Mathematics for Political Science

Day 1 – Introduction & Foundations

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*Thanks to Dave Ohls and Brad Jones for past years' teaching materials!

Mathematics in Political Science

- Empirical research
  - How do we measure or represent political phenomena?
  - What patterns do we observe in the data?
  - How sure are we that these patterns are meaningful (that is, would not have occurred by chance)?
  - How do we change our beliefs about the occurrence of phenomena based on these data?*
  - Does this pattern reflect a *causal* process?*
- Formal theory
  - Who are the key "players" (decision-makers) in a political setting?
  - What interests or goals do these players have?
  - How can they best achieve their goals, given tradeoffs and perhaps limited information?

Math Camp Objectives

- Introduce and/or review fundamentals and mathematical concepts that are directly used or serve as the basis for political science methods
- Reinforce an intuitive understanding of these concepts and how they apply in political science
- Prepare you with skills to successfully complete department course offerings in statistical and formal methodology

But...

- Not a replacement for 812/813, 835/836, etc., and also not an advanced methodology course
- Not proof-driven, not pure mathematics
- Not a historical or theoretical treatment of quantitative social science

Course Outline

- Monday: Fundamentals & Algebra
  - Notation, functions, common operators
  - Basic and multivariate algebra, factoring
- Tuesday: Calculus I & II
  - Limits and derivatives with a single variable
  - Multivariate calculus, partial derivatives, integrals
- Wednesday: Probability & Random Variables—Looking Ahead
  - Basics of probability, random variables, and distributions
- Thursday: R I
  - Data entry and manipulation, basic math operations
- Friday: R II
  - Loading and cleaning external data, visualization

Course Structure

- Mornings:
  - Lectures: 10 am – 12.30 pm (with a break*)
- Afternoons:
  - Office Hours: 1.00 – 3.00 pm**
- Learning Environment:
  - Intensive treatment of a lot of material, conscious of varying backgrounds
  - Safe space to ask questions and build confidence

Let me know how I can make this work best for you

Course Resources

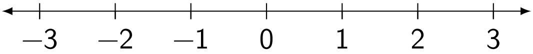
- Online Resources
  - http://bouchat.github.io/
- Book
  - Jeff Gill, *Essential Mathematics for Political and Social Research*

Purpose of Mathematics

- Precise definition of “mathematics” is contentious, but practically, math serves as a tool to clearly and concisely express ideas (logicist/formalist)
  - Use to characterize actors or situations in terms of of quantity and classification
- With precision, we can make specific and *falsifiable* statements—building blocks of knowledge
  - Instrumental and fundamental to the philosophy of science
  - One way, but not the only way, of addressing complexity in social science
  - Helpful to remember that math is also an academic discipline/field of study, not scripture

Numbers

The set of all real numbers is denoted $\mathbb{R}$ or **R**



$\mathbb{R}^+$ (or $\mathbb{R}_{>0}$) and $\mathbb{R}^-$ (or $\mathbb{R}_{<0}$) refer to only positive or negative real numbers, respectively

$\mathbb{R}^n$ refers specifically to real numbers in n dimensions

$\mathbb{Z}$ refers to only integers

Special Numbers

- 0 and 1: Commonly occur, but also different. Used for identities of multiplication (1) and addition (0).
- π: Approximately 3.1415926... The ratio between the circumference of a circle and its diameter. Useful for anything involving angles or circles.
- e: Approximately 2.7182818... The unique number with derivative of e^x being equal to 1 at $x = 0$. Many very convenient properties in calculus.
- i: The *imaginary number* defined as $\sqrt{-1}$. Necessary for complex numbers and useful for some advanced algebra.
- ∞: Infinity. Expresses the concept of quantity without bound (not technically a number).

Variables

A *variable* is a symbol representing a number or group of numbers that may take on different values.

A single value is a *scalar*. Usually denoted in regular typeface:

$$x$$

An ordered series of numbers is a *vector*. Usually denoted in bold face:

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]$$

A rectangular array of numbers is a *matrix*. A vector is a 1-dimensional matrix. Usually denoted in bold face capitals:

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

Data

Data is a collection of information represented as a set of values or observations. A group of data is often referred to as a *dataset*. A singular piece of information is a *datum* or *data point*.

	Coffee	WiFi	Atmosphere	Area	Notes
Aldo's	9	1	good	Campus	no plugs
Ancora (Capitol)	8.5	1	good	Capitol	closes early
Barriques (Capitol)	8.7	1	good	Capitol	wine
Coffeebytes	7.0	1	poor	Campus	convenient
Colectivo (Capitol)	4	1	fair	Capitol	bad wifi
Espresso Royale	5.8	1	fair	Campus	small tables
EVP	6.3	1	poor	Isthmus	fairly small
Fair Trade	7.5	1	good	Campus	good pastries
Madison Sourdough	8	0	good	Willy St.	great brunch
Michelangelo's	7.5	1	fair	Capitol	open late
Mother Fool's	4	1	fair	Willy St.	vegan
Starbucks (State)	5.2	1	poor	Campus	undergrads
Steep and Brew	8.5	1	fair	Campus	weird music

Classifying Data

Quantitative:

- Measurable/quantifiable data used for statistical description, analysis, or inference

Qualitative:

- In-depth, non-quantifiable data used for rich description and understanding

Generally...

- Large-N
- Small-n

Classifying Data

Discrete:

- Variables that take on only specific values from a finite set, sometimes with "gaps"
- atmosphere, number of committee seats, number of parties

Dichotomous*:

- Binary variables (0/1, yes/no)
- wireless, war/peace, election winner/loser

Continuous

- Variables that take any value on an infinite (limited or unlimited) continuum, no gaps
- coffee quality, military spending*, unemployment percentage

Classifying Data

Categorical (Nominal):

- Categorization with no meaningful dimension or order
- neighborhood, continent, political party*

Ordinal:

- Clear ordering, no meaning to the "distance" between ranks*
- atmosphere, agree/unsure/disagree surveys, levels of democratization

Interval:

- Ordered values, distance has significance
- feeling thermometer surveys, time in years

Ratio:

- Ordered values with significance to the intervals and 0 as a meaningful point
- coffee, votes for a candidate, war deaths

Indexing

When expressing the values that a variable takes on for different individuals in a dataset, it is often convenient to *index* those individuals using the subscript i (x_i) for the general case and specific (letter or number) identifiers for each individual.

	Collegiate Math Classes
x_1	3
x_2	0
x_3	1
x_4	1
$\vdots$	$\vdots$
x_{12}	4

Intervals

Intervals or ranges of numbers are expressed like so:

$[0, 1]$	$0 \leqslant x \leqslant 1$	Square brackets [and] imply that the end-points are included (closed interval)
$(0, 1)$	$0 < x < 1$	Parenthesis (and) indicate that the end-points are not included (open interval)
$\{ \}$		Used to indicate listed elements, not strictly an interval

The number of minutes for today's class will be on the interval $[60, 180]$.

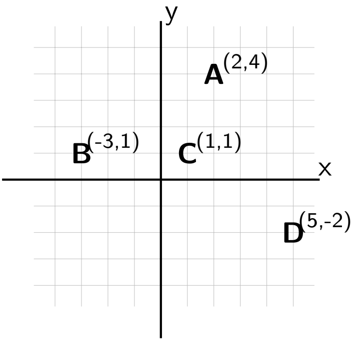
The number of years you will spend in graduate school will be on the interval $[0, \infty)$.

Coordinate System

When working with more than one variable, one can conceive of individual data points as representing a specific spot on a spatial coordinate system, where positive numbers go one way (right, up, etc) and negative numbers go the opposite.

A two dimensional representation of the following individuals in (x, y) space may look like this:

individual	x	y
A	2	4
B	-3	1
C	1	1
D	5	-2



Conceptually, the coordinate system can incorporate any number of dimensions.

Symbols

A number of symbols are commonly used as shorthand in mathematical expressions:

$\neg$	not
$\sim$	not
$/$	not (through symbol)
$\forall$	for all
$\exists$	there exists
$\mid$	given that
$\in$	is an element of

$\approx$	approximately equal to
$\equiv$	equivalent to
$>$ $(<)$	greater (less) than
$\geqslant$ $(\leqslant)$	greater (less) than or equal to
$\propto$	proportional to
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if

Functions

A *function* is an operation or rule of assignment that takes some quantity or quantities as input and maps it onto exactly one (unique) output.

- input(s): *argument(s)*
- output: *value*

Often denoted $f(x)$ or $g(x)$, though other letters or symbols may be used particularly when working with multiple functions at once.

Analogies: machine, black box, routinized process

Functions in Action

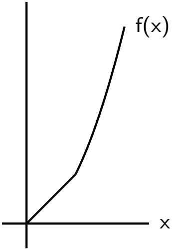
Some data, and a few simple functions:

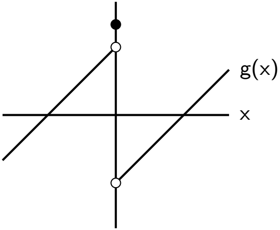
x	y	z	$f(x, y) = x - y$	$g(z) = 2z - 1$	$h(x, y, z) = \frac{x+y}{z}$
5	0	5	5	9	1
2	5	8	-3	15	.875
0	3	9	-3	17	.333
3	2	2	1	3	2.5
8	4	2	4	3	6
1	2	4	-1	7	.75

Multiple Component Functions

Functions may also prescribe different operations on different intervals (or precise points) of their input(s).

$$f(x) = \begin{cases} x & x \in [0,1] \\ x^2 & x \in (1,2] \end{cases} \qquad g(x) = \begin{cases} x+3 & x \in (-\infty, 0) \\ 4 & x = 0 \\ x-3 & x \in (0, \infty) \end{cases}$$





Nested Functions

Functions can also be nested inside one another, e.g. $f(g(x))$. The inside function is applied to the data first, then the output is taken as the input for the outside function.

Suppose $x = 3$, $f(x) = 2x + 5$, and $g(x) = x - 7$.

$$
\begin{aligned}
f(g(x)) &= 2(x - 7) + 5 \\
&= 2(3 - 7) + 5 \\
&= 2(-4) + 5 \\
&= -8 + 5 \\
&= -3 \\
g(f(x)) &= (2x + 5) - 7 \\
&= (2 * 3 + 5) - 7 \\
&= 11 - 7 \\
&= 4
\end{aligned}
$$

Functions

Sketch graphs of the following functions:

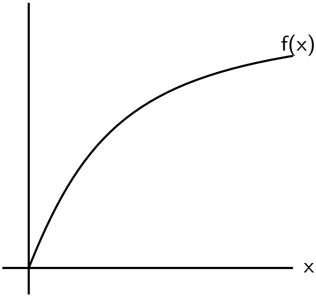
- $p(x) = 2x - 1$, on the interval $[-2, 2]$.
- $q(x) = x * (-1)^x$, for integers $\{0,1,2,3,4,5\}$
- $r(x) = 2x^2 - 3x + 4$, on the interval $(0, 4)$.

Continuity

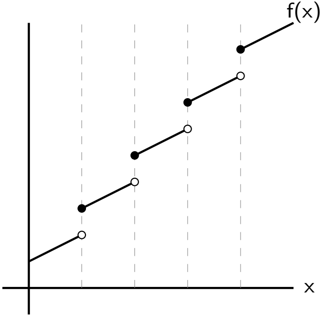
A function is *continuous* if it has no gaps or jumps. Small changes in input produce small changes in output.

Continuous

$$\lim_{x \to a} f(x) = f(a)$$



continuous



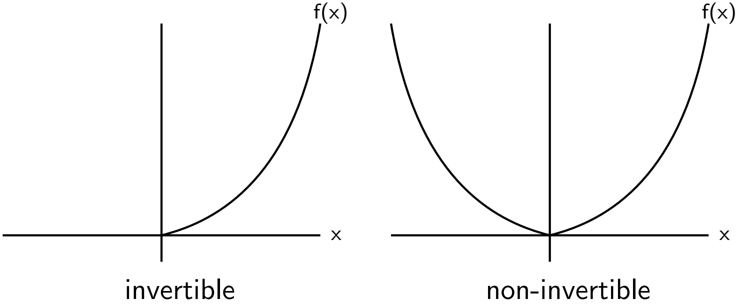
discontinuous

Invertibility

A function is *invertible* if a reverse function exists that maps the functions output back onto its input.

Invertible

$$f^{-1}(y) = x \text{ such that } f^{-1}(f(x)) = x$$

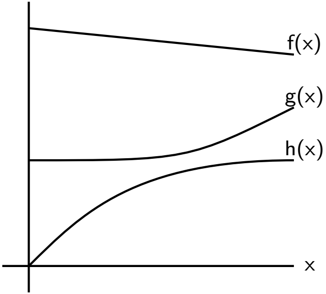


Monotonicity

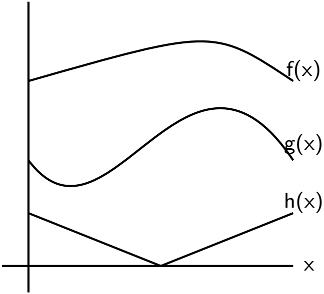
A function is *monotonic* if it either always rises (monotonically increasing) or always falls (monotonically decreasing).

Monotonic

$$x_1 > x_2 \Rightarrow f(x_1) > f(x_2) \quad \text{or} \quad x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$$



monotonic



non-monotonic

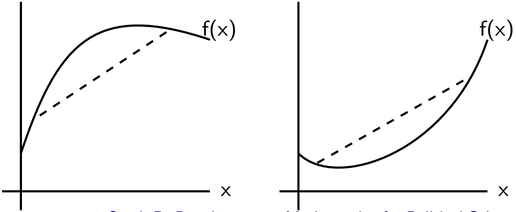
Concavity and Convexity

Functions are *concave* if the output of the average for any two points is greater than the average of the output (open facing down). Functions are *convex* if the output of the average for any two points is less than the average of the output (open facing up).

Concave and Convex

$$\text{Concave:} \quad \frac{f(x_1 + x_2)}{2} \geqslant \frac{f(x_1) + f(x_2)}{2}$$

$$\text{Convex:} \quad \frac{f(x_1 + x_2)}{2} \leqslant \frac{f(x_1) + f(x_2)}{2}$$



Functions

Sketch a function which is:

- continuous, invertible, and monotonic
- discontinuous, non-invertible, and non-monotonic
- invertible and non-monotonic

Operators

Operators are components of functions that tell you what to do with the inputs to produce the outputs. Several common ones occur quite frequently as the building blocks of mathematical functions.

Order of operations when functions incorporate more advanced operators or multiple operators:

- Operations on individual variables
- Operations within parentheses
- Exponents
- Multiplication and Division (working left to right)
- Addition and Subtraction (working left to right)

Summations

The *summation* operator (denoted with a capital sigma $\sum$) adds up the results of the function inside it for all numbers indexed within the range indicated.

Summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n$$

If no range is indicated ($\sum x_i$), this implies all observations are included.

Summations

Suppose we have data:

x_1	x_2	x_3	x_4	x_5
3	4	1	0	2

$$\begin{aligned}\sum_{i=1}^{5} x_i &= x_1 + x_2 + x_3 + x_4 + x_5 \\ &= 3 + 4 + 1 + 0 + 2 \\ &= 10\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{3} (x_i^2 + 3) &= (x_1^2 + 3) + (x_2^2 + 3) + (x_3^2 + 3) \\ &= (3^2 + 3) + (4^2 + 3) + (1^2 + 3) \\ &= 35\end{aligned}$$

Products

The *product* operator (denoted with a capital pi $\prod$) multiplies the results of the function inside it for all numbers indexed within the range indicated.

Product

$$\prod_{i=1}^{n} x_i = x_1 * x_2 * \ldots * x_n$$

If no range is indicated ($\prod x_i$), this implies all observations are included.

Products

Consider again the data:

x_1	x_2	x_3	x_4	x_5
3	4	1	0	2

$$\begin{aligned}\prod_{i=1}^{5} x_i &= x_1 * x_2 * x_3 * x_4 * x_5 \\ &= 3 * 4 * 1 * 0 * 2 \\ &= 0\end{aligned}$$

$$\begin{aligned}\prod_{i=1}^{3} (x_i^2 + 3) &= (x_1^2 + 3) * (x_2^2 + 3) * (x_3^2 + 3) \\ &= (3^2 + 3) * (4^2 + 3) * (1^2 * 3) \\ &= 12 * 19 * 4 \\ &= 912\end{aligned}$$

Products and Summations

Given the data to the right, find:

- $\sum x_i$ and $\sum y_i$
- $\sum (x_i + y_i)$
- $\prod x_i$ and $\prod y_i$
- $\prod (x_i * y_i)$

i	x_i	y_i
1	7	2
2	1	5
3	3	4
4	2	4

Show that, generally:

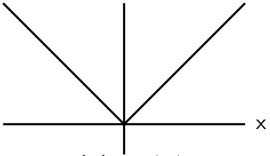
- $\sum (t_i + C) = \sum (t_i) + N * C$
- $\sum (C * t_i) = C * \sum (t_i)$
- $\prod (C * t_i) = C^N * \prod (t_i)$

Absolute Values

The *absolute value* operator returns the positive representation of the number.

Absolute Value

$$|x| = \begin{cases} \text{x} & \text{if x is positive } (x > 0) \\ \text{-x} & \text{if x is negative } (x < 0) \end{cases}$$



$f(x) = |x|$

Absolute Value

Let $f(x) = |x|$:

$$f(10) = 10$$
$$f(-3) = 3$$

Let $g(x) = |x + 4|$:

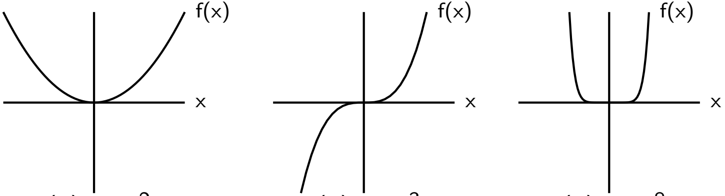
$$\begin{aligned} g(10) &= |10 + 4| \\ &= |14| \\ &= 14 \\ g(-3) &= |-3 + 4| \\ &= |1| \\ &= 1 \\ g(-8) &= |-8 + 4| \\ &= |-4| \\ &= 4 \end{aligned}$$

Exponents

The *exponent* operator multiplies a number by itself the number of times indicated in the exponent.

Exponent - N^{th} Power

$$x^n = x * x * ... * x \quad \text{(n times)}$$



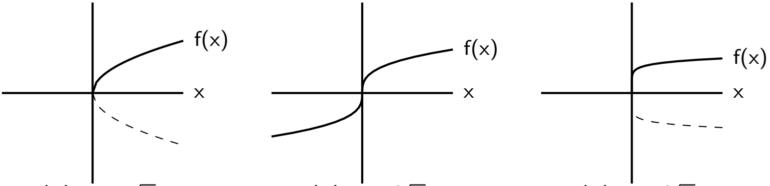
$f(x) = x^2$ $f(x) = x^3$ $f(x) = x^8$

Roots

Root operators return the number that, when multiplied by itself the number of times indicated in the root, is equal to the input. When no number is given, that indicates the *square root*.

Root - N^{th} Root

$$x = \sqrt[n]{x} * \sqrt[n]{x} \ldots \sqrt[n]{x} \quad (\text{n times})$$



$f(x) = \sqrt{x}$ $f(x) = \sqrt[3]{x}$ $f(x) = \sqrt[8]{x}$

Exponents and Roots

All roots can be expressed as exponents and have the same properties:

$$\sqrt[n]{x} \equiv x^{\frac{1}{n}}$$

Key properties for manipulating exponents (and roots):

0^{th} Power	$x^0 = 1$
Negative Powers	$x^{-n} = \frac{1}{x^n}$
Distribution of Powers (Multiplication)	$(x * y)^n = x^n * y^n$
Distribution of Powers (Division)	$(\frac{x}{y})^n = \frac{x^n}{y^n}$
Product of Powers	$x^n * x^m = x^{n+m}$
Nested Powers	$(x^n)^m = x^{n*m}$

Exponents and Roots

Let $f(x) = x^3$:

$$\begin{aligned} f(5) &= 5 * 5 * 5 \\ &= 125 \end{aligned}$$

Let $g(x) = x^4 * x^1$:

$$\begin{aligned} g(6) &= 6^4 * 6^1 \\ &= 6^5 \\ &= 7776 \end{aligned}$$

Let $h(x) = (x^3)^2$:

$$\begin{aligned} h(2) &= (2^3)^2 \\ &= 2^6 \\ &= 64 \end{aligned}$$

Exponents and Roots

Solve:

- $\left(\frac{5}{7}\right)^3$
- $y^3 * y^2 * (x^2)^4 + x^3$

Multiplying Polynomials

Distribute: every element of each polynomial must be multiplied by every element of each other polynomial. Then group the resulting elements according to powers of the variable.

$$\begin{aligned} ax * (bx^2 + c) &= (ax * bx^2) + (ax * c) \\ &= abx^3 + acx \end{aligned}$$

FOIL: first-outside-inside-last (multiply the first term in each, then the outside terms, then the inside terms, then the last term in each).

$$\begin{aligned} (ax + b) * (cx + d) &= (ax * cx) + (ax * d) + (b * cx) + (b * d) \\ &= acx^2 + adx + bcx + bd \\ &= acx^2 + (ad + bc)x + bd \end{aligned}$$

Multiplying Polynomials

Longer polynomials work the same way, just keep track of all the terms.

$$\begin{aligned}(ax^2 + bx + c) * (dx + e) &= (ax^2 * dx) + (ax^2 * e) + (bx * dx) + \\ &\quad (bx * e) + (c * dx) + (c * e) \\ &= adx^3 + aex^2 + bdx^2 + bex + cdx + ce \\ &= adx^3 + (ae + bd)x^2 + (be + cd)x + ce\end{aligned}$$

$$\begin{aligned}(2x^4 + 5x^3) * (8x^2 + x + 3) &= (2x^4 * 8x^2) + (2x^4 * x) + (2x^4 * 3) + \\ &\quad (5x^3 * 8x^2) + (5x^3 * x) + (5x^3 * 3) \\ &= 16x^6 + 2x^5 + 6x^4 + 40x^5 + 5x^4 + 15x^3 \\ &= 16x^6 + 42x^5 + 11x^4 + 15x^3\end{aligned}$$

Multiplying Polynomials

Expand the following products of polynomials:

- $(x^2 + 3) * (x - 2)$
- $(3p + 4q) * (p - 2q)$

Logarithms

The *logarithm* (*log*) of some value x with a given base b equals the power y (exponent) to which that base would need to be raised to equal that value. A logarithm "undoes" exponentiation.

Logarithm

$$\log_b x = y \equiv b^y = x$$

Special logarithms:

- log x is often used to mean $\log_{10}$ (the "common log"), but in 812 and more generally, it is increasingly common for this to imply $\log_e$ or a natural log
- ln x implies a base e. e is a mathematical constant, also known as *Euler's number*, roughly equivalent to 2.71828. $\ln(x)$ is known as a "natural log."

Logarithms

Key properties for manipulating logarithms:

Log of 0	$\log(1) = 0$
Multiplication	$\log(x * y) = \log(x) + \log(y)$
Division	$\log(\frac{x}{y}) = \log(x) - \log(y)$
Exponentiation	$\log(x^a) = a * \log(x)$
Basis	$\log_b(b^x) = x$

All of these properties apply to all logarithms, regardless of their base.

Logarithms

Let $f(x) = \log_{10}(x)$:

$$\begin{aligned} f(1000) &= \log_{10}(1000) \\ &= 3 \end{aligned}$$

Let $g(x) = \ln(\frac{x}{e})$:

$$\begin{aligned} g(4) &= \ln(\frac{4}{e}) \\ &= \ln(4) - \ln(e) \\ &\approx 1.386 - 1 \\ &\approx 0.386 \end{aligned}$$

Let $h(x) = \log_2(x^5)$:

$$\begin{aligned} h(4) &= \log_2(4^5) \\ &= 5 * \log_2(4) \\ &= 5 * 2 \\ &= 10 \end{aligned}$$

Logarithms

Solve:

- $\log_2(4^3)$
- $\ln\left(\frac{x}{y} * q^4 * e\right)$

Factorials

The *factorial* operator (denoted with an exclamation mark !) returns the product of an integer with all lesser integers.

Factorial

$$x! = x * (x-1) * (x-2) * ... * 2 * 1$$

Special factorials:

$$1! = 1$$
$$0! = 1$$

Factorials: Examples

Let $f(x) = x!$:

$$f(6) = 6 * 5 * 4 * 3 * 2 * 1$$
$$= 720$$

Combinations

Combinations calculate the number of unique sub-groups of size k that could be selected from a larger group n. They are denoted with large parentheses and read 'n choose k'.

Combination

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Key properties:

- Groups are unordered - any group with the same individuals is the same as any other group with those individuals, regardless of the order in which they were selected.
- There is no replacement - once an individual is picked for the group, they cannot be picked again.

Combinations

Let $f(x) = \binom{10}{x}$:

$$\begin{aligned} f(6) &= \frac{10!}{6! * 4!} \\ &= \frac{10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1}{(6 * 5 * 4 * 3 * 2 * 1) * (4 * 3 * 2 * 1)} \\ &= \frac{3628800}{720 * 24} \\ &= 210 \end{aligned}$$

Let $g(n, k) = \binom{n}{k}$:

$$\begin{aligned} g(5, 2) &= \frac{5!}{2! * 3!} \\ &= \frac{120}{12} \\ &= 10 \end{aligned}$$

Combinations

Solve or simplify:

- $\binom{y}{3}$
- $\binom{p}{q}$ where $p = q + 2$