Math 20: Probability

Homework 2 Solution

July 13, 2020

Problem 1

4 pts

Chapter 1.2 Exercise 6

A dice is loaded in such a way that the probability of each face turning up is proportional to the number of dots on that face. (For example, a six is three times as probable as a two.) What is the probability of getting an even number in one throw?

Assume that the probability of getting 1 is p. Then the probability of getting n is np, where $n \in \{1, 2, 3, 4, 5, 6\}$.

Notice that $\sum_{n=1}^{6} np = 1$. That is, $21p = 1$ and we get $p = \frac{1}{21}$.

To get an even number, we need $n = 2, 4$ or 6. Therefore, the probability is

$$(2 + 4 + 6) \cdot \frac{1}{21} = \frac{4}{7}.$$

Problem 2

4 pts

Chapter 1.2 Exercise 14

Let X be a random variable with distribution function $m_X(x)$ defined by

$$m_X(-1) = \frac{1}{5}, \qquad m_X(0) = \frac{1}{5}, \qquad m_X(1) = \frac{2}{5}, \qquad m_X(2) = \frac{1}{5}.$$

(a) Let Y be the random variable defined by the equation $Y = X + 3$. Find the distribution function $m_Y(y)$ of Y.

(b) Let Z be the random variable defined by the equation $Z = X^2$. Find the distribution function $m_Z(z)$ of Z.

(a)

$$m_Y(y) = \begin{cases} \frac{1}{5}, & Y = 2, 3 \,\text{or}\, 5 \\ \frac{2}{5}. & Y = 4 \end{cases}$$

(b)

$$m_Z(z) = \begin{cases} \frac{1}{5}, & Z = 0 \\ \frac{3}{5}, & Z = 1 \\ \frac{1}{5}. & Z = 4 \end{cases}$$

Problem 3

4 pts

Chapter 1.2 Exercise 22

A dice is rolled until the first time that a six turns up. We shall see that the probability that this occurs on the nth roll is $(\frac{5}{6})^{n-1}\frac{1}{6}$. Using this fact, describe the appropriate infinite sample space and distribution function for the experiment of rolling a dice until a six turns up for the first time. Verify that for your distribution function $\sum_{\omega} m(\omega) = 1$.

- Sample space: $\Omega = \{1, 2, \cdots, +\infty\}$
- Distribution function: $m(\omega) = (\frac{5}{6})^{\omega-1}\frac{1}{6}$

$$\sum_{\omega=1}^{+\infty} m(\omega) = \sum_{\omega=1}^{+\infty} (\frac{5}{6})^{\omega-1}\frac{1}{6} = \frac{1}{6} \cdot \frac{1}{1-\frac{5}{6}} = 1.$$

Problem 4

4 pts

Chapter 1.2 Exercise 26

Two cards are drawn successively from a deck of 52 cards. Find the probability that the second card is higher in rank than the first card. Hint: Show that $1 = P(\text{higher}) + P(\text{lower}) + P(\text{same})$ and use the fact that $P(\text{higher}) = P(\text{lower})$.

The possible outcomes of drawing two cards from a deck of 52 cards are

- the first card is higher in rank than the second card,
- the second card is higher in rank than the first card, and
- the ranks of the two cards are the same.

Therefore, $1 = P(\text{higher}) + P(\text{lower}) + P(\text{same})$.

By symmetry, we know that $P(\text{higher}) = P(\text{lower})$. That is, to get the probability that the second card is higher in rank than the first card, we only need to figure out $P(\text{same})$, which is

$$13 \cdot \frac{4 \cdot 3}{52 \cdot 51} = \frac{1}{17}.$$

And hence

$$P(\text{higher}) = P(\text{lower}) = \frac{1}{2} \cdot (1 - \frac{1}{17}) = \frac{8}{17}.$$

Problem 5

5 pts

Chapter 2.2 Exercise 2

Suppose you choose a real number X from the interval $[2, 10]$ with a density function of the form

$$f(x) = Cx,$$

where C is a constant.

(a) Find C.

(b) Find $P(E)$, where $E = [a, b]$ is a subinterval of $[2, 10]$.

(c) Find $P(X > 5)$, $P(X < 7)$, and $P(X^2 - 12X + 35 > 0)$.

(a) Given that

$$\int_2^{10} f(x)dx = \int_2^{10} Cxdx = \frac{C}{2}x^2|_2^{10} = 48C = 1,$$

we obtain $C = \frac{1}{48}$.

(b)

$$P(E) = P([a, b]) = \int_a^b f(x)dx = \int_a^b \frac{1}{48}xdx = \frac{b^2 - a^2}{96}.$$

(c)

$$P(X > 5) = \int_5^{10} f(x)dx = \int_5^{10} \frac{1}{48}xdx = \frac{75}{96} = \frac{25}{32}.$$

$$P(X < 7) = \int_2^{7} f(x)dx = \int_2^{7} \frac{1}{48}xdx = \frac{45}{96} = \frac{15}{32}.$$

Solving the inequality $X^2 - 12X + 35 = (X-5)(X-7) > 0$ yields $X < 5$ or $X > 7$. Hence

$$\begin{aligned} P(X^2 - 12X + 35 > 0) &= P(X < 5) + P(X > 7) \\ &= 1 - P(X > 5) + 1 - P(X < 7) \\ &= 1 - \frac{25}{32} + 1 - \frac{15}{32} \\ &= \frac{3}{4}. \end{aligned}$$

Problem 6

4 pts

Chapter 2.2 Exercise 5

Suppose you are watching a radioactive source that emits particles at a rate described by the exponential density

$$f(t) = \lambda e^{-\lambda t},$$

where $\lambda = 1$, so that the probability $P(0, T)$ that a particle will appear in the next T seconds is $P([0, T]) = \int_0^T \lambda e^{-\lambda t} dt$. Find the probability that a particle (not necessary the first) will appear

(a) within the next second.

(b) within the next 3 seconds.

(c) between 3 and 4 seconds from now.

(d) after 4 seconds from now.

(a)

$$P([0,1]) = \int_0^1 \lambda e^{-\lambda t} dt = -e^{-\lambda t}|_0^1 = 1 - e^{-\lambda} = 1 - e^{-1} = 1 - \frac{1}{e}.$$

(b)

$$P([0,3]) = \int_0^3 \lambda e^{-\lambda t} dt = -e^{-\lambda t}|_0^3 = 1 - e^{-3\lambda} = 1 - e^{-3} = 1 - \frac{1}{e^3}.$$

(c) Notice the phrasing "**not necessary the first**".

$$P([0, 4-3]) = P([0,1]) = 1 - \frac{1}{e}.$$

(d) Similar to the last part,

$$P([0, +\infty - 4)) = P([0, +\infty)) = 1.$$

Problem 7

4 pts

Chapter 2.2 Exercise 6

Assume that a new light bulb will burn out after t hours, where t is chosen from $[0, \infty)$ with an exponential density

$$f(t) = \lambda e^{-\lambda t}.$$

In this context, λ is often called the **failure rate** of the bulb.

(a) Assume that $\lambda = 0.01$, and find the probability that the bulb will not burn out before T hours. This probability is often called the **reliability** of the bulb.

(b) For what T is the reliability of the bulb $= \frac{1}{2}$.

(a)

$$P([T, +\infty)) = \int_T^{+\infty} f(t) dt = \int_T^{+\infty} \lambda e^{-\lambda t} dt = e^{-\lambda T} = e^{-T/100}.$$

(b) Since $P([T, +\infty)) = e^{-T/100} = \frac{1}{2}$, we have $T = 100 \ln(2)$.

Problem 8

4 pts

Chapter 5.2 Exercise 2

Choose a number U from the interval $[0, 1]$ with uniform distribution. Find the cumulative distribution and density for the random variables

(a) $Y = \frac{1}{U+1}$.

(b) $Y = \log(U+1)$.

(a) The cumulative distribution function is

$$\begin{aligned} F_Y(y) = P(Y \leq y) &= P(\frac{1}{U+1} \leq y) \\ &= P(U \geq \frac{1}{y} - 1) \\ &= 2 - \frac{1}{y}, \end{aligned}$$

where $\frac{1}{2} \leq y \leq 1$.
Further we have

$$f(y) = \frac{d}{dy} F_Y(y) = \frac{1}{y^2}.$$

(b) The cumulative distribution function is

$$\begin{aligned} F_Y(y) = P(Y \leq y) &= P(\log(U+1) \leq y) \\ &= P(U \leq e^y - 1) \\ &= e^y - 1, \end{aligned}$$

where $0 \leq y \leq \log(2)$.
Further we have

$$f(y) = \frac{d}{dy} F_Y(y) = e^y.$$

Problem 9

4 pts

Chapter 5.2 Exercise 10

Let U, V be random numbers chosen independently from the interval $[0, 1]$. Find the cumulative distribution and density for the random variables

(a) $Y = \max(U, V)$.

(b) $Y = \min(U, V)$.

(a)

$$\begin{aligned} F_Y(y) = P(Y \leq y) &= P(\max(U, V) \leq y) \\ &= P(U \leq y \,\text{and}\, V \leq y) \\ &= P(U \leq y \cap V \leq y) \\ &= P(U \leq y) \cdot P(V \leq y) \\ &= y^2, \end{aligned}$$

where $0 \leq y \leq 1$.
Further we have

$$f(y) = \frac{d}{dy} F_Y(y) = 2y.$$

(b)

$$\begin{aligned} 1 - F_Y(y) = P(Y > y) &= P(\min(U, V) > y) \\ &= P(U > y \,\text{and}\, V > y) \\ &= P(U > y \cap V > y) \\ &= P(U > y) \cdot P(V > y) \\ &= (1 - y)^2, \end{aligned}$$

where $0 \leq y \leq 1$. And hence

$$F_Y(y) = 1 - (1 - y)^2.$$

Further we have

$$f(y) = \frac{d}{dy} F_Y(y) = 2(1 - y).$$