MATH 20: PROBABILITY

Markov Chain

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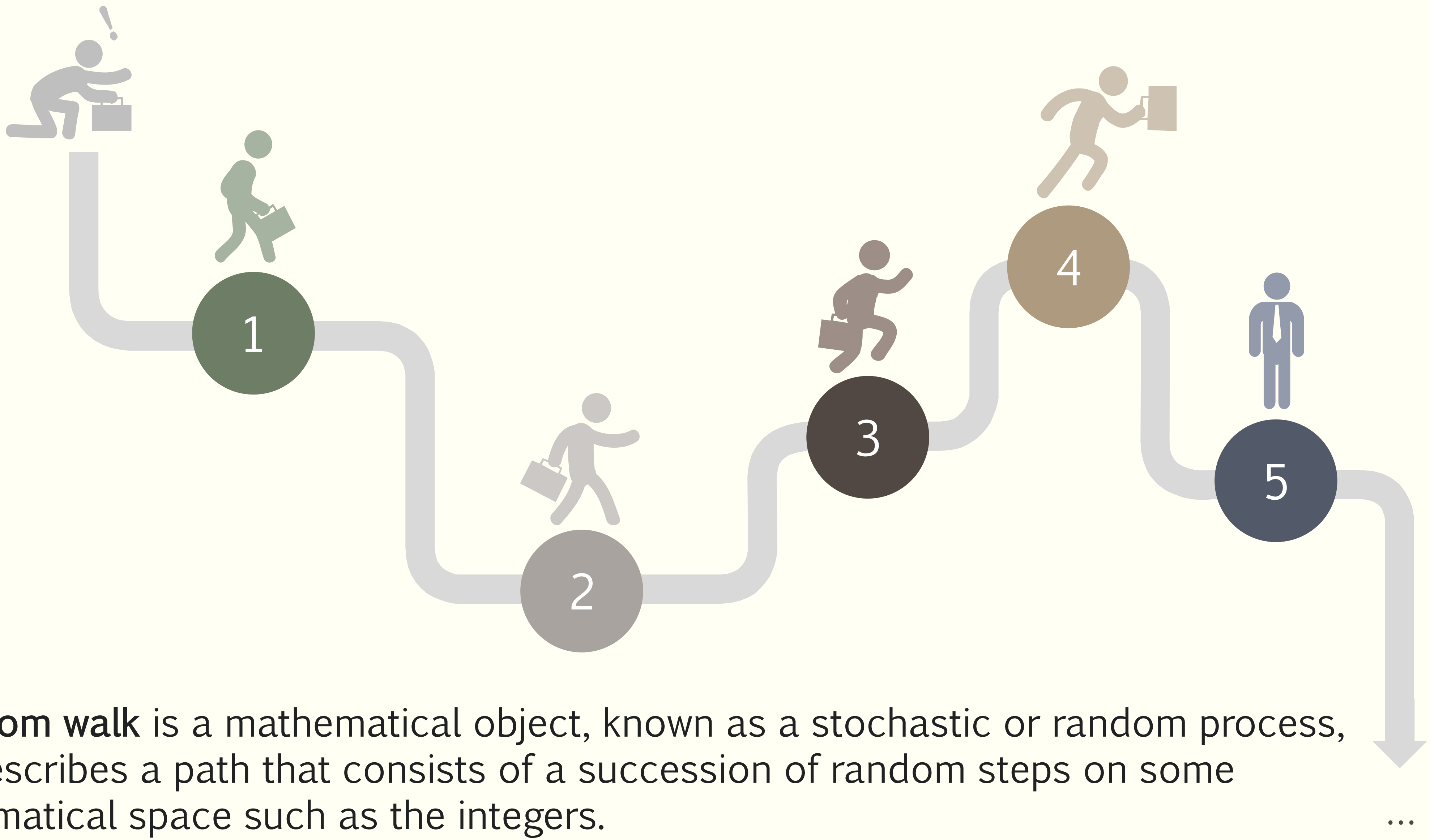
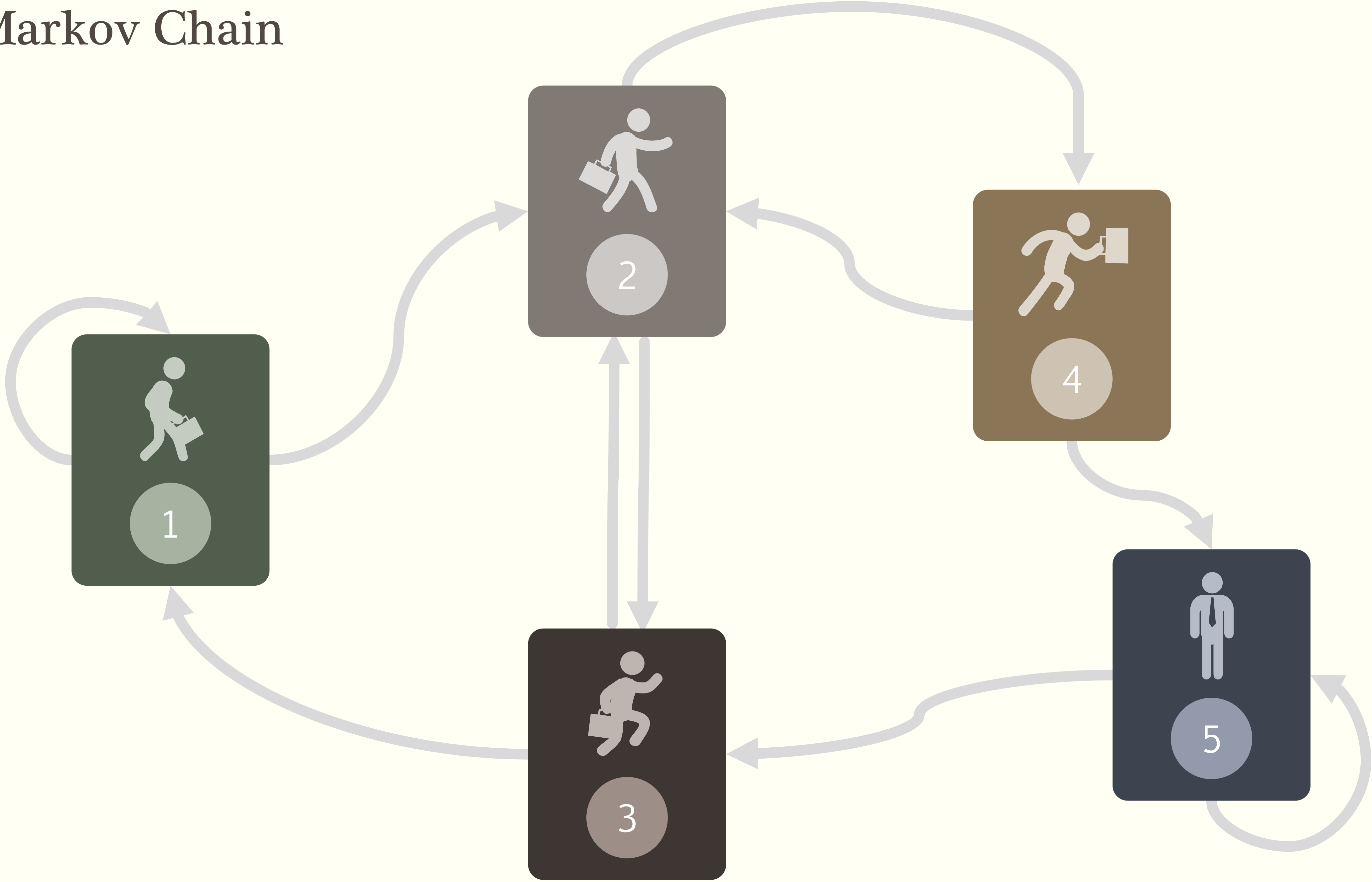
Random Walk



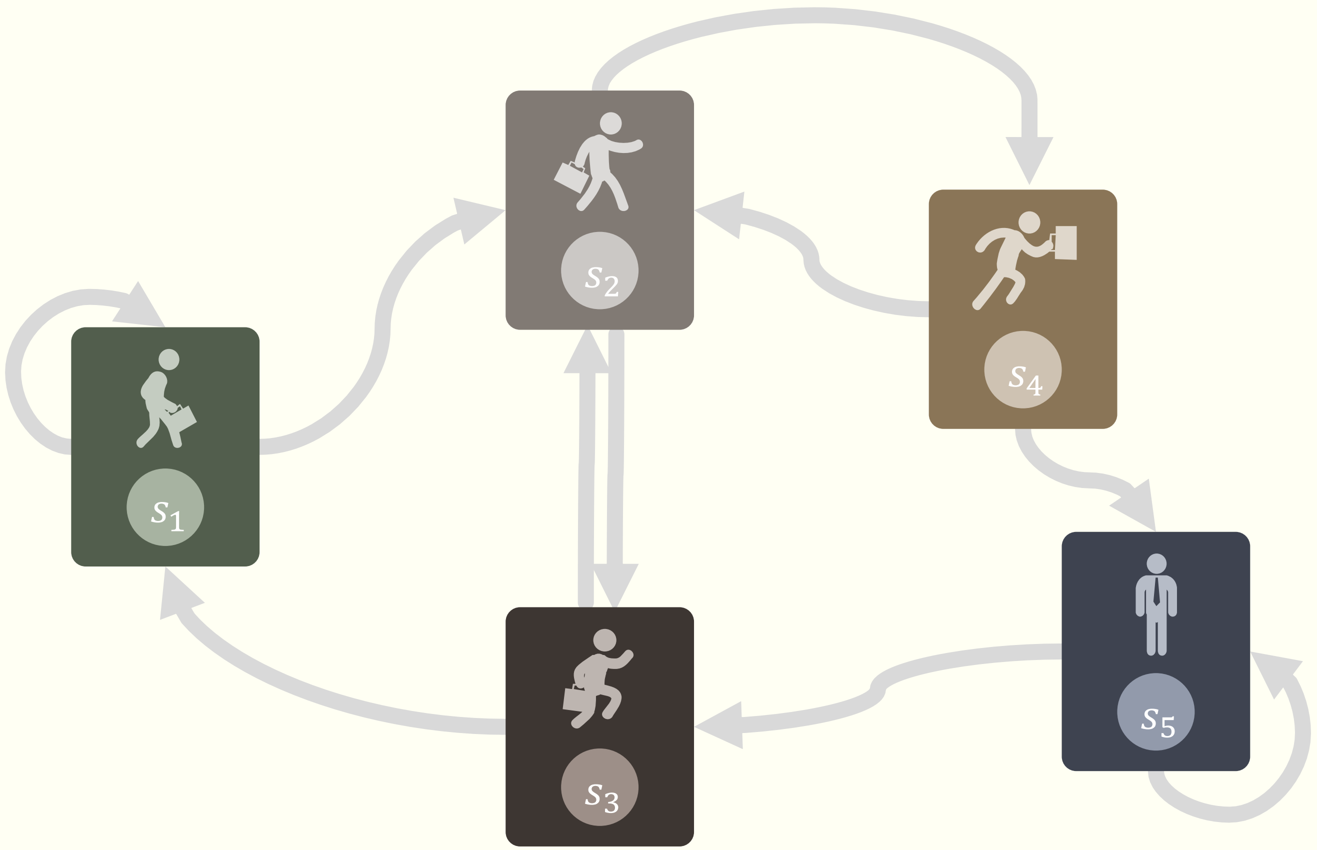
A **random walk** is a mathematical object, known as a stochastic or random process, that describes a path that consists of a succession of random steps on some mathematical space such as the integers.

Markov Chain

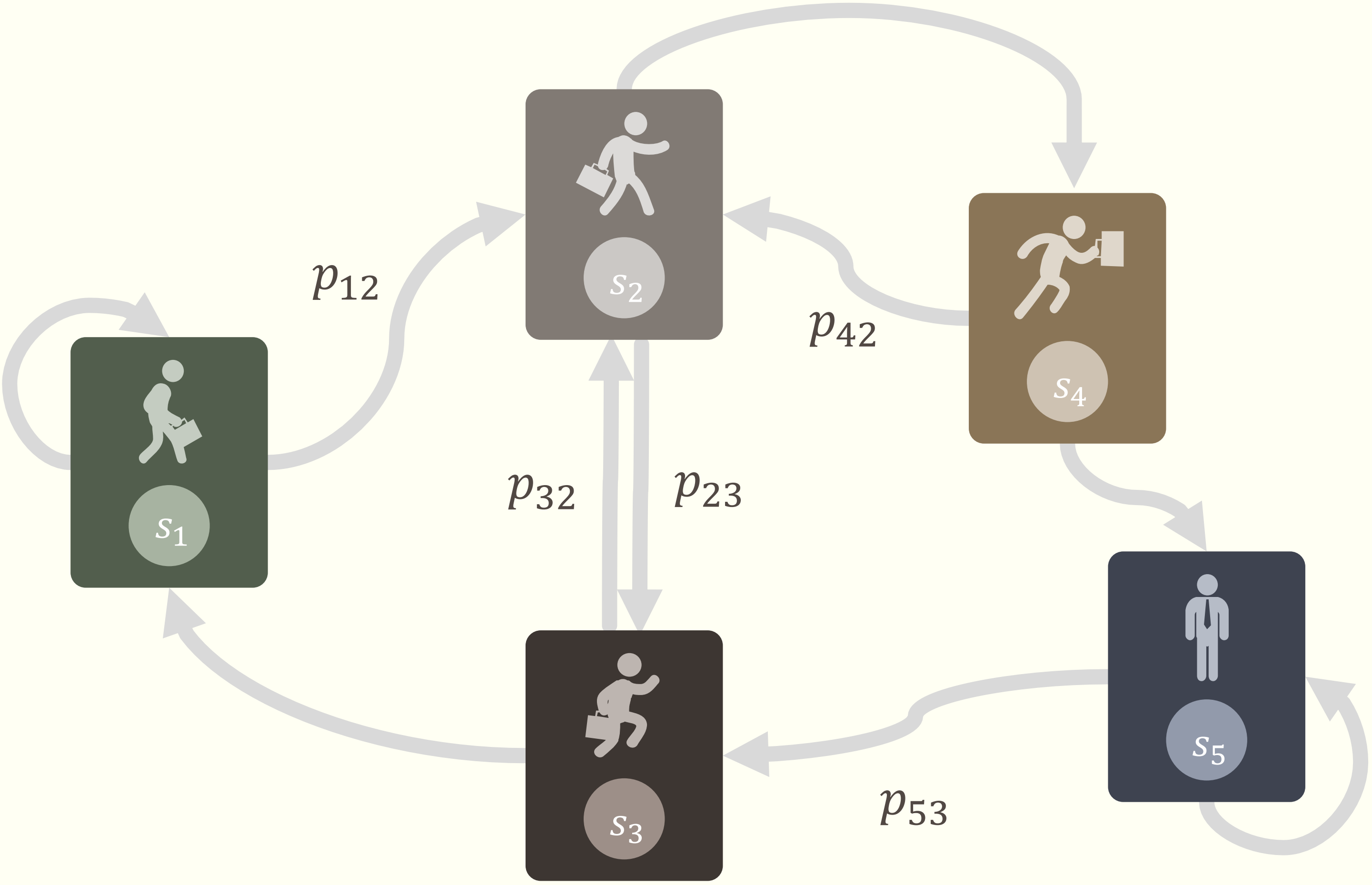


Specifying a Markov Chain

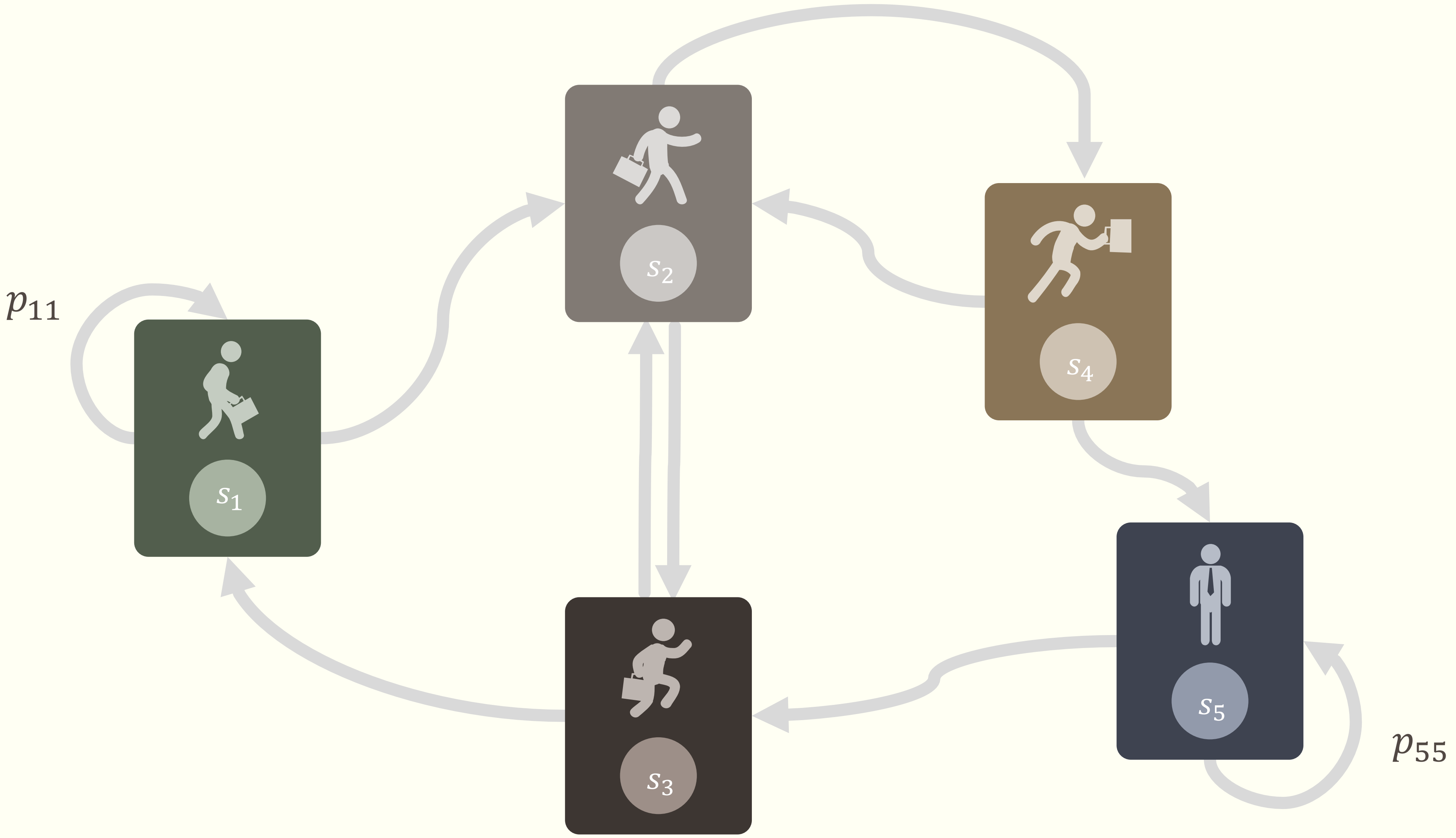
- We describe a Markov chain as follows: We have a set of states, $S = \{s_1, s_2, \cdots, s_r\}$.
- The process starts in one of these states and moves successively from one state to another. Each move is called a step.



- If the chain is currently in state s_i, then it moves to state s_j at the next step with a probability denoted by p_{ij}.
- The probability p_{ij} does not depend upon which states the chain was in before the current state.
- These probabilities are called transition probabilities.



- The process can remain in the state it is in, and this occurs with probability p_{ii}.

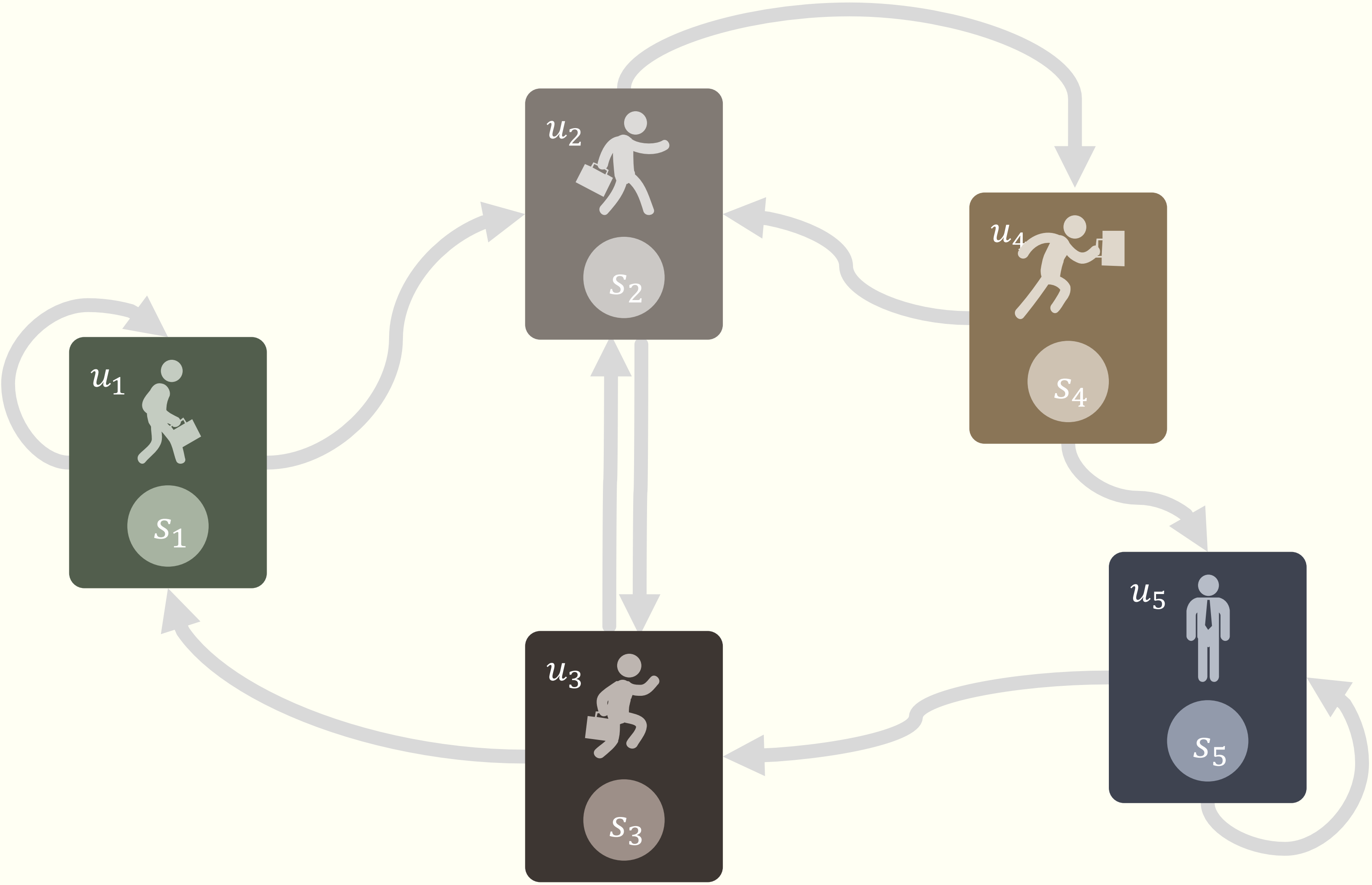


- An initial probability distribution, defined on S, specifies the starting state. Usually this is done by specifying a particular state as the starting state.

$$u = [u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5]$$

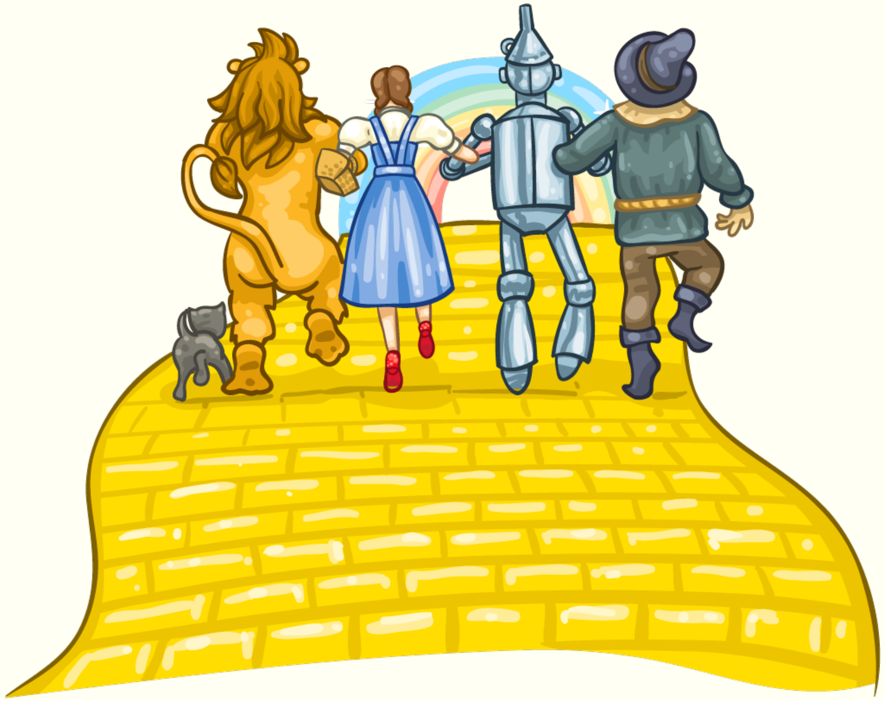
$$[0 \quad 1 \quad 0 \quad 0 \quad 0]$$

$$\sum_{i=1}^{n} u_i = 1$$



THE LAND OF OZ

- the Land of Oz is blessed by many things, but not by good weather.
- They never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day.
- If they have snow or rain, they have an even chance of having the same the next day.
- If there is change from snow or rain, only half of the time is this a change to a nice day.



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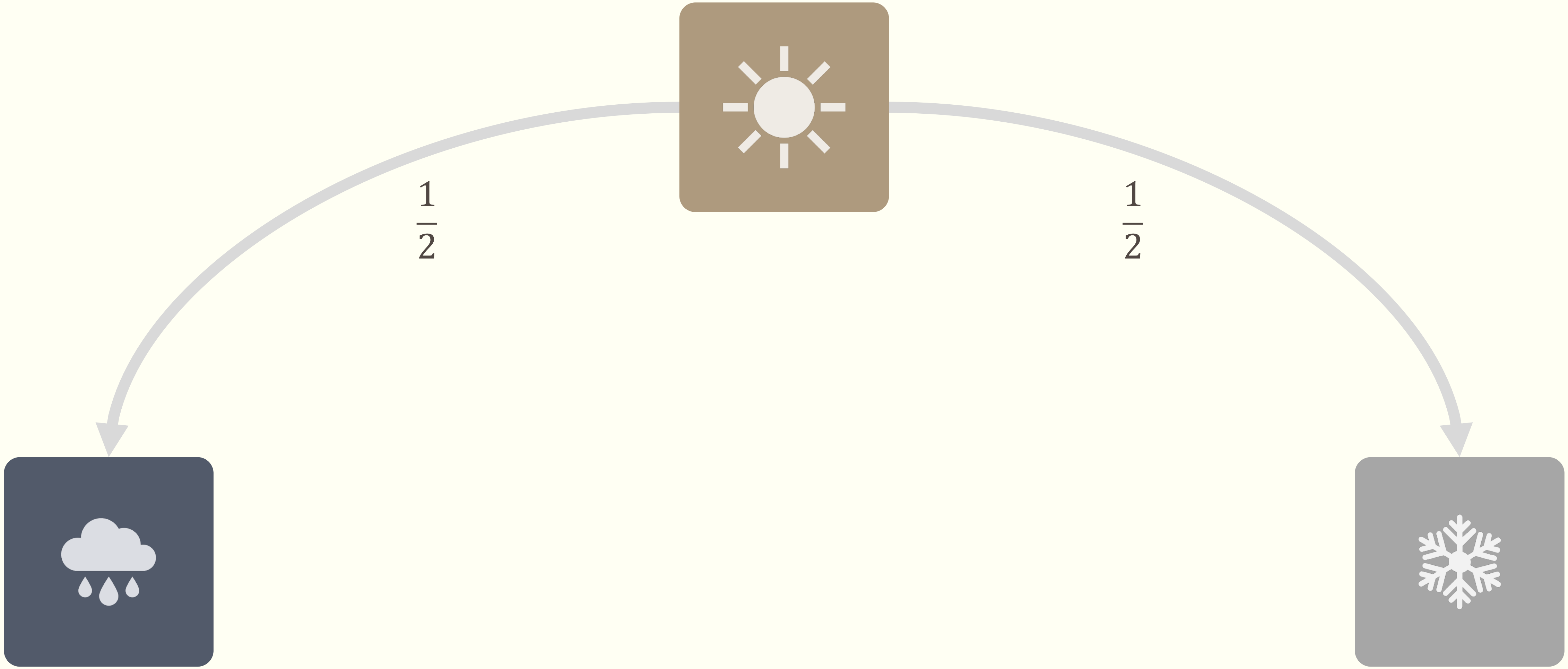
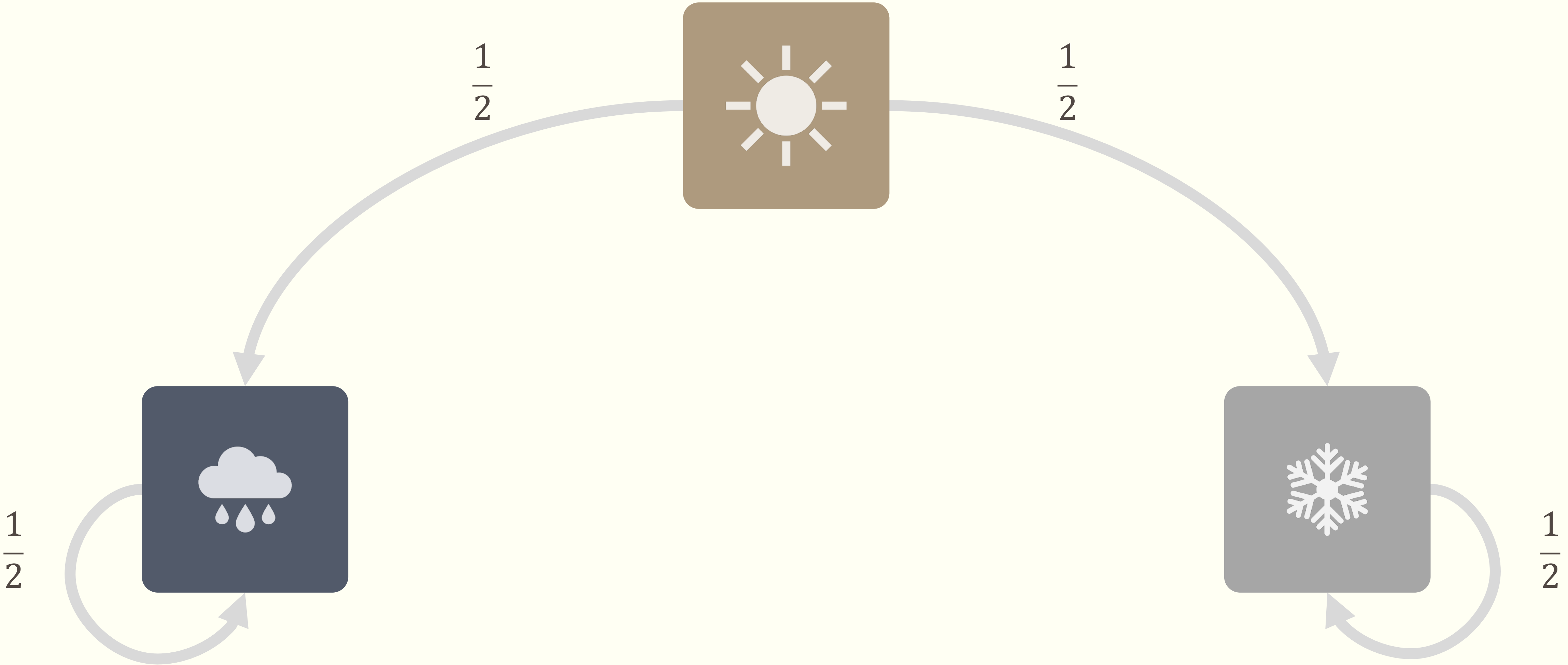
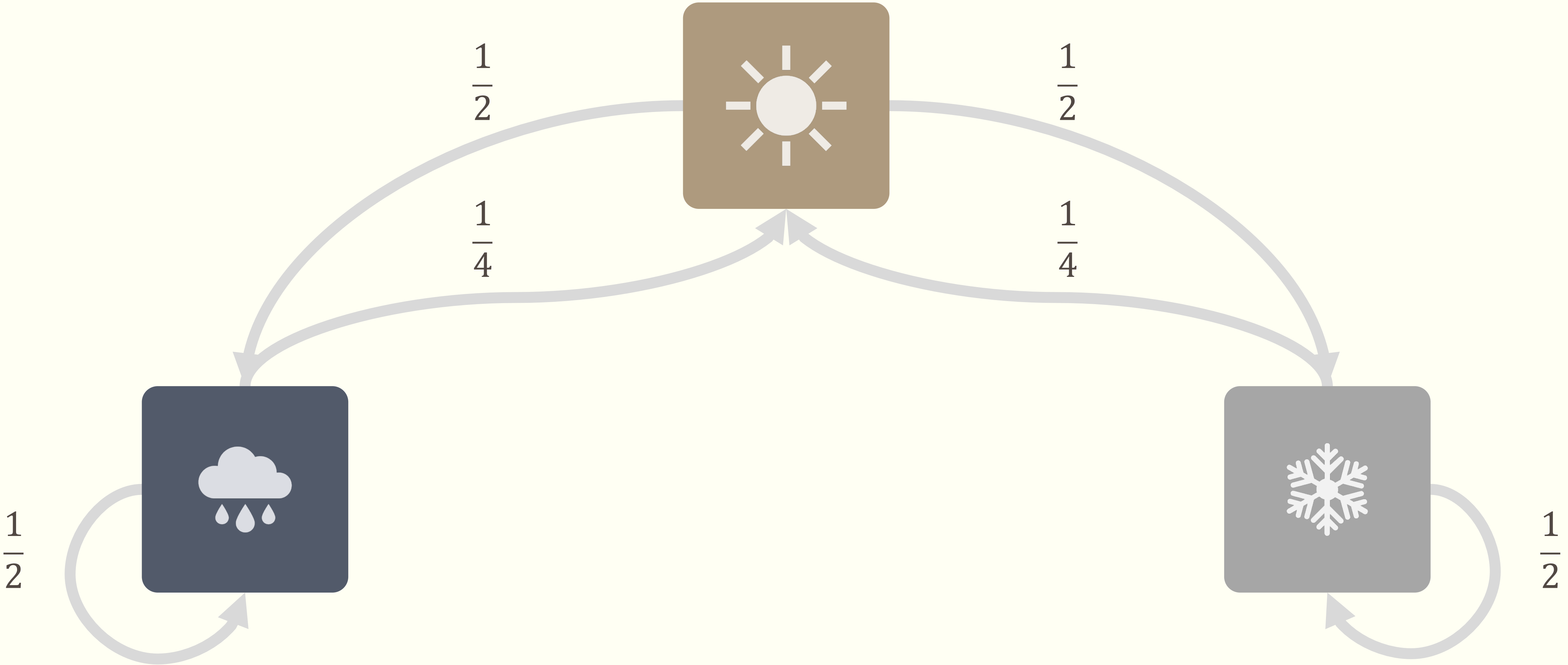


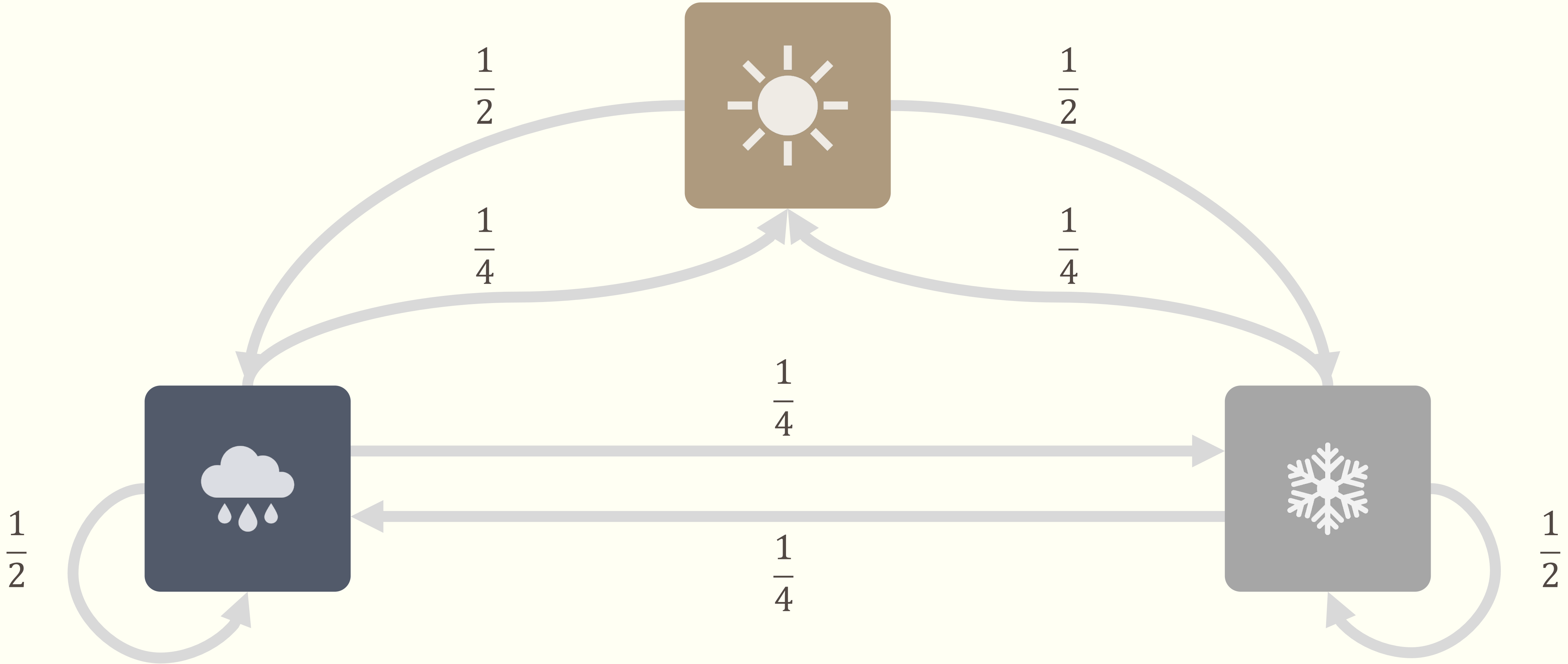
- If they have snow or rain, they have an even chance of having the same the next day.



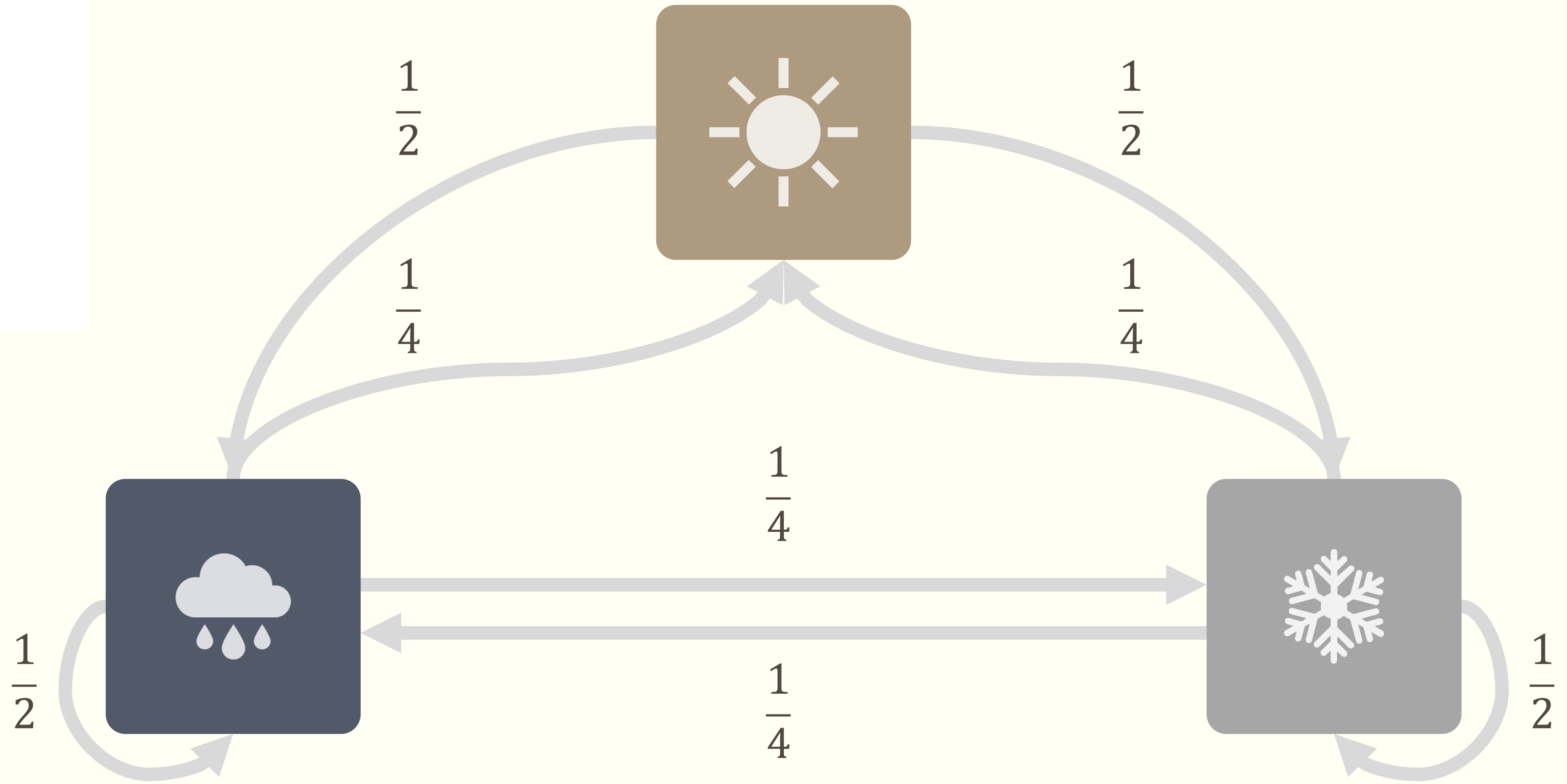
- If there is change from snow or rain, only half of the time is this a change to a nice day.



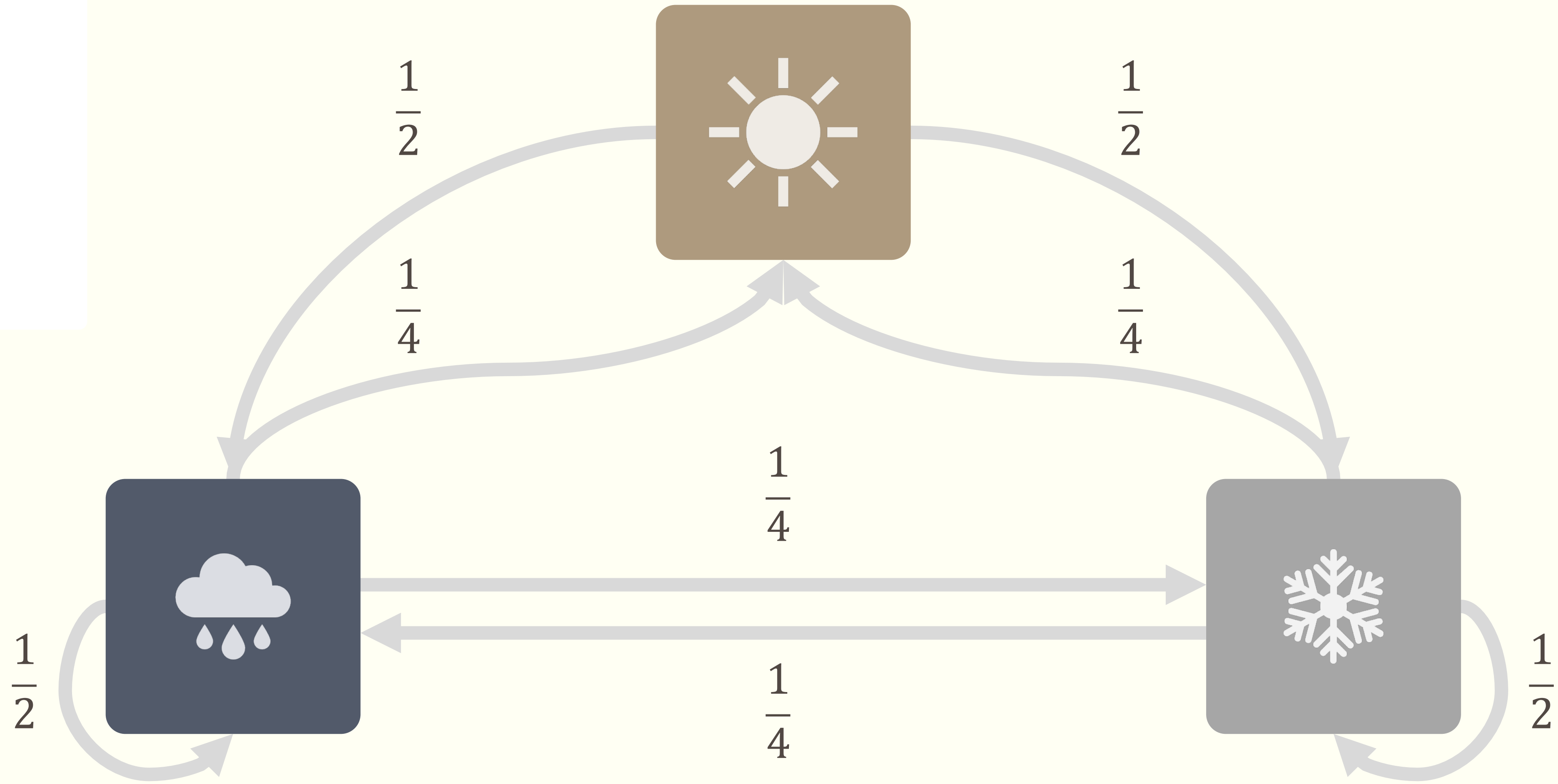
- If there is change from snow or rain, only half of the time is this a change to a nice day.



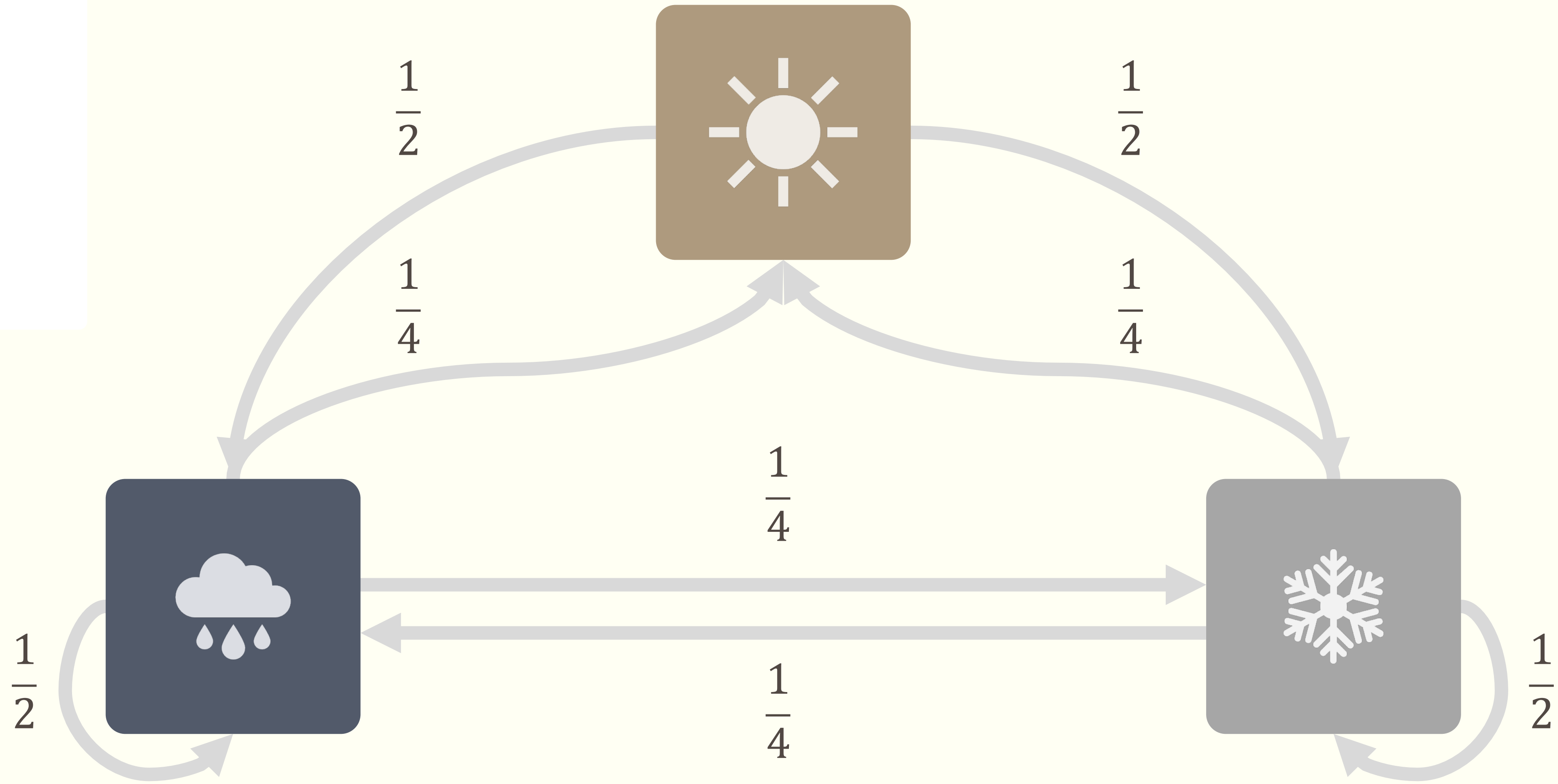
- $$\begin{array}{c|ccc} \nearrow & R & N & S \\ \hline R & & & \\ N & & & \\ S & & & \end{array}$$



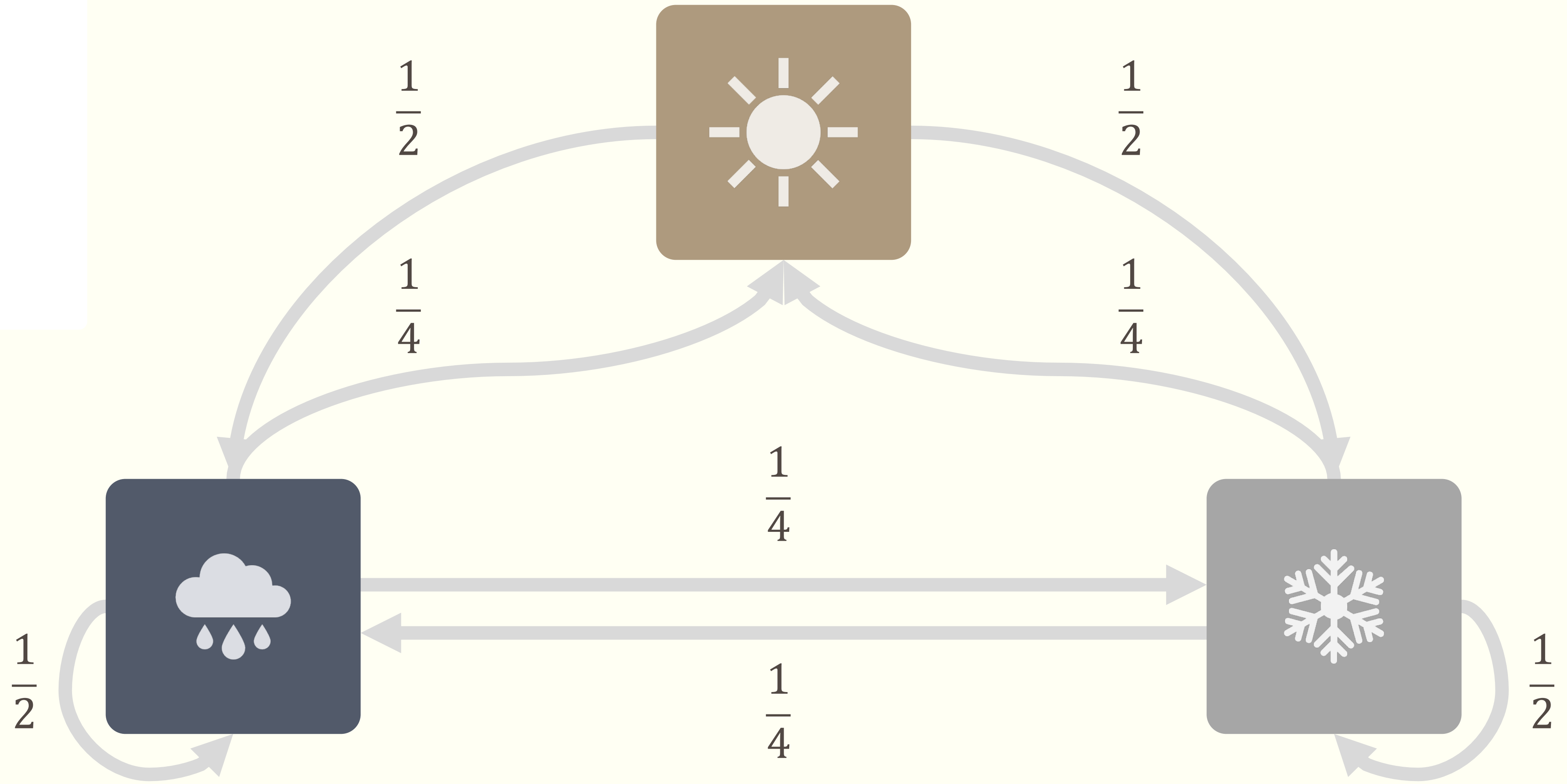
- $$\begin{array}{cc} \nearrow & \begin{array}{ccc} R & N & S \end{array} \\ \begin{array}{c} R \\ N \\ S \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ & & \\ & & \end{bmatrix} \end{array}$$



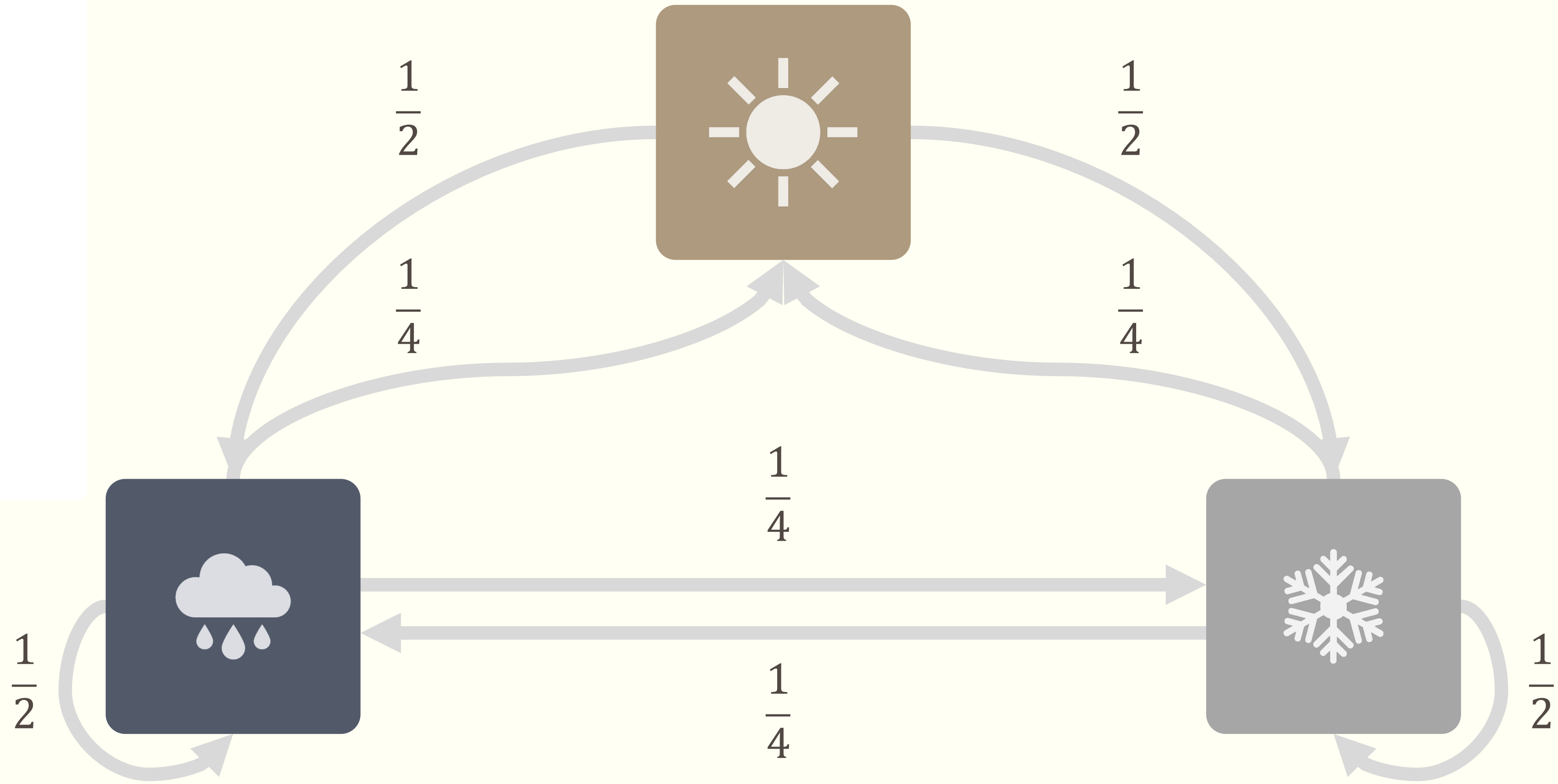
- $\begin{array}{cc} \nearrow & \begin{array}{ccc} \mathrm{R} & \mathrm{N} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{R} \\ \mathrm{N} \\ \mathrm{S} \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ & & \end{bmatrix} \end{array}$



- $$\begin{array}{c|ccc} \nearrow & R & N & S \\ \hline R & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ N & \frac{1}{2} & 0 & \frac{1}{2} \\ S & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array}$$



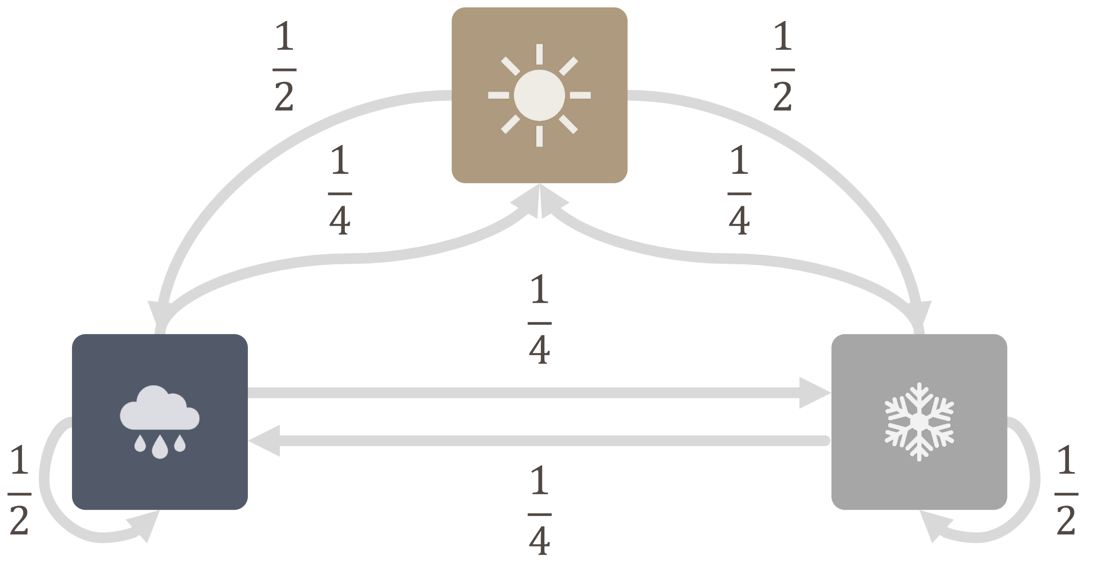
- States:
    - s_1: rain
    - s_2: nice
    - s_3: snow
- $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$



Transition Matrix

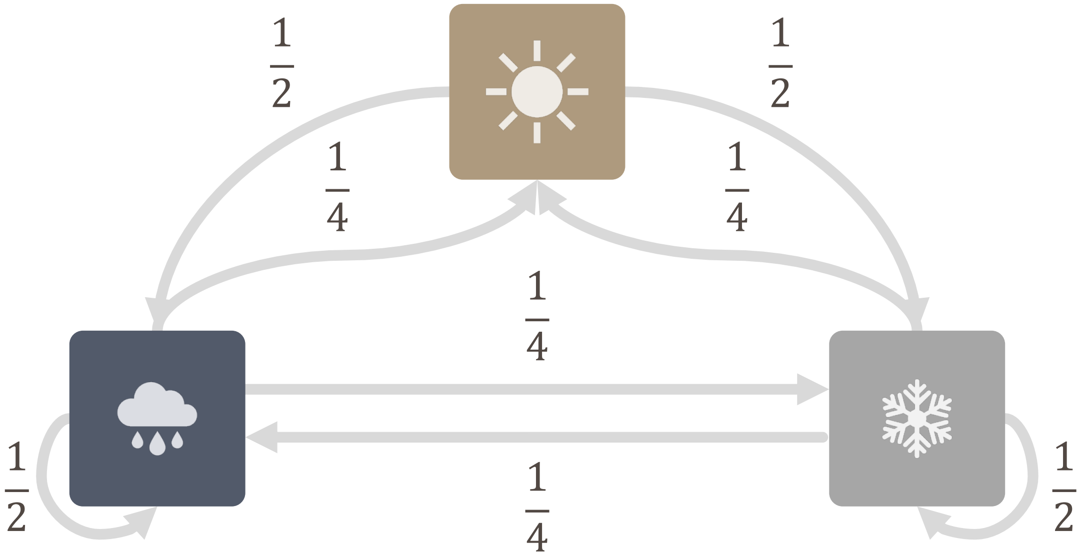
- The entries in the first row of the matrix P in the example represent the probabilities for the various kinds of weather following a rainy day.
- Similarly, the entries in the second and third rows represent the probabilities for the various kinds of weather following nice and snowy days, respectively.
- Such a square array is called the matrix of transition probabilities, or the transition matrix.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$



- States:
  - s_1: rain
  - s_2: nice
  - s_3: snow

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

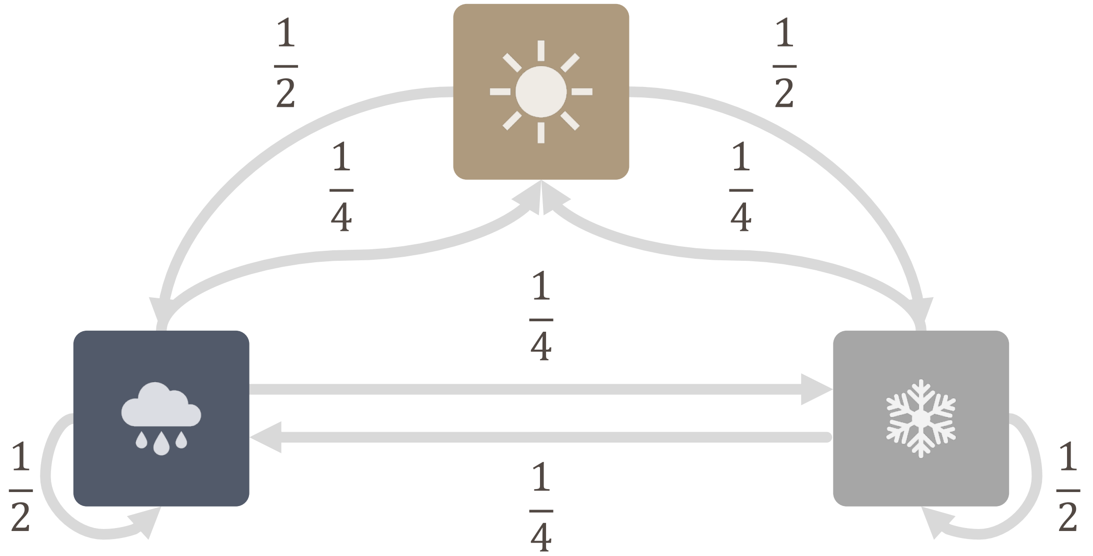


the probability that, given the chain is in state i today, i will be in state j tomorrow

?

- States:
  - s_1: rain
  - s_2: nice
  - s_3: snow

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$



the probability that, given the chain is in state i today, it will be in state j tomorrow

?

$$p_{ij}^{(1)} = p_{ij}$$

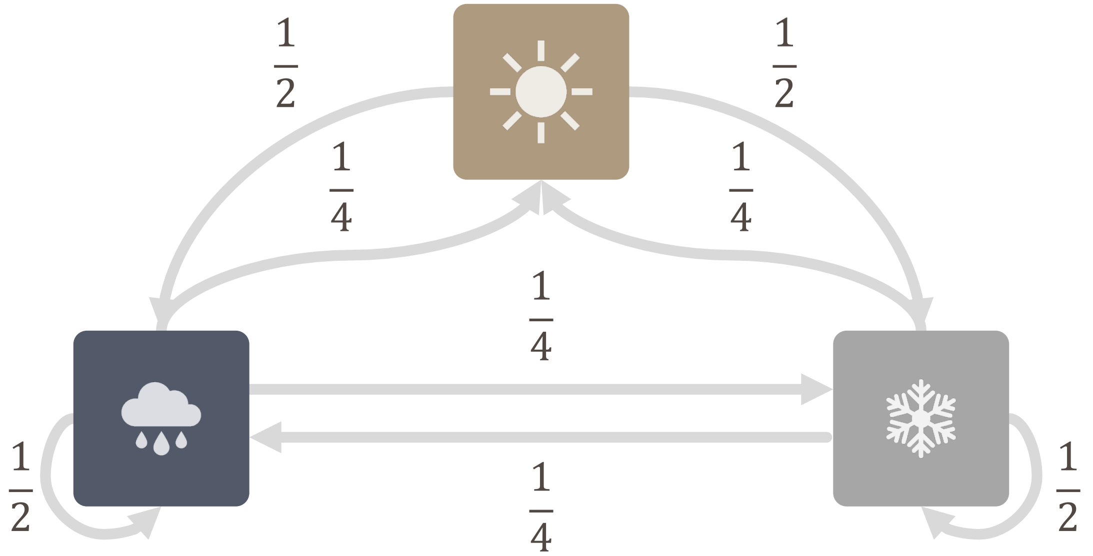
the probability that, given the chain is in state i today, it will be in state j the day after tomorrow

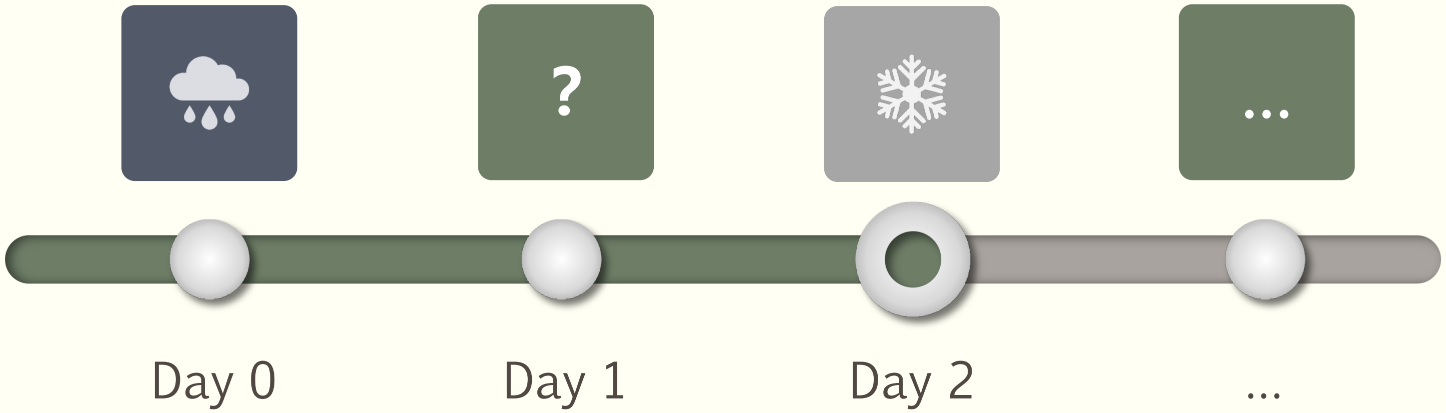
?

$$p_{ij}^{(2)} = \cdots$$

- States:
  - s_1: rain
  - s_2: nice
  - s_3: snow

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

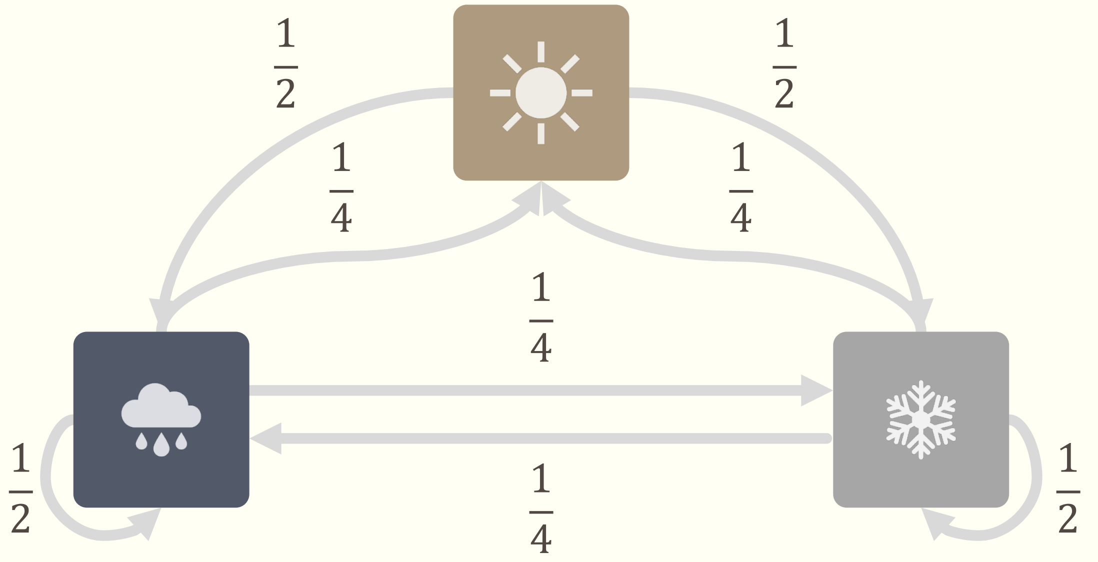


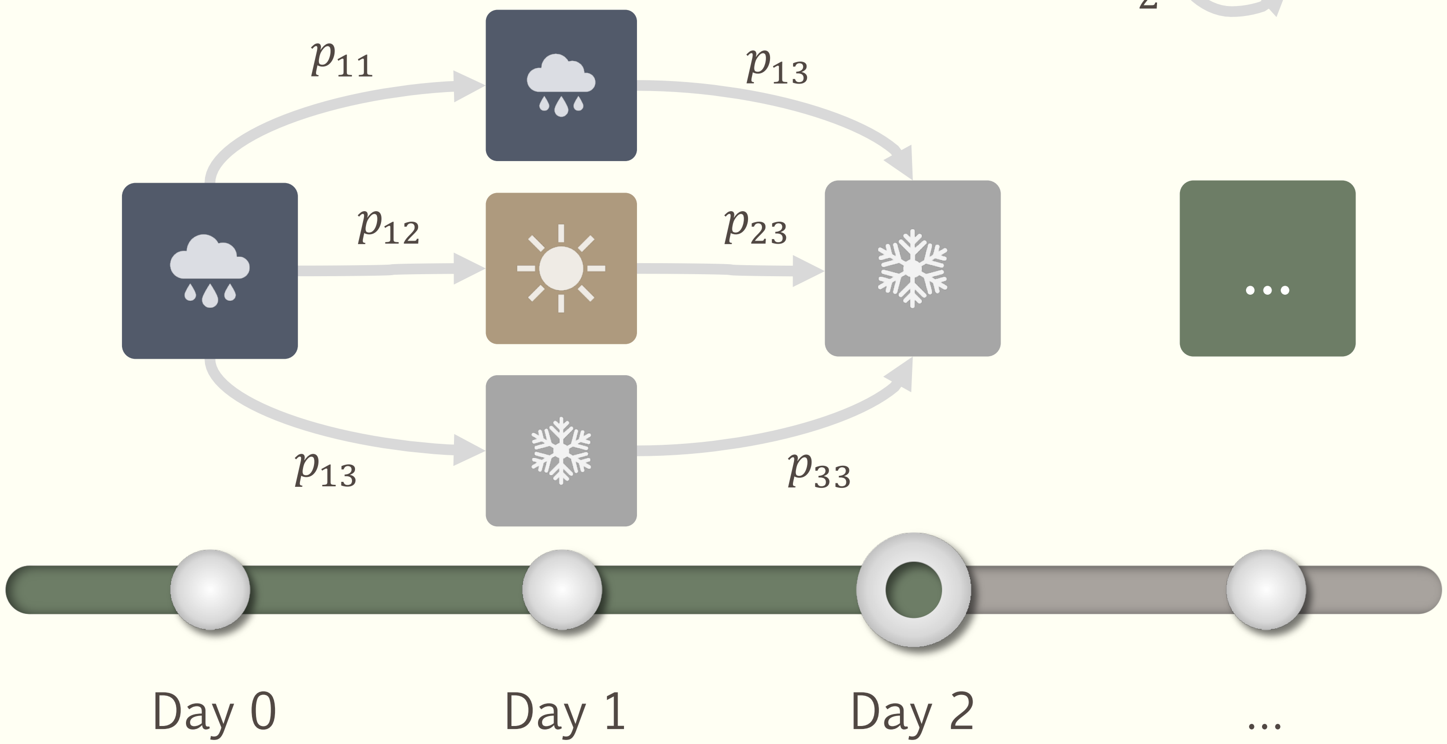


$$p_{13}^{(2)} = \cdots$$

- States:
  - s_1: rain
  - s_2: nice
  - s_3: snow

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$





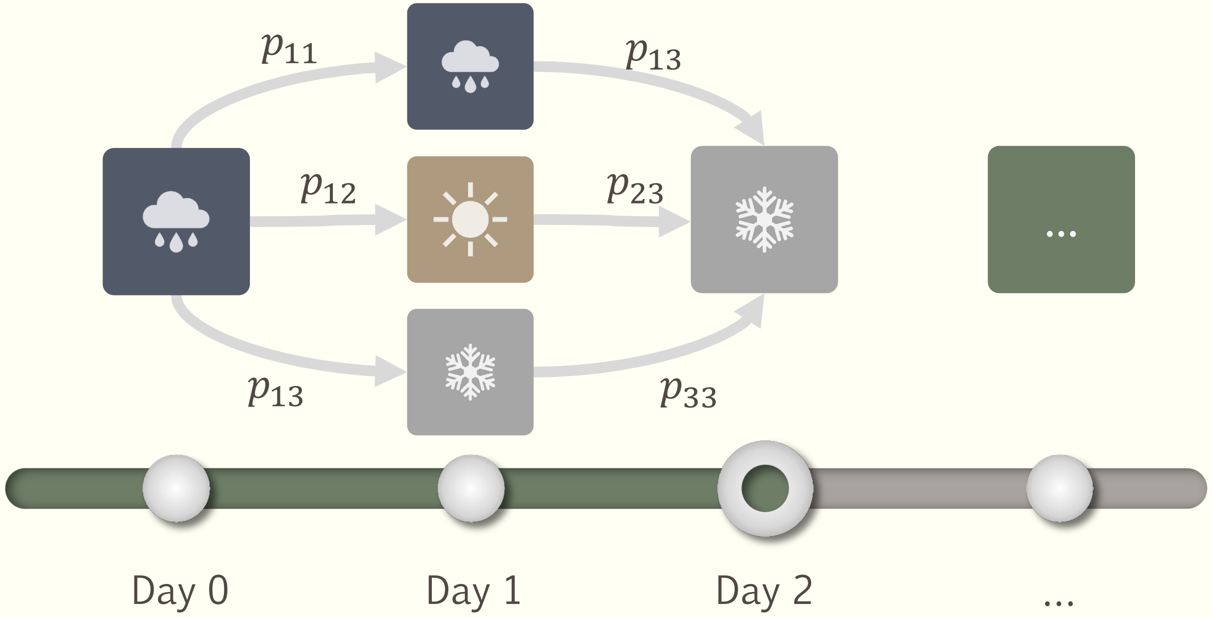
$$p_{13}^{(2)} = p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33}$$

- States:
  - s_1: rain
  - s_2: nice
  - s_3: snow

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

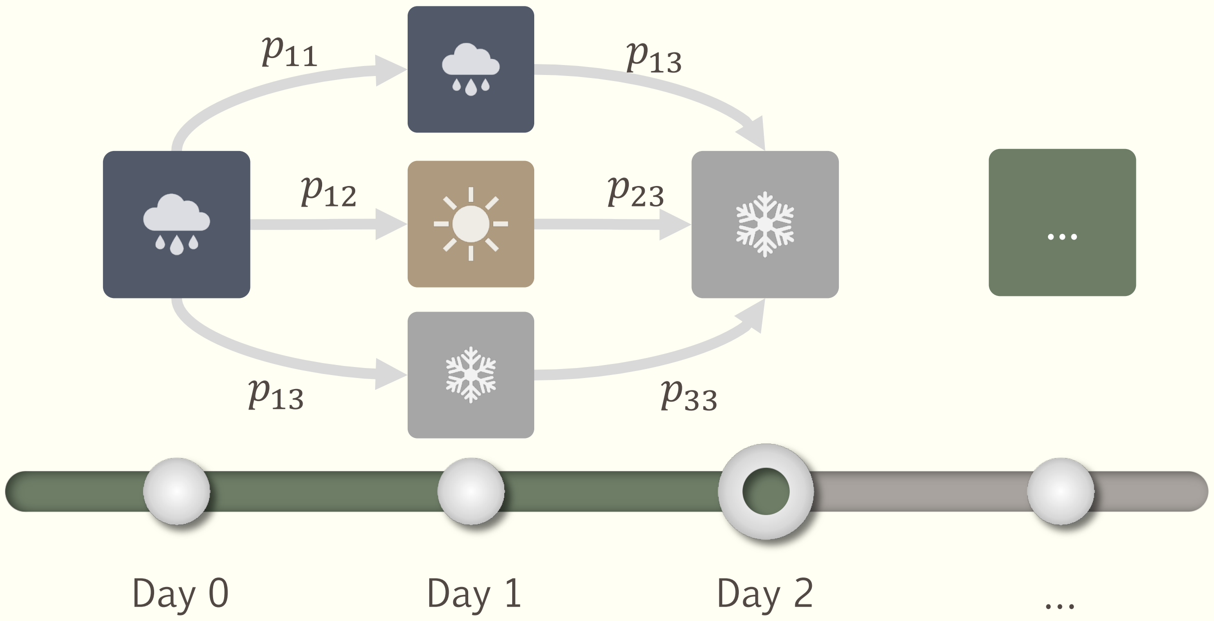
$$p_{13}^{(2)} = p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33}$$

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \end{bmatrix} \bullet \begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix}$$

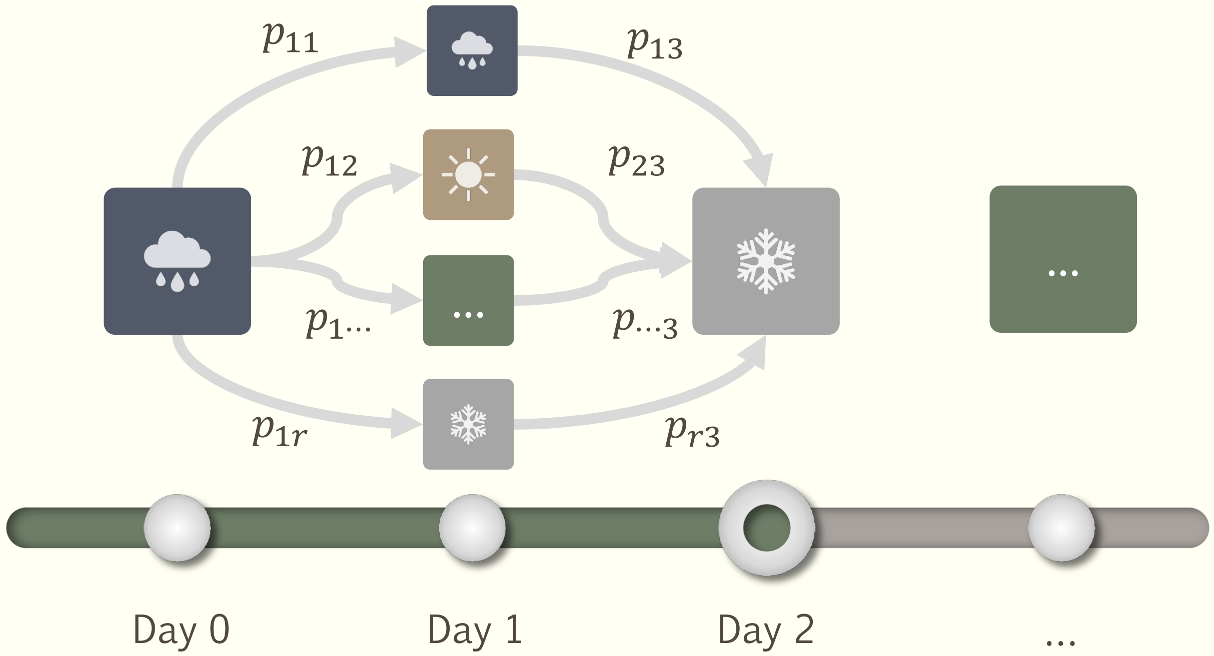


$$P^2 = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

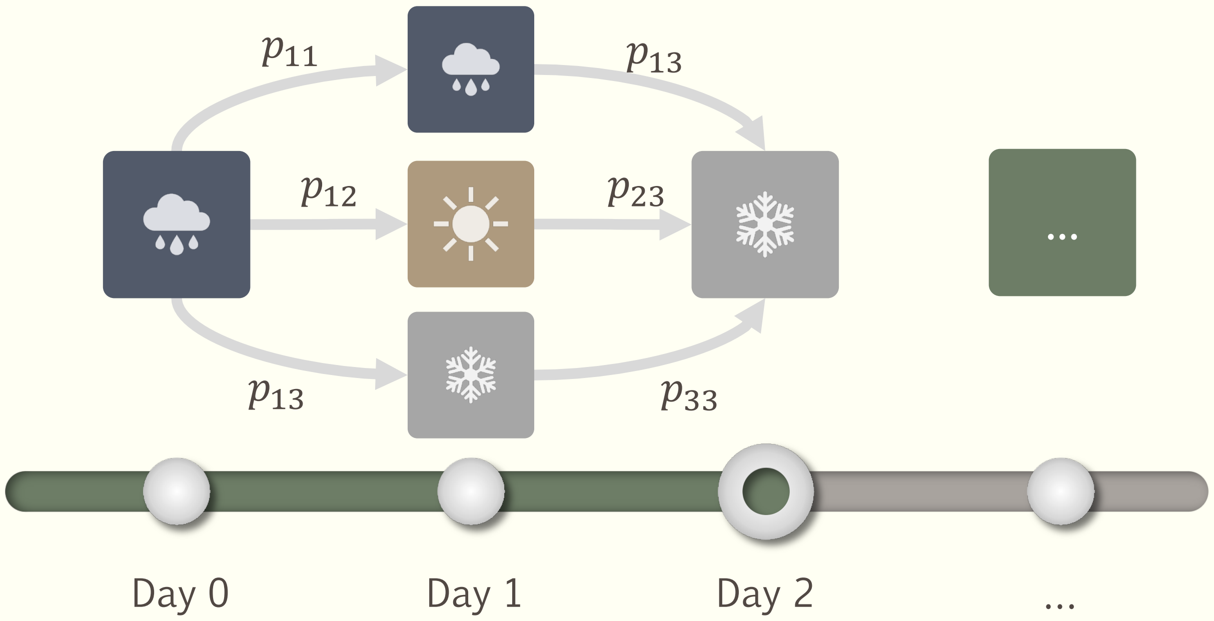
$$= \begin{bmatrix} & & p_{13}^{(2)} \\ & & \\ & & \end{bmatrix}$$



$$
\begin{aligned}
p_{13}^{(2)} &= p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} \\
&= \sum_{k=1}^{3} p_{1k}p_{k3}
\end{aligned}
$$

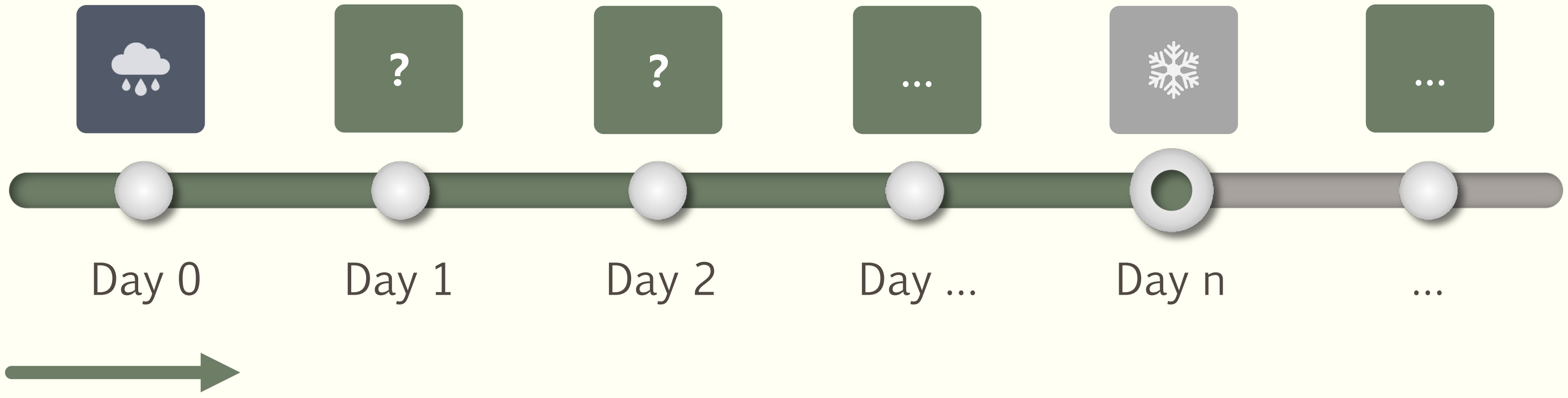


$$
\begin{aligned}
p_{13}^{(2)} &= p_{11}p_{13} + p_{12}p_{23} + \cdots + p_{1r}p_{r3} \\
&= \sum_{k=1}^{r} p_{1k}p_{k3}
\end{aligned}
$$



$$P^2 = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \cdot \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

$$= \begin{bmatrix} & & p_{13}^{(2)} \\ & & \\ & & \end{bmatrix}$$

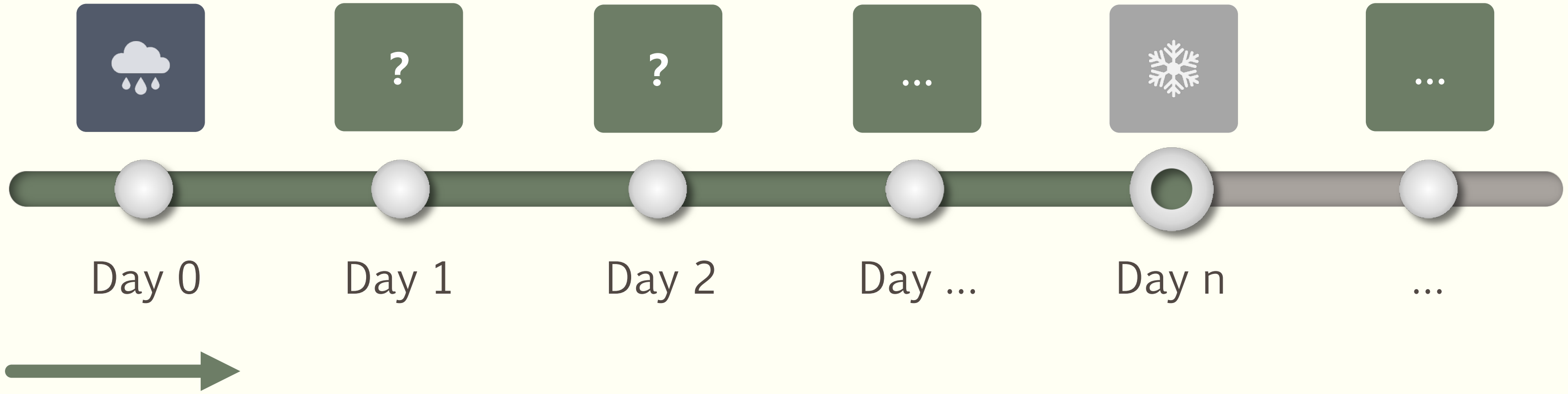


$$P^n = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}^n$$

$$= \begin{bmatrix} & & p_{13}^{(n)} \\ & & \\ & & \end{bmatrix}$$

Transition Matrix

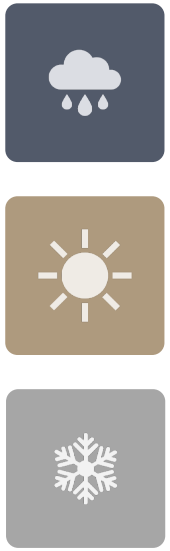
- Let P be the transition matrix of a Markov chain.
- The ijth entry p_{ij} of the matrix P^n gives the probability that the Markov chain, starting in state s_i, will be in state s_j after n steps.



$$P^n = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}^n$$

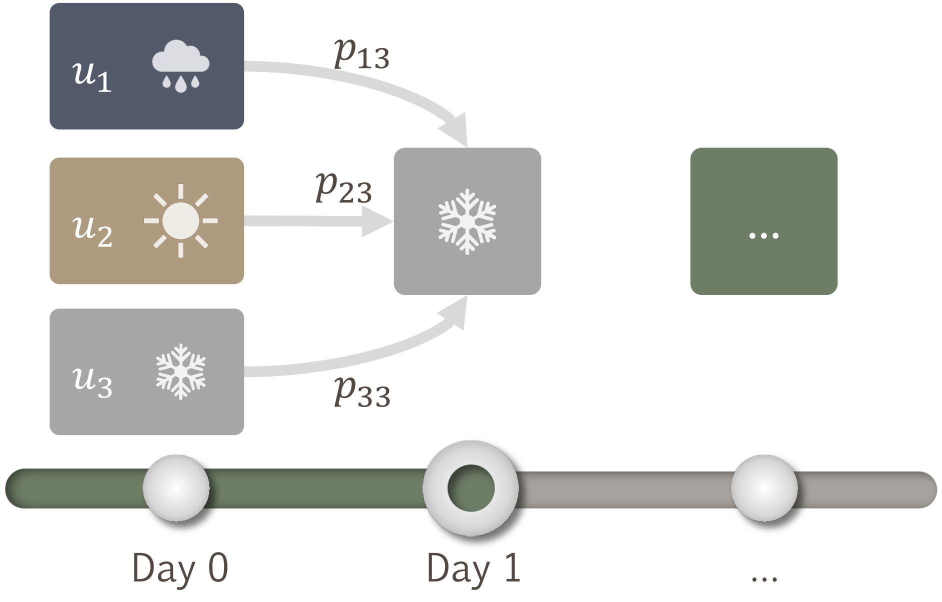
$$= \begin{bmatrix} & & p_{13}^{(n)} \\ & & \\ & & \end{bmatrix}$$

- Starting states:
  - rain: u_1
  - nice: u_2
  - snow: u_3



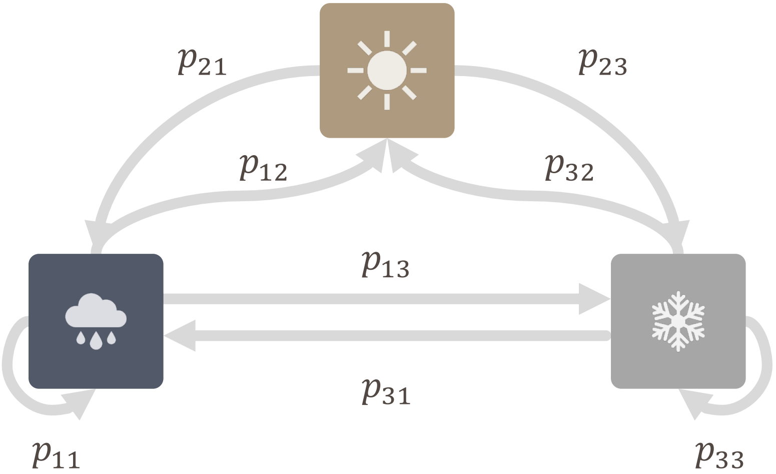
$$u = [u_1 \quad u_2 \quad u_3]$$

- the probability that the chain is in state s_k after n steps:
  - $k = 3$
  - $n = 1$



- Transition matrix:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

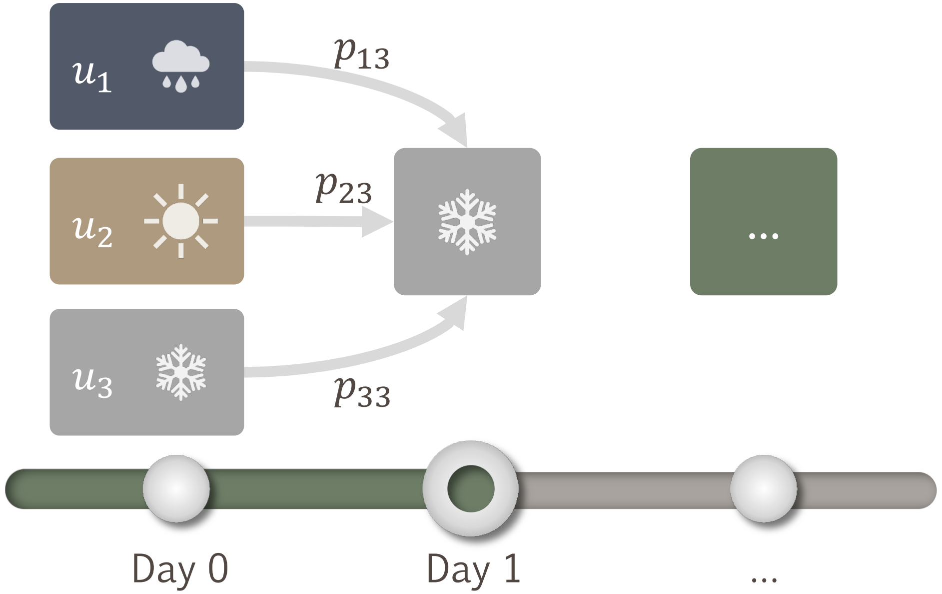


$$u_3^{(1)} = u_1 p_{13} + u_2 p_{23} + u_3 p_{33}$$

$$[u_1 \quad u_2 \quad u_3] \bullet \begin{bmatrix} p_{11} & p_{12} & \boxed{p_{13}} \\ p_{21} & p_{22} & \boxed{p_{23}} \\ p_{31} & p_{32} & \boxed{p_{33}} \end{bmatrix}$$

$$u^{(1)} = [u_1^{(1)} \quad u_2^{(1)} \quad u_3^{(1)}] = uP$$

- the probability that the chain is in state s_k after n steps:
  - $n = 1$

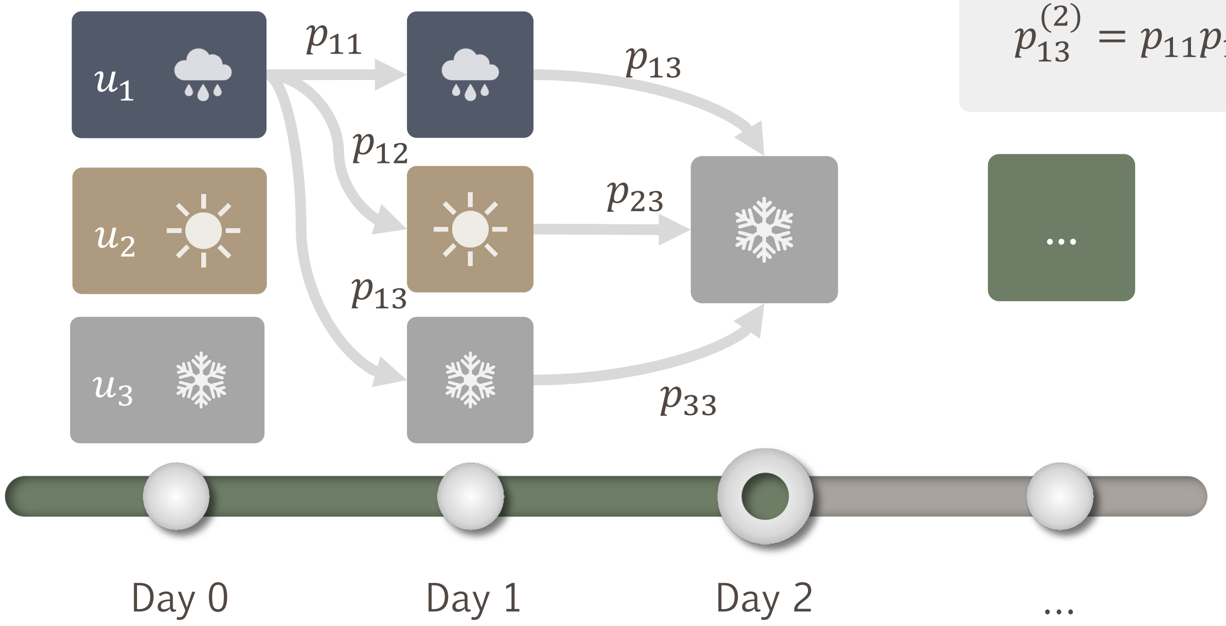


$$u_3^{(1)} = u_1 p_{13} + u_2 p_{23} + u_3 p_{33}$$

$$u_k^{(1)} = u_1 p_{1k} + u_2 p_{2k} + u_3 p_{3k}$$

$$u^{(1)} = \begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} \end{bmatrix} = uP$$

- the probability that the chain is in state s_k after n steps:
  - $n = 2$



$$p_{13}^{(2)} = p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} = \sum_{k=1}^{3} p_{1k}p_{k3}$$

$$u_3^{(2)} = u_1 p_{13}^{(2)} + u_2 p_{23}^{(2)} + u_3 p_{33}^{(2)}$$

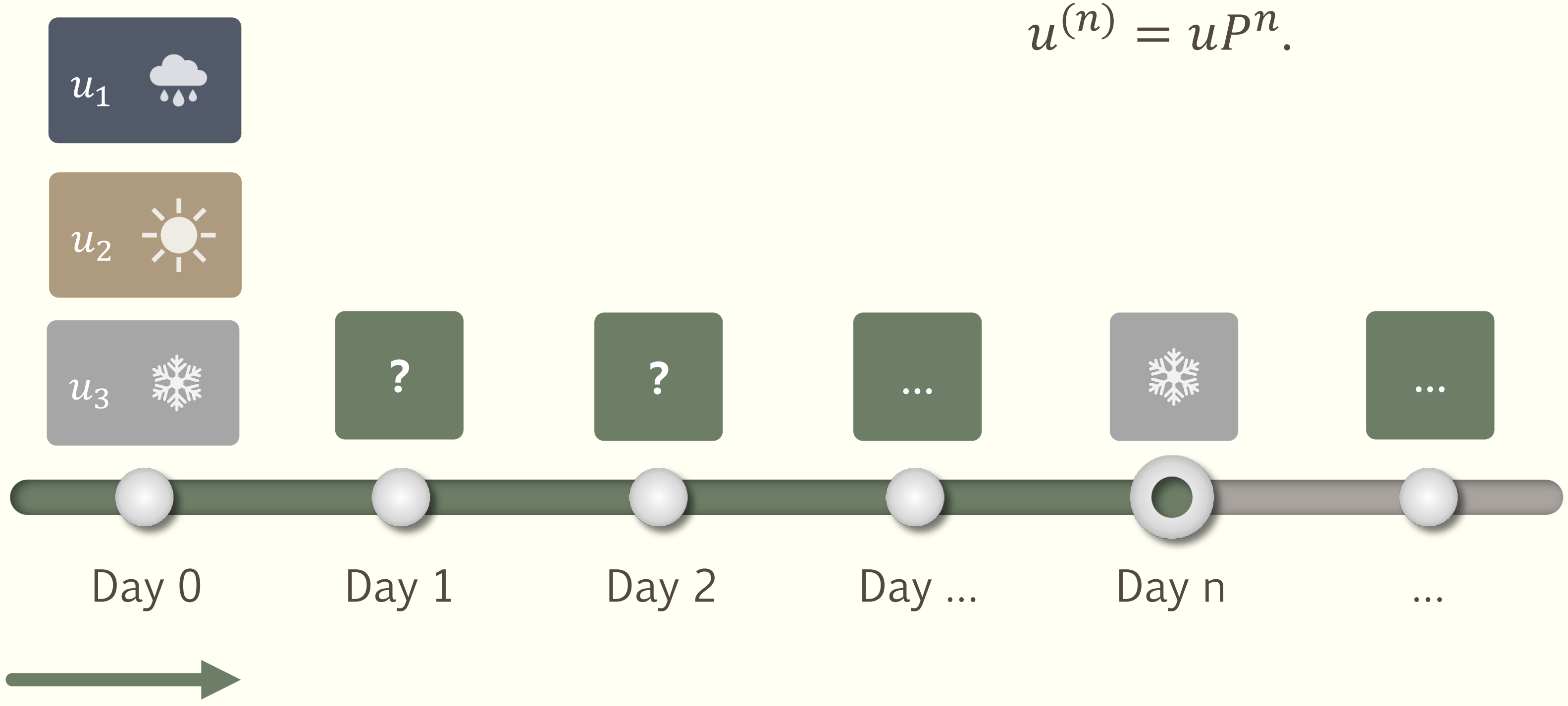
$$u_k^{(2)} = u_1 p_{1k}^{(2)} + u_2 p_{2k}^{(2)} + u_3 p_{3k}^{(2)}$$

$$u^{(2)} = \begin{bmatrix} u_1^{(2)} & u_2^{(2)} & u_3^{(2)} \end{bmatrix} = uP^2$$

Transition Matrix

- Let P be the transition matrix of a Markov chain, and let u be the probability vector which represents the starting distribution.
- Then the probability that the chain is in state s_k after n steps is the kth entry in the vector

$$u^{(n)} = uP^n.$$



$$u^{(n)} = uP^n$$

$$= u \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}^n$$

$$= \begin{bmatrix} u_1^{(n)} & u_2^{(n)} & u_3^{(n)} \end{bmatrix}$$

August 2020

Sun	Mon	Tue	Wed	Thu	Fri	Sat
26	27	28	29	30	31	01
02	03	04	05	06	07	08
09	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31	01	02	03	04	05

Final

Open book

Scope: Mostly Chapters 7, 8, 9, 10, and 11

- sum of random variables
- LLN and CLT
- generating functions
- Markov chains

Materials: **slides**, **homework**, **quizzes**, textbook

Date & Time: 12:00 pm – 10: 00 pm, August 30

Office hours: August 27, 28

Homework due: 11:00 pm August 28