Math 20: Probability

Homework 6

August 12, 2020

Problem 1

4 pts

Chapter 5.2 Exercise 5

Choose a number U from the interval $[0,1]$ with uniform distribution. Find the cumulative distribution and density for the random variables

(a) $Y = |U - \frac{1}{2}|$

(b) $Y = (U - \frac{1}{2})^2$

Note: You need to specify the range of Y.

2 pts

We have

$$F(y) = P(Y \leq y) = P(|U - \frac{1}{2}| \leq y) = 2y.$$

Therefore,

$$f(y) = \frac{d}{dy}F(y) = 2,$$

where $y \in [0, \frac{1}{2}]$.

2 pts

We have

$$F(y) = P(Y \leq y) = P((U - \frac{1}{2})^2 \leq y) = P(|U - \frac{1}{2}| \leq \sqrt{y}) = 2\sqrt{y}.$$

Therefore,

$$f(y) = \frac{d}{dy}F(y) = \frac{1}{\sqrt{y}},$$

where $y \in [0, \frac{1}{4}]$.

Problem 2

4 pts

Chapter 5.2 Exercise 26 (modified)

Bridies' Bearing Works manufactures bearing shafts whose diameters are normally distributed with parameters $\mu = 1$, $\sigma = 0.003$. The buyer's specifications require these diameters to be 1.000 ± 0.006 cm. What fraction of the manufacturer's shafts are likely to be rejected? If the manufacturer improves her quality control, she can reduce the value of σ. What value of σ will ensure that no more than 0.3 percent of her shafts are likely to be rejected?

Note: The numbers have been modified.

2 pts

We have

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}.$$

Here $\mu = 1$ and $\sigma = 0.003$.

Notice that $0.006 = 2 \times 0.003$. We would like to figure out the probability

$$P(|X - \mu| \geq 2\sigma) = 1 - 95\% = 5\%.$$

That is, 5 percent of the shafts are likely to be rejected.

2 pts

Since no more than 0.3 percent of the shafts are likely to be rejected, we know that at least 99.7 percent of them will have diameters within 1.000 ± 0.006 cm.

Recall that 99.7 percent corresponds to the range of 3σ. Hence 3σ is at most 0.006 cm and σ should be at most 0.002.

Problem 3

4 pts

Chapter 5.2 Exercise 29

Suppose that the time (in hours) required to repair a car is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$. What is the probability that the repair time exceeds 4 hours? If it exceeds 4 hours what is the probability that it exceeds 8 hours?

2 pts

$$P(T > 4) = e^{-4\lambda} = e^{-2}.$$

2 pts

$$P(T > 8 \,|\, T > 4) = P(T > 4) = e^{-4\lambda} = e^{-2}.$$

Problem 4

4 pts

Chapter 5.2 Exercise 31

Let U be a uniformly distributed random variable on $[0, 1]$. What is the probability that the equation $x^2 + 4Ux + 1 = 0$ has two distinct real roots x_1 and x_2.

4 pts

To guarantee that the equation has two distinct real roots, we need

$$\Delta = b^2 - 4ac = 16U^2 - 4 > 0.$$

That is, $U^2 > \frac{1}{4}$ and hence $U > \frac{1}{2}$.

The probability is simply $\frac{1}{2}$.

Problem 5

4 pts

Chapter 5.2 Exercise 34

Jones puts in two new lightbulbs: a 60 watt bulb and a 100 watt bulb. It is claimed that the lifetime of the 60 watt bulb has an exponential density with average lifetime 200 hours ($\lambda = 1/200$). The 100 watt bulb also has an exponential density but with average lifetime of only 100 hours ($\lambda = 1/100$). Jones wonders what is the probability that the 100 watt bulb will outlast the 60 watt bulb.

- method 1

4 pts

Let X and Y denote the lifetime of the 60 watt bulb and the 100 watt bulb, respectively. We have

$$f_X(x) = \frac{1}{200}e^{-x/200}, \qquad f_Y(y) = \frac{1}{100}e^{-x/100},$$

and

$$F_X(x) = 1 - e^{-x/200}, \qquad F_Y(y) = 1 - e^{-y/100}.$$

What we are after is

$$\begin{aligned}
P(Y > X) &= \int_{x=0}^{\infty}\int_{y=x}^{\infty} f_X(x)f_Y(y)dydx \\
&= \int_{x=0}^{\infty} f_X(x)(1 - F_Y(x))dx \\
&= \int_{x=0}^{\infty} \frac{1}{200}e^{-x/200}e^{-x/100}dx \\
&= \int_{x=0}^{\infty} \frac{1}{200}e^{-3x/200}dx \\
&= \frac{1}{3}.
\end{aligned}$$

- method 2

4 pts

Let X and Y denote the lifetime of the 60 watt bulb and the 100 watt bulb, respectively. We have

$$f_X(x) = \frac{1}{200}e^{-x/200}, \qquad f_Y(y) = \frac{1}{100}e^{-x/100},$$

and

$$F_X(x) = 1 - e^{-x/200}, \qquad F_Y(y) = 1 - e^{-y/100}.$$

What we are after is

$$\begin{aligned} P(Y > X) &= \int_0^\infty P(Y > X \mid X = u) f_X(u) du \\ &= \int_0^\infty P(Y > u) f_X(u) du \\ &= \int_0^\infty (1 - F_y(u)) f_X(u) du \\ &= \int_0^\infty e^{-u/100} \frac{1}{200} e^{-u/200} du \\ &= \int_0^\infty \frac{1}{200} e^{-3u/200} du \\ &= \frac{1}{3}. \end{aligned}$$

Problem 6

4 pts

Chapter 5.2 Exercise 37 (modified)

Let X be a random variable having a **standard** normal density and consider the random variable $Y = e^X$. Then Y has a *log normal* density. Find this density of Y.

Note: Now we only consider when X is a **standard** normal random variable.

Hint: **Leibniz integral rule**. For instance, $\frac{d}{dx}\int_a^{x^2} \sin(t)dt = 2x\sin(x^2)$.

4 pts

We first calculate the cumulative distribution function.

$$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln(y)) = \int_{-\infty}^{\ln(y)} f(x)dx.$$

The density function would then be

$$f(y) = \frac{d}{dy}F_Y(y) = \frac{1}{y}f(\ln(y)) = \frac{1}{\sqrt{2\pi}y}e^{-\frac{\ln^2(y)}{2}},$$

where $y > 0$.

Problem 7

4 pts

Chapter 6.1 Exercise 6

A dice is rolled twice. Let X denote the sum of the two numbers that turn up and Y the difference of the numbers (specifically, the number on the first roll minus the number on the second). Show that $E(XY) = E(X)E(Y)$. Are X and Y independent?

Hint: There is a quick (pain-free) method to solve this problem which does not need the distribution functions of X and Y. We can, as a matter of fact, rewrite X and Y as functions of simpler random variables.

To our grader Maria: please give some bonus credits to those students who can use the method I provided in the solutions.

The distribution function of X is

$$m(X=2) = m(X=12) = \frac{1}{36}, \qquad m(X=3) = m(X=11) = \frac{2}{36},$$
$$m(X=4) = m(X=10) = \frac{3}{36}, \qquad m(X=5) = m(X=9) = \frac{4}{36},$$
$$m(X=6) = m(X=8) = \frac{5}{36}, \qquad m(X=7) = \frac{6}{36}.$$

That of Y is

$$m(Y=-5)=m(Y=5)=\frac{1}{36}, \qquad m(Y=-4)=m(Y=4)=\frac{2}{36},$$

$$m(Y=-3)=m(Y=3)=\frac{3}{36}, \qquad m(Y=-2)=m(Y=2)=\frac{4}{36},$$

$$m(Y=-1)=m(Y=1)=\frac{5}{36}, \qquad m(Y=0)=\frac{6}{36}.$$

4 pts

Let U and V the outcomes of the first and the second rolls, respectively. Then we know that $X=U+V$ and $Y=U-Y$.
Therefore,

$$E(XY)=E((U+V)(U-V))=E(U^2-V^2)=E(U^2)-E(V^2),$$

and

$$\begin{aligned} E(X)E(Y) &= E(U+V)E(U-V) \\ &= [E(U)+E(V)][E(U)-E(V)] \\ &= E^2(U)-E^2(V). \end{aligned}$$

Notice that the distribution functions for U and V are the same. Hence $E(U)=E(V)$, $E(U^2)=E(V^2)$ and we have

$$E(XY)=E(X)E(Y)=0.$$

Notice that $P(X=2 \,|\, Y=0)=\frac{1}{6} \neq P(X=2)=\frac{1}{36}$. The two random variables are not independent.

Problem 8

4 pts

Chapter 6.1 Exercise 7

Show that, if X and Y are random variables taking on only two values each, and if $E(XY)=E(X)E(Y)$, then X and Y are independent.

Note: To make your life easier in this hot summer, let's just assume that X can only take values 0 and 1 and so does Y (why we can simply the original problem in this way?).

4 pts

Define the distribution functions of X and Y as

$$m(X) = \begin{cases} x_1, & \text{with probability } p_x \\ x_2. & \text{otherwise} \end{cases},$$

and

$$m(Y) = \begin{cases} y_1, & \text{with probability } p_y \\ y_2. & \text{otherwise} \end{cases}.$$

Let

$$U = \frac{X - x_2}{x_1 - x_2}, \qquad V = \frac{Y - y_2}{y_1 - y_2}.$$

Then both U and V take only values 0 and 1.

Given that $E(XY) = E(X)E(Y)$, we have

$$E(UV) = E(\frac{X - x_2}{x_1 - x_2} \times \frac{Y - y_2}{y_1 - y_2}) = E(\frac{X - x_2}{x_1 - x_2}) \times E(\frac{Y - y_2}{y_1 - y_2}) = E(U)E(V).$$

And if U and V are independent, so are X and Y.

Thus it is sufficient to prove independence for U and V.

Now

$$E(UV) = P(U = 1, V = 1) = E(U)E(V) = P(U = 1)P(V = 1)$$

,
and

$$\begin{aligned} P(U = 1, V = 0) &= P(U = 1) - P(U = 1, V = 1) \\ &= P(U = 1)(1 - P(V = 1)) \\ &= P(U = 1)P(V = 0). \end{aligned}$$

Similarly, $P(U = 0, V = 1) = P(U = 0)P(V = 1)$ and $P(U = 0, V = 0) = P(U = 0)P(V = 0)$.

Therefore, U and V are independent and hence X and Y are also independent.

Problem 9

4 pts

Chapter 6.1 Exercise 13

You have 80 dollars and play the following game. An urn contains two white balls and two black balls. You draw the balls out one at a time without replacement until all the balls are gone. On each draw, you bet half of your present fortune that you will draw a white ball. What is your expected final fortune?

4 pts

There are $\binom{4}{2} = 6$ possible outcomes. That is, the distribution function of the outcomes is $m(x) = \frac{1}{6}$. And the dollars I have in the end may be

- white, white, black, black: 45,
- white, black, white, black: 45,
- white, black, black, white: 45,
- black, white, white, black: 45,
- black, white, black, white: 45,
- black, black, white, white: 45.

Hence the expected fortune is 45.