Math 20: Probability

Homework 3 Solution

July 10, 2020

Please specify whether you complete the homework independently or cooperate with (get help from) the TA, your classmates, friends, online resources, etc.

For every problem, show the process and necessary details.

A pdf file is required for submission.

Problem 1

4 pts

Chapter 3.1 Exercise 6

In arranging people around a circular table, we take into account their seats relative to each other, not the actual position of any one person. Show that n people can be arranged around a circular table in $(n-1)!$ ways.

If we consider arranging people in a row, it can be done in $n!$ ways.

Notice that for a circular table, the n permutations

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{pmatrix}, \qquad \ldots, \qquad \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}$$

are equivalent for any permutation $a = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \end{pmatrix}$.

Therefore, n people can be arranged around a circular table in $\frac{n!}{n} = (n-1)!$ ways.

Problem 2

4 pts

Chapter 3.1 Exercise 8

A finite set Ω has n elements. Show that if we count the empty set and Ω as subsets, there are 2^n subsets of Ω.

Hint: Consider the number of ways to generate a subset: for every single element, we need to decide whether to include it in the subset.

Assume that $\Omega = a_1, a_2, \cdots, a_n$ and we need to generate a subset of Ω. For every single element a_k, there are two possibilities: a_k is in the subset and a_k is not in the subset. That is, for n elements, there are in total 2^n possibilities.

More to say about this problem:

We also know that there are $\binom{n}{k}$ ways to choose a subset of size k. Hence if we include the empty set and Ω as subsets, there are altogether

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

subsets. In the light of the above analysis, we can obtain the identity

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Problem 3

4 pts

Chapter 3.1 Exercise 10

A deck of ordinary cards is shuffled and 13 cards are dealt. What is the probability that the last card dealt is an ace?

Hint: There are 13 ranks in total for a deck of ordinary cards. By symmetry, the probability that the last card dealt is an ace is the same as the probability that the last card is a $2/3/\cdots/$King.

Let random variable X be the rank of the last card dealt. The sample space for X is $\Omega = \{1(\text{Ace}), 2, 3, \cdots, \text{Jack}, \text{Queen}, \text{King}\}$. And we have

$$\sum_{x\in\Omega} P(X = x) = 1.$$

By symmetry, we also have

$$P(X = 1(\text{Ace})) = P(X = 2) = \cdots = P(X = \text{King}).$$

Combining the above two equations, we get

$$P(X = 1(\text{Ace})) = \frac{1}{13}.$$

Problem 4

4 pts

Chapter 3.1 Exercise 13

A certain state has license plates showing three numbers and three letters. How many different license plates are possible

(a) if the numbers must come before the letters?

(b) if there is no restriction on where the letters and numbers appear?

Hint: a single number can be $0, 1, \cdots, 9$ and a single letter can be $A, B, \cdots, Z$ (we only consider capitalized letters).

(a) $10^3 \times 26^3$

(b) $\binom{6}{3} \times 10^3 \times 26^3$

Problem 5

4 pts

Chapter 3.2 Exercise 7

Show that

$$b(n, p, j) = \frac{p}{q}(\frac{n - j + 1}{j})b(n, p, j - 1),$$

for $j \geq 1$. Use this fact to determine the value or values of j which give $b(n, p, j)$ its greatest value.

Hint: Consider the successive ratios as j increases.

Assume that $q = 1 - p$.

We know that

$$b(n,p,j) = \binom{n}{j} p^j q^{n-j} = \frac{n!}{j!(n-j)!} p^j q^{n-j}$$

and that

$$\begin{aligned}
b(n,p,j-1) &= \binom{n}{j-1} p^{j-1} q^{n-j+1} \\
&= \frac{n!}{(j-1)!(n-j+1)!} p^{j-1} q^{n-j+1} \\
&= \frac{jq}{(n-j+1)p} \frac{n!}{j!(n-j)!} p^j q^{n-j}.
\end{aligned}$$

Therefore

$$b(n,p,j) = \frac{p}{q}\left(\frac{n-j+1}{j}\right) b(n,p,j-1).$$

Problem 6

4 pts

Chapter 3.2 Exercise 8

A dice is rolled 30 times. What is the probability that a 6 turns up exactly 5 times? What is the most probable number of times that a 6 will turn up?

Hint: Use Problem 5 to solve the second question. Notice that if k is the most probable number, we will always have

$$b(n,p,k-1) \leq b(n,p,k), \qquad b(n,p,k+1) \leq b(n,p,k).$$

In this problem, the parameters are

$$n = 30, \qquad p = \frac{1}{6}, \qquad q = \frac{5}{6}.$$

(a) $\binom{30}{5}(\frac{1}{6})^5(\frac{5}{6})^{25}$.

(b) Let k be the most probable number. It follows that

$$b(n,p,k-1) \leq b(n,p,k), \qquad b(n,p,k+1) \leq b(n,p,k),$$

which correspond to

$$\frac{p}{q}\left(\frac{n-k+1}{k}\right) \geq 1, \qquad \frac{p}{q}\left(\frac{n-k}{k+1}\right) \leq 1.$$

That is,

$$\frac{1}{5}(\frac{31-k}{k}) \geq 1, \qquad \frac{1}{5}(\frac{30-k}{k+1}) \leq 1.$$

Solving the above two inequalities yields

$$k \leq \frac{31}{6}, \qquad k \geq \frac{25}{6},$$

which further yields

$$k = 5.$$

Problem 7

4 pts

Chapter 3.2 Exercise 11

A restaurant offers apple and blueberry pies and stocks an equal number of each kind of pie. Each day ten customers request pie. They choose, with equal probabilities, one of the two kinds of pie. How many pieces of each kind of pie should the owner provide so that the probability is about 0.95 that each customer gets the pie of his or her own choice?

Hint: Still, it is a binomial distribution problem.

Given that there are ten customers, we have $n = 10$. And since the customers choose one of the two kinds of pie with equal probabilities, if we define **choosing apple pie** as **success**, we get $p = \frac{1}{2}$ and the probability of k customers choosing apple pies will be $b(n, p, k)$.

Suppose that the restaurant provide m pieces of each kind of pie. When there are k customers choosing apple pies, naturally the rest $n - k$ customers choose blueberry pies. As long as $k \leq m$ and also $n - k \leq m$, each customer can get the pie of his or her own choice.

We need to consider the probability

$$p = \sum_{k=n-m}^{m} \binom{n}{k} p^k q^{n-k},$$

which can be further specified as

$$p = \sum_{k=10-m}^{m} \binom{10}{k} (\frac{1}{2})^{10} = .$$

Solving the inequality

$$p \geq 0.95$$

is equivalent to solve

$$1 - p = \sum_{k=0}^{9-m} \binom{10}{k} (\frac{1}{2})^{10} + \sum_{k=m+1}^{10} \binom{10}{k} (\frac{1}{2})^{10} \leq 0.05$$

.

Notice that we always have $\binom{n}{k} = \binom{n}{n-k}$, the above inequality can be simplified as

$$1 - p = 2 \sum_{k=0}^{9-m} \binom{10}{k} (\frac{1}{2})^{10} = \sum_{k=0}^{9-m} \binom{10}{k} (\frac{1}{2})^{9} \leq 0.05$$

.

Using a computer, we find that $9 - m \leq 1$ and thus $m \geq 8$.

The owner has to provide at least 8 pieces of each kind.

Problem 8

4 pts

Chapter 3.2 Exercise 16

The Siwash University football team plays eight games in a season, winning three, losing three, and ending two in a tie. Show that the number of ways that this can happen is

$$\binom{8}{3}\binom{5}{3} = \frac{8!}{3!3!2!}.$$

In this problems, we have 8 games to arrange.

First we can consider choose 3 games out of the 8 games as winning games. There are $\binom{8}{3}$ ways to do that.

Then for the rest $8 - 3 = 5$ games, we need to choose 3 games out of them as losing games. There are $\binom{5}{3}$ ways to do that.

The remaining $5 - 3 = 2$ games are naturally ties.

Therefore, we have

$$\binom{8}{3}\binom{5}{3} = \frac{8!}{3!5!}\frac{5!}{3!2!} = \frac{8!}{3!3!2!}$$

possible ways.

Problem 9

4 pts

Chapter 3.2 Exercise 21

A lady wishes to color her fingernails on one hand using at most two of the colors red, yellow, and blue. How many ways can she do this?

Hint: It is very straightforward to calculate the number of ways to color her fingernails using one color. Now doing this in two colors. If we specify that the two colors are red and yellow, how many possible ways are there to color the five fingers on one hand? Recall Problem 2. Here, the five fingers are the elements in the set Ω. Consider a subset of Ω and every finger in the subset will be colored red. How many subsets does Ω has? Do we count the empty set and Ω as subsets in this problem?

The number of ways to color her fingernails using one color is

$$\binom{3}{1} = 3.$$

The number of ways to color her fingernails using two colors is

$$\binom{3}{2} \times (2^5 - 2) = 3(2^5 - 2) = 90.$$

The total number of ways using at most two colors is

$$3 + 90 = 93.$$