Mathematics for Political Science

Lecture 3 – Calculus I

Sarah B. Bouchat

University of Wisconsin–Madison*

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Calculus: Overview

Calculus evaluates the behavior of functions:

- Increasing/decreasing
- Rate of change
- Change in rate of change
- Area of the region they define

Concepts from calculus underlie a wide variety of mathematics, particularly in the applied math that we use in political science.

Calculus in Political Science

- Finding the fit line with the minimal distance between predicted and observed data
- Calculating the probability density in regions of continuous distributions
- Choosing parameter values that maximize the likelihood of generating observed data
- Solving for the choice that maximizes a decision maker's utility

Agenda

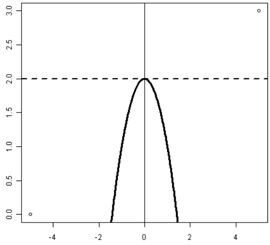
(1) Limits

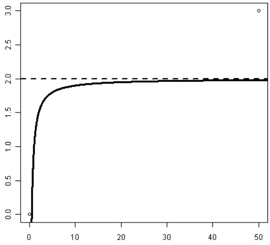
(2) Derivatives

- Notation
- Simple Derivatives
- Derivatives of Sums/Differences
- Derivatives of Products/Quotients
- Derivatives of Nested Functions, Exponents, and Logarithms

Limits

The *limit* of a function characterizes its behavior given a certain input/as input value changes.





$$\lim_{x \to 0} (2 - x^2) = 2$$

x	-.2	-.1	0	.1	.2
f(x)	1.96	1.99	2	1.99	1.96

$$\lim_{x \to \infty} (2 - \frac{1}{x}) = 2$$

x	0	5	10	20	50
f(x)	1	1.8	1.9	1.96	1.98

Limits

Formally, a limit is defined as:

Limit

$$\lim_{x \to c} f(x) = L \Rightarrow$$

$$\forall\ \epsilon > 0, \exists\ \delta > 0$$

$$\text{such that } (L - \epsilon) < f(x) < (L + \epsilon)$$

$$\text{if either } (c - \delta) < x < c \text{ or } c < x < (c + \delta)$$

Intuition:

- Pick some small number ϵ...
- ... there will exist some number δ...
- ... such that keeping x within δ of c
- ... assures that $f(x)$ is within ϵ of L

General Tips on Finding Limits

Simplify as much as possible.

- Separate out into operations on limits of distinct elements.
- Move constants outside the limit operator.

Consider the behavior of different sub-parts of the function as it approaches the limit.

- Do any components grow very large or very small?
- Does anything become zero?
- Are these in the numerator or denominator of a fraction?

Evaluate the function at the limit.

- For functions that are well-behaved (continuous), the limit as x approaches a finite point is generally the value of the function at that point (if it exists).

Take it to the Limit

$\lim_{x \to 0}(2 - x^2)$

Simplify/Separate:

$$\begin{aligned}\lim_{x \to 0}(2 - x^2) &= \lim_{x \to 0} 2 - \lim_{x \to 0} x^2 \\ &= 2 - 0 \\ &= 2\end{aligned}$$

Sub-behaviors:

The 2 component remains constant regardless of x, while the x^2 component grows very small as x approaches 0.

Evaluate at the Limit:

$$\begin{aligned}\lim_{x \to 0}(2 - x^2) &= 2 - x^2 \text{ where } x = 0 \\ &= 2 - 0^2 \\ &= 2\end{aligned}$$

Take it to the Limit

Consider $\frac{4x^4+7x^2+8}{3x^4}$ ... as $x \to \infty$...

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x^4+7x^2+8}{3x^4} &= \lim_{x\to\infty} \frac{4x^4}{3x^4} + \lim_{x\to\infty} \frac{7x^2}{3x^4} + \lim_{x\to\infty} \frac{8}{3x^4} \\
&= \lim_{x\to\infty} \frac{4}{3} + \lim_{x\to\infty} \frac{7}{3x^2} + \lim_{x\to\infty} \frac{8}{3x^4} \\
&= \frac{4}{3} + 0 + 0 \\
&= \frac{4}{3}
\end{aligned}$$

As x increases without bound, the effects of the highest order coefficients (the x^4 terms) outweigh the other terms, and the limit is their ratio.

Take it to the Limit

Consider $\frac{4x^4+7x^2+8}{3x^4}$ ... as $x \to 0$...

$$\begin{aligned}\lim_{x\to 0} \frac{4x^4+7x^2+8}{3x^4} &= \frac{\lim_{x\to 0} 4x^4+7x^2+8}{\lim_{x\to 0} 3x^4} \\ &= \frac{8}{0} \\ &= \infty\end{aligned}$$

As x approaches 0, the function retains some positive value in the numerator while all other terms (including the entire denominator) approach 0. Since you cannot divide by 0, this function increases without bound.

Derivatives

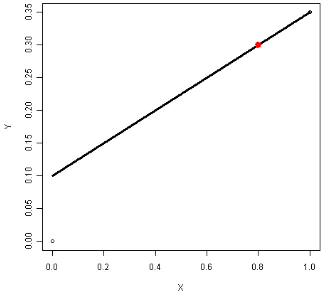
The *derivative* of a function is its rate of change in the output as the value of its input changes.

The instantaneous slope of the line at any given point.

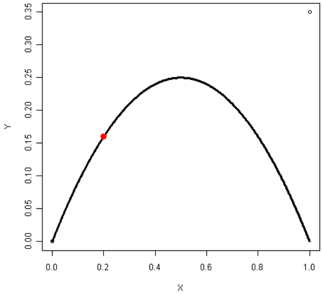
The slope of the line tangent to the function at any given point.

Slopes

The *slope* of a function is how much the output changes as a result of changes in the input. Using Δ to signify 'change', this is $\frac{\Delta f(x)}{\Delta x}$.



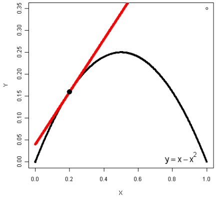
$$f(x) = .25x + .1$$
$$\frac{\Delta f(x)}{\Delta x} = .25$$

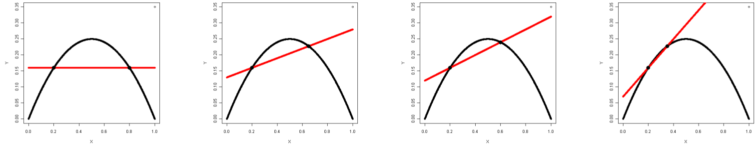


$$f(x) = x - x^2$$
$$\frac{\Delta f(x)}{\Delta x} = ???$$

Derivative as a Limit

Approximate the slope by picking a point nearby on the line and finding the slope of the straight line connecting them





When the interval is wide, not a good approximation. But increasingly close as the interval grows smaller and smaller.

Derivative as a Limit

As you reduce the interval size to 0, this line converges in the limit to the line that lies tangent to the curve at that point. Recall that $f(x) = x - x^2$ and consider a very small interval ϵ...

$$\frac{\Delta f(x)}{\Delta x} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{(x+\epsilon) - x}$$

$$\frac{df(x)}{dx} = \lim_{\epsilon \to 0} \frac{[(x+\epsilon) - (x+\epsilon)^2] - [(x) - (x)^2]}{(x+\epsilon) - x}$$

$$\frac{df(x)}{dx} = \lim_{\epsilon \to 0} \frac{[(x+\epsilon) - (x^2 + 2x\epsilon + \epsilon^2)] - [(x) - (x)^2]}{(x+\epsilon) - x}$$

$$\frac{df(x)}{dx} = \lim_{\epsilon \to 0} \frac{x + \epsilon - x^2 - 2x\epsilon - \epsilon^2 - x + x^2}{x + \epsilon - x}$$

Derivative as a Limit

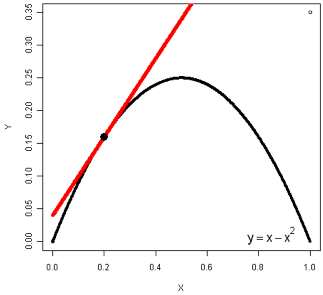
$$\frac{df(x)}{dx} = \lim_{\epsilon \to 0} \frac{x + \epsilon - x^2 - 2x\epsilon - \epsilon^2 - x + x^2}{x + \epsilon - x}$$

$$\frac{df(x)}{dx} = \lim_{\epsilon \to 0} \frac{\epsilon - 2x\epsilon - \epsilon^2}{\epsilon}$$

$$\frac{df(x)}{dx} = \lim_{\epsilon \to 0} 1 - 2x + \epsilon$$

$$\frac{df(x)}{dx} = 1 - 2x$$

Derivative as a Limit



Using this formula, the slope of the curve at $x = .2$ is exactly:

$$\begin{aligned} \text{slope} &= 1 - 2(.2) \\ &= .6 \end{aligned}$$

Alternatively, we could find the point at which the slope is exactly 0:

$$\begin{aligned} 0 &= 1 - 2x \\ 2x &= 1 \\ x &= .5 \end{aligned}$$

Derivatives

Derivative

$$\frac{d[f(x)]}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{(x+\epsilon) - x}$$

Using this approach, we can find:

- A general equation for the slope at *any* point
- The *exact value* of the slope at a given point
- *The point* that has a given slope

Derivative Notation

Different notational systems for expressing derivatives:

First derivative:	$\frac{d[f(x)]}{dx}$	$\frac{d}{dx}[f(x)]$	$f'^{(x)}$	$f^1(x)$
Second derivative:	$\frac{d^2[f(x)]}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$	$f''(x)$	$f^2(x)$

Cautionary Notes on Derivatives

A few assumptions in using this approach to find the slope:

- The function is continuous.
  - no gaps or jumps
- The derivative exists (the limit of the slope is the same from the left as it is from the right).
  - no sharp corners

For nearly all political science applications, these are fine assumptions. But it is important to state them explicitly and be aware that they're there.

Straightforward Derivatives

Fortunately, it is not necessary to take limits with deltas and epsilons every time. A few rules generate the derivatives of most functions.

Power Rule

$$\frac{d[ax^n]}{dx} = nax^{n-1}$$

Special cases of this rule:

- The derivative of a straight line is a constant: $\frac{d[ax]}{dx} = a$
- The derivative of a constant is zero: $\frac{d[C]}{dx} = 0$

Straightforward Derivatives

$\frac{d[ax^n]}{dx} = nax^{n-1}$

Let $f(x) = 3x^2$:
$$\frac{d[f(x)]}{dx} = (2 * 3)x^{2-1} = 6x$$

Let $g(x) = x^5$:
$$\frac{d[g(x)]}{dx} = (5 * 1)x^{5-1} = 5x^4$$

Let $h(x) = 7x$:
$$\frac{d[h(x)]}{dx} = (1 * 7)x^{1-1} = 7$$

Straightforward Derivatives

$\frac{d[ax^n]}{dx} = nax^{n-1}$

Find the following derivatives, and calculate the instantaneous slope of the curves at the point $x = 2$:

- $f(x) = \frac{1}{4}x^4$
- $g(x) = 2x^{-3}$
- $h(x) = 4x^{\frac{5}{2}}$

Derivative of a Sum (or Difference)

The derivative of a sum is the sum of the derivatives.

Sum (Difference) Rule

$$\frac{d[f(x) + g(x)]}{dx} = \frac{d[f(x)]}{dx} + \frac{d[g(x)]}{dx}$$

Derivative of a Sum (or Difference) $\frac{d[f(x)+g(x)]}{dx} = \frac{d[f(x)]}{dx} + \frac{d[g(x)]}{dx}$

Let $f(x) = 4x^2$ and $g(x) = 5x^3$:

$$\begin{aligned}\frac{d[4x^2 + 5x^3]}{dx} &= \frac{d[4x^2]}{dx} + \frac{d[5x^3]}{dx} \\ &= 8x + 15x^2\end{aligned}$$

Let $f(x) = x$ and $g(x) = x^2$:

$$\begin{aligned}\frac{d[x - x^2]}{dx} &= \frac{d[x]}{dx} - \frac{d[x^2]}{dx} \\ &= 1 - 2x\end{aligned}$$

Derivative of a Sum (or Difference)

Find the derivative of the function below:

- $f(x) = 2x^5 - 3x^4 + 7x^2 + 12x + 5$

Derivative of a Product

The derivative of a product two functions is:

- the first times the derivative of the second...
- ... plus the second times the derivative of the first

Product Rule

$$\frac{d[f(x)g(x)]}{dx} = f(x)\frac{d[g(x)]}{dx} + g(x)\frac{d[f(x)]}{dx}$$

Derivative of a Product

$\frac{d[f(x)g(x)]}{dx} = f(x)\frac{d[g(x)]}{dx} + g(x)\frac{d[f(x)]}{dx}$

Let $f(x) = 2x^3 + 5$ and $g(x) = 6x - 2$:

$$\begin{aligned}
\frac{d[(2x^3+5)(6x-2)]}{dx} &= (2x^3+5)\frac{d[6x-2]}{dx} + (6x-2)\frac{d[2x^3+5]}{dx} \\
&= (2x^3+5)(6) + (6x-2)(6x^2) \\
&= 12x^3 + 30 + 36x^3 - 12x^2 \\
&= 48x^3 - 12x^2 + 30
\end{aligned}$$

This is easy to check by multiplying out the polynomial:
$(2x^3 + 5)(6x - 2) = 12x^4 - 4x^3 + 30x - 10$

$$\frac{d[12x^4 - 4x^3 + 30x - 10]}{dx} = 48x^3 - 12x^2 + 30$$

Derivative of a Quotient

The derivative of a quotient is:

- the bottom times the derivative of the top...
- ... minus the top times the derivative of the bottom...
- ... all divided by the bottom, squared.

Quotient Rule

$$\frac{d[\frac{f(x)}{g(x)}]}{dx} = \frac{g(x)\frac{d[f(x)]}{dx} - f(x)\frac{d[g(x)]}{dx}}{(g(x))^2}$$

This can also be derived from the product rule, since dividing by $g(x)$ is equivalent to multiplying by the reciprocal $\frac{1}{g(x)}$.

Derivative of a Quotient

$$\frac{d[\frac{f(x)}{g(x)}]}{dx} = \frac{g(x)\frac{d[f(x)]}{dx} - f(x)\frac{d[g(x)])}{dx}}{(g(x))^2}$$

Let $f(x) = 8x - 1$ and $g(x) = 3x^2 + 2$:

$$\frac{d[(\frac{8x-1}{(3x^2+2)})]}{dx} = \frac{(3x^2+2)(8) - (8x-1)(6x)}{(3x^2+2)^2}$$

$$= \frac{(24x^2+16) - (48x^2 - 6x)}{(9x^4 + 12x^2 + 4)}$$

$$= \frac{-24x^2 + 6x + 16}{9x^4 + 12x^2 + 4}$$

Derivative of Products and Quotients

Find the derivative of the following expressions:

- $(3x^2 - 4x + 2)(x^3 - x^2 + x - 1)$
- $\frac{4x+1}{3x^2-2}$

Derivative of Nested Functions

The derivative of one function nested inside another is:

- the derivative of the outside with respect to the inside...
- ... times the derivative of the inside function

Chain Rule

$$\frac{d[f(g(x))]}{dx} = \frac{d[f(g(x))]}{d(g(x))}\frac{d[g(x)]}{dx}$$

This looks messy, but is actually fairly straightforward and extremely useful as a way to find derivatives of complex functions by treating them as nested chains of functions.

Derivative of Nested Functions

$\frac{d[f(g(x))]}{dx} = \frac{d[f(g(x))]}{d(g(x))} \frac{d[g(x)]}{dx}$

Let $h(x) = 6(3x^2 + 2)^4$. Observe that this can be thought of as two nested functions, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $h(x) = f(g(x))$:

$$\begin{aligned} \frac{d[6(3x^2+2)^4]}{dx} &= 24(3x^2+2)^3 6x \\ &= 144x(3x^2+2)^3 \end{aligned}$$

Let $k(x) = 3(6x^4)^2 + 2$. Observe that this can be thought of the same two functions nested in the reverse order, such that $k(x) = g(f(x))$:

$$\begin{aligned} \frac{d[3(6x^4)^2+2]}{dx} &= 6(6x^4)24x^3 \\ &= 864x^7 \end{aligned}$$

Derivative of Nested Functions

Express the functions below as the nested result of two simpler functions, and use the chain rule to find the derivative:

- $(3x-1)^4$
- $2(x^4+x^3)+7$

Derivative of Exponents

When the variable itself is in the exponent, the derivative is found according to the formula below.

Exponent Rule

$$\frac{d[n^x]}{dx} = \ln(n)n^x$$

Note the special case (because $\ln(e) = 1$):

$$\frac{d[e^x]}{dx} = e^x$$

Derivative of Exponents

$\frac{d[n^x]}{dx} = \ln(n)n^x$

Let $f(x) = 4^x$:

$$\frac{d[4^x]}{dx} = \ln(4)4^x$$

Let $g(x) = 2^{3x}$ (using the Chain Rule):

$$\frac{d[2^{3x}]}{dx} = \ln(2)2^{3x} * 3$$

Let $h(x) = 4e^x$:

$$\frac{d[4e^x]}{dx} = 4e^x$$

Derivative of Logarithms

When the variable is inside a logarithm function, the derivative is found according to the formula below.

Logarithm Rule

$$\frac{d[\log_b(x)]}{dx} = \frac{1}{x(\ln(b))}$$

A very useful case of this is the derivative of the natural log:

$$\frac{d[\ln(x)]}{dx} = \frac{1}{x}$$

Derivative of Logarithms

$\frac{d[\log_b(x)]}{dx} = \frac{1}{x(\ln(b))}$

Let $f(x) = \log_{10}(x)$:

$$\frac{d[\log_{10}(x)]}{dx} = \frac{1}{x \ln 10}$$

Let $g(x) = \ln(5x^2 + 2)$:

$$\begin{aligned} \frac{d[\ln(5x^2+2)]}{dx} &= \frac{1}{5x^2+2} 10x \\ &= \frac{10x}{5x^2+2} \end{aligned}$$