise 2

A radioactive material emits α-particles at a rate described by the density function

$$f(t) = 0.1e^{-0.1t}.$$

Find the probability that a particle is emitted in the first 10 seconds, given that

(a) no particle is emitted in the first second.

$$P(T \leq 10 \,|\, T > 1) = 1 - P(T > 10 \,|\, T > 1) = 1 - P(T > 9) = P(T \leq 9) = 1 - e^{-0.9}.$$

(b) no particle is emitted in the first 5 seconds.

$$P(T \leq 10 \,|\, T > 5) = 1 - P(T > 10 \,|\, T > 5) = 1 - P(T > 5) = P(T \leq 5) = 1 - e^{-0.5}.$$

(c) a particle is emitted in the first 3 seconds.

$$P(T \leq 10 \,|\, T \leq 3) = 1.$$

(d) a particle is emitted in the first 20 seconds.

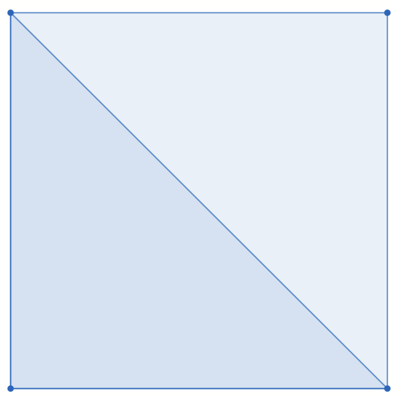
$$P(T \leq 10 \,|\, T \leq 20) = \frac{P(T \leq 10 \cap T \leq 20)}{P(T \leq 20)} = \frac{P(T \leq 10)}{P(T \leq 20)} = \frac{1 - e^{-1}}{1 - e^{-2}} = \frac{e^2 - e}{e^2 - 1} = \frac{e}{e + 1}.$$

Problem 9

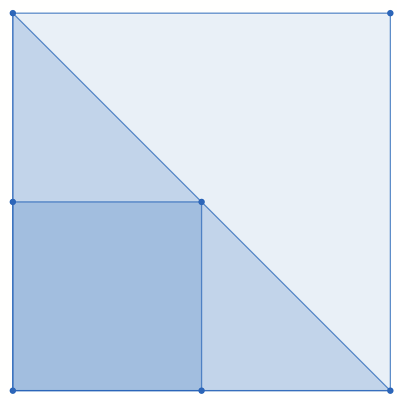
4 pts

Chapter 4.2 Exercise 5 (a) and (c)

Suppose you choose two numbers x and y, independently at random from the interval $[0, 1]$. Given that their sum lies in the interval $[0, 1]$, find the probability that



(a)



(c)

(a) $|x - y| < 1$.

$$P(|x - y| < 1 \,|\, 0 \leq x + y \leq 1) = 1.$$

(b)

(c) $\max\{x, y\} < \frac{1}{2}$.

$$P(\max\{x, y\} < \frac{1}{2} \,|\, 0 \leq x + y \leq 1) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

(d)

(e)