Math Camp 2012: Algebra, Notation and Functions

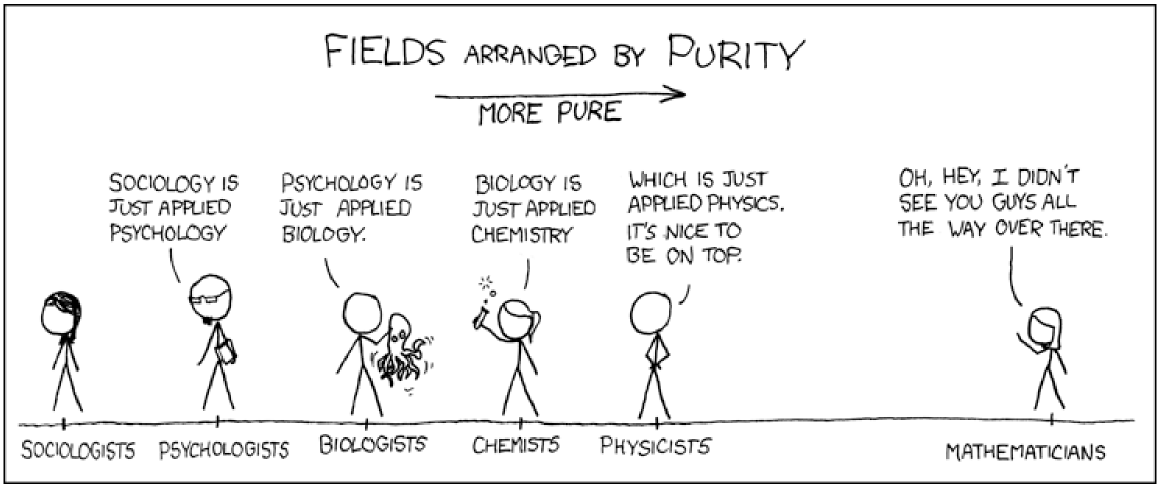
August 2012

Today's Topics*

- Introduction:
  - Why are you doing this to us?
  - Tips and suggestions
- Notation and algebra refresher
  - Common math notation
  - Exponents
  - Logarithms
  - Euler's constant
  - Order of operations
  - Cartesian coordinates/coordinate geometry
  - Lines
- Functions
  - Examples
  - Domain and Range
  - Inverse functions
  - Roots
- Neighborhoods and Sets

*These notes were prepared by Jacob Montgomery. Much of the material and examples for this lecture are taken from Harvard "Math (P)refresher" class notes whose authors are listed here and the Math Camp lecture notes prepared by Rebecca Nugent at the University of Washington.

1 Introduction



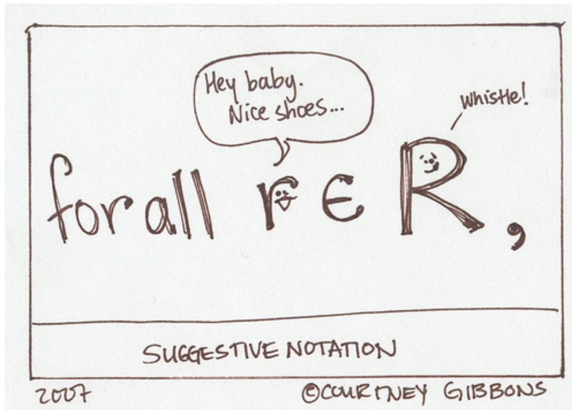
1.1 Why math?

- Science requires precision.
  - Deductive methods (e.g., game theory)
  - Inductive methods (e.g., statistics)
- You like learning things.
- You want to have mad skills.
  - "You know, like nunchuk skills, bow hunting skills, computer hacking skills... Girls only want boyfriends who have great skills." – Napolean Dynamite
  - Graduate school is all about research skills: writing, reading, quantitative analysis, formal and computational modeling, interviewing, research design, archival work, survey design, data management, time management, presentational skills, ...
  - Math is a skill that feeds into several aspects of research and that helps you answer the questions you care about. Having skills will make you a better political scientist and make you more attractive to your future employers.
- Go open any leading political science journal.
  - You need math skills to *read* contemporary political science. Without basic math skills you cannot fully engage the literature.
  - You improve your chances of publishing in these journals if you have a larger skill set.

1.2 Zen and the art of methods training

- Work from the assumption that you can learn this material. If you don't trust yourself, then trust us. You would not have been admitted if you could not cut it.
- Look at the person to the left of you and the person on the right of you. These are your allies. Graduate school (and methods training in particular) is a positive-sum game. No one cares if you are better or worse than your classmates. No one cares about your grades in your classes. The only thing your professors and future employers care about is that you understand these materials and can use them in your own research.
- Graduate school is your job. That's why you get paid to do it. Take a professional attitude towards your job.
- Your job is *not* about getting good grades. If you struggle in your methods classes – no one cares. If you learn the material without even trying – no one cares. No one has ever gotten a Ph.D for doing well in their classes. For the first couple of years your job is to acquire skills and knowledge. That means that you need to know more about game theory, statistics, etc. at the end of your classes than at the beginning. Work hard – try not to stress about external validation.
- If this was easy to do, then your pay would be terrible (or ... more terrible). Don't expect all of this stuff to come easy. Just keep working, and ask for help when you need it.
- Ask questions. No, really. Ask questions. Ask lots of them to lots of people. Of course you don't know how to do things. That's why you are going to school for 5 years.
- This is going to be a *very* long week if you do not engage. No one wants to see me do a dramatic reading of a linear algebra book. Actively engage your textbook. Question the relevance of the material. Demand clearer explanations. One of the skills you need is the ability to follow a lecture, find flaws, and ask on-point questions. Start now.
- Go to the gym. Take up knitting. Find a bar. If you feel like you don't have time for any of that, it means you really *need a hobby.*

2 Notation and algebra



2.1 Common math notation

- $a,\ b,\ c,\ d$: Real numbers
  - Examples: 4, $\sqrt{2}$, $\frac{2}{3}$, 3.14159265
  - The set of real numbers is denoted $\mathbf{R}$ or $\mathbf{R}^1$ and includes any number ranging from $-\infty$ to $+\infty$.
  - You will often see the expression $a \in \mathbf{R}^1$, which means that a is a real number. More correctly, it means that a is in the set of real numbers.
- $i,\ j,\ k,\ l$: Integers (whole numbers)
  - Examples: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
  - The set of integers is denoted $\mathcal{I}$. Positive integers are denoted $\mathcal{I}^+$. Negative integers are $\mathcal{I}^-$.
- $x,\ y,\ z$: Variables that can take on varying values.
- $f,\ g,\ h$: Functions of some variable (e.g., $f(x)$)
- n: Commonly denotes some non-specified positive integer. For example, it usually represents the sample size.
- These are often used in combination:
  - Indexing: $a_1 + a_2 + \ldots + a_i$
  - Functions: $f(x) = a + bx$
- Special combinations:
  - Summation: $\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n$

1. $\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$
2. $\sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$
3. $\sum_{i=1}^{n} c = nc$
4. $\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
5. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
6. $\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2 = (\sum_{i=0}^{n} i)^2$

– Product: $\prod_{i=1}^{n} x_i = x_1 \times x_2 \times \ldots \times x_n$

1. $\prod_{i=1}^{n} cx_i = c^n \prod_{i=1}^{n} x_i$
2. $\prod_{i=1}^{n}(x_i + y_i) =$ a mess
3. $\prod_{i=1}^{n} c = c^n$

2.2 Exponents

- $a^3 = a \times a \times a$: "a to the power of 3" or "a to the 3rd".
- $a^n = a \times a \times \ldots \times a = \prod_{i=1}^{n} a$

Some basic rules that apply at all times:

1. $a^1 = a$
2. $a^0 = 1^\dagger$
3. $(a^k)^l = a^{kl}$
4. $(ab)^k = a^k \times b^k$
5. $(\frac{a}{b})^k = \frac{a^k}{b^k}$
6. $a^{-1} = \frac{1}{a}$
7. $a^{\frac{1}{2}} = \sqrt{a}$
8. $\sqrt[k]{a} = a^{\frac{1}{k}}$

Rules that are true only when $a = b$ (each part has the same "base"):

† Really? Doesn't this seem a bit strange? Ask about it.

- $a^k \times b^l = a^{k+l} = b^{k+l}$
- $\frac{a^k}{b^l} = a^{k-l} = b^{k-l}$

2.3 Logarithms

- Logarithms are the power required to raise a base to a given number: [‡]

$$y = log_a(x) \quad \Longrightarrow \quad a^y = x$$

- It is helpful to think of the log as the inverse of exponential functions.

$$log_a(a^x) = x$$

$$a^{log_a(x)} = x$$

- In almost all cases you will see logarithms are base 10 or base e, where e is $euler'sconstant$.

  1. Base 10: $b = log_{10}(a) \iff 10^b = a$
     The base 10 logarithm is often simply written as "$log(x)$" with no base denoted.
  2. Base e: $y = log_e(x) \iff e^y = x$
     The base e logarithm is referred to as the "natural" logarithm and is usually written as $ln(x)$.
  3. In statistics, you will almost always be working with $ln(x)$.

- Some properties of logs:

  1. $log_c(ab) = log(a) + log(b)$ Why?:

$$\begin{aligned} x = log_c(ab) &\iff c^x = ab \\ \Longrightarrow \quad c^{x_1+x_2} &= ab, \text{ where } x = x_1 + x_2 \\ \Longrightarrow \quad c^{x_1}c^{x_2} = ab \quad &\Longrightarrow \quad c^{x_1} = a; c^{x_2} = b \\ \Longrightarrow \quad x_1 &= log_c(a); x_2 = log_c(b) \\ \Longrightarrow \quad log_c(ab) &= log_c(a) + log_c(b) \end{aligned}$$

  2. $log(1/x) = -log(x)$
  3. $log(x/y) = log(x) - log(y)$
  4. $log(x^y) = ylog(x)$
  5. $log(1) = 0$
  6. $\ln(e) = 1$

- You can switch bases as necessary using the following equation: $log_b(x) = \frac{log_a(x)}{log_a(b)}$

[‡]Do you really need to know this? When will you ever use this information?

- If you see a product of an exponent, you might want to use a log to change it into sums:

$$ln(\prod_{i=1}^{n} ae^{x_i}) = ln(ae^{x_1} \cdot ae^{x_2} \cdot ... \cdot ae^{x_n}) = ln(a^n \cdot e^{\sum x_i})$$

$$ln(a^n) + ln(e^{\sum x_i}) = nln(a) + \sum x_i ln(e) = nln(a) + \sum x_i$$

2.4 Euler's constant

- Euler's constant is denoted e and is equal to $2.71828\ldots$.
- Like π, it appears in a surprising number of places in math, probability, and statistics.
  - It is the only number such that the derivative of e^x is equal to itself.
  - $e = \lim_{x \to \infty} (1 + \frac{1}{x})^x$
  - $e = \sum_{x=0}^{\infty} \frac{1}{x!}$

2.5 Order of operations

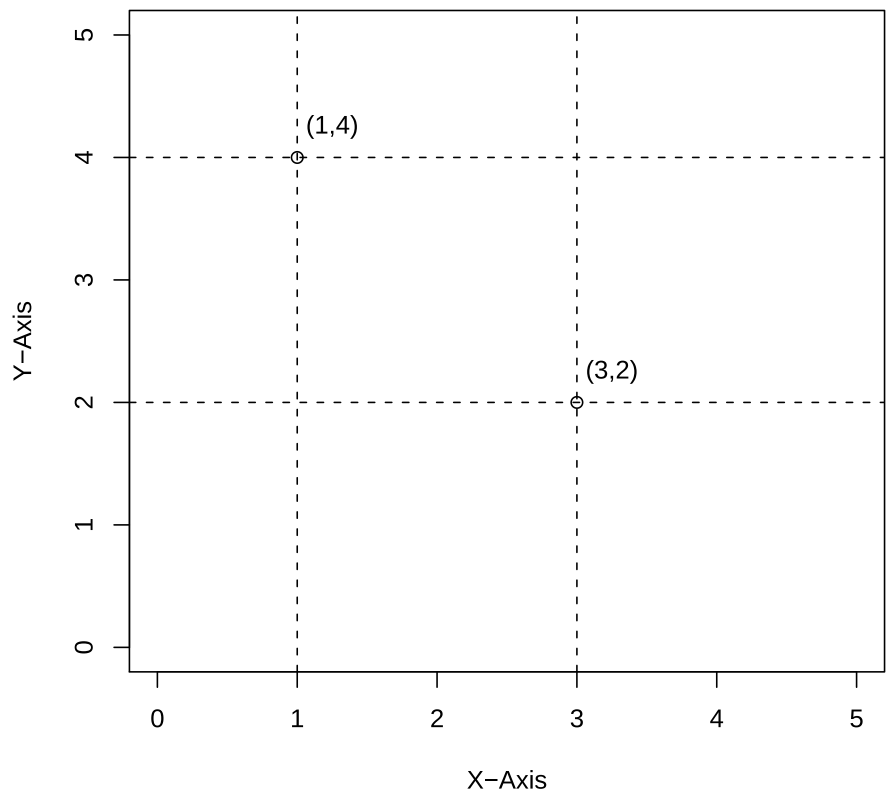
- **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally
- **P**arentheses **E**xponents **M**ultiplication **D**ivision **A**ddition **S**ubtraction
- Example:

$$((1 + 3)^3)^2 \times 2 + 5 = (4^3)^2 \times 2 + 5$$

$$= (64)^2 \times 2 + 5 = 4096 \times 2 + 5 = 8197$$

2.6 Coordinate Geometry

- Pairs of numbers represent points: $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$
- The plot has two axes:
  - x-axis is the horizontal axis at the pottom of the plot
  - y-axis is the verticle line on the left hand of the plot
- To plot some point (x_i, y_i):
  - Move along the x-axis to the point x_i from zero
  - Move up or down to the position of y_i

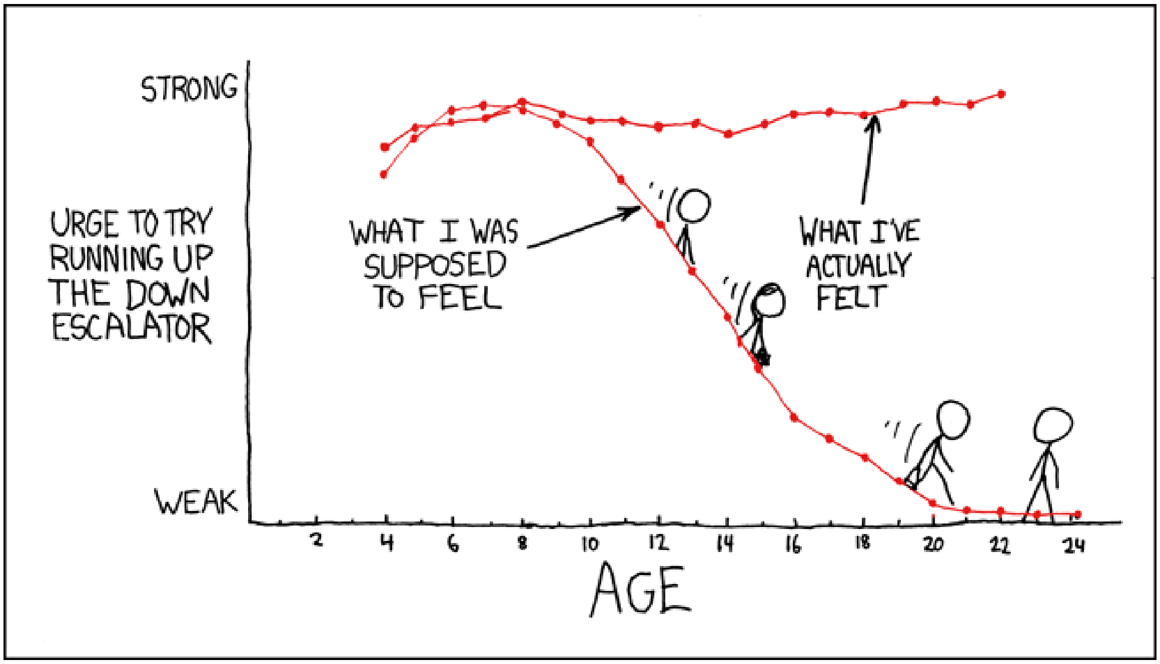


2.7 Lines

- The equation for a line will most often be written as:
  1. $y = mx + b$
  2. $y = a + bx$
  3. $y = \beta_0 + \beta_1 x$
- Hardly a day will go by for the next five years when you do not encounter (implicitly or explicitly) an equation that looks like this.
- From any two points we can calculate the "slope" (e.g., m) and the "intercept."
  - The slope is the ratio of the differences in the two y pairs and the two x pairs (i.e., rise over run)
  - The intercept is the value of the function where $x = 0$. In other words, it is the point where the line hits the y-axis.
- **Example:** $[2, 1], [3, 5]$
  - $m = \frac{5-1}{3-2} = 4$

$$-\ 1 = 4(2) + b \iff b = 1 - 8 = -7$$

3 Functions



3.1 $\mathbf{R}^1$ and $\mathbf{R}^n$

- $\mathbf{R}^1$ is the set of all **real** numbers extending from $-\infty$ to $+\infty$ — i.e., the real number line.
- $\mathbf{R}^n$ is an n-dimensional space (often referred to as Euclidean space), where each of the n axes extends from $-\infty$ to $+\infty$.
- Examples:
    1. $\mathbf{R}^1$ is a line.
    2. $\mathbf{R}^2$ is a plane.
    3. $\mathbf{R}^3$ is a 3-D space.
    4. $\mathbf{R}^4$ could be 3-D plus time.
- Points in $\mathbf{R}^n$ are ordered n-tuples, where each element of the n-tuple represents the coordinate along that dimension.

3.2 Interval Notation for $\mathbf{R}^1$

- **Open interval:** $(a, b) \equiv \{x \in \mathbf{R}^1 : a < x < b\}$

- **Closed interval:** $[a, b] \equiv \{x \in \mathbf{R}^1 : a \leq x \leq b\}$
- Half open, half closed: $(a, b] \equiv \{x \in \mathbf{R}^1 : a < x \leq b\}$

3.3 Introduction to Functions

- A **function** (in $\mathbf{R}^1$) is a rule or relationship or mapping or transformation that assigns one and only one number in $\mathbf{R}^1$ to each number in $\mathbf{R}^1$.
- Mapping notation examples
  1. Function of one variable: $f : \mathbf{R}^1 \rightarrow \mathbf{R}^1$
  2. Function of two variables: $f : \mathbf{R}^2 \rightarrow \mathbf{R}^1$
- Examples:
  1. $f(x) = x + 1$
     For each x in $\mathbf{R}^1$, $f(x)$ assigns the number $x + 1$.
  2. $f(x, y) = x^2 + y^2$
     For each ordered pair (x, y) in $\mathbf{R}^2$, $f(x, y)$ assigns the number $x^2 + y^2$.
- Often use one variable x as input and another y as output.
  Example: $y = x + 1$
- Input variable also called **independent** variable. Output variable also called **dependent** variable.

3.4 Domain and Range

- Some functions are defined only on proper subsets of $\mathbf{R}^n$.[§]
- **Domain**: the set of numbers in X at which $f(x)$ is defined.
- **Range**: elements of Y assigned by $f(x)$ to elements of X, or

$$f(X) = \{y : y = f(x), x \in X\}$$

  Most often used when talking about a function $f : \mathbf{R}^1 \rightarrow \mathbf{R}^1$.
- Examples:
  1. $f(x) = \frac{3}{1+x^2}$
     Domain $X = \mathbf{R}^1$
     Range $f(X) = (0, 3]$

[§]What is the point of this?

2. $f(x) = \begin{cases} x+1, & 1 \le x \le 2 \\ 0, & x = 0 \\ 1-x & -2 \le x \le -1 \end{cases}$

   Domain $X = [-2,-1] \cup \{0\} \cup [1,2]$
   Range $f(X) = [2,3] \cup \{0\}$

3. $f(x) = 1/x$

   Domain $X = \mathbf{R}^1 - \{0\}$
   Range $f(X) = \mathbf{R}^1 - \{0\}$

4. $f(x,y) = x^2 + y^2$

   Domain $X = \mathbf{R}^2$
   Range (or Image) $f(X,Y) = \mathbf{R}^1_+$

3.5 Some General Types of Functions

- **Monomials**: $f(x) = ax^k$
  a is the coefficient. k is the degree.
  Examples: $y = x^2$, $y = -\frac{1}{2}x^3$

- **Polynomials**: sum of monomials.
  Examples: $y = -\frac{1}{2}x^3 + x^2$, $y = 3x + 5$
  The degree of a polynomial is the highest degree of its monomial terms. Also, it's often a good idea to write polynomials with terms in decreasing degree.

- **Linear**: polynomial of degree 1.
  Example: $y = mx + b$, where m is the slope and b is the y-intercept.

- **Nonlinear**: anything that isn't constant or polynomial of degree 1.
  Examples: $y = x^2 + 2x + 1$, $y = \sin(x)$, $y = \ln(x)$, $y = e^x$

- **Quadratic**: a second degree polynomial function.
  Example: $f(x) = ax^2 + bx + c$

- Always, always, always, graph your function.

3.6 Inverse Functions

- Sometimes we're given a function $y = f(x)$ and we want to find how x varies as a function of y. That is, we are trying to "solve for" x.

- If f is a one-to-one mapping, then it has an inverse. This will usually be denoted as $f^{-1}(x)$. That means that $f^{-1}(f(x)) = x$.

- A good way to do this is to use algebra to isolate x (your independent variable in $f(x)$) on one side of the equation.

- Examples: (we want to solve for x)
  1. $y = 3x + 2 \implies y - 2 = 3x \implies x = \frac{1}{3}(y - 2)$
  2. $y = 3x - 4z + 2 \implies y + 4z - 2 = 3x \implies x = \frac{1}{3}(y + 4z - 2)$
  3. $y = e^x + 4 \implies y - 4 = e^x \implies \ln(y - 4) = \ln(e^x) \implies x = \ln(y - 4)$
- Note: the inverse may not not exist. This is especially likely in non-linear functions.
- Example: We're given the function $y = x^2$ (a parabola). Solving for x, we get $x = \sqrt{y}$ and $x = -\sqrt{y}$ — for each value of y, there are two values of x.

3.7 Roots

- You are going to be spending a lot of time finding **roots** of functions: those values where $f(x) = 0$.
  - Decision theory/game theory
  - Dynamic systems
  - Maximum likelihood
- Procedure: Given $y = f(x)$, set $y = 0$. Solve for x.
- **x-intercepts**: Where does the line $f(x) = a + bx$ cross the x-axis?
  $a + bx = 0 \implies a = -bx \implies x = -\frac{a}{b}$
- **The "quadratic equation"**: What is the root of $f(x) = ax^2 + bx + c$
  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- **Factoring** $f(x) = x^2 + 3x - 4 = 0 \implies (x - 1)(x + 4) = 0 \implies x = \{1, -4\}$.
  - **FOIL** (**F**irst **O**utside **I**nside **L**ast)
  - The middle term (e.g., $3x$) is the sum of the constants. The final term is the product.
  - If you have $ax^2 - c$, then the factors are of the form $(\sqrt{a}x + \sqrt{c})(\sqrt{a}x - \sqrt{c})$
  - If you cannot figure out how to factor, you may need to "complete the square." (Google it!)

4 Neighborhoods: Intervals, Disks, and Balls

- In many areas of math, we need a formal construct for what it means to be "near" a point $\mathbf{c}$ in $\mathbf{R}^n$. This is generally called the **neighborhood** of $\mathbf{c}$ and is represented by an open interval, disk, or ball, depending on whether $\mathbf{R}^n$ is of one, two, or more dimensions, respectively. Given the point c, these are defined as

1. ϵ-**interval** in $\mathbf{R}^1$: $\{x : |x - c| < \epsilon\}$
   The open interval $(c - \epsilon, c + \epsilon)$.
2. ϵ-**disk** in $\mathbf{R}^2$: $\{\mathbf{x} : ||\mathbf{x} - \mathbf{c}|| < \epsilon\}$
   The open interior of the circle centered at $\mathbf{c}$ with radius ϵ.
3. ϵ-**ball** in $\mathbf{R}^n$: $\{\mathbf{x} : ||\mathbf{x} - \mathbf{c}|| < \epsilon\}$
   The open interior of the sphere centered at $\mathbf{c}$ with radius ϵ.

5 Sets, Sets, and More Sets

- **Interior Point**: The point $\mathbf{x}$ is an interior point of the set S if $\mathbf{x}$ is in S and if there is some ϵ-ball around $\mathbf{x}$ that contains only points in S. The **interior** of S is the collection of all interior points in S. The interior can also be defined as the union of all open sets in S.
  Example: The interior of the set $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 < 4\}$.
- **Boundary Point**: The point $\mathbf{x}$ is a boundary point of the set S if every ϵ-ball around $\mathbf{x}$ contains both points that are in S and points that are outside S. The **boundary** is the collection of all boundary points.
  Example: The boundary of $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 = 4\}$.
- **Open**: A set S is open if for each point $\mathbf{x}$ in S, there exists an open ϵ-ball around $\mathbf{x}$ completely contained in S.
  Example: $\{(x, y) : x^2 + y^2 < 4\}$
- **Closed**: A set S is closed if it contains all of its boundary points.
  Example: $\{(x, y) : x^2 + y^2 \leq 4\}$
- Note: a set may be neither open nor closed.
  Example: $\{(x, y) : 2 < x^2 + y^2 \leq 4\}$
- **Complement**: The complement of set S is everything outside of S.
  Example: The complement of $\{(x, y) : x^2 + y^2 \leq 4\}$ is $\{(x, y) : x^2 + y^2 > 4\}$.
- **Closure**: The closure of set S is the smallest closed set that contains S.
  Example: The closure of $\{(x, y) : x^2 + y^2 < 4\}$ is $\{(x, y) : x^2 + y^2 \leq 4\}$
- **Bounded**: A set S is bounded if it can be contained within an ϵ-ball.
  Examples: Bounded: any interval that doesn't have ∞ or $-\infty$ as endpoints; any disk in a plane with finite radius. Unbounded: the set of integers in $\mathbf{R}^1$; any ray.
- **Compact**: A set is compact if and only if it is both closed and bounded.